Neighbourhood sequences and character recognition by Walsh transformation

doktori (PhD) értekezés tézisei

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Contents

Introduction

This PhD thesis contains new results from the fields of digital image processing and digital geometry. In the first part we examine the properties of neighbourhood sequences which provide a tool to generalise the classical neighbourhood relations that are used in many places in digital geometry and topology. Our results are also published in [15], [17], [18] and [19]. In the second part of the thesis we describe an own character recognition algorithm based on Walsh transformation. Our algorithm and its application for compressing typeset documents are also published in the papers [14] and [13], respectively. We give the introduction to these two parts separately.

I. Neighbourhood sequences

In digital applications it is often very useful to have an appropriate (digital) distance function on the digital grid the application is based on. See the survey paper [23] of MELTER for an account. In two dimensions MELTER and TOMESCU [24] investigated path generated digital metrics, while HARARY et al. [20] graph generated digital distances. For three and higher dimensional analysis we can mention the paper of OKABE et al. [25] about the properties of paths, and the paper of BORGEFORS [1] with certain generalizations of former two dimensional results. YAMASHITA and HONDA [32] found a condition for a sequence of neighbourhoods to generate digital metrics. YAMASHITA and IBARAKI [33] extended and generalized the results of [32] using a general neighbourhood definition. Investigating a special case of a concept studied in [33], DAS, CHAKRABARTI, CHATTERJI and MUKHERJEE published several papers [3]–[10] in this topic. One of the most essential tasks in this area is to give a convenient digital metric which approximates the Euclidean metric well. Many of the above papers also contain such results, like [3], [7], [8], [9], [10], and [33]. A detailed description about the basic concepts of digital topology can be found in [21] or [31].

The theory of neighbourhood sequences is an important part of this topic. By the help of such sequences, we can describe planar (or higher dimensional) movements. The classical digital – cityblock and chessboard – motions in $\mathbb{Z}^2$ were introduced by ROSENFELD and PFALTZ [26]. The cityblock motion allows movements only in horizontal and vertical directions, while the chessboard motion in diagonal directions, as well. Based on these two types of motions, the authors in [26] defined two distances. The d_4 or d_8 distance of two points is the number of steps required to reach one point from the other, where only cityblock or chessboard motions can be used, respectively. To obtain a better approximation of the Euclidean distance, ROSENFELD and PFALTZ recommended the alternate use of these motions, which defines the distance d_{oct}. By allowing arbitrary periodic mixture of the cityblock and chessboard motions, DAS et al. [5] introduced the concept of periodic neighbourhood sequences, and generalised it to arbitrary dimension. Moreover, the authors in [5] established a formula for calculating the distance of two points in a finite dimensional digital space, determined by such a periodic neighbourhood sequence. Using this formula, DAS in [4] showed that on the set of periodic 2D-neighbourhood sequences a "natural" partial ordering relation can be introduced. Furthermore, he investigated the structure of this set and some of its subsets with respect to this ordering. More precisely, he proved that under this ordering, the set of l-periodic 2D-neighbourhood sequences forms a distributive lattice. Recently, FAZEKAS in [12] has proved that a similar partial ordering can also be introduced for neighbourhood sequences in 3D.

In the first part of the thesis we generalise the concept of neighbourhood sequences by allowing not periodic sequences only. We show that the results of DAS [4] and FAZEKAS [12] about ordering the set of periodic neighbourhood sequences, can be extended to arbitrary dimension, even in this more general case. Furthermore, we investigate the structure of the set and some subsets of these general neighbourhood sequences in finite dimension under this ordering. As some of the sets considered do not have nice structural properties, instead of this "natural" partial ordering we propose another relation which is in close connection with the original one. We extend all these results to ∞D, which is the most interesting case theoretically. The lattice obtained in ∞D under the new ordering relation is the closure of the union of the finite dimensional lattices, in some sense. These results are also presented in [15], joint with A. FAZEKAS and L. HAJDU.

The "natural" partial ordering fails to be a complete ordering on the set of neighbourhood sequences. However, in certain applications it can be useful to compare any two neighbourhood sequences, i.e. to decide which one spreads "faster". For this purpose we introduce a norm-like concept, called velocity, and investigate its properties. This concept has to be introduced in a way to fit the ordering relation, so we need some preliminaries before defining velocity. Furthermore, we define a metric for neighbourhood sequences. We also work out a possible application scheme of these notions for distributing information. The obtained results are also contained in [19] which is joint with L. HAJDU.

DAS and CHATTERJI [9] studied the geometric properties of the octagons occupied by a periodic neighbourhood sequence during "spreading" on the 2D plane. We extend the geometric description of 2D periodic neighbourhood sequences in [9] to general neighbourhood sequences, and embed the former results into this more general structure. Moreover, we perform our analysis in arbitrary finite dimension instead of $\mathbb{Z}^2$. We give the coordinates of vertices of polyhedra occupied by neighbourhood sequences, and investigate the symmetry and convexity of these bodies. Since 2D and 3D digital grids play a very important role in digital image processing, we study the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$ in detail. This geometrical analysis is also presented in [17].

DAS in [3] determined distance functions induced by periodic neighbourhood sequences that provide good approximations of the Euclidean distance in a certain sense. This analysis is a special case of the one presented in [33], moreover, DAS restricted his investigations to the so called simple metrics. We perform approximations of the 2D Euclidean distance by distance functions based on general neighbourhood sequences. The whole set of neighbourhood sequences is considered in our investigations. Interestingly, the best approximating sequences we obtain are (mostly) Beatty sequences, thus they can be constructed very easily. We give sequences such that the corresponding distance functions are metrics on $\mathbb{Z}^2$. In particular, we determine the digital metric that best approximates the Euclidean one from below "uniformly", i.e. independently of the sense of approximation. Our results also can be found in [18], joint with L. HAJDU.

II. Character recognition by Walsh transformation

The history of character recognition by tools of digital image processing dates back to the 1950's (see e.g. [30]). A character recognition process usually consists of a scanning step, preprocessing (binarization – segmentation), feature extraction (skeletonization – contour extraction), the actual recognition and classification, sometimes postprocessing, and verification. Most of the character recognition algorithms can be applied to typeset texts, and there exist procedures to process handwritten characters. Algorithms for recognizing both typeset and handwritten characters can be based on several methods, like projection histograms [29], zoning [11], or on the Fourier descriptors [30]. Naturally, the recognition of handwritten characters is a more difficult problem than the recognition of printed text.

In the second part of the thesis we present a character recognition algorithm which basically supports the recognition of typeset documents. We focused on segmentation, classification, and made some experiments to verify the reliability of our character recognition method. To classify a character, usually a feature vector is composed and the actual recognition is achieved by searching for the closest prototype feature vector. The dimension of the feature vector can vary in different applications, and a suitable metric also should be chosen to measure the difference between feature vectors. In our investigations we tried to find a method which generates easily separable feature vectors, that is the prototype vectors have large distance from each other. We found that the feature vectors generated by Walsh transformation can be separated more effectively, than using other popular character recognition methods like zoning or projection histograms. Using Walsh transformation with underdetermined feature vectors, we also obtain a noise filtering effect. Feature extraction is not needed now, since the classification step of our algorithm is based on the Walsh transforms of the image, and they can be calculated without any modifications. We built our character recognition algorithm into an application by which we can recognize and compress typeset documents for further transmission. The results of the above studies are also presented in [13] and [14] which are joint with A. FAZEKAS.

I. Neighbourhood sequences

We give the basic definitions and notation needed later. From now on, n will denote an arbitrary positive integer.

Definition 1. Let p be a point in $\mathbb{Z}^n$. The i-th coordinate of p is indicated by $\mathrm{Pr}_i(p)$ $(i = 1, \ldots, n)$. Let M be an integer with $0 \leq M \leq n$. The points $p, q \in \mathbb{Z}^n$ are M-neighbours, if the following two conditions hold:

- $|\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq 1 \quad (1 \leq i \leq n)$,
- $\sum\limits_{i=1}^{n} |\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq M$.

Definition 2. The sequence $A = (A(i))_{i=1}^{\infty}$, where $A(i) \in \{1, \ldots, n\}$ for all $i \in \mathbb{N}$, is called an n-dimensional (shortly nD) neighbourhood sequence. If for some $l \in \mathbb{N}$, $A(i+l) = A(i)$ $(i \in \mathbb{N})$, then A is periodic with period l. In this case we briefly write $A = (A(1), A(2), \ldots, A(l))$. The set of the nD-neighbourhood sequences will be denoted by S_n.

Remark 1. We note that the above concept is a generalization of the notion of neighbourhood sequences introduced in [5]. The authors in [3], [4], [5], [6], [9] and [12] dealt only with periodic sequences.

Definition 3. Let $p,\ q \in \mathbb{Z}^n$ and $A \in S_n$. The point sequence $p = p_0, p_1, \ldots, p_m = q$, where p_{i-1} and p_i are $A(i)$-neighbours in $\mathbb{Z}^n$ $(1 \leq i \leq m)$, is called an A-path from p to q of length m. The A-distance $d(p, q; A)$ of p and q is defined as the length of the shortest A-path(s) between them. As a brief notation, we also use $d(A)$ for the A-distance.

Notation 1. Let $p, q \in \mathbb{Z}^n$, and $A = (A(i))_{i=1}^{\infty} \in S_n$. Let

$$||p - q|| = \sum_{h=1}^{n} |\mathrm{Pr}_h(p) - \mathrm{Pr}_h(q)|,$$

and for every $i \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$ put

$$A^{(j)}(i) = \min(A(i), j) \quad \text{and} \quad f_j^A(i) = \sum_{k=1}^{i} A^{(j)}(k).$$

Let

$$x = (x(1), x(2), \ldots, x(n))$$

be the nonincreasing ordering of the numbers $|\operatorname{Pr}_i(p) - \operatorname{Pr}_i(q)|$ with $i = 1, \ldots, n$, that is, $x(i_1) \geq x(i_2)$ if $i_1 < i_2$. For $j = 1, \ldots, n$ put

$$a_j = \sum_{k=1}^{n-j+1} x(k).$$

Finally, for any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x, and $\lceil x \rceil$ the smallest integer which is greater than or equal to x.

DAS et al. in [5] provided an algorithm to calculate $d(p, q; A)$, where $p, q \in \mathbb{Z}^n$ and A is a periodic nD-neighbourhood sequence. By the following lemma we extend this result to any neighbourhood sequence belonging to S_n.

Lemma 1. *Let $p, q \in \mathbb{Z}^n$, and $A \in S_n$. Write $c = ||p - q||$, and let*

$$g_j^A(i) = f_j^A(c) - f_j^A(i-1) - 1, \quad i = 1, \ldots, c.$$

Then the A-distance of p and q can be calculated as

$$d(p, q; A) = \max_{j=1}^{n} d_j^A(p, q),$$

where

$$d_j^A(p, q) = \sum_{i=1}^{c} \left\lfloor \frac{a_j + g_j^A(i)}{f_j^A(c)} \right\rfloor.$$

With the following definition we introduce a natural ordering relation on S_n. We note that this relation is identical with those given by DAS [4] and FAZEKAS [12] in 2D and 3D, respectively, for periodic neighbourhood sequences.

Definition 4. For any $A, B \in S_n$ we define the relation $\sqsupseteq^*$ in the following way:

$$A \sqsupseteq^* B \iff d(p, q; A) \leq d(p, q; B) \text{ for all } p, q \in \mathbb{Z}^n.$$

Beside S_n, we investigate the structure of all those sets under the ordering relation $\sqsupseteq^*$ which were studied by DAS [4] in the periodic case. These subsets of S_n are given below.

Notation 2. Let S'_n, $S'_n(l_{\geq})$ and $S'_n(l)$ be the sets of periodic, at most l-periodic, and l-periodic ($l \in \mathbb{N}$) nD-neighbourhood sequences, respectively.

In our investigations we also need the well-known concept of L_p metrics.

Definition 5. Let $q, r \in \mathbb{R}^n$. The L_p ($p > 0$) distance of q and r is

$$L_p(q, r) = \left(\sum_{i=1}^{n} |\operatorname{Pr}_i(q) - \operatorname{Pr}_i(r)|^p \right)^{\frac{1}{p}}, \quad \text{and} \quad L_\infty(q, r) = \max_{i=1}^{n} \left(|\operatorname{Pr}_i(q) - \operatorname{Pr}_i(r)| \right).$$

We also investigate the way a neighbourhood sequence spreads in the digital space starting from a point of $\mathbb{Z}^n$. This spreading is translation-invariant, so for simplicity we may choose the origin $\mathbf{0}$ of $\mathbb{Z}^n$ as the starting point.

Definition 6. Let $A \in S_n$, and for every $k \in \mathbb{N}$ put

$$\widehat{A_k} = \{q \in \mathbb{Z}^n \,:\, d(\mathbf{0}, q; A) \leq k\}.$$

That is, $\widehat{A_k}$ is the region occupied by A after k steps. Let $H(\widehat{A_k})$ be the convex hull of $\widehat{A_k}$ in $\mathbb{R}^n$.

In 2D, we will compare the regions occupied by neighbourhood sequences with the corresponding Euclidean disks. For that purpose, we introduce the following notation.

Definition 7. For every $k \in \mathbb{N}$, let

$$O_k = \{q \in \mathbb{Z}^2 \,:\, L_2(\mathbf{0}, q) \leq k\}, \quad \text{and} \quad G_k = \{q \in \mathbb{R}^2 \,:\, L_2(\mathbf{0}, q) \leq k\}$$

be the Euclidean disks of radius k in $\mathbb{Z}^2$ and $\mathbb{R}^2$, respectively.

The number of the values i $(i = 1, \ldots, n)$ occurring among the first k elements of a neighbourhood sequence $A \in S_n$ is also used in our investigations.

Definition 8. Let $A = (A(i))_{i=1}^{\infty} \in S_n$. For every $i = 1, \ldots, n$ and $k \in \mathbb{N}$ put

$$A(i,k) = \#\{A(j) \,:\, A(j) = i,\ 1 \leq j \leq k\}.$$

For convenience, write $A(i, 0) = 0$.

We use the densities of the elements of the neighbourhood sequences to study the asymptotic behaviour of the occupied regions.

Definition 9. Let $A \in S_n$. The density of the value i $(i = 1, \ldots, n)$ in A is defined as

$$D^A(i) = \lim_{k \to \infty} \frac{A(i,k)}{k},$$

if this limit exists.

1. Algebraic properties of neighbourhood sequences

First we recall some definitions and remarks from lattice theory that will be needed later on. These basic concepts also can be found e.g. in [2].

Definition 10. Let $(P, \leq)$ be a partially ordered set, and $S \subseteq P$. An element $a \in P$ is the least upper bound (or greatest lower bound) of S, if for all $x \in S$, $a \geq x$ (or $a \leq x$), and $b \geq a$ (or $b \leq a$) for every upper bound (or lower bound) b of S. Moreover, if every pair of elements (x, y) with $x, y \in P$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$, then $(P, \leq)$ is called a lattice.

Definition 11. The lattice $(P, \leq)$ is distributive if for all $x, y, z \in P$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Definition 12. The lattice $(P, \leq)$ is complete if its every subset $S \subseteq P$ has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$.

Definition 13. Let $(P, \leq)$ be a complete lattice and $S \subseteq P$. The set $S^c = \{x \in P : x \leq \bigvee S\}$ is called the closure of S.

Lattice of neighbourhood sequences in nD

In our investigations we can take advantage of a fundamental connection between the functions $f_j^A(i)$ and the relation $\sqsupseteq^*$. Such a connection was shown by DAS [4], in case of periodic 2D-neighbourhood sequences. A similar result was proved by FAZEKAS in 3D (see [12]). Together with A. FAZEKAS and L. HAJDU, in [15] we extended these results to nD with arbitrary $n \in \mathbb{N}$, to general nD-neighbourhood sequences. Clearly, this case also includes the periodic one. In particular, our result is new even for $n = 2$ and 3.

Theorem 1. *For any nD-neighbourhood sequences $A = (A(i))_{i=1}^{\infty}$ and $B = (B(i))_{i=1}^{\infty}$*

$$d(p,q;A) \leq d(p,q;B), \text{ for all } p,q \in \mathbb{Z}^n \iff f_j^A(i) \geq f_j^B(i), \text{ for all } i \in \mathbb{N},\ j \in \{1,\dots,n\}.$$

Remark 2. By this theorem, $A \sqsupseteq^* B$, if and only if for every $i, j \in \mathbb{N}$, $f_j^A(i) \geq f_j^B(i)$ holds.

Now we formulate our results about the lattice structure of S_n (and its subsets) under $\sqsupseteq^*$ (see also [15]).

Theorem 2. *$(S_2, \sqsupseteq^*)$ is a complete distributive lattice.*

We show that unfortunately $\sqsupseteq^*$ has some unpleasant structural properties, too.

Proposition 1. *The following structures are not lattices:*
- $(S_n, \sqsupseteq^*)$, *for* $n \geq 3$,
- $(S'_n, \sqsupseteq^*)$, *for* $n \geq 2$,
- $(S'_2(l_{\geq}), \sqsupseteq^*)$, *for* $l \geq 5$,
- $(S'_n(l_{\geq}), \sqsupseteq^*)$, *for* $l \geq 2$, $n \geq 3$,
- $(S'_n(l), \sqsupseteq^*)$, *for* $l \geq 2$, $n \geq 3$.

We introduce another ordering relation, which is in close connection with $\sqsupseteq^*$. Moreover, S_n and its subsets considered above, form much nicer structures under this new relation.

Definition 14. For any $A = (A(i))_{i=1}^{\infty}$, $B = (B(i))_{i=1}^{\infty} \in S_n$ we define the relation $\sqsupseteq$ in the following way:

$$A \sqsupseteq B \iff A(i) \geq B(i), \quad \text{for every } i \in \mathbb{N}.$$

Remark 3. It is clear that $\sqsupseteq^*$ is a proper refinement of $\sqsupseteq$ in S_n, S'_n, $S'_n(l_{\geq})$ and $S'_n(l)$.

Proposition 2. *$(S_n, \sqsupseteq)$ is a complete distributive lattice with greatest lower bound $\bigwedge S_n = (1)$ and least upper bound $\bigvee S_n = (n)$.*

Proposition 3. *$(S'_n, \sqsupseteq)$ is a distributive lattice with greatest lower bound $\bigwedge S'_n = (1)$ and least upper bound $\bigvee S'_n = (n)$.*

However, the ordering relation $\sqsupseteq$ has worse properties in S'_n than in S_n. This is shown by the following "negative" result.

Proposition 4. *For $n \geq 2$, $(S'_n, \sqsupseteq)$ is not a complete lattice.*

The next proposition shows that $S'_n(l_{\geq})$ is not a "natural" subset of S_n, in the sense that it does not form a nice structure even under $\sqsupseteq$.

Proposition 5. *$(S'_n(l_{\geq}), \sqsupseteq)$ is not a lattice for any $n \geq 2$ and $l \geq 6$.*

Remark 4. We note that $(S'_n(l_{\geq}), \sqsupseteq)$ is a distributive lattice if $1 \leq l \leq 5$, for every $n \in \mathbb{N}$.

Proposition 6. *$(S'_n(l), \sqsupseteq)$ is a distributive lattice for every $n, l \in \mathbb{N}$.*

Lattice of neighbourhood sequences in ∞D

Beside the finite dimensional digital spaces we generalize our investigations also to $\mathbb{Z}^\infty = \{(z_i)_{i=1}^\infty : z_i \in \mathbb{Z}\}$. Our purpose is to extend the results concerning $\mathbb{Z}^n$ to this case. First we give some definitions that are natural generalizations of the finite dimensional concepts.

Definition 15. The i-th coordinate of a point $p \in \mathbb{Z}^\infty$ is indicated by $\mathrm{Pr}_i(p)$. The points $p, q \in \mathbb{Z}^\infty$ are called M-neighbours for some $M \in \mathbb{N} \cup \{\infty\}$, if

- $|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq 1 \quad (i \in \mathbb{N})$,
- $\sum\limits_{i=1}^{\infty} |\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq M$.

Definition 16. A sequence $A = (A(i))_{i=1}^\infty$, where $A(i) \in \mathbb{N} \cup \{\infty\}$ is called an ∞D-neighbourhood sequence. If for some $l \in \mathbb{N}$, $A(i) = A(i+l)$ holds for every $i \in \mathbb{N}$, then A is called periodic, with period l, or simply l-periodic. In this case we will use the abbreviation $A = (A(1), \dots, A(l))$. The set of the ∞D-neighbourhood sequences will be denoted by S_∞.

Definition 17. Let $p, q \in \mathbb{Z}^\infty$ and $A \in S_\infty$. The point sequence $p = p_0, p_1, \dots, p_m = q$, where p_{i-1} and p_i are $A(i)$-neighbours in $\mathbb{Z}^\infty$ $(1 \leq i \leq m)$, is called an A-path from p to q of length m. If such a path exists, then the A-distance of p and q is defined as the common length of the shortest A-paths from p to q. It will be denoted by $d(p, q; A)$ or briefly $d(A)$. If there is no A-path from p to q, then we put $d(p, q; A) = \infty$.

Remark 5. Observe that for any $p, q \in \mathbb{Z}^\infty$, the following two statements are equivalent:

- $d(p, q; A) = \infty$, for every $A \in S_\infty$,
- the set $\{|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| : i \in \mathbb{N}\}$ is unbounded.

The function $d(p, q; A)$ has some "symmetry" properties, that is, it depends only on the differences of the coordinates of the points, i.e. on the numbers $|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)|$, $i \in \mathbb{N}$. We note that by Lemma 1, the same is also true in nD for every $n \in \mathbb{N}$. In the next theorem we describe how $d(p, q; A)$ can be calculated.

Theorem 3. *Let p and q be two distinct points in $\mathbb{Z}^\infty$ such that the set $\{|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| : i \in \mathbb{N}\}$ is bounded and let $A = (A(i))_{i=1}^\infty \in S_\infty$. For $c \geq 1$ let $H_c = \{i : |\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \geq c\}$, and put $k = \min\{c : \#H_c < \infty\}$ and $h = \#H_k$. Let $B(i) = \min(h, A(i))$, and $B = (B(i))_{i=1}^\infty$ $(i \in \mathbb{N})$. Moreover, put $r = (\mathrm{Pr}_i(p))_{i \in H_k}$ and $s = (\mathrm{Pr}_i(q))_{i \in H_k}$. Let t be defined by the following properties:*

- $A(t) = \infty$,
- $\#\{i : i \leq t \text{ and } A(i) = \infty\} = k - 1$.

If such t does not exist, then put

$$t = \begin{cases} 0, & \text{if } k = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Now the following equality holds:

$$d(p,q;A) = \max(d_h(r,s;B),t),$$

where for $h \geq 1$, $d_h(r,s;B)$ *is the* h*-dimensional* B*-distance of* r *and* s, *and* $d_0(r,s;B) = 0$.

Remark 6. Combining the above result with the formula provided for $d(p,q;A)$ in Lemma 1, it is possible to calculate explicitly the distance of two points in $\mathbb{Z}^\infty$, determined by an ∞D-neighbourhood sequence. On the other hand, if we take $p,q \in \mathbb{Z}^\infty$ such that they differ only at finitely many places, then we have $k = 1$ whence $t = 0$ in Theorem 3. This shows that the distance defined in $\mathbb{Z}^\infty$ is in fact a generalization of the distances introduced in the finite dimensional cases.

Now we examine the structure of the ∞D-neighbourhood sequences. We study two ordering relations on them, which are the extensions of the finite dimensional orderings to this general case.

Definition 18. Let $A,B \in S_\infty$. We write $A \sqsupseteq^* B$, if and only if for every $p,q \in \mathbb{Z}^\infty$, $d(p,q;A) \leq d(p,q;B)$.

The following result is the extension of Theorem 1 to $\mathbb{Z}^\infty$.

Theorem 4. *Let* $A,B \in S_\infty$ *with* $A = (A(i))_{i=1}^\infty$ *and* $B = (B(i))_{i=1}^\infty$. *For* $i,j \in \mathbb{N}$, *put* $f_j^A(i) = \sum_{k=1}^i \min(A(k),j)$ *and* $f_j^B(i) = \sum_{k=1}^i \min(B(k),j)$. *Then*

$$d(p,q;A) \leq d(p,q;B), \text{ for all } p,q \in \mathbb{Z}^\infty \iff f_j^A(i) \geq f_j^B(i), \text{ for all } i \in \mathbb{N}, j \in \mathbb{N}.$$

Remark 7. As in the finite dimensional case, $A \sqsupseteq^* B$ if and only if for every $i,j \in \mathbb{N}$, $f_j^A(i) \geq f_j^B(i)$ holds.

Remark 8. Let S'_∞, $S'_\infty(l_\geq)$ and $S'_\infty(l)$ be the sets of periodic, at most l-periodic and l-periodic ($l \in \mathbb{N}$) ∞D-neighbourhood sequences, respectively. It is clear that $\sqsupseteq^*$ is an antisymmetric, transitive relation, i.e. a partial ordering on S_∞ and on all its above subsets. However, just as in the finite dimensional case, $(S_\infty,\sqsupseteq^*)$, $(S'_\infty,\sqsupseteq^*)$, $(S'_\infty(l_\geq),\sqsupseteq^*)$ and $(S'_\infty(l),\sqsupseteq^*)$ with $l \geq 2$ are not lattices.

Definition 19. For $A = (A(i))_{i=1}^\infty$ and $B = (B(i))_{i=1}^\infty$ in S_∞, we write $A \sqsupseteq B$ if and only if $A(i) \geq B(i)$, for every $i \in \mathbb{N}$.

It turns out that $\sqsupseteq$ has much more pleasant properties than $\sqsupseteq^*$ in case of ∞D-neighbourhood sequences, too.

Proposition 7. *For any* $l \in \mathbb{N}$, $(S_\infty,\sqsupseteq)$, $(S'_\infty,\sqsupseteq)$ *and* $(S'_\infty(l),\sqsupseteq)$ *are distributive lattices, with greatest lower bound* (1) *and least upper bound* (∞). *Moreover, the first and third lattices are complete, while the second one is not.*

Proposition 8. $(S'_\infty(l_\geq),\sqsupseteq)$ *is a complete distributive lattice for* $1 \leq l \leq 5$. *If* $l \geq 6$, *then* $(S'_\infty(l_\geq),\sqsupseteq)$ *is not a lattice.*

The following result provides some information about the structures of several other subsets of S_∞ under $\sqsupseteq$.

Proposition 9. *Let* $S_\infty^* = \{A : A = (A(i))_{i=1}^\infty \text{ with } A(i) \in \mathbb{N}\}$, $S_\infty^+ = \bigcup_{n=1}^\infty S_n$ *and* $S_\infty^- = \bigcup_{n=1}^\infty S'_n$. *Then* $(S_\infty^*,\sqsupseteq)$, $(S_\infty^+,\sqsupseteq)$ *and* $(S_\infty^-,\sqsupseteq)$ *are non-complete distributive lattices (sublattices of* S_∞*) with greatest lower bound* (1), *but having no least upper bounds.*

Finally, we show that the lattice $(S_\infty, \sqsupseteq)$ (and in some special cases $(S'_\infty(l_\geq), \sqsupseteq)$ or $(S'_\infty(l), \sqsupseteq))$ can be considered as the closure of the union of the finite dimensional lattices. We also include the lattices $(S'_\infty, \sqsupseteq)$ and $(S^*_\infty, \sqsupseteq)$ into this consideration.

Proposition 10. *Let* $S^-_\infty(l_\geq) = \bigcup_{n=1}^\infty S'_n(l_\geq)$ *and* $S^-_\infty(l) = \bigcup_{n=1}^\infty S'_n(l)$ $(l \in \mathbb{N})$. *For any* $l \in \mathbb{N}$, *in* S_∞

$$(S_\infty, \sqsupseteq) = (S'_\infty, \sqsupseteq)^c = (S^*_\infty, \sqsupseteq)^c = (S^+_\infty, \sqsupseteq)^c =$$

$$(S^-_\infty, \sqsupseteq)^c = (S^-_\infty(l_\geq), \sqsupseteq)^c = (S^-_\infty(l), \sqsupseteq)^c$$

holds. Moreover, for any $l \in \mathbb{N}$, *in* $S'_\infty(l)$ *we have*

$$(S'_\infty(l), \sqsupseteq) = (S^-_\infty(l), \sqsupseteq)^c.$$

Finally, for $1 \leq l \leq 5$, *in* $S'_\infty(l_\geq)$

$$(S'_\infty(l_\geq), \sqsupseteq) = (S^-_\infty(l_\geq), \sqsupseteq)^c.$$

2. Metrical properties of neighbourhood sequences

As we could see, $\sqsupseteq^*$ is not a complete ordering on the set of neighbourhood sequences, and the structure obtained is not even a lattice in higher dimensions. However, in certain applications it can be useful to compare any two neighbourhood sequences, i.e. to decide which one spreads "faster". For this purpose, we introduce a norm-like concept, called velocity, on the set of neighbourhood sequences, and investigate its properties. This concept should be introduced in a way to fit the relation $\sqsupseteq^*$, so we need some preliminaries before defining velocity. Furthermore, we introduce a metric for neighbourhood sequences.

Preliminaries to introduce velocity

By defining velocity, we assign a positive real number to every neighbourhood sequence. We give some natural conditions which should be met by this concept.

(i) *Velocity must be sensitive for the behaviour of the sequences in different dimensions.*
(ii) *The elements of a sequence should contribute to its velocity by different weights.*
(iii) *Velocity must be defined in a way to fit the natural ordering.*

Assigning velocity to neighbourhood sequences

According to (ii), first we give the concept of a weight system, which will be appropriate in our further investigations.

Definition 20. Let $n \in \mathbb{N}$. The set of functions $\delta_j : \mathbb{N} \to \mathbb{R}$ $(j = 1, \ldots, n)$ is called a weight system, if the following three conditions hold:

- $\delta_j(i) \geq 0$ $(i \in \mathbb{N},\ j = 1, \ldots, n)$,
- $\sum\limits_{i=1}^{\infty} \delta_j(i) < \infty$ $(j = 1, \ldots, n)$,
- δ_j is monotone decreasing $(j = 1, \ldots, n)$.

In order to meet (i), we introduce the concept of velocity in two steps. First, we assign an n-tuple to every neighbourhood sequence. The elements of this n-tuple reflect the "velocity" of the given neighbourhood sequence in the subspaces of $\mathbb{Z}^n$ of dimensions from 1 to n. Then, we define one descriptive velocity value.

Definition 21. Let $A \in S_n$, and $\delta_j \quad (j = 1, \dots, n)$ be a weight system. The j-dimensional velocity of A is defined as

$$v_j^A = \sum_{i=1}^{\infty} A^{(j)}(i)\delta_j(i).$$

The velocity of A is given by

$$v^A = \frac{1}{n}\sum_{j=1}^{n} v_j^A.$$

Now we show how (i), (ii) and (iii) are met. The vector $(v_1^A, v_2^A, \dots, v_n^A)$, thus also v^A is obviously sensitive for the behaviour of the sequence A in subspaces of $\mathbb{Z}^n$ of dimensions from 1 to n. Thus (i) is satisfied. As we use a weight system to define $(v_1^A, v_2^A, \dots, v_n^A)$ and v^A, the requirements of (ii) are also met. The following theorem verifies that our velocity concept satisfies (iii) as well.

Theorem 5. *Let $A, B \in S_n$ with $A \sqsupseteq^* B$, and let $\delta_j \quad (j = 1, \dots, n)$ be a weight system. Then, $v_j^A \geq v_j^B$ for every $j = 1, \dots, n$.*

Remark 9. By the definition of velocity, this theorem implies that if $A \sqsupseteq^* B$, then $v^A \geq v^B$.

Remark 10. The monotonity of δ_j is necessary to have Theorem 5.

As one can easily see, it can happen that with some weight system δ_j, $v_j^A \geq v_j^B$ for every $j = 1, \dots, n$, but $A \sqsupseteq^* B$ does not hold. However, in some sense we can reverse Theorem 5. More precisely, we have

Theorem 6. *Let $A, B \in S_n$. If for any weight system δ_j $(j = 1, \dots, n)$, $v_j^A \geq v_j^B$ holds for all $j = 1, \dots, n$, then $A \sqsupseteq^* B$.*

Examples of weight systems

Let $c > 1$, and put

$$\delta_j(i) = \frac{1}{c^{i-1}}\ , \text{ for every } j = 1, \dots, n \text{ and } i \in \mathbb{N}.$$

Obviously, δ_j is a weight system with

$$\sum_{i=1}^{\infty} \delta_j(i) = \frac{c}{c-1} \quad (j = 1, \dots, n).$$

Consider the nD-neighbourhood sequences

$$A = (h,\ 1,\ 1,\ 1,\ 1,\ \dots), \text{ and } B = (1,\ n,\ n,\ n,\ n,\ \dots), \text{ where } 2 \leq h \leq n.$$

Then

$$v^A = v_j^A = \frac{1}{c-1} + h, \text{ and } v^B = v_j^B = \frac{n}{c-1} + 1 \quad (j = 1, \dots, n).$$

Clearly, the sequences A and B cannot be compared by the ordering $\sqsupseteq^*$. We show how the relation between the velocity values of A and B changes according to the choice of the parameter c. First, suppose that $c > n$. Then, we have

$$v^A = v_j^A = \frac{1}{c-1} + h \geq \frac{1}{c-1} + 2 = \frac{c}{c-1} + 1 > \frac{n}{c-1} + 1 = v_j^B = v^B.$$

Using this weight system we obtain a very strong condition, namely that $v^A > v^B$ if and only if A precedes B lexicographically. Now, let $c = 2$. In this case we have

$$v^A = v_j^A = 1 + h \leq 1 + n = v_j^B = v^B,$$

with equality only for $h = n$. Finally, set $c < 2$. By a simple calculation, we get $v^B = v_j^B > v_j^A = v^A$ in this case.

An application for distributing information

We present an application scheme of neighbourhood sequences and velocity in a network model, where the members of the network are the points of $\mathbb{Z}^2$. The network model has an information source at the center (origin) which distributes information to the other members (clients) of the network. The system is based on priority, that is if a client is "closer" to the origin than another one, it has greater priority, and receives the information earlier. In this model we use 2D-neighbourhood sequences to deliver the information to the clients. Suppose that the cost of distributing information decreases with the number of 2-s in the chosen neighbourhood sequence. Knowing the importance of the information, we have to choose one of the cheapest sequences, which is still "fast" enough. That is, we take a neighbourhood sequence, whose velocity fits the importance of the information to be sent. By choosing an appropriate weight system, we can increase and decrease the initial priority of the clients in the network. This network model can be easily extended to $\mathbb{Z}^3$. In this case, we can take more advantage of the behaviour of neighbourhood sequences in lower dimensional subspaces. If we know in advance that a special type of information is important only for a group of clients, we can place these clients onto or close to the $[x, y]$, $[y, z]$ and $[x, z]$ planes.

Metric spaces of neighbourhood sequences

We introduce a metric on S_n in a similar fashion as we did it for velocity.

Definition 22. Let $\Delta = \{\delta_j : j = 1, \ldots, n\}$ be a weight system and $A, B \in S_n$. The distance ϱ_Δ of these sequences is defined by

$$\varrho_\Delta(A, B) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{\infty} |A^{(j)}(i) - B^{(j)}(i)| \; \delta_j(i).$$

Remark 11. For any weight system Δ, the function ϱ_Δ is a metric on S_n.

In what follows, we establish some useful and interesting properties of these metric spaces.

Theorem 7. *For any weight system Δ, (S_n, ϱ_Δ) is a complete metric space.*

Definition 23. A sequence $(A_k)_{k=1}^{\infty}$ is monotone increasing (resp. decreasing), if $A_{i+1} \sqsupseteq^* A_i$ (resp. $A_i \sqsupseteq^* A_{i+1}$) holds for every $i \in \mathbb{N}$.

Theorem 8. *Every monotone increasing or decreasing sequence $(A_k)_{k=1}^{\infty}$, with $A_k \in S_n$ $(k \in \mathbb{N})$ is convergent.*

The next result shows that the Bolzano-Weierstrass theorem is true in the constructed metric spaces.

Theorem 9. *For any weight system Δ, every subset of (S_n, ϱ_Δ) of infinite cardinality has an accumulation point.*

The following result shows that periodic neighbourhood sequences form a dense subset of (S_n, ϱ_Δ). As the set of periodic neighbourhood sequences is countable, this also yields that (S_n, ϱ_Δ) is a separable metric space.

Theorem 10. *For any weight system Δ, the set of periodic neighbourhood sequences is dense in (S_n, ϱ_Δ).*

3. Geometric properties of neighbourhood sequences

DAS and CHATTERJI [9] investigated some geometric behaviour of the regions occupied by 2D periodic neighbourhood sequences. Now we extend these results to arbitrary finite dimensions. Moreover, we use general neighbourhood sequences in our analysis.

Geometric properties of nD-neighbourhood sequences

We start our geometric investigations in the general digital space $\mathbb{Z}^n$. Since neighbourhood sequences spread in an "isotropic" way, the occupied regions are symmetric objects.

Theorem 11. *Let $A \in S_n$ and $k \in \mathbb{N}$. If a point $p \in \mathbb{Z}^n$ with coordinates $(p_1, p_2, \dots, p_n)$ belongs to $\widehat{A_k}$, then the points with coordinates $(\lambda_1 p_{i_1},\ \lambda_2 p_{i_2},\ \dots\ ,\ \lambda_n p_{i_n})$ also belong to $\widehat{A_k}$. Here $\lambda_j = \pm 1$ $(j = 1, \dots, n)$, and $(i_1, i_2, \dots, i_n)$ is an arbitrary permutation of $(1, 2, \dots, n)$.*

Remark 12. It is easy to verify that the theorem also holds for the points of $H(\widehat{A_k})$.

Using the above results we can find hyperplanes, for which the regions occupied by neighbourhood sequences are symmetric.

Remark 13. Let $A \in S_n$ and $k \in \mathbb{N}$. Then $H(\widehat{A_k})$ is symmetric to those $(n-1)$D hyperplanes that contain $(n-1)$ coordinate axes (this implies that the coordinate values can change sign), and to their rotations by 45° around any axis they contain (this implies that coordinates can be permuted).

Now we calculate the coordinates of vertices of polyhedra occupied by neighbourhood sequences.

Theorem 12. *Let $A \in S_n$ and $k \in \mathbb{N}$. The vertices of $H(\widehat{A_k})$ are exactly those points, whose coordinates are the permutation of the values*

$$\left(\lambda_1 \sum_{i=1}^{n} A(i,k),\ \lambda_2 \sum_{i=2}^{n} A(i,k),\ \dots\ ,\ \lambda_n A(n,k)\right),$$

where $\lambda_j = \pm 1$, for every $j = 1, \dots, n$, and λ_j can be different in every permutation.

Remark 14. Using the above theorem we obtain that the maximal number of vertices of $H(\widehat{A_k})$ is $n! \cdot 2^n$. The polyhedron $H(\widehat{A_k})$ can be degenerate if some of the elements $(1, \dots, n)$ do not occur in A, causing the decrease of the number of the vertices.

We investigate the existence of some convergence limits related to the sequence of polyhedra $(H(\widehat{A_k}))_{k=1}^{\infty}$. We use the densities of the $1,\ldots,n$ elements in A to calculate the corresponding limits. In the general n-dimensional case only the asymptotic behaviour of the diameter of $H(\widehat{A_k})$ is considered, where we use the usual definition of diameter, that is $diam(H(\widehat{A_k})) = \sup\limits_{p,q\in H(\widehat{A_k})} L_2(p,q)$.

Theorem 13. *Let $A \in S_n$, and suppose that $D^A(i)$ exists for every $i = 1,\ldots,n$. Then*

$$\lim_{k\to\infty} \frac{diam(H(\widehat{A_k}))}{k} = 2\left(\sum_{i=1}^{n} iD^A(i)^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} iD^A(i)D^A(j)\right)^{\frac{1}{2}}.$$

Geometric properties of 2D- and 3D-neighbourhood sequences

We present a detailed investigation of the geometry of neighbourhood sequences in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. In [9] DAS and CHATTERJI showed that for every 2D periodic neighbourhood sequence A, $H(\widehat{A_k})$ is always an octagon. They calculated the coordinates of the vertices of the octagon, and also the length of its sides based on the relative occurrence of the 1 and 2 values in a period of A. Using general neighbourhood sequences we obtain similar results, but in this case we have to work with the densities of the 1 and 2 values in the neighbourhood sequence.

Remark 15. If we start any 2D-neighbourhood sequence A from the origin, then for every $k \in \mathbb{N}$ the octagon $H(\widehat{A_k})$ is symmetric to both coordinate axes, and to their $45°$ rotations.

In what follows, we calculate some geometric parameters of these octagons, namely the length of their sides, their perimeters and areas. We give these data in terms of $A(1,k)$ and $A(2,k)$.

Definition 24. Let A be a 2D-neighbourhood sequence. Let $x^A(k)$ be the length of the horizontal and $y^A(k)$ be the length of the inclined sides of the octagon $H(\widehat{A_k})$. Moreover, let $P_{2D}^A(k)$ be the perimeter and $V_{2D}^A(k)$ be the area of this octagon.

Proposition 11. *Using the above notation, the following relations hold:*

- $x^A(k) = 2A(2,k), \quad y^A(k) = \sqrt{2}A(1,k),$
- $P_{2D}^A(k) = 4\left(\sqrt{2}A(1,k) + 2A(2,k)\right), \quad V_{2D}^A(k) = 2\left(A^2(1,k) + 4A(1,k)A(2,k) + 2A^2(2,k)\right).$

Our purpose is to describe the asymptotic behaviour of the sequences $x^A(k)/k$, $y^A(k)/k$, $P_{2D}^A(k)/k$, and $V_{2D}^A(k)/k^2$. If the densities $D^A(1)$ and $D^A(2)$ exist, we can obtain these limits by their help, while if they do not exist, we still can use the lower and upper densities instead. See the dissertation or [17] for details.

Theorem 14. *Let $A \in S_2$. If $D^A(1)$ exists, then we have*

- $\lim\limits_{k\to\infty} \frac{x^A(k)}{k} = 2(1 - D^A(1)), \quad \lim\limits_{k\to\infty} \frac{y^A(k)}{k} = \sqrt{2}D^A(1),$
- $\lim\limits_{k\to\infty} \frac{P_{2D}^A(k)}{k} = 8 + (4\sqrt{2} - 8)D^A(1), \quad \lim\limits_{k\to\infty} \frac{V_{2D}^A(k)}{k^2} = 2(2 - D^A(1)^2).$

Remark 16. We can formulate the opposite statement as well. Namely, for any $A \in S_2$ the density $D^A(1)$ exists, if and only if any of the convergence limits formulated in Theorem 14 exist.

The extension of the above investigations to 3D is quite straightforward, and can be done without considerable difficulties. See [17] or the dissertation for details.

4. Approximating the Euclidean distance by digital metrics

We discuss some possibilities of approximating the Euclidean distance in $\mathbb{Z}^2$, and perform such an analysis using digital metrics induced by neighbourhood sequences. In our approximation process we compare the regions $\widehat{A_k}$ occupied by a 2D-neighbourhood sequence with the Euclidean disks O_k. The best approximating sequences we obtain are (mostly) Beatty sequences.

Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$, and let $A = (A(i))_{i=1}^{\infty}$, $B = (B(i))_{i=1}^{\infty}$ be sequences of 1-s and 2-s, defined by

$$A(i) = \lfloor i\alpha \rfloor - \lfloor (i-1)\alpha \rfloor + 1, \quad B(i) = \lceil i\alpha \rceil - \lceil (i-1)\alpha \rceil + 1 \quad (i \in \mathbb{N}).$$

The sequences A, B are called Beatty sequences on the letters 1, 2. Clearly, for every $k \in \mathbb{N}$ we have

$$A(2,k) = \lfloor k\alpha \rfloor \quad \text{and} \quad B(2,k) = \lceil k\alpha \rceil.$$

For more properties of Beatty sequences and their generalizations see [22].

Three approximation problems

A natural approximation approach could be to choose the number of integer points in the symmetric difference $\widehat{A_k} \bigtriangledown O_k$ as an error function. However, there is no exact formula for the number of integer points inside O_k, so we follow a slightly different method. Namely, we compare the sets $H(\widehat{A_k})$ and G_k, and choose A to minimize the area of $H(\widehat{A_k}) \bigtriangledown G_k$. Of course, it can be done only separately for each $k \in \mathbb{N}$. However, surprisingly it turns out that for every k the very same A can be chosen to minimize this area. So this neighbourhood sequence A can be regarded as the one that approximates L_2 best (in the above sense).

According to these principles, we will investigate the function

$$TE_A(k) = \text{Area}(H(\widehat{A_k}) \bigtriangledown G_k),$$

called the total error of the approximation at the k-th step. We will also use the relative error at the k-th step, defined as

$$RE_A(k) = \frac{TE_A(k)}{k^2\pi},$$

and the limit relative error (if it exists)

$$RE_A = \lim_{k\to\infty} RE_A(k).$$

We perform several types of approximation. Our aims can be summarized by the following problems. The first problem concerns the general case.

Problem 1. *Find a neighbourhood sequence $\Lambda \in S_2$ (if exists) such that for every $A \in S_2$ and $k \in \mathbb{N}$*

$$\text{Area}(H(\widehat{\Lambda_k}) \bigtriangledown G_k) \leq \text{Area}(H(\widehat{A_k}) \bigtriangledown G_k).$$

We consider separately the case when the octagons $H(\widehat{A_k})$ cover G_k for every $k \in \mathbb{N}$, that is the corresponding function $d(A)$ minorates L_2.

Problem 2. *Find a neighbourhood sequence $\Phi \in S_2$ (if exists) such that $H(\widehat{\Phi_k}) \supseteq G_k$ for every $k \in \mathbb{N}$, and for every $A \in S_2$, $H(\widehat{A_k}) \supseteq G_k$ implies that*

$$\mathrm{Area}(H(\widehat{\Phi_k}) \setminus G_k) \leq \mathrm{Area}(H(\widehat{A_k}) \setminus G_k).$$

Note that it does not make sense to consider a problem with $H(\widehat{A_k}) \subseteq G_k$. Indeed, observe that $H(\widehat{A_k})$ is contained in G_k if and only if the first k elements of A are all 1-s. This is the reason why we do not take up the problem of majorating L_2 by digital metrics $d(A)$.

We will construct two neighbourhood sequences, satisfying the requirements of Problems 1 and 2, respectively. Moreover, we will give a sequence such that the corresponding distance function $d(A)$ is a *metric*, and it can be considered as the digital metric which approximates L_2 best in the sense of Problem 1. We also investigate the following "discrete" version of Problem 2. Note that Problem 1 does not have a similar variant.

Problem 3. *Find a neighbourhood sequence $\Psi \in S_2$ (if exists) such that $\widehat{\Psi_k} \supseteq O_k$ for every $k \in \mathbb{N}$, and if $A \in S_2$ with $\widehat{A_k} \supseteq O_k$, then $\widehat{A_k} \supseteq \widehat{\Psi_k}$.*

Observe that the sequence Ψ has the nice property that the corresponding distance function $d(\Psi)$ is "uniformly" the best one to approximate L_2 from below. That is, for every $A \in S_2$, if $d(q,r;A) \leq L_2(q,r)$ for any $q,r \in \mathbb{Z}^2$, then $d(q,r;A) \leq d(q,r;\Psi)$ for any $q,r \in \mathbb{Z}^2$. In Theorem 16 we will solve Problem 3, by constructing the sequence Ψ having the desired property. Interestingly, it will turn out that the distance function $d(\Psi)$ is a metric on $\mathbb{Z}^2$. To show this, the following lemma will be useful.

Lemma 2. *Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$. Let $A = (A(i))_{i=1}^{\infty} \in S_2$ be the unique sequence with $A(2,k) = \lfloor k\alpha \rfloor$ for every $k \in \mathbb{N}$. Then $d(A)$ is a metric.*

The solution of the approximation problems

We construct "extremal" sequences described in Problems 1, 2, and 3, starting with Problem 2, being the simplest to handle. For this purpose we consider only neighbourhood sequences A, with G_k contained in $H(\widehat{A_k})$ for all $k \in \mathbb{N}$. As we mentioned, it means that the corresponding distance function $d(A)$ minorates L_2. The next result gives a solution to Problem 2.

Theorem 15. *Let $\Phi = (\Phi(i))_{i=1}^{\infty}$ be the unique 2D-neighbourhood sequence defined by $\Phi(2,k) = \lceil k(\sqrt{2}-1) \rceil$ $(k \in \mathbb{N})$, that is*

$$\Phi(i) = \lceil i(\sqrt{2}-1) \rceil - \lceil (i-1)(\sqrt{2}-1) \rceil + 1 \qquad (i \in \mathbb{N}).$$

Then $H(\widehat{\Phi_k}) \supseteq G_k$ for any $k \in \mathbb{N}$. Moreover $A \in S_2$, and $H(\widehat{A_k}) \supseteq G_k$ implies that

$$\mathrm{Area}(H(\widehat{\Phi_k}) \setminus G_k) \leq \mathrm{Area}(H(\widehat{A_k}) \setminus G_k).$$

Remark 17. The octagons $H(\widehat{\Phi_k})$ are almost regular. (For regularity we should have $A(2,k) = k(\sqrt{2}-1)$, which is impossible.) Obviously, the ratio of the inclined and horizontal (or vertical) sides of $H(\widehat{\Phi_k})$ tends to 1 as $k \to \infty$.

Remark 18. For the k-th total error of the approximation of L_2 with $d(\Phi)$ we get

$$TE_{\Phi}(k) = (4-\pi)k^2 - 2\left(k - \Phi(2,k)\right)^2.$$

Thus for the k-th relative error and for the relative error we obtain

$$RE_{\Phi}(k) = \frac{4-\pi}{\pi} - \frac{2}{\pi}\left(1 - \frac{\Phi(2,k)}{k}\right)^2, \quad \text{and} \quad RE_{\Phi} = \frac{8(\sqrt{2}-1)-\pi}{\pi} = 0.054786175\ldots.$$

By the following theorem we solve Problem 3.

Theorem 16. *Let* $\Psi = (\Psi(i))_{i=1}^{\infty} \in S_2$ *be the unique sequene defined by* $\Psi(2,k) = \lfloor k(\sqrt{2}-1) \rfloor$ $(k \in \mathbb{N})$*, that is*

$$\Psi(i) = \lfloor i(\sqrt{2}-1) \rfloor - \lfloor (i-1)(\sqrt{2}-1) \rfloor + 1 \quad (i \in \mathbb{N}).$$

Then for every $k \in \mathbb{N}$, $O_k \subseteq \widehat{\Psi_k}$. *Moreover, if* $A \in S_2$ *such that* $O_k \subseteq \widehat{A_k}$ *for some* $k \in \mathbb{N}$, *then* $\widehat{\Psi_k} \subseteq \widehat{A_k}$.

Remark 19. By Lemma 2, $d(\Psi)$ is a metric on $\mathbb{Z}^2$. That is, among the digital metrics corresponding to neighbourhood sequences, $d(\Psi)$ is the best one to approximate L_2 from below in $\mathbb{Z}^2$.

Now we solve Problem 1.

Theorem 17. *Let the neighbourhood sequence* $\Lambda = (\Lambda(i))_{i=1}^{\infty}$ *be defined by*

$$\Lambda(i) = \begin{cases} 1, & \text{if } E\left(\frac{\Lambda(2,i-1)}{i}\right) < E\left(\frac{\Lambda(2,i-1)+1}{i}\right), \\ 2, & \text{otherwise,} \end{cases}$$

where the function $E : [0,1] \to \mathbb{R}$ *is given by*

$$E(y) = \begin{cases} -y^2 + 2y, & \text{if } y \geq \sqrt{2}-1, \\ 2\arccos(y(y+2)) - 2(y+1)\sqrt{1-2y-y^2} - y^2 + 2y, & \text{otherwise.} \end{cases}$$

Then for any $A \in S_2$ *and* $k \in \mathbb{N}$,

$$\operatorname{Area}(H(\widehat{\Lambda_k}) \bigtriangledown G_k) \leq \operatorname{Area}(H(\widehat{A_k}) \bigtriangledown G_k).$$

Remark 20. We have $\Lambda = (2,1,1,1,2,1,2,1,1,2,1,1,\dots)$. For the k-th total and relative errors of Λ we obtain

$$TE_\Lambda(k) = k^2(2 - \pi + 2E(y)), \quad \text{and} \quad RE_\Lambda(k) = \frac{2-\pi+2E(y)}{\pi}.$$

By a simple calculation we get

$$RE_\Lambda = \frac{2-\pi}{\pi} + \frac{2}{\pi}\left(2\arccos\left(\frac{3+8\sqrt{6}}{25}\right) + \frac{4\sqrt{6}-11}{5}\right) = 0.046525347\dots.$$

Clearly, $d(\Lambda)$ is not a metric on $\mathbb{Z}^2$. Another unpleasant feature of Λ is that it is not easy to generate: to obtain its k-th element, we have to calculate the first $k-1$ elements previously. Now we give two sequences which are easy to construct, and for every $k \in \mathbb{N}$, one of them is also the "best" to approximate G_k.

Corollary 1. *For* $j = 1,2$ *and* $i \in \mathbb{N}$ *put*

$$C^{[j]}(i) = \begin{cases} j, & \text{if } i = 1, \\ \lfloor i\frac{2\sqrt{6}-3}{5} \rfloor - \lfloor (i-1)\frac{2\sqrt{6}-3}{5} \rfloor + 1, & \text{if } i > 1, \end{cases}$$

and write $C^{[1]} = (C^{[1]}(i))_{i=1}^{\infty}$ *and* $C^{[2]} = (C^{[2]}(i))_{i=1}^{\infty}$. *Then for every* $A \in S_2$ *and* $k \in \mathbb{N}$,

$$\min\left(TE_{C^{[1]}}(k),\ TE_{C^{[2]}}(k)\right) \leq TE_A(k).$$

Remark 21. As $C^{[1]}(2,k) = \lfloor k\frac{2\sqrt{6}-3}{5}\rfloor$ for every $k \in \mathbb{N}$, by Lemma 2, $d(C^{[1]})$ is a metric on $\mathbb{Z}^2$. Thus in a sense, $d(C^{[1]})$ can be considered to be the best metric (coming from a neighbourhood sequence) to approximate the Euclidean distance on $\mathbb{Z}^2$. Note that $RE_{C^{[1]}} = RE_{\Lambda}$.

Comparing approximation results

We compare our results with those of DAS in [3]. He used an error function which measures the average difference between the Euclidean distance and the "simple metric value" generated by a neighbourhood sequence. DAS concluded that the periodic neighbourhood sequence $S = (1,1,2,1,2)$, which generates a "simple metric", should be used to approximate L_2. Note that for every $k \in \mathbb{N}$, $H(\widehat{S_k}) \not\supseteq G_k$. As we propose to use the sequence $C^{[1]}$ defined in Corollary 1 to approximate L_2, we compare S and $C^{[1]}$ here. Since we used a different error function than Das in [3], we chose a third one to compare our results. We examined how the k-disks $\widehat{A_k}$ approximate G_k in digital sense, and we obtained that the sequence $C^{[1]}$ behaves better. See the dissertation for details.

II. Character recognition by Walsh transformation

In the second part of the thesis we present a character recognition process which is based on Walsh transformation. Walsh transformation is frequently applied in several fields of digital image processing. Since this transformation is magnification-invariant and preserves geometry nicely, it is well applicable for character recognition, as well. Now we describe briefly the steps of our algorithm, and the investigations we performed.

In the first step of our algorithm a segmentation procedure divides the original binary image into smaller segments which are stored in a chained list. The segments are actually rectangles, and beside the image information, the coordinates of the upper left pixel and the size of the rectangle are also stored. For every segment a 64-dimensional feature vector is composed according to the first 64 Walsh transforms of the segment. After calculating the feature vector of a given segment, we judge whether the segment contains text information (letter, number, etc.) or an unrecognizable symbol. We used commonly applied character sets (OCR-A, OCR-B, CMR, TIMES NEW ROMAN and license plate characters) to test our algorithm from several points of view, like computation speed, noise sensitivity, or recognition failure. We made other statistical investigations for correlation analysis as well. The above results are also published in [14] which is joint with A. FAZEKAS. To prove the practical applicability of our character recognition algorithm, together with A. FAZEKAS we built our method into an application for compressing typeset documents [13].

1. Segmentation

The first step of the algorithm performs a segmentation procedure on the binary image. We try to determine minimal storing rectangles for those subsets of the image foreground which can be separated by horizontal and vertical lines. These segments can be determined by calculating their size and the coordinates of their upper left pixel. Using storing rectangles, our segmentation procedure does not extract the connected foreground components in the case of a ligature. Our algorithm is preconditioned for this phenomenon, and can be trained to recognize ligatures. In our experiments we also used CMR character set which allows ligatures beside the ligature-free sets OCR-A and OCR-B (which were composed for optical character recognition). The segmentation can be made parallel easily, and the whole procedure can be performed as an alternating recursive sequence of the following horizontal and vertical segmentation steps.

Horizontal segmentation step: We have a pixel running from left to right, starting from the upper left corner of every segment we have already extracted. If we find an object (foreground) point in the given row then we go one pixel down and start running a pixel from the beginning of this row. The procedure continues till the running pixel reaches the right side of the segment (we find a row which does not contain object points). In this case we obtain a new segment. The top row of the new segment will be the uppermost row of the original segment which contains an object point. The bottom row of the new segment will be the lowermost row of the original segment which has been already processed and contains an object point. After defining the new segment, we go on with processing the original segment, starting from that row which did not contain object points.

Vertical segmentation step: The vertical segmentation procedure is analogous to the horizontal one, but here the pixel runs from top to bottom, starting from the upper left corner of a segment. We go one pixel right till a column is found which does not contain object points. In this case a new segment is defined.

Horizontal segmentation steps are followed by vertical segmentation steps, and vice versa. Alternating these two steps, each of the existing segments is divided into smaller segments. If the number of the segments does not change during a segmentation step, the algorithm stops. The result of the first (horizontal) step is a line of text if the original binary image is a text document. The top of the line is determined by the highest character, and the bottom is determined by the lowest character. The second (vertical) segmentation step divides the text line into characters, but the rectangles that store the characters contain relatively large background components which can be eliminated with the following (horizontal) segmentation step. If the document contains some graphic parts then the number of the necessary segmentation steps depends on the complexity of the graphics.

The segmentation of accented characters is an interesting and difficult problem. In the case of English text after three segmentation steps (horizontal – vertical – horizontal) the storing rectangles cannot be reduced any more, while in the case of Hungarian text (which contains accented characters), we have the same situation only after the fourth segmentation step. It is rather difficult to recognize accented characters, since the accent is segmented separately. Analysing the placement of the segments of small size can help to recognize these characters. For example, it can be useful to examine if a vowel takes place below a segment with small size. Our system was not trained to recognize accented characters.

The input image (a text document) can be distorted in many ways: it can be rotated, corrupted by noise, etc. The image might be rotated if e.g. the document was inserted into the scanner improperly. Our method tolerates rotation with small degrees. If the degree of the rotation is too large, the horizontal segmentation is not able to separate the document lines in the first step, since the lowest pixel of a line is "under" the highest pixel of the next line. The degree of the maximal tolerable rotation α can be obtained as

$$\alpha = \arctan \frac{\text{line-space}}{\text{paper width} - \text{horizontal margins}} \quad .$$

2. Character recognition

Walsh transformation was applied for several purposes in digital image processing, but never to character recognition. We found that this transformation also gives a verbose description of the image, like symmetrical relations, placement of the foreground and background pixels, and so on. Before we explain in detail how Walsh transformation can be applied here, we give a brief overview on its theory.

Walsh transformation

The Walsh transformation $W(u,v)$ in 2D is a transformation of the form

$$W(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) g(x,y,u,v).$$

Here f is the binary image of size $N \times N$, where N is supposed to be a power of 2. The intensity of the pixel (x,y) is denoted by $f(x,y)$, and $u,v = 0,\ldots,N-1$. The kernel function g of the transformation is defined by

$$g(x,y,u,v) = \frac{1}{N} \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-i-1}(u)+b_i(y)b_{n-i-1}(v)},$$

where $b_i(w)$ is the i-th bit in the binary expansion of w, and $n = \log_2 N$.

We compute N^2 Walsh transforms altogether which can be organized into the N^2-dimensional feature vector

$$(W(0,0), W(0,1), \ldots, W(0,N-1), W(1,0), W(1,1), \ldots, W(N-1,N-1)).$$

The Walsh transform is injective in the sense that if we consider two different binary images, then the corresponding feature vectors are also different. If we compose a feature vector which does not contain all the Walsh transform values, then this vector can be the same for two different images. The kernel function of the Walsh transformation is separable:

$$g(x,y,u,v) = g_1(x,u)\ g_2(y,v).$$

Moreover, the equality

$$g(x,y,u,v) = g_1(x,u)\ g_1(y,v)$$

also holds, thus the Walsh transformation is symmetric, as well. With these two properties the computation of the 2D transforms can be made considerably faster, since it can be simplified to the computation of two 1D Walsh transformations, and the symmetry makes the computation even faster. All of these properties of the Walsh transformation are well-known from the literature, see e.g. [16]. More analytical properties of Walsh functions can be found in [28].

Application of the Walsh transformation

To perform Walsh transformation, first we have to magnify the original image to the size of $2^n \times 2^n$ for some $n \in \mathbb{N}$. This does not mean a considerable modification, since the Walsh transformation is invariant for magnification. In our algorithm the image size was fixed as

32×32 and we use linear transformation to magnify the segments. However, we compute only the following 64 Walsh transforms instead of the $32 \times 32 = 1024$ ones:

$$(W(0,0), W(0,1), W(0,2), \dots, W(0,7), W(1,0), W(1,1), \dots, W(7,7)).$$

We have the following reasons to reduce the number of Walsh transforms:

- to save computation time,
- these 64 values describe the global features and symmetric relations well,
- to filter out some noise, since the computation of less Walsh transforms results in blurring.

In our investigations we used the metric L_1 for measuring the distance of the feature vectors. The distance of the 64D feature vectors of any two different characters is significant, so the recognition is quite reliable. For example, we compared the feature vectors of prototype OCR digits, and experienced strong differences which fortifies our conception to calculate only a 64 dimensional feature vector for each character.

Magnifying the segments to the same size 32×32 can cause a problem in character recognition. Though the lower and upper cases of the characters usually look different, some characters have similar lower and upper cases (e.g. "w" and "W") which become identical during magnification. The proper case can be restored by examining the side lengths of the original storing rectangle.

We showed how the 2D Walsh transformation can be performed by two 1D transformations. In our algorithm we used a faster method and computed the transforms directly from tables, since the value

$$N \cdot g(x,y,u,v) = \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-i-1}(u)+b_i(y)b_{n-i-1}(v)}$$

in the kernel function can be ± 1 only.

Comparing our method with other algorithms

Our algorithm was compared with two classical character recognition methods: one of them is based on projection histograms, the other one is based on zoning. The reason why we involved these character recognition algorithms into our analysis is that both of them use feature vectors to classify the recognizable characters and assign a 64D feature vector to every recognizable segment, similarly to our algorithm. We worked with several font types: OCR-A, OCR-B, CMR, and TIMES NEW ROMAN in this comparism. By fixing a prototype alphabet (letters, digits, punctuation marks, and other symbols), first we computed the average distance values for every symbol from the rest of the alphabet. The most significant difference values occurred in the case of Walsh transformation which results in better recognition performance.

Character recognition – feature and prototype vectors

We obtain a 64 dimensional feature vector by computing some of the Walsh transforms for a segment. This feature vector is compared with prototype vectors which contain the same 64 Walsh transforms of prototype characters. For a given feature vector we find the closest prototype vector by using a suitable distance function. If we assume that the size of the character segments lie in an interval, we can exclude the segments of too small (noise) or too large (graphic parts) size from the recognition process before the decision step. The recognition is based on the distance between the feature vector of the analysed character and the prototype vector.

Decision and training

To make the algorithm more flexible, we inserted a decision step into the recognition process to handle unrecognizable segments. We have the following two possibilities to make a decision about the recognizability of a segment.

2-level decision: With this step the given segment is always recognized which means that the algorithm finds the prototype character whose feature vector has the minimal distance from the feature vector of the given segment.

3-level decision: With this step we classify the segments as recognizable, or uncertain ones. In the uncertain cases the algorithm has "doubts" about the recognizability of the given segments. It happens when the minimal distance is larger than a threshold value.

During character recognition we create a 64D feature vector for every segment, then calculate the minimal distance between this vector and the prototype vectors. The decision steps above are based on this distance value. In the 3-level decision step we use one critical value. The decision can be made according to the relation between the minimal distance value and the critical value. In the 3-level case, if the minimal distance value is larger than the critical value the segment is considered unrecognizable. The critical values can be given globally or separately for every prototype vector.

There is a training part inserted into the algorithm which is independent of the recognition step. Training can be performed in two possible ways. If we have the font in electronic form, we can easily compose an artificial document containing the whole alphabet without any noise. Scanning through this document we can train the algorithm for every symbol of the alphabet. If we cannot compose such an artificial document, then the algorithm must be trained by using 3 to 10 samples for every symbol from scanned material.

3. Experimental results

We calculated the average computation time the algorithm took to process one page of printed text which contained 27 rows. It took 3.5 seconds to segment the document and additional 4.5 seconds to perform the recognition step. The test was executed on a Personal Computer at a moderate performance level (Celeron 433 processor). The segmentation step of the algorithm takes approximately the same time to finish as the actual recognition step. The performance of the algorithm can be improved highly by making the procedure parallel.

We inserted a noise generator step into the character recognition process after the segmentation. This way we corrupted the image with different noise types (global, contour) at different levels before executing the recognition step. We applied uniformly distributed additive noise corruption and the level of the corruption is defined as the percentage of the pixels which are affected by the noise corruption. Global noise corruption means that all the points of the binary image are involved in the noise corruption, while in the case of contour noise corruption only the contour points are affected.

To perform a comparative analysis, we made some tests for our recognition algorithm, and for the methods based on projection histograms, and zoning. Synthetic images were generated with the character sets CMR, OCR-A, OCR-B. These prototype documents were used to train the algorithms, so the prototype feature vectors were calculated. We composed some one page test documents containing regular text, and generated 100 noisy images for every input document. For noise corruption we used global and contour noise separately and parallely, as well. Moreover, tests were made on actually scanned printed material which is equivalent to a small (approximately 1%) global and a moderate (approximately 20%) contour noise corruption. Using these samples, we made an analysis according to the recognition accuracy of our method for each symbol of the alphabets. We also calculated an average accuracy value to measure the recognition efficiency of the above algorithms, and got the best result in the case of ours.

Our experiments indicated strong correlation among the levels of the noise corruption, the feature vector of the segment, and its distance from the corresponding prototype vector. At the significance level 0.001 the correlation coefficient is $r = 0.7875$.

If the recognition is restricted only to digits then the dimension of the feature space can be reduced. According to a factor analysis the dimension of the feature space can be reduced from 64 to 48 without violating the accuracy of the recognition. We can have a moderate recognition accuracy (at the level of 90%) if we use only a 16-dimensional feature space.

For more details about our experiments see [14] and the dissertation.

4. An application – compressing typeset documents

As a practical application we built our character recognition method into an information loss compressive algorithm, see [13]. The original digital images are supposed to contain basically text information – with a small number of symbols – recorded in a typographically fixed form, and optionally some graphic parts. During the compression our main purpose is to preserve the visual information of the document. The graphic parts of the document are compressed by an information preserving compression program, while the text information is recognized and the characters are encoded with their character code. The characters can be magnified as the algorithm is invariant for magnification. The experimental analysis indicated that the algorithm tolerates noise corruption quite well. The noise sensitivity of the method can be reduced further by lexical analysis. The algorithm was tested in several cases, and proved to be pretty efficient and reliable for simple documents.

The segmentation procedure also means an efficient compression, since it eliminates the large background components from the compressed data structure. We used an LZW based (see, e.g. [27] Chapter 5, p. 127-132) compression program and investigated the efficiency of the compression, when our method was also applied. We experienced a highly better compression performance, when our method was applied before LZW.

Szomszédsági szekvenciák és karakterfelismerés Walsh transzformációval

Jelen PhD tézisek a digitális képfeldolgozás területére eső új eredményeket tartalmaznak. A dolgozat két részre tagolódik. Az első részben a szomszédsági szekvenciák elméletével kapcsolatos eredményeinket ismertetjük, míg a második részben egy Walsh transzformáción alapuló karakterfelismerő eljárást mutatunk be.

I. Szomszédsági szekvenciák

A klasszikus digitális – 4-szomszédos (cityblock) és 8-szomszédos (chessboard) – mozgásokat $\mathbb{Z}^2$-ben ROSENFELD és PFALTZ [26] vezette be. Egy 4-szomszédos lépéssel csak vízszintesen vagy függőlegesen, míg 8-szomszédos lépéssel átlósan is mozoghatunk. Ezekhez a lépésekhez kötődően ROSENFELD és PFALTZ két távolságfüggvényt definiált. Két pont d_4, illetve d_8 távolsága az ahhoz szükséges 4-, illetve 8-szomszédos lépések minimális száma, hogy az egyik pontból a másikba jussunk. Az euklideszi metrika egy jobb közelítéséhez ROSENFELD és PFALTZ a 4- és 8-szomszédos lépések felváltva történő alkalmazását javasolta, amely a d_{oct} oktagonális távolságot adja. A szerzők [26]-ban azt is megemlítették, hogy a közelítés tovább javítható a 4- és 8-szomszédos lépések további kombinációival.

A 4- és 8-szomszédos lépések tetszőleges periodikus sorozatba való rendezésével DAS és szerzőtársai [5]-ben bevezették a $\mathbb{Z}^2$-beli periodikus szomszédsági szekvencia fogalmát, majd kiterjesztették azt tetszőleges véges dimenzióra. A témakör részletes irodalmi áttekintését a disszertáció 1. fejezete tartalmazza.

1. Szomszédsági szekvenciák srukturális vizsgálata

Legyen n egy pozitív egész szám. Egy $p \in \mathbb{Z}^n$ pont i-edik ($i = 1, \dots, n$) koordinátáját jelölje $\mathrm{Pr}_i(p)$. Legyen M egy olyan egész, hogy $0 \le M \le n$. A $p, q \in \mathbb{Z}^n$ pontokat M-szomszédoknak mondjuk, ha kielégítik a következő két feltételt:

- $|\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \le 1 \quad (1 \le i \le n)$,
- $\sum\limits_{i=1}^{n} |\,\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \le M$.

Az $A = (A(i))_{i=1}^{\infty}$ sorozatot, ahol $A(i) \in \{1, \dots, n\}$ minden $i \in \mathbb{N}$-re, n-dimenziós (röviden nD) általános szomszédsági szekvenciának nevezzük. Ha létezik olyan $l \in \mathbb{N}$, melyre $A(i+l) = A(i)$ $(i \in \mathbb{N})$, akkor A periodikus l periódussal. Az nD általános szomszédsági szekvenciák halmazát S_n-nel jelöljük. Megjegyezzük, hogy az általános szomszédsági szekvencia az [5]-ben bevezetett periodikus szomszédsági szekvencia általánosítása.

Legyen $p, q \in \mathbb{Z}^n$ és $A \in S_n$. A $p = p_0, p_1, \dots, p_m = q$ pontsorozatot, ahol p_{i-1} és p_i $A(i)$-szomszédok $\mathbb{Z}^n$-ben $(1 \le i \le m)$, egy p-t q-val összekötő m hosszú A-útnak nevezzük. A p és q pont $d(p, q; A)$-val jelölt A-távolságán a pontok között vezető legrövidebb A-út hosszát értjük. Az A-távolságot röviden $d(A)$-val jelöljük. Minden $i \in \mathbb{N}$ és $j \in \{1, \dots, n\}$ esetén vezessük be az alábbi jelöléseket:

$$A^{(j)}(i) = \min(A(i), j) \quad \text{és} \quad f_j^A(i) = \sum_{k=1}^{i} A^{(j)}(k).$$

Sokszor fontos két pont A-távolságának meghatározása. Ezt DAS szerzőtársaival [5]-ben megtette periodikus szekvenciákra nézve. A disszertációban (lásd 2. fejezet, 2.9. Lemma) ezt kiterjesztjük az általános esetre.

Legyen $A, B \in S_n$ tetszőleges. Definiáljuk a $\sqsupseteq^*$ relációt az alábbi módon:

$$A \sqsupseteq^* B \quad \Longleftrightarrow \quad d(p,q;A) \leq d(p,q;B), \text{ minden } p,q \in \mathbb{Z}^n \text{ esetén.}$$

Ez a reláció a DAS [4] által periodikus esetben bevezetett rendezés általánosítása. A disszertáció 3. fejezetében (lásd 3.7. Tétel) megmutatjuk, hogy

$$d(p,q;A) \leq d(p,q;B),\ \forall p,q \in \mathbb{Z}^n\text{-re} \quad \Longleftrightarrow \quad f_j^A(i) \geq f_j^B(i),\ \forall i \in \mathbb{N}\text{-re},\ j \in \{1,\ldots,n\}\text{-re.}$$

Az értekezés 3.1. szakaszában azt is megvizsgáljuk, hogy a szomszédsági szekvenciák halmazai milyen hálóelméleti struktúrát alkotnak a fenti rendezésre nézve. S_n mellett annak ugyanazon típusú részhalmazait elemezzük, mint DAS [4]-ben a periodikus esetben. Mivel a $\sqsupseteq^*$ rendezés ritkán indukál szép hálóstruktúrákat, ezért bevezetjük annak egy durvább változatát is, amelyet használva szebb strukturális eredményekhez jutunk. Minden $A = (A(i))_{i=1}^{\infty}$, $B = (B(i))_{i=1}^{\infty} \in S_n$ esetén definiáljuk a $\sqsupseteq$ relációt az alábbi módon:

$$A \sqsupseteq B \quad \Longleftrightarrow \quad A(i) \geq B(i), \quad \text{minden } i \in \mathbb{N} \text{ esetén.}$$

A fenti strukturális eredményeinket a ∞ dimenziós digitális térre is kiterjesztjük a disszertáció 3.2. szakaszában. Az ott kapott hálóstruktúrák bizonyos értelemben a véges dimenziós hálók uniói lezártjának tekinthetők. Az értekezés 2. és 3. fejezete mellett ezek az eredmények [15]-ben is közlésre kerültek FAZEKAS ATTILA és HAJDU LAJOS társszerzőkkel.

2. Szomszédsági szekvenciák metrikus vizsgálata

A $\sqsupseteq^*$ rendezés csak féligrendezés a szomszédsági szekvenciák halmazán, ám egyes esetekben szükség lehet bármely két szekvencia „gyorsaságának” összehasonlítására. Ezen probléma megoldására a disszertáció 4. fejezetében bevezetünk egy norma jellegű sebességfüggvényt a szekvenciák halmazán, és megvizsgáljuk annak tulajdonságait. A sebesség definiálásához a következő feltételeket vesszük alapul:

(i) *A sebességnek érzékenynek kell lennie a szekvencia egyes alterekben való viselkedésére.*
(ii) *A szekvencia elemei különböző mértékben járuljanak hozzá a sebességhez.*
(iii) *A sebességnek összhangban kell lennie a $\sqsupseteq^*$ rendezéssel.*

A fentieknek megfelelően a sebességet a következőképpen adjuk meg. A $\delta_j : \mathbb{N} \to \mathbb{R}$ $(j = 1,\ldots,n)$ függvényhalmazt súlyrendszernek nevezzük, ha teljesül az alábbi három feltétel:

- $\delta_j(i) \geq 0 \quad (i \in \mathbb{N},\ j = 1,\ldots,n)$,
- $\sum\limits_{i=1}^{\infty} \delta_j(i) < \infty \quad (j = 1,\ldots,n)$,
- δ_j monoton csökkenő $\ (j = 1,\ldots,n)$.

Legyen $A \in S_n$ és $\delta_j \ (j = 1,\ldots,n)$ egy súlyrendszer. Az A szekvencia j-dimenziós sebessége

$$v_j^A = \sum_{i=1}^{\infty} A^{(j)}(i)\delta_j(i),$$

az A sebessége pedig

$$v^A = \frac{1}{n}\sum_{j=1}^{n} v_j^A.$$

Mivel $\sqsupseteq^*$ csak féligrendezés, ezért (iii) csak az egyik irányban teljesülhet. A másik irány megvalósításához még egy további feltétel szükséges:

$v_j^A \geq v_j^B$, minden δ_j-re $(j = 1, \dots, n) \implies A \sqsupseteq^* B$, minden $j = 1, \dots, n$-re.

A sebesség bevezetéséhez hasonlóan a disszertáció 4.5. szakaszában metrikát is definiálunk a szomszédsági szekvenciák halmazán, és elvégezzük a szokásos vizsgálatokat. Legyen $\Delta = \{\delta_j : j = 1, \dots, n\}$ egy súlyrendszer és $A, B \in S_n$. Ezen szekvenciák ϱ_Δ távolságát a következőképpen definiáljuk:

$$\varrho_\Delta(A, B) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{\infty} |A^{(j)}(i) - B^{(j)}(i)| \ \delta_j(i).$$

Az értekezés 4.5. szakaszában megmutatjuk azt is, hogy minden Δ súlyrendszer esetén (S_n, ϱ_Δ) teljes metrikus tér. Továbbá, a metrikus tér minden korlátos monoton sorozata konvergens, és minden korlátos végtelen részhalmazának létezik torlódási pontja. A periodikus sorozatok halmaza a metrikus tér egy sűrű részhalmazát adja. A sebesség és metrika bevezetésével kapcsolatos eredményeket és gyakorlati példákat a disszertáció 4. fejezete és a HAJDU LAJOSSAL közös [19] tartalmazza.

3. Szomszédsági szekvenciák geometriai vizsgálata

A disszertáció 5. fejezetében a szomszédsági szekvenciák által elfoglalt tartományokat vizsgáljuk meg. A szekvenciák „terjedése” eltolásinvariáns, ezért az egyszerűség kedvéért a $\mathbb{Z}^n$ tér $\mathbf{0}$ origójából indulunk. Legyen $A \in S_n$ és minden $k \in \mathbb{N}$-re

$$\widehat{A_k} = \{q \in \mathbb{Z}^n : d(\mathbf{0}, q; A) \leq k\}$$

az A által az első k lépésben elfoglalt tartomány. Jelölje $H(\widehat{A_k})$ az $\widehat{A_k}$ konvex burkát $\mathbb{R}^n$-ben.

Approximációs módszereinkhez a szomszédsági szekvenciákhoz és a klasszikus euklideszi metrikához tartozó „körök” összehasonlítása szolgál alapul. Minden $k \in \mathbb{N}$ esetén legyen

$$O_k = \{q \in \mathbb{Z}^2 : L_2(\mathbf{0}, q) \leq k\}, \qquad \text{illetve} \qquad G_k = \{q \in \mathbb{R}^2 : L_2(\mathbf{0}, q) \leq k\}$$

a k sugarú euklideszi körlap $\mathbb{Z}^2$-ben, illetve $\mathbb{R}^2$-ben.

Számításainkban gyakran használjuk egy $A \in S_n$ szomszédsági szekvencia első k elemében előforduló azonos elemek számát. Legyen $A = (A(i))_{i=1}^{\infty} \in S_n$ és minden $i = 1, \dots, n$ és $k \in \mathbb{N}$ esetén

$$A(i, k) = \#\{A(j) : A(j) = i, \ 1 \leq j \leq k\}.$$

Az elfoglalt tartományok aszimptotikus tanulmányozásához a szomszédsági szekvencia egyes elemeinek sűrűségét (amennyiben ez nem létezik, alsó és felső sűrűségét) használjuk a disszertáció 5. fejezetében.

DAS és CHATTERJI [9] az euklideszi metrika szomszédsági szekvenciákkal való közelítésének hatékonyságát vizsgálta. Nevezetesen, azon nyolcszögek geometriai tulajdonságait elemezték, amelyeket egy szomszédsági szekvencia az origóból kiindulva a síkban való terjedésekor elfoglal. Általánosított szomszédsági szekvenciákat használva az euklideszi metrika jobban közelíthető. Ennek megfelelően a disszertáció 5. fejezetében kiterjesztjük DAS és CHATTERJI [9] 2D-ben elvégzett geometriai vizsgálatait általános szomszédsági szekvenciákra, és beágyazzuk a korábbi eredményeket az új környezetbe. $\mathbb{Z}^2$ helyett ezeket a vizsgálatainkat tetszőleges véges dimenzióra elvégezzük. Egy sok helyen jól felhasználható eredményként (lásd disszertáció 5. fejezet, 5.4. Tétel) először megadjuk az elfoglalt poliéderek csúcspontjait. Legyen $A \in S_n$ és $k \in \mathbb{N}$. A $H(\widehat{A_k})$ poliéder csúcsai pontosan azok a pontok, melyeknek koordinátái az alábbiak egy permutációja

$$\left(\lambda_1 \sum_{i=1}^{n} A(i, k), \ \lambda_2 \sum_{i=2}^{n} A(i, k), \ \dots, \ \lambda_n A(n, k) \right),$$

ahol $\lambda_j = \pm 1$ $(j = 1, \ldots, n)$, és λ_j eltérhet az egyes permutációkban.

Az n-dimenziós esetben, a disszertáció 5.1. szakaszában meghatározzuk az elfoglalt poliéderek átmérőjét, és vizsgáljuk az átmérő és a lépésszám hányadosának aszimptotikus viselkedését. Mivel a 2D-s és 3D-s eset kiemelt jelentősséggel bír a képfeldolgozásban, ezért a $\mathbb{Z}^2$ és $\mathbb{Z}^3$ tereket behatóan tanulmányozzuk a disszertáció 5.2. szakaszában. A szomszédsági szekvencia elemeinek sűrűségével (illetve alsó/felső sűrűségével) meghatározzuk az elfoglalt tartományok oldalhosszának, területének és térfogatának aszimptotikus viselkedését. Geometriai eredményeink a disszertáció 5. fejezete mellett [17]-ben is közlésre kerültek.

4. Az euklideszi metrika közelítése digitális metrikákkal

Az euklideszi metrika digitális távolságokkal való közelítése széleskörben kutatott terület. Das [3]-ban olyan szomszédsági szekvenciák által indukált távolságfüggvényeket adott meg, amelyek egy bizonyos értelemben az euklideszi metrika jó közelítéseit adják. Mi is végrehajtunk egy hasonló, de általánosabb vizsgálatot az euklideszi metrika közelítésére szomszédsági szekvenciák segítségével a disszertáció 6. fejezetében. Approximációs modellünk geometriai megközelítésen alapszik, nevezetesen az euklideszi metrika és a szomszédsági szekvenciák távolságfüggvényeihez tartozó különböző sugarú köröket hasonlítjuk össze. Szemben Das [3] eredményeivel, periodikus helyett általános szekvenciákkal dolgozunk, és lehetőség nyílik a legjobban közelítő szomszédsági szekvencia explicit, zárt alakban való megadására is. Vizsgálatunk a szomszédsági szekvenciák egész halmazára kiterjed, és megadjuk a legjobban közelítő, metrikát generáló szekvenciákat is.

Három approximációs problémát fogalmazunk meg, amelyek megoldása a legjobban közelítő szomszédsági szekvenciákat szolgáltatja. Az alábbiakban $T(H)$ a H síkbeli tartomány területét jelöli.

1. probléma. Keressük meg azt a $\Lambda \in S_2$ szomszédsági szekvenciát, amelyre minden $A \in S_2$ és $k \in \mathbb{N}$ esetén

$$T(H(\widehat{\Lambda_k}) \bigtriangledown G_k) \leq T(H(\widehat{A_k}) \bigtriangledown G_k).$$

2. probléma. Keressük meg azt a $\Phi \in S_2$ szomszédsági szekvenciát, amelyre minden $k \in \mathbb{N}$ esetén $H(\widehat{\Phi_k}) \supseteq G_k$, továbbá ha $A \in S_2$ és $H(\widehat{A_k}) \supseteq G_k$, akkor

$$T(H(\widehat{\Phi_k}) \setminus G_k) \leq T(H(\widehat{A_k}) \setminus G_k).$$

A disszertáció 6.2. szakaszában megadunk két, a fenti problémákat kielégítő szomszédsági szekvenciát. Ezen kívül meghatározunk egy szekvenciát, amely olyan digitális metrikát indukál, amely az L_2-t az első probléma értelmében legjobban közelíti. A következő kérdés a második probléma diszkrét változatára vonatkozik.

3. probléma. Keressük meg azt a $\Psi \in S_2$ szomszédsági szekvenciát, amelyre $\widehat{\Psi_k} \supseteq O_k$ minden $k \in \mathbb{N}$ esetén, és ha $A \in S_2$ úgy, hogy $\widehat{A_k} \supseteq O_k$, akkor $\widehat{A_k} \supseteq \widehat{\Psi_k}$.

A probléma megoldásához ugyancsak egy olyan szomszédsági szekvenciát tudunk megadni a disszertáció 6.2. szakaszában, ami metrikát generál. A megoldás során ú.n. Beatty-sorozatokat kapunk, így a legjobban approximáló szomszédsági szekvenciákat zárt alakban fel tudjuk írni. Approximációs eredményeink az értekezés 6. fejezetében és a Hajdu Lajos társszerzővel közös [18] cikkben is megtalálhatók.

II. Karakterfelismerés Walsh transzformációval

A disszertáció második részében egy Walsh transzformáción alapuló, saját fejlesztésű karakterfelismerőt mutatunk be, amely elsősorban nyomdatechnikailag rögzített szövegek felismeréséhez használható. Az algoritmus elkészítése során a szegmentálásra és osztályozásra fektettük a hangsúlyt, és kísérletekkel ellenőriztük rendszerünk megbízhatóságát. A digitális képfeldolgozáson belül gyakran használt Walsh transzformációt nagyításinvarianciája és geometriát megőrző tulajdonsága teszi alkalmassá karakterfelismerésre. A felismerni kívánt karakterekhez Walsh transzformáltakból álló sajátságvektorokat rendelünk, amelyeket etalonvektorok alapján osztályozunk. A Walsh etalonvektorok meglehetősen jól szeparálhatók, ezért a tévesztés esélye kisebb, mint más klasszikus (például projekciós hisztogramon vagy zónázáson alapuló) módszerek esetében. Algoritmusunkban nem számolunk ki minden transzformáltat, ami csökkenti a számítási időt és fokozza a zajjal szembeni toleranciát. Módszerünk hatékonyságának és megbízhatóságának bizonyítékaképpen bemutatjuk eljárásunk zajtűrő képességét, továbbá más felismerőkkel való összehasonlításának eredményeit.

Az algoritmusunk első lépésként a kiindulási bináris képet kisebb szegmensekre bontja, amelyeket egy láncolt listában tárol el. A szegmensek téglalapok, melyek bal felső pontjainak koordinátái és méretei kerülnek tárolásra a képi információval együtt. A szegmentációs algoritmus részletes leírása az értekezés 9.1. szakaszában található meg. Karakterfelismerő eljárásunk a disszertáció 9.2. szakaszában leírt Walsh transzformáción alapul. Minden szegmenshez egy 64D-s sajátságvektort készítünk a szegmens első 64 Walsh transzformáltjából. A sajátságvektor elkészítése után eldöntjük, hogy a szegmens felismerhető karaktert vagy egyéb képadatot tartalmaz-e. Az értekezés 9.3. szakasza tartalmazza a különböző szinteken végrehajtható döntési folyamatot és az algoritmus taníthatóságának feltételeit. A hatékonyság ellenőrzésére, a zajérzékenységre és a felismerés pontosságára vonatkozó tesztjeinkbe olyan gyakran használt betűtípusokat vontunk be, mint az OCR-A, OCR-B, CMR, TIMES NEW ROMAN és a magyar rendszámtáblákon szereplő karakterek. Ezeken kívül más statisztikai teszteket, például korrelációs analízist is végeztünk. Kísérleti tapasztalatainkat a disszertáció 9.4. szakasza rögzíti. A vizsgálat eredményei a FAZEKAS ATTILA társszerzővel közös [14] publikációban ugyancsak megtalálhatók.

Karakterfelismerő eljárásunknak egy gyakorlati alkalmazását is adtuk: az eljárást beépítettük egy géppel írt dokumentumok tömörítésére használható, karakterfelismerésen alapuló rendszerbe. Az alkalmazás leírását és a végrehajtott teszteket az értekezés 9.5. szakasza és a FAZEKAS ATTILÁVAL közös [13] is tartalmazza.

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