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A note on the distributions in quantum mechanical systems

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Abstract. In this paper, we study the distributions and the affine distributions of the quantum mechanical system. Also, we discuss the controllability of the quantum mechanical system with the related question concerning the minimum time needed to steer a quantum system from a unitary evolution $U(0) = I$ of the unitary propagator to a desired unitary propagator U_f . Furthermore, the paper introduces a description of a $\mathfrak{k} \oplus \mathfrak{p}$ sub-Finsler manifold with its geodesics, which equivalents to the problem of driving the quantum mechanical system from an arbitrary initial state $U(0) = I$ to the target state U_f , some illustrative examples are included. We prove that the Lie group G on a Finsler symmetric manifold G/K can be decomposed into KAK .

1. Introduction

The research field that links geometric theories and quantum mechanics is yet very active research. Geometric quantum mechanics is an attempt to reformulate quantum mechanics theory in the context of differential geometry. Differential geometry enters the quantum mechanics theory in at least two ways, the space of states and the space of time evolutions are the differential manifolds, see [6] for an outstanding review of geometric quantum mechanics. In the last two decades, there has been a resurgent interest in the quantum mechanics systems with sub-Riemannian (sub-Finsler) geometries. Excellent work can be found in [1, 12, 20, 21]. The objective of the controllability of quantum systems is to drive a dynamic system from an arbitrary initial state into a desired final state [9, 13], the interesting question about the controllability of quantum systems is steering the system in a minimum time [20, 21].

Let M be a smooth n -dimensional manifold, and $k \leq n$. Suppose that for any $x \in M$, we specify a subspace of the tangent space of dimension k , i.e. $\mathcal{D}_x \subset T_x M$. Thus, for a neighbourhood $\mathcal{U}_x \subset M$ of x there exist k linearly independent smooth vector fields X_1, \dots, X_k . Moreover, for each point $y \in \mathcal{U}_x$,

$$\mathcal{D}_y = \text{span}\{X_1(y), \dots, X_k(y)\}.$$

Let \mathcal{D} refer to the collection of all the linear k -dimensional subspaces \mathcal{D}_x for all $x \in M$, then \mathcal{D} is a *distribution* of rank k , or a *k-plane distribution* on M . The set of smooth vector fields $\{X_1, \dots, X_k\}$ is said to be a local basis of \mathcal{D} . Moreover, if now X_1, \dots, X_k, Y are k linearly independent (pointwise) vector fields on M , and the points of the distribution as a subset $\mathcal{D}_x \subset T_x M$ are given by

$$\mathcal{D}_x^+ = Y(x) + \mathcal{D}_x,$$



in other words,

$$\mathcal{D}_x^+ = Y(x) + \text{span}\{X_1(x), X_2(x), \dots, X_k(x)\}.$$

Then we label \mathcal{D}_x^+ as an *affine distribution* of rank $k \leq n$, simply denoted by \mathcal{D}_f . The affine distribution \mathcal{D}_f describes a control system since it catches all the possible directions of motion available at a definite point $x \in M$, up to a parametrization by control inputs (for more details, see [23]). In our recent work [4], we studied in detail the distribution of the tangent and cotangent bundles. We also investigated the sub-Finslerian setting with nonholonomic mechanics. The study of affine distributions is more recent and less extensive, Clelland et al [12] and Pappas et al [23] provided a detailed exposition of affine distribution in the quantum mechanics systems.

This paper is organized as follows. In section 2, we have compiled some basic facts on the geometric description of the sub-Finsler manifolds and algebraic description of Lie groups and Lie algebras in which we introduce some notions which will play an essential role in this paper. Next, the third section is devoted to the study of a homogeneous sub-Finsler manifold and a $\mathfrak{k} \oplus \mathfrak{p}$ sub-Finsler manifold. Also, we show the decomposition of the Lie group G as KP if G and its subgroup K are connected such that $P = \exp(\mathfrak{p})$, and as KAK on a Finsler symmetric manifold G/K . Then, in section 4, we review in more general settings the quantum mechanics system with some examples. After that, section 5, is intended to motivate our investigation of the controllability of the quantum mechanical system in such a way we devoted to the study the controllability of quantum mechanical system in three cases: Firstly, the system has no drift hamiltonian. Secondly, a quantum system has a drift. Thirdly, the controllability of linear systems. It is also shown that each case mentioned above has a different distribution and how these distributions linked by the sub-Riemannian (sub-Finsler) geometries.

2. Preliminaries

Here, we define the geometric and algebraic notions that will be used throughout the paper.

2.1. Geometric description of the sub-Finsler manifolds

Definition 1. A *sub-Finsler metric* is a generalisation of a sub-Riemannian metric on an n -dimensional smooth manifold M . Let $F : \mathcal{D} = \mathcal{D} \setminus \{0\} \rightarrow [0, +\infty)$ be a non-negative function on a distribution \mathcal{D} of rank $k \leq n$, which is a smooth (vector) subbundle $\mathcal{D} \subset TM$ of the tangent bundle.

F is called a sub-Finsler metric in M if it satisfies the following conditions:

- (1) Smoothness: F is C^∞ on $\tilde{\mathcal{D}}$;
- (2) Positive homogeneity: $F(\lambda v) = \lambda F(v)$ for all $v \in \tilde{\mathcal{D}}$ and $\lambda > 0$;
- (3) Strong convexity: For any non-zero vector $v \neq 0$, the Hessian matrix formed by following

$$g_{ij}(x, v) = \frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v), \quad \text{for all } x \in M, v \in \tilde{\mathcal{D}}_x,$$

is positively definite. Equivalently, the corresponding indicatrix

$$\Sigma_x = \{v \mid v \in \mathcal{D}_x, F(x, v) = 1\}$$

is strictly convex.

A differentiable manifold M equipped with a sub-Finsler metric F defined on a subbundle \mathcal{D} of rank k of a tangent bundle is called a *sub-Finsler manifold* or sub-Finsler space denoted by (M, \mathcal{D}, F) .

The Hessian matrix for a sub-Finsler metric F defines an inner product at the non-zero vector $v \neq 0 \in \tilde{\mathcal{D}}_x$, namely,

$$\langle u, w \rangle_v = g_{ij}(x, v)u^i w^j = \frac{\partial^2 F^2(v + tu + sw)}{\partial t \partial s} \Big|_{t,s=0},$$

for any $u = u^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$ in $\tilde{\mathcal{D}}_x$, the above inner product (bilinear form) on $\tilde{\mathcal{D}}_x$ is called the *fundamental tensor*. We note that if $\mathcal{D} = TM$, then this is the usual definition of a Finsler manifold [5].

Let (M, \mathcal{D}, F) be a sub-Finsler manifold and $x, y \in M$. A piecewise smooth curve (trajectory) $\gamma : [0, T] \rightarrow M$ is said to be *horizontal*, or *admissible* if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0, T]$, that is $\gamma(t)$ is tangent to \mathcal{D} . The length of a piecewise smooth horizontal curve γ is defined by

$$\ell(\gamma) = \int_0^T F(\gamma(t), \dot{\gamma}(t)) dt. \quad (1)$$

We can use the length to define the sub-Finslerian distance $d(x, y)$ between two points $x, y \in M$ as in Finsler geometry:

$$d(x, y) = \inf\{\ell(\gamma) \mid \gamma \text{ a piecewise smooth horizontal curve joining } x \text{ to } y\}.$$

Furthermore, the sub-Finslerian distance is infinite, i.e. $d(x, y) = \infty$ if there are no horizontal curves between x and y .

Definition 2. The horizontal curve $\gamma : [0, T] \rightarrow M$ is called a *length minimizing* (or simply a *minimizing*) geodesic if it realizes the distance between its end points, that is, $\ell(\gamma) = d(\gamma(0), \gamma(T))$.

Chow's theorem, also called the Chow-Rashevskii theorem, [11, 19], answered a fundamental question of how we know these geodesic exist. More precisely, given two points x and y in a sub-Finslerian manifold, is there a geodesic joins them. The answer for the above question depends whether the distribution is a bracket generating or not. To be more specific, if we have a bracket generating distribution on a connected manifold M then any two points in M can be joined by a horizontal path. A distribution \mathcal{D} is called *bracket generating* if any local frame X_i of \mathcal{D} , together with all of its iterated Lie brackets spans the whole tangent bundle TM , see [19].

2.2. Algebraic description

Throughout this the paper, we assume G a Lie group, which is a connected group (smooth manifold) with the properties that the map of the group multiplication and the inverse map are both smooth maps. If G is a Lie group, then $e \in G$ will be the group identity element (we use I to denote the identity matrix when working with the matrix representation of the group) and denote by $\mathfrak{g} = T_e G$ or $\text{Lie}(G)$ to be the Lie algebra of G .

Let K be a (smooth submanifold) compact closed subgroup of a Lie group G . Cartan's theorem asserts that every closed subset K of the Lie group G , which is its subgroup, is the Lie subgroup of the Lie group G . Suppose $\text{Lie}(K) = \mathfrak{k}$ represent the Lie algebra of K , then the direct sum decomposition of subalgebras \mathfrak{k} and \mathfrak{p} , denoted by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \quad (2)$$

We call the pair $(\mathfrak{g}, \mathfrak{k})$ *symmetric pair* if there exists a Lie algebra automorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, such that $\theta^2 = I$, where θ is the Cartan involution, and which has \mathfrak{k} as its 1-eigenspace. The direct sum decomposition (2) well known as *Cartan decomposition* if $\mathfrak{p} \subseteq \mathfrak{k}^\perp$ and $\mathfrak{k} \subseteq \mathfrak{p}^\perp$ with

respect to the metric induced by the killing form (7). However, if \mathfrak{g} a semisimple Lie algebra, namely, a Lie algebra \mathfrak{g} is semisimple if the only Abelian ideal in \mathfrak{g} is $\{0\}$ or the killing form is non-degenerate on \mathfrak{g} , then $\mathfrak{p} = \mathfrak{k}^\perp$ and $\mathfrak{k} = \mathfrak{p}^\perp$, i.e. \mathfrak{g} is the orthogonal sum of \mathfrak{k} and \mathfrak{p} . Moreover, Cartan decomposition must satisfy the commutation relations, namely

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \quad (3)$$

In addition, a pair $(\mathfrak{k}, \mathfrak{p})$ is called a *Cartan pair* of \mathfrak{g} .

As an example for the Cartan decomposition one can take the Lorentz group $SO_0(n, 1)$, precisely, its Lie algebra $\mathfrak{so}(n, 1)$. The Lie algebra of the Lorentz group can be easily obtained by calculating the tangent vectors to curves $t \mapsto A(t)$, where $A(t)$ is a matrix, on $SO_0(n, 1)$ through the identity I . Since $A(t)$ satisfies

$$A^\top J A = J, \quad J = I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

by differentiate and use the fact that $A(0) = I$, we obtain

$$A'^\top J + J A' = 0.$$

Consequently

$$\mathfrak{so}(n, 1) = \{A \in M_{n+1}(\mathbb{R}) \mid A^\top J + J A = 0\}.$$

It follows that $J A$ is skew-symmetric accordingly to $J = J^\top$, and so

$$\mathfrak{so}(n, 1) = \left\{ \begin{pmatrix} B & \lambda \\ \lambda^\top & 0 \end{pmatrix} \in M_{n+1}(\mathbb{R}) \mid \lambda \in \mathbb{R}^n, B^\top = -B \right\}.$$

It is appropriate to write $A^\top J + J A$ equivalent with $A^\top = -J A J$ since $J^2 = I$. Note that we can write each matrix $A \in \mathfrak{so}(n, 1)$ uniquely as

$$\begin{pmatrix} B & \lambda \\ \lambda^\top & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \lambda \\ \lambda^\top & 0 \end{pmatrix},$$

such that $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is skew-symmetric and $\begin{pmatrix} 0 & \lambda \\ \lambda^\top & 0 \end{pmatrix}$ is symmetric, further, both matrices are still in $\mathfrak{so}(n, 1)$. Therefore, It is normal to identify that

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \mid B \in M_n(\mathbb{R}), B^\top = -B \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \lambda \\ \lambda^\top & 0 \end{pmatrix} \mid \lambda \in \mathbb{R}^n \right\}.$$

One can see at once clearly that both \mathfrak{k} and \mathfrak{p} are subspaces (as vectors) of $\mathfrak{so}(n, 1)$. However, on the one hand, \mathfrak{k} is a Lie subalgebra isomorphic to $\mathfrak{so}(n)$, on the other hand, since \mathfrak{p} is not closed under the Lie bracket, so it is not a Lie subalgebra of $\mathfrak{so}(n, 1)$. Nevertheless, (3) still valid. Therefore, the direct sum decomposition for the Lie algebra $\mathfrak{so}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition.

A *maximal Abelian subalgebra*, denoted by \mathfrak{h} , of the Lie subalgebra \mathfrak{p} is called a *Cartan subalgebra* of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$, and the common dimension of all the maximal subalgebras

is called the *rank* of the decomposition (2). As an example, suppose that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition such that $\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{k} = \mathfrak{so}(n)$ and

$$\mathfrak{p} = \{iS | S \text{ is } n \times n \text{ traceless real symmetric matrix}\},$$

then

$$\mathfrak{h} = \{iD | D \text{ is } n \times n \text{ traceless diagonal matrix}\}$$

is a maximal abelian subalgebra contained in \mathfrak{p} , therefore

$$SU(n) = \{k_1 \exp(a)k_2 | k_1, k_2 \in SO(n), a \in \mathfrak{h}\}.$$

Let us turn to illustrate the action of Lie group G on its Lie algebra \mathfrak{g} by giving a brief exposition of its definition as follows: Let G be a Lie group and \mathfrak{g} its Lie algebra, the automorphism map on G

$$\varphi_g : G \rightarrow G,$$

given by the inner automorphism (conjugation)

$$X \mapsto gXg^{-1}, \text{ for all } X \in \mathfrak{g} \text{ and } g \in G.$$

In addition, if we take the differential of φ at the identity e we get an automorphism of the Lie algebra

$$\text{Ad}_g = (\varphi_{g*})e : \mathfrak{g} \rightarrow \mathfrak{g}.$$

Then for any Lie group G , the *adjoint representation* is defined by

$$\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}).$$

Now, let G be a Lie group with Lie algebra \mathfrak{g} , then the exponential map defined by

$$\exp : \mathfrak{g} \rightarrow G.$$

Given $g \in G$, $X \in \mathfrak{k}$, and consider the one-parameter subgroup $t \mapsto \exp tX$ of K

$$\text{Ad}_g(X) = gXg^{-1} \Big|_{t=0},$$

since $\exp(\text{Ad}_g(X)) = g(\exp X)g^{-1}$, it follows that

$$\text{Ad}_g(\exp tX) = \frac{d}{dt} g(\exp tX)g^{-1} \Big|_{t=0}, \quad (4)$$

such that $XY - YX = [X, Y] = \text{ad}(X)Y$ for any $X, Y \in \mathfrak{g}$. Therefore,

$$\text{ad}_g(X) = \frac{d}{dt} \text{Ad}_g(\exp tX), \quad (5)$$

where $\text{ad}_g(X)$ is the *adjoint representation* of \mathfrak{g} . The proof of the next theorem can be found in [17], Chapter V.

Theorem 3. *If \mathfrak{h} and \mathfrak{h}' are two maximal Abelian subalgebras contained in \mathfrak{p} . Then*

- (1) *There exists an element $X \in \mathfrak{h}$ whose centralizer in \mathfrak{p} is equal to \mathfrak{h} .*
- (2) *There is an element $k \in K$ such that $\text{Ad}_k(\mathfrak{h}) = \mathfrak{h}'$.*
- (3) $\mathfrak{p} = \bigcup_{k \in K} \text{Ad}_k(\mathfrak{h})$.

By the above theorem, the maximal Abelian subalgebras of \mathfrak{p} are all conjugate by Ad_k . Moreover, any two maximal flat totally geodesic submanifolds of M must be congruent under the group of isometries of M [14]. Also, if we assume G a connected and compact Lie group such that \mathfrak{g} its Lie algebra, then Theorem 3 asserts that any two Cartan subalgebras of a complex semisimple Lie algebra are congruent under the group of automorphisms.

Definition 4. Let G be a Lie group and K be a compact closed subgroup of G , the pair (G, K) is called a *Klein geometry* where the (left) coset space G/K is connected. For any Klein geometry (G, K) , the quotient space $M = G/K$ is a smooth manifold of dimension

$$\dim M = \dim G - \dim K.$$

In particular, there is a natural smooth left action of G on M such that the canonical mapping (projection) $\pi : G \rightarrow M$ is smooth, given by

$$L_g : G \times M \rightarrow M, \quad g_1 \cdot g_2 K := (g_1 g_2) K \text{ where } g_1, g_2 \in G,$$

is transitive. Moreover, K acts on G from the right by

$$\nu : G \times K \rightarrow G, \quad g \times k \mapsto gk, \quad g \in G, k \in K.$$

The quotient space M can be done by the equivalence relation

$$g_1 \sim g_2 \Leftrightarrow g_1 = \nu(g_2, k), \quad \text{for some } k \in K.$$

The largest subgroup \mathcal{K} of K that is normal in G is called the *kernel* of a Klein geometry. The Klein geometry (G, K) is called *locally effective* if the kernel is discrete while is called an *effective* if the kernel is trivial, therefore, K is compact. In fact, K is compact by the Ascoli-Arzelà theorem, as it is the stabilizer of the origin in G/K . The kernel of a Klein geometry as defined above is well-defined, and if we have a Klein geometry (G, K) with kernel \mathcal{K} , then there is an associated effective Klein geometry $(G/\mathcal{K}, K/\mathcal{K})$ which gives the same smooth manifold, i.e.

$$M = (G/\mathcal{K}, K/\mathcal{K}) \cong (G/K).$$

An effective Klein geometry is frequently called a *homogeneous manifold*, i.e. if G a Lie group and $K \subset G$ a compact closed subgroup, we call the elements $g_1, g_2 \in G$ a congruent modulo K if $g_1 K = g_2 K$ such that this is an equivalence relation to the equivalence classes being left cosets modulo K and by this equivalence relation the quotient (G/K) is called a *homogeneous space*.

Choose a G -invariant subbundle \mathcal{D} (our hypothesis is always based on \mathcal{D} is a bracket generating) of the tangent bundle TM of $M = G/K$, that is \mathcal{D} of vector subspaces $\mathcal{D}_p \in T_p M$ for every $p \in M$ satisfying

$$(d\mu_g)_p \mathcal{D}_p \subset \mathcal{D}_{\mu_g(p)}, \quad \text{for all } g \in G \text{ and } p \in M,$$

such that the diffeomorphism map $\mu_g : M \rightarrow M$ defined by the induced (isomorphism) tangent map $(d\mu_g)_p : T_p M \rightarrow T_{\mu_g(p)} M$ at $p \in M$ and $g \in G$. If the G -invariant subbundle \mathcal{D} of TM , where $M = G/K$ is a homogeneous space, then under this assumption we can say that \mathcal{D} is analytic (see, [17]). Moreover, assume that $g \in G$, then the diffeomorphism given as

$$\psi_g : G/K \rightarrow G/K, \quad kK \mapsto gkK. \quad (6)$$

In general, K will be called the *isotropy subgroup*.

The choice of the subbundle $\mathcal{D} \subset TM$ can be interpreted as follows. First of all, we can see an one-to-one- correspondence between $\text{Ad}(K)$ -invariant subspaces $V \in \text{Lie}(G)$ containing $\text{Lie}(K)$, and K -invariant subspaces $\mathcal{D}_K \subset T_K M$. Consider a subspace $\mathcal{D}_K \subset T_K M$, consequently, a corresponding V in $\text{Lie}(G)$. Thus, the subbundle \mathcal{D} is defined as

$$\mathcal{D}_{gK} := (d\psi_g)_K \mathcal{D}_K,$$

for any $gK \in G/K$, where ψ_g is given in (6). Since \mathcal{D}_K is K -invariant therefore the above expression does not depend on the representative in gK which means the subbundle is well defined. Let V of $\text{Lie}(G)$ be the subspace that is associated with \mathcal{D}_K satisfies the property that $\text{Lie}(G)$ is the smallest Lie subalgebra of $\text{Lie}(G)$ containing V , then V is called a *bracket generating*. Consider there exist a G -invariant norm on \mathcal{D} if K is compact, such that if we choose a seminorm on V that is $\text{Ad}(K)$ -invariant with the property that the kernel is \mathfrak{k} . The natural projection $\pi : G \rightarrow G/K$ allows a G -invariant norm on \mathcal{D}_K denoted by $\|\cdot\|$. Therefore, we have an induced G -invariant norm on \mathcal{D} as follows

$$\|v\| = \|(d\psi_{g^{-1}})_{gK} v\|, \quad \text{for all } v \in \mathcal{D}_{gK}.$$

The above equation is independent of the choice of the representative in gK because the initial norm is $\text{Ad}(K)$ -invariant, see [15].

3. Homogeneous sub-Finsler manifolds

Definition 5. The triple (M, \mathcal{D}, F) is said to be a *homogeneous sub-Finsler manifold* where $M = G/K = \{gK : g \in G\}$ a smooth manifold such that G/K is a homogeneous space formed by a Lie group G modulo a compact subgroup K . Then we have a bracket generating G -invariant subbundle $\mathcal{D} \subset TM$, and G -invariant norm defined on \mathcal{D} .

One can consider a sub-Finslerian metric by choosing a (semi) sub-Finslerian metric on a bracket generating subspace V that is $\text{Ad}(K)$ -invariant and for which the kernel is \mathfrak{k} . We deduce that any sub-Finsler homogeneous manifold has the form G/K with G a Lie group and K a compact subgroup, according to the fact that the isometry group of a homogeneous manifold is a Lie group.

As in general case, we call the homogeneous sub-Finsler manifold G/K *reductive* if there exists a subspace \mathfrak{p} of the Lie algebra \mathfrak{g} such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where } \text{Ad}(k)\mathfrak{p} \subset \mathfrak{p}, \forall k \in K,$$

and we say that the above decomposition is a *reductive decomposition*. Further to the previous settings and their applications in the sub-Finslerian manifold. A $\mathfrak{k} \oplus \mathfrak{p}$ *sub-Finsler manifold* is a simple real Lie group G of matrices with associated Lie algebra \mathfrak{g} provided with such a sub-Finslerian structure. For instance, on $G = SO_0(n, 1)$, consider the G -invariant subbundle $\mathcal{D} \subset \mathfrak{so}(n, 1)$ endowed with the G -invariant norm defined on \mathcal{D} . In this case, we call the triple $(SO_0(n, 1), \mathcal{D}, \|\cdot\|)$ a $\mathfrak{k} \oplus \mathfrak{p}$ sub-Finsler manifold.

If \mathfrak{k} satisfies the Cartan decomposition conditions, then it is a symmetric subalgebra. The corresponding subgroup K of G is a symmetric subgroup and the coset space G/K a symmetric space. Furthermore, if we have an inner product, given by the (bilinear) killing form

$$B(X, Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(Y)), \quad X, Y \in \mathfrak{g} \text{ and } \text{Tr} \text{ is the trace}, \quad (7)$$

on \mathcal{D} that is invariant under the action of K since it is a compact, then the metric is a sub-Riemannian or a Riemannian (see for instance [19]). Consequently, we have a *Riemannian symmetric space*.

Theorem 6. [10] Let M be a Riemannian symmetric space, G be the largest connected group of isometries of M , K be the stabilizer of the group G at the point $o \in M$. Then any G -invariant intrinsic metric on G/K will be Finsler.

Proposition 7. [23] An affine distribution $\mathcal{D}_f = X + \mathcal{D}$ on M is invariant under a vector field Z if and only if for every $Y \in \mathcal{D}$:

$$[Z, Y] \subseteq Y.$$

Lemma 8. Assume $K \subset G$ where K is a subgroup of the Lie group G with finite centre and their Lie algebra are \mathfrak{k} and \mathfrak{g} , respectively. If G and K are connected, then $G = KP$, such that $P = \exp(\mathfrak{p})$.

Proof. By using the action of the involutive automorphism $\theta : G \rightarrow G$, we can define \mathfrak{k} and \mathfrak{p} (and hence K and P) as follows: For any $X \in \mathfrak{g}$ and consider the +1 and -1 eigenspaces of the derivative $d\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, defined by

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid d\theta(X) = X\},$$

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid d\theta(X) = -X\},$$

such that $\theta \neq I$ and $\theta^2 = I$. Applying the exponential map, these conditions become $\theta(k) = k$, $k \in K$. Further, for each $p \in P$

$$\theta(P) = \theta(\exp \mathfrak{p}) = \exp(\theta(\mathfrak{p})) = \exp(-\mathfrak{p}) = p^{-1},$$

on the involutive automorphism θ on G . Therefore, G satisfies the decomposition $G = KP$. \square

It is easy to see an automorphism on the Lie algebra $\mathfrak{so}(n, 1)$ of the Lorentz Group $SO_0(n, 1)$ that given earlier, as follows

$$\mathfrak{k} = \{A \in \mathfrak{so}(n, 1) \mid d\theta(A) = A\},$$

and

$$\mathfrak{p} = \{A \in \mathfrak{so}(n, 1) \mid d\theta(A) = -A\},$$

where the Cartan involution is $d\theta(A) = -A^\top = JAJ$ and involution $d\theta$ is the derivative at I of the involutive isomorphism θ of the group $SO_0(n, 1)$ also given by $\theta(A) = JAJ$ and $A \in SO_0(n, 1)$.

Theorem 9. Let G/K be a Finsler symmetric manifold. If \mathfrak{h} be a Cartan subalgebra of the pair $(\mathfrak{g}, \mathfrak{k})$ and define $A := \exp(\mathfrak{h}) \subset G$. Then $G = KAK$.

Proof. G/K is a Finsler symmetric manifold by Theorem 6, in Lemma 8, we decompose G to K and P . However, we can decompose K and P further. Consider $(K)_0$ is the identity component of K , by the completeness in the Finslerian metric (for more details we refer the reader to [5], Chapter 6), with the help of Lemma 8 implies that

$$P = \exp(\mathfrak{p}).$$

Moreover, use Theorem 3 (III), we get

$$P = \exp\left(\bigcup_{k \in K} \text{Ad}_k(\mathfrak{h})\right) = \bigcup_{k \in K} \text{Ad}_k(\exp(\mathfrak{h})) = \bigcup_{k \in K} \text{Ad}_k(A) = \text{Ad}_K(A),$$

for a fix cartan subalgebra A . By (4) $P \subset KAK$, i.e. $p = k_1 g k_1^{-1}$ for any $k_1 \in K$ and $g \in A$. Thus $G = KP = KAK = KAK$. \square

Observe that, the space G/K induced by a union of maximal Abelian subgroups $\text{Ad}_k(A)$, called *maximal tori*, [20]. The set

$$\text{Ad}_K(X_d) = \{\text{Ad}_{k_1}(X_d) = k_1 X_d k_1^{-1} | k_1 \in K\} \in \mathfrak{p}$$

is called the *adjoint orbit* of X_d , which representee the directions in G/K , which we can choose to move directly.

Assume we have the following decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{h} \subset \mathfrak{p}$ represent the maximal Abelian subalgebra containing X_d . Let $\mathfrak{W}_{X_d} = \mathfrak{h} \cap \text{Ad}_K(X_d)$ denote the maximal commuting set contained in the adjoint orbit of X_d , where the *Weyl orbit* of X_d is the set of \mathfrak{W}_{X_d} . We define the convex hull of the Weyl orbit of X_d with vertices given by the elements of the Weyl orbit of X_d as follows:

$$\mathfrak{c}(X_d) = \left\{ \sum_{i=1}^n \beta_i X_i \mid \beta_i \geq 0, \sum \beta_i = 1, X_i \in \mathfrak{W}_{X_d} \right\}.$$

Theorem 10. *If $\mathfrak{h} \subset \mathfrak{p}$ and $\Gamma : \mathfrak{p} \rightarrow \mathfrak{h}$ is the orthogonal projection with respect to the G -invariant norm. Then for any $X \in \mathfrak{h}$, we have $\Gamma(\text{Ad}_K X) = \mathfrak{c}(W \cdot X)$, where \mathfrak{c} denotes convex hull.*

Proof. The proof is similar to Kostant's proof [22], so we omit it. □

4. Quantum mechanical systems

In general, the physical systems are dynamic, i.e. they evolve over time. In non-relativistic quantum mechanics, the time evolution of a quantum system (e.g., an atom, a molecule, or a system of particles with spin) can be described by a map $\psi : \mathbb{R} \rightarrow \mathbb{S}$, where the domain \mathbb{R} is the set of all real numbers and the range \mathbb{S} a unit sphere in a complex separable Hilbert \mathcal{H} , which is a complex valued function of the real variable called a *wavefunction* [16]. The starting point is to let $\psi(0)$ be the initial state of the system, and $\psi(t)$ be the state vector at some other time t . Now, the unitary state evolution of a quantum system is given by

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle,$$

where $U(t)$ unitary (propagator) transformation.

In order to implement a certain quantum information processing task in a controlled quantum system, we consider the dynamics of the system, i.e. the time evolution operator can be defined through the time-dependent Schrödinger equation

$$\frac{dU}{dt}(t) = -iH(t)U(t), \quad U(0) = I, \quad (8)$$

where $H(t)$ is a (Hamiltonian) self-adjoint operator acting on a complex separable Hilbert space. We can separate the Hamiltonian as

$$H = H_d + \sum_{j=1}^m u_j H_j, \quad m < n, \quad (9)$$

such that H_d is the internal Hamiltonian part of the system and it is called the *drift* or *free Hamiltonian* and the second part of the Hamiltonian $\sum_{j=1}^m u_j H_j$ represent coherent manipulations from outside called the *control Hamiltonian*, where $u_j \in \mathbb{R}$ are controls that can be switched on and off, see [13]. Moreover, H_d and H_j are traceless Hermitian matrices, see the example below.

Example 11. Consider the group of unitary unimodular 2×2 complex matrices

$$SU(2) = \left\{ \begin{pmatrix} \mu & \nu \\ -\bar{\nu} & \bar{\mu} \end{pmatrix} \in M_2(\mathbb{C}) \mid |\mu|^2 + |\nu|^2 = 1 \right\},$$

it is compact and simply connected. Its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ of antiHermitian traceless 2×2 complex matrices described by

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\mu & \nu \\ -\bar{\nu} & -i\bar{\mu} \end{pmatrix} \in M_2(\mathbb{C}) \mid \mu \in \mathbb{R}, \nu \in \mathbb{C} \right\}.$$

The generators (basis) of $\mathfrak{su}(2)$ are $\{-i\sigma_x, -i\sigma_y, -i\sigma_z\}$ where σ_x, σ_y , and σ_z are matrices called Pauli matrices, given by

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \sigma_y = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_z = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (10)$$

they are traceless Hermitian, where i is the imaginary unit, the matrices σ_x, σ_y , and σ_z satisfy the commutation relations, namely,

$$[\sigma_x, \sigma_y] = \sigma_z, \quad [\sigma_y, \sigma_z] = \sigma_x, \quad [\sigma_z, \sigma_x] = \sigma_y$$

which completely describe the Lie algebra $\mathfrak{su}(2)$, the matrix commutator defined as before (5), specifically $[X, Y] = XY - YX$. The choice of the subspaces

$$\mathfrak{k} = \text{span}\{\sigma_z\}, \quad \mathfrak{p} = \text{span}\{\sigma_x, \sigma_y\}$$

allows a Cartan decomposition for $\mathfrak{su}(2)$. Furthermore, $\{\sigma_x, \sigma_y\}$ is an orthonormal frame, with respect to the Riemannian metric g [3], for the inner product (see Definition 1) restricted to \mathfrak{p} . Now, on $G = SU(2)$, define the G -invariant subbundle $\mathcal{D}_x = x\mathfrak{p}$ endowed with the G -invariant norm defined on it. Therefore, the triple $(SU(2), \mathcal{D}, \|\cdot\|)$ is a $\mathfrak{k} \oplus \mathfrak{p}$ sub-Finsler manifold.

Turn back to the quantum system, Khaneja et al [20, 21] calculated the minimum time it takes to steer this system (8) from a unitary evolution $U(0) = I$ of the unitary propagator to a desired unitary propagator U_f . In other words, if the Hamiltonian does not change in time, then the time evolution operator for time t is the unitary operator $U(t)$. Note that if we have a bounded unitary operator $U(t)$, there exists a unique unitary operator $U^\dagger(t)$ called the *adjoint* of $U(t)$ acting on \mathcal{H} . However, a unitary operator satisfies the relation $U(t)U^\dagger(t) = 1$.

The problem we are ultimately interested in is to find the minimum time required to transfer the density matrix from the initial state $\rho(0)$ to a final state ρ_f . Thus, we will be interested in computing the minimum time required to steer the system

$$\frac{dU}{dt}(t) = -i \left(H_d + \sum_{j=1}^m u_j(t) H_j \right) U(t), \quad (11)$$

from identity, $U(0) = I$, to a final propagator U_f , this system defines a bilinear control system, as it is linear both in the unitary operator $U(t)$ and in the control amplitudes $u_j(t) \in \mathbb{R}$. Observe that, if you have a finite H_d , then it will take infinite time to get very far place, however, if you fix a final terminal point then there will be a finite time to get there. System (11) is exactly (respectively, approximately) controllable if every point of \mathbb{S} can be steered to (respectively, steered arbitrarily close to) any other point of \mathbb{S} , by a horizontal curve of (11).

For a finite dimensional quantum mechanical system, if n is the number of energy levels we have $\mathcal{H} = \mathbb{C}^n$ and the state space \mathbb{S} is the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. In this setting, one can naturally associated with (8) (and consequently (11)) is its lift on the unitary group $SU(n)$, such that the solution is of the form

$$U(t) = g(t)U(I), \quad \text{with } g(t) \in SU(n).$$

Moreover, we can write (8) as

$$\frac{d}{dt}g(t) = -iH(t)g(t), \quad U(0) = I, \quad (12)$$

where $-iH(t)$ is a traceless skew Hermitian matrix, that belongs to the Lie algebra $\mathfrak{su}(n)$ [9]. The problem of controllability is understandable nowadays, more precisely, the typical problem one is indeed interested in a quantum control is that one wants to steer the state of a quantum system from a given initial to a target state for each couple of points in $SU(n)$. Furthermore, we can actuate or manipulate the system by modulating the hamiltonian as a function of time. In fact, the system is controllable if and only if the Hörmander's condition satisfies

$$\text{Lie}\{iH_0, iH_1, \dots, iH_m\} = \mathfrak{su}(n).$$

Let us consider the following example of the subalgebra is $\mathfrak{su}(2)$ which is spanned by the multiples of the Pauli matrices (10). Recall that the system Lie algebra H of (11) is defined as the smallest Lie subalgebra of \mathfrak{g} containing H_d, H_1, \dots, H_m , i.e. H is the smallest linear subspace $\{H_d, H_1, \dots, H_m\}$ of \mathfrak{g} , that contains H_d, H_1, \dots, H_m together with all the iterated Lie brackets

$$[H_d, H_i], [H_i, H_j], [H_d, [H_i, H_j]], \dots$$

It is easily seen, that the associated Lie group $\exp(H)$ is the smallest subgroup of G that contains \mathbb{S} , i.e. $\exp(H)$ is equal to the system group G .

5. Controllability in a quantum system

Controllability is a major matter in system analysis before a control strategy is applied, and is used to judge whether it is possible to control or stabilize the system. In other words, it is the ability to steer a quantum system from a given initial state to any final state, infinite time, using the available controls. The controllability of the system (11) depends on some conditions that we will clarify in detail during this section.

First of all, the case of the quantum system that referred to earlier in (11) which has no drift hamiltonian H_d instead it is just evolution in the unitary group that can be switched on and off certain generators or hamiltonian H_j , such that the form of the system is as follows

$$\frac{dU}{dt} = -i \left(\sum_{j=1}^m u_j H_j \right) U \in \text{span}\{iH_1U, \dots, iH_mU\}, \quad (13)$$

then if these generators were rich enough to span the tangent space at each point of the unitary group, then basically the problem of the controllability will be trivial which is better than we can steer the system to the desired target point or not. Now, if we assume that our unitary group is connected, then we can take two points and connect them by a curve and follow this curve, more precisely, follow its tangent by choosing generators appropriately with the required way. The problem will become more interesting when this number of generators H_j that we have is actually much smaller than the tangent space of the unitary group, however, in this case

we can switch on and off these hamiltonian, one can produce a commutator or a new generator of these hamiltonian as in the following procedure

$$\begin{aligned} U(\delta t) &= \exp(iH_2\delta t) \exp(iH_1\delta t) \exp(-iH_2\delta t) \exp(-iH_1\delta t) \\ &= -(\delta t)^2 [iH_1, iH_2] \approx I - (\delta t)^2 \underbrace{[iH_1, iH_2]}_{\text{new generator}}, \end{aligned} \quad (14)$$

such that the backward evolution $\exp(-iH_{1,2}\delta t)$ is generated by letting the forward map $\exp(iH_{1,2}\delta t)$ evolve for sufficient period of time, which might not lie in the span of these origin generators that one had at one's disposal, which gives us a new direction of motion. Therefore, we can gen-generalized by taking such all iterate commutators and generates the lie algebra such that if these all iterate commutators span the tangent space (the Lie algebra) for the unitary group then one can show the system is controllable, these results goes by Chow's theorem [11].

Concerting u_j , note that we have the freedom to choose them positive and negative, i.e. they can be made arbitrary large. So, if the control amplitude unbounded, then the point that can be reached, it can be reached with no time under our considering that u_j was chosen large enough. Consequently, if the system (13) is controllable then its controllable in arbitrary small amount of time. To sum up the controllability of the system (11), if the Lie Algebra $\{-iH_j\}_{\mathfrak{g}}$ generated by $\{-iH_j\}$ span the Lie algebra of the unitary group, then the system (11) is controllable. The controllability of a driftless control system can described as a nonholonomic control system, for more details and examples inspired by robotics, we refer the reader to [4].

Second case, the controllability of a quantum system has a drift, i.e H_0 , that mostly takes the following form

$$\frac{dU}{dt} = -i \left(H_0 + \sum_{j=1}^m u_j H_j \right) U \in -iH_0U + \text{span}\{iH_1U, \dots, iH_mU\}, \quad (15)$$

if the Lie algebra $\{-iH_0, -iH_j\}_{\mathfrak{g}}$ span the Lie algebra of the unitary group, then the system is controllable. However, one can fundamentally generate these commutators between H_0 and H_j by getting evolution negative in a direction of H_0 by forwarding enough evolution since the group being compact. If the Lie algebra span by H_j is not a full unitary group, then the system is controllable this is due to the fact that we can not change the strength of H_0 . Keep in mind, no matter how long your control is, there is a minimum time to reach anywhere, that time can not be shrunk down to be zero. Regarding the system (15), one of the assumption to make the time is minimum for going from the identity to the desired state by making the control u_j arbitrary large in addition to a good approximation in the setting when the strength of control Hamiltonian can be made much larger than the drift Hamiltonian in the system. If G is the unitary matrix group $SU(2^n)$. Then the problem of finding the most efficient radio-frequency (rf) pulse train required to evolve the system to the desired state is therefore equivalent to the problem of finding a time optimal curve from $U(0) = I$ to the desired U_f . We can generate a subgroup K from the control Hamiltonian H_j defined by

$$K \subset SU(2^n) = \exp(\{H_j\}_{\mathfrak{g}}), \quad (16)$$

where $\{H_j\}_{\mathfrak{g}} = \mathfrak{k}$ is the Lie algebra generated by elements $\{iH_0, iH_1, \dots, iH_m\}$, the subgroup K is a collection of unitary propagators which can be generated as long as the drift Hamiltonian H_d is removed from the system (11). We suppose the strength of Hamiltonian controls could be arbitrarily made large.

If we assume that \mathfrak{g} represent the Lie algebra of unitary group and \mathfrak{k} be a subalgebra of \mathfrak{g} , which correspond to the generator that our control Hamiltonian can be produced. Then the

problem of finding a minimum time to produce a propagator can be characterized this minimum time in special settings in which if we have the Cartan's decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

and this decomposition satisfy the commutation relations, specifically,

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

So, we can define an inner product by the natural killing form such that \mathfrak{p} and \mathfrak{k} are orthogonal to each other. Consequently, one has a sub-Riemannian metric or sub-Finsler metric, see Example 11; additionally, the example below shows the expression of geodesics.

Example 12. For $\mathfrak{k} \oplus \mathfrak{p}$ sub-Finsler manifolds, the strict abnormal extremals are never optimal [1, 8]. If we write the distribution at a point $x \in G$ and $\mathcal{D}_x = x\mathfrak{p}$, the Hamiltonian system given by the Pontryagin's maximum principle is integrable concerning elementary functions. In addition, we have the following expression for geodesics parametrized by arclength, starting at time zero from x_0 ,

$$x(t) = x_0 e^{(A_k + A_p)t} e^{-(A_k)t} \quad (17)$$

such that $A_k \in \mathfrak{k}, A_p \in \mathfrak{p}$, as well as, $\langle A_p, A_p \rangle = \|A_p\|^2 = 1$. This is a well known formula in the community [1, 8, 9, 24]. Note that the controls whose corresponding trajectories starting from x_0 are the normal Pontryagin extremals.

Based on Example 11, μ and ν are complex, the manifold of $SU(2)$ is the sphere \mathbb{S}^3 in \mathbb{R}^4 . So, we have

$$SU(2) \simeq \mathbb{S}^3 = \left\{ \begin{pmatrix} \mu \\ \nu \end{pmatrix} \in \mathbb{C}^2 \mid |\mu|^2 + |\nu|^2 = 1 \right\},$$

by using the isomorphism

$$\eta : SU(2) \rightarrow \mathbb{S}^3, \quad \begin{pmatrix} \mu & \nu \\ -\bar{\nu} & \bar{\mu} \end{pmatrix} \mapsto \begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

Now, by utilizing formula (17), we calculate the explicit expression of geodesics as follows:

First, consider an initial covector in $\mathfrak{su}(2)$ as $A_k \in \mathfrak{k}, A_p \in \mathfrak{p}$, and

$$\lambda = \lambda(\theta, c) = \cos(\theta)\sigma_x + \sin(\theta)\sigma_y + c\sigma_z, \quad \theta \in \mathbb{S}^1, c \in \mathbb{R}.$$

The corresponding exponential map for all $t \in \mathbb{R}$ is

$$\text{Exp}(\theta, c, t) := \text{Exp}(\lambda(\theta, c), t) = e^{(\cos(\theta)\sigma_x + \sin(\theta)\sigma_y + c\sigma_z)t} e^{-(c\sigma_z)t} = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

such that

$$\begin{aligned} \mu &= \frac{c \sin(\frac{ct}{2}) \sin(\sqrt{1+c^2}\frac{ct}{2})}{\sqrt{1+c^2}} + \cos(\frac{ct}{2}) \cos(\sqrt{1+c^2}\frac{t}{2}) \\ &+ i \left(\frac{c \cos(\frac{ct}{2}) \sin(\sqrt{1+c^2}\frac{t}{2})}{\sqrt{1+c^2}} - \sin(\frac{ct}{2}) \cos(\sqrt{1+c^2}\frac{t}{2}) \right), \\ \nu &= \frac{\sin(\sqrt{1+c^2}\frac{t}{2})}{\sqrt{1+c^2}} \left(\cos(\frac{ct}{2} + \theta) + i \sin(\frac{ct}{2} + \theta) \right). \end{aligned}$$

To compute the explicit expression of geodesics for the shortened Lorentz group $SO_0(2, 1)$, we choose an initial covector $\mathfrak{so}(2, 1)$ as $A_k \in \mathfrak{k}, A_p \in \mathfrak{p}$, and

$$\kappa = \kappa(\theta, c) = \cos(\theta)\sigma_x + \sin(\theta)\sigma_y - c\sigma_z, \quad \theta \in \mathbb{S}^1, c \in \mathbb{R}.$$

Using formula (17), the corresponding exponential map for all $t \in \mathbb{R}$ is

$$\text{Exp}(\theta, c, t) := \text{Exp}(\kappa(\theta, c), t) = e^{(\cos(\theta)\sigma_x + \sin(\theta)\sigma_y - c\sigma_z)t} e^{(c\sigma_z)t} = \begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

where μ and ν are given above. Obviously, the geodesics for both groups, $SU(2)$ and the shortened Lorentz group $SO_0(2, 1)$, are the same.

Let us now take a look at case of the controllability of linear systems with unbounded controls, given by the form

$$\frac{dU}{dt} = AU + Bu, \quad (18)$$

such that U is a vector in \mathbb{R}^n (not a unitary propagator), A and B are skew Hermitian matrices, and u is a vector amplitude in such a way we can rewrite it as $u_j b_j$, so the system becomes

$$\frac{dU}{dt} = AU + \sum_j^m u_j b_j \in AU + \text{span}\{b_1, \dots, b_m\},$$

where u_j number multiplying by the columns b_j of the matrix B . This system is said to be a completely controllable if there exists an unconstrained control law that can steer the system U from the identity to a target. Even if the system has a drift, i.e. the system is controllable, it takes arbitrary small time to steer the system U between points of interest. The solution for such a system

$$U(t) = e^{At}U(0) + \int e^{A(t-s)}B(s)u(s)ds,$$

where $U(0) = I$ is the vector at no time (time zero).

We will be interested in the Lie algebra $\mathfrak{u}(n)$ of skew-Hermitian $n \times n$ matrices considered as a Lie algebra over the real field. For example, all matrices $-i(\sum_{j=1}^m u_j(t)H_j)$ in (13) are in $\mathfrak{u}(n)$. The subalgebra $\mathfrak{su}(n)$ of $\mathfrak{u}(n)$ will play an essential role. It includes the matrices in $\mathfrak{u}(n)$ with zero trace.

According to the above setting, we have the ability to module these controls, i.e. in the sense we can make them positive and negative values. Thus, it gives us the possibility to go anywhere exactly. In fact, we want to find out what is the minimum time takes to do this, to simplify, the minimum time will only depend on H_0 and the target state.

Corollary 13. *The quantum systems given above, namely, (13), (15) and (18) have the following distributions*

$$\mathcal{D} = \text{span}\{iH_1U, \dots, iH_mU\}; \quad (19)$$

$$\mathcal{D}_f = -iH_0U + \text{span}\{iH_1U, \dots, iH_mU\}; \quad (20)$$

$$\mathcal{D}_f = AU + \text{span}\{b_1, \dots, b_m\}, \quad (21)$$

respectively, where \mathcal{D} is the distribution and \mathcal{D}_f are the affine distributions.

One of the most significant roles of the sub-Finsler geometry being in understanding the optimal trajectories as sub-Finsler geodesics, precisely, it can suitably be called sub-Randers geodesics. Now comes the importance of the previous Corollary 13, through which we can easily construct a sub-Finslerian metric (consequently, a sub-Finsler geometry) on the distribution

$$\mathcal{D} = \text{span}\{iH_1U, \dots, iH_mU\}$$

that produce from the system (13), (see for instance, [4]). Russell et al [24] studied the Hörmander's condition and showed that by using the fact that $\sum_{j=1}^m u_j(t)H_j$ is constrained such that

$$h\left(i \sum_{j=1}^m u_j(t)H_j, i \sum_{j=1}^m u_j(t)H_j\right) = 1, \quad \forall t, \quad (22)$$

for some inner product $h : \mathfrak{su} \times \mathfrak{su} \rightarrow \mathbb{R}$ on $\mathfrak{su}(n)$, the time optimal trajectories of the time evolution operator $U(t)$ are exactly the geodesics of the following right invariant Randers metric on $SU(n)$, i.e, the time independent is the constraint and valid for all time. In addition, using some constraints (e.g. (22)) the optimal trajectories for $U(t)$ are geodesics of Randers metric restricted to an affine distribution, i.e. \mathcal{D}_f on $SU(n)$. The affine distribution $\mathcal{D}_f \subset TSU(n)$ consisting of vectors of the expression (Corollary (22))

$$\mathcal{D}_f = -iH_0U + \text{span}\{iH_1U, \dots, iH_mU\},$$

such that $\{iH_1U, \dots, iH_mU\} \subset \mathfrak{su}(n)$ span the subset of $\mathfrak{su}(n)$ which is h -orthogonal to the span of the subset of $\mathfrak{su}(n)$ spanning the 'forbidden directions'. In the sense that $\mathcal{D}_U = \mathcal{D}U$ the distribution \mathcal{D} is right invariant, i.e. the optimal trajectories are the length minimizing geodesics that joining the given endpoints according to a sub-Finsler metric in the sense of restrictions distribution \mathcal{D} , and which are parallel to the distribution \mathcal{D} . Such geodesic $V(t)$ is parallel to the distribution \mathcal{D} means that $\frac{dV(t)}{dt} = \mathcal{D}_{V(t)}$, $\forall t$. Under consideration that the distribution \mathcal{D} is bracket generating, we can say the system is controllable, that is, every unitary gate could be implemented. This provides an exact condition for controllability in the presence of additional constraints. Khaneja et al in [20] consider a quantum system with a drift (15), left invariant on some compact group G . However, all vector fields H_j belong to \mathfrak{k} , a Lie subalgebra of the Lie algebra \mathfrak{g} of G . Moreover, they chose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with the standard Cartan's commutation relations, and so, the Lie algebra generated by the H_j 's, $j > 1$ is not equal to \mathfrak{g} (it is only \mathfrak{k}). Therefore, in their case, to move from a point in a coset K_0 to another point in a coset K_1 requires the use of the drift H_0 and hence requires a bounded speed. This implies that, even for unbounded controls there is a minimum time, which is strictly larger than zero (and not attained in general), while Boscain et al in [7], show that if we relax the constraint $u_1^2 + u_2^2 \leq 1$ then the minimum time is zero (also not attained in general).

Now, if we consider G as the unitary group. In the system given by (11), we denoted by $\mathcal{R}(I, t)$ the *reachable set* of all $\hat{U} \in G$ that can be achieved from identity $U(0) = I$ at the time t . Moreover, the time given by

$$t^*(U_f) = \inf\{t \geq 0 \mid U_f \in \overline{\mathcal{R}(I, t)}\},$$

$$t^*(KU_f) = \inf\{t \geq 0 \mid kU_f \in \overline{\mathcal{R}(I, t)}, k \in K\},$$

is said to be *infimizing time*, for producing the propagator $U_f \in G$, where $\overline{\mathcal{R}(I, t)}$ is the closure of the set $\mathcal{R}(I, t)$. Note that the reachable set from the origin is only the origin. In addition, we define the following notation

$$\mathcal{R}(I, T) := \bigcup_{0 \leq t \leq T} \mathcal{R}(I, t),$$

$$\mathcal{R}(I) := \bigcup_{0 \leq t \leq \infty} \mathcal{R}(I, t).$$

Remark 14. A system (11) is said to be controllable if $\mathcal{R}(I, t) = G$ for all $I \in G$. However, the system is accessible but not controllable if the system semigroup $\mathcal{S} := \mathcal{R}(I)$ has an interior point in G .

The following example show that system (11) is an accessible but not controllable:

Example 15. Assume that $M = \mathbb{R}^2$ such that the system given by

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} q^2 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The reachable set from a point $r \in \mathbb{R}^2$ is

$$\mathcal{R}(r) = \{s \in \mathbb{R}^2 \text{ such that } s_1 > r_1\} \cup \{r\}.$$

therefore, the system is not controllable from the initial point $(0, 0)$, however accessible from the same point because the point not in the interior.

Let us define the adjoint system given by

$$\frac{dU}{dt} = -i \left(X_d + \sum_{j=1}^m u_j(t) X_j, \right) U, \quad X \in \text{Ad}_K X_d, \quad (23)$$

where $U(0) = I$ and $\text{Ad}_K(X_d)$ denotes the adjoint orbit of $X_d \in \mathfrak{g}$ under K induced by conjugation as follows

$$\text{Ad}_K(X_d) = \{\text{Ad}_k(X_d) = kX_dk^{-1}, \text{ such that } k \in K\} \subseteq \mathfrak{g}.$$

On the homogeneous sub-Finsler manifold G/K , let us define a right invariant control system for any $P \in G$

$$\dot{P} = XP, \quad P(0) = U(0), \quad X \in \text{Ad}_K X_d, \quad (24)$$

we call such a control system an *adjoint control system*.

Theorem 16. *The infimizing time $t^*(U_f)$ for steering the system (23) from $U(0) = I$ to U_f is the same as the minimum coset time needed for steering the adjoint system*

$$\dot{P} = XP, \text{ where } P \in G \text{ and } X \in \text{Ad}_K(-X_d), \quad (25)$$

from $P(0) = I$ to KU_f [20].

Remark 17. Let $U_f \in G/K$ be an arbitrary target state, we can replace U_f by any other point

$$kU_fk^{-1} = kU_f \in G/K, k \in K$$

in a manner, the control problem does not change essentially. Since the set of control variables, i.e. $\text{Ad}_K(H_d)$ is invariant under conjugation in K therefore the optimal trajectory after such a conjugation remains optimal. In addition, following the general properties of symmetric spaces, we have

$$\text{Ad}_K U_f = \{kU_f \in G/K \text{ where } k \in K\}$$

has nonempty intersection with $T \subseteq G/K$ such that T is a maximal torus in G . To be more specific, the following equality holds

$$\text{Ad}_K U_f \cap T = \exp(\mathfrak{W}_X), \quad \forall X \in \mathfrak{h}, \text{ with } \exp(X) \in \text{Ad}_K U_f.$$

6. Conclusion

The system (11) guarantees that all admissible motions in G/K are generated by the drift Hamiltonian H_d . To pick out an initial direction in G/K it enough to choose a convenient $k \in K$ by means of the controls. Practically speaking, for each point $KU_\gamma \in G/K$ there is a proper subspace \mathcal{D}_{KU_γ} of the directions generated by H_d can be chosen by the controls. Recall that, thanks to the Chow's theorem, we ensure that every point in G/K can be reached from any given starting point by a piecewise smooth curve determined by the controls since the subspaces \mathcal{D}_{KU_γ} is bracket generating. The set of the adjoint orbit $Ad_K(-iH_d)$ of $-iH_d$ under the action of the subgroup K , in general is not the whole of \mathfrak{p} , is the set of all possible directions in G/K , that means not all the directions in G/K are directly accessible, unless we achieved that by a back and forth motion in directions we can directly access. To produce new directions of motion we can apply the noncommuting generators that mentioned in (14).

This kind of problem, where one has to find the shortest curve between its endpoints in a manifold subject to the constraint that the tangent to the curve always belong to a subset of all possible directions, have been well studied in the seance sub-Riemannian and sub-Finslerian geometries which are known as (sub-Riemannian) sub-Finslerian geodesics [2, 3, 18, 19]. If the set of accessible directions is the set $Ad_K(-iH_d)$, then the the problem of finding time optimal control laws minimize to find sub-Finsler geodesics in coset G/K [21]. Khaneja et al in [20] and [21], show that the sub-Riemannian geodesics were computed for the space $SU(4)/SU(2) \otimes SU(2)$, in the framework of optimal control of coupled two-spin and three-spin systems, in particular, if $n = 2$ spins-1/2 then G/K takes the form of a Riemannian symmetric space. Thus time-optimal trajectories between points in G correspond to Riemannian geodesics, but if $n > 2$ the coset G/K are no longer Riemannian symmetric spaces, so the time-optimal trajectories in G denote sub-Riemannian geodesics.

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