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Full Length Article

Pollutant dynamics in a subterranean estuary (Waquoit Bay, MA, USA) via mathematical modeling

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ABSTRACT

Minerals and nutrients are vital to life, but their excess can harm health and sometimes lead to life risks. Harmful impact of nutrients in water can be a threat to marine life as well. Waquoit Bay linked with its eight rivers and ponds is becoming the reason for excess nutrient passage into the Bay. Excess amounts of nutrients were found in Waquoit Bay from coastal areas and therefore analyzed in this work by a system of nine differential equations. The main objective of this paper is to develop a mathematical model to assess the situation of pollution in Waquoit Bay and suggest some solutions using optimal control theory with Hamiltonian system. Mathematical Model of water pollution will be evaluated numerically, with different input models to track environmental contamination in a water body. To prove the accuracy of this model, its numerical results will be compared with real data and the impact of control strategies i.e. nutrient control from housing area and bio dredging will be studied. Benefit of developing such mathematical models can be a decision support for planning restrictions in coastal areas near Waquoit Bay.

1. Introduction

Nitrogen, the fifth most important element in the universe, was discovered by Daniel Rutherford (a Scottish Scientist) in 1772. It constitutes about 78% of the earth's atmosphere, but if its composition is somehow increased more than a certain desired level in the earth's soil or water, it will be detrimental to the survival of life and living creatures. Nitrogen is undoubtedly very elementary to life and the productivity of coastal ecosystems but as the excess of everything is bad, so does its surplus. Excess amounts of phosphorus, silicon, iron, nitrogen, and other minerals cause nutrient pollution [1]. Local waters, downstream waters, coastal waters, streams, or watersheds are the most affected areas by a nutrient imbalance in water or so-called water pollution. Excess nutrients act as a fertilizer for sea plants, resulting in excessive growth of coral reefs or macro-algae and phytoplankton, etc. [2]. Plenty of reasons can lead to incrementing the quantity of nitrogen in soil and water. These reasons may be due to an increase in urbanization near coastal areas, deforestation, or use of fertilizers on agricultural lands [3].

Almost a decade ago, a Bay named Waquoit Bay in the U.S. was observed to be affected by the increase in the number of nutrients [5]. Waquoit Bay is a major national estuary, usually used as a refuge for research exploration. It is part of Nantucket Sound and is located in Massachusetts, USA, on the southern shore of Cape Cod. This Bay defines the boundary of Falmouth and Mashpee, Massachusetts towns. Waquoit comes from the word "Weeqayut" (Waquoit) from Wampanoag, meaning "Place of Light." The research teams conducted several studies and found out that the changes in the ecological system and geochemistry in that area and Bay where due to a continuous increase in the nutrients. Due to this, the quality of life was being affected near ponds,

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coastal rivers, and harbors in many communities in southeastern Massachusetts. In the Towns of Mashpee and Falmouth, the problems in coastal waters include [6]: loss of eelgrass beds, undesirable increments in full-scale green growth, reductions in the decent variety of benthic creature populaces, periodic green growth blossoms. Without appropriate administration, more extreme issues may create, including: periodic fish killing, unpleasant smells and filth and Benthic people group decreased to the most pressure tolerant species, or in the most pessimistic scenarios, close to loss of the benthic animal networks.

Waquoit Bay is a particularly good example of shallow bays across the northeastern U.S. Analysis here increases the overall understanding of these areas and encourages coastal sustainability across the country. Due to changes in biogeochemistry and food chains related to this area, Waquoit Area has been debated in various research papers since the 1980s. Coastal communities, including Mashpee and Falmouth, rely on safe, healthy, and aesthetically pleasing marine and estuarine waters for tourism, recreational swimming, fishing, shell fishing, and commercial fin fishing. Failure to minimize and monitor N loads would result in the complete replacement of Eelgrass with macro-algae. Higher extreme levels decrease in concentrations of dissolved oxygen and fish kills, widespread incidence of unpleasant odors and noticeable scum, and complete loss of benthic macroinvertebrates in most embayments. Because of these impacts on the environment, industrial and recreational uses of the coastal waters of Waquoit Bay will be greatly reduced and could eventually cease. Due to the tourism effect and residential interest in this Bay area. It is becoming the attention of tourists; hence buildings and structures are being formed, and an increase in nutrients is natural. Nitrogen streams into the waters of coastal embayments from the following sources [6]: River valley, sub-surface sewage collection structures on-site, natural environment, groundwater, sewage treatment facilities, atmospheric pollution, nutrient-rich bottom sediments in embayments.

To analyze this situation, a project was launched in 2002 by the U.S. government [10], [11]. This report presents the results generated from implementing the Massachusetts Estuaries Project's Linked Watershed-Embayment Approach to the Waquoit Bay embayment system. The primary products of this initiative are: the latest quantitative nutrient-related health evaluation of the Waquoit Bay embayment, the detection of all nitrogen sources (and their corresponding N loads) in embayment waters, the preservation of Massachusetts Water Quality Requirements in embayment waters, watershed nitrogen load mitigation study to achieve N threshold concentrations in embayment waters, operational optimized and validated Linked Watershed-Embayment modeling method that can be easily used to test alternatives to nitrogen management (to be built by the municipalities) for the restoration of the embayment system in Waquoit Bay.

Research conducted on this issue not only found traces of nutrients but radon was also discovered [12] and hence the alarming situation was tried to be controlled [13], [14]. To evaluate the health of 89 ecosystems in the coastal embayment of southern Massachusetts in 2001, the Massachusetts Estuaries Project (MEP) was established. The MEP's goal was to establish nitrogen levels and nutrient reductions necessary to support healthy ecosystems. They published their results very recently that give a detailed report on the watershed of the upper cape: Waquoit Bay in Falmouth, Mashpee, and Sandwich. Hence many geological and ecological studies have been done on the increasing amount of nutrients in Waquoit Bay and its surroundings. The actual data depicts the abnormal amount of nutrients continuously increasing, which has side effects on Eels, Jellyfish, etc. Hence all these studies show that historical land-use developments on coastal watersheds have raised soil-derived nitrogen loading levels in estuaries and modified their biogeochemistry and food chains since the 1930s. Around 1938 and 1990, the amount of nitrogen in Waquoit Bay more than tripled. The primary nitrogen source applied to the Bay shifted during the 1980s from atmospheric deposition to sewage disposal, reflecting Cape Cod's increased population growth. Larger nitrogen loads increased nitrogen concentrations in the waters, influencing primary producers' assembly and resulting in estuary water pollution. Phytoplankton and macroalgae biomass increased, and areal eelgrass cover (Zostera marina) reduced with raised nitrogen supply. The rise in N ha^{-1} $year^{-1}$ nitrogen load from 15 to 30 kg nearly eliminated the grasslands of Eelgrass. Therefore, land-use modifications caused by urbanization can be related to significant changes in water quality and receiving water algae blooms. All such modifications are not only harmful for marine life but also for human lives. Humans can be exposed to the chemicals that are caused by nutrient pollution by drinking this water or accidentally swallowing while swimming. Water that is affected by harmful algal can cause serious illness if swallowed such as liver or stomach problems and skin rashes. Therefore to study this complex problem we use mathematical modeling.

These days, mathematical models are being developed that describe the complex real-life processes in almost every field very efficiently such as analyze the elimination of Direct Red 81 using zero-valent iron nanoparticles from aqueous solutions that are harmful for the environment [24], [25], [26]. Mathematical models helps the researchers to predict or forecast the outcomes of certain phenomenons such as reduction of energy consumption and its effect on the environment, or to analyze the blood flow of Covid 19 patients with atherosclerotic etc [31], [32], [33], [34], [4]. Mathematically, specifically water contamination, industrial and ecological phenomena were also studied such as the harmful dyes and pharmaceutical products that contaminate the environment can be eliminated from aqueous solutions see [27], [28]. Among such mathematical models is the water pollution model of [7] and [8]. Their model was modified [9]. He utilized those concepts to show the rate of change of pollutants in an interconnected three-lake system. [22] presented a mathematical model to show the impact of pollution in a river and improvement in pollutants after aeration. [23] developed a mathematical model based on advection-diffusion equations. This model discussed the pollutants amount present in lowland Barnaulka river in Russian federation. [29] developed a mathematical model that studied the pollutant flow rate in small lowland rivers. [30] also developed a mathematical model that studied the transmission of water pollution in large water bodies.

In this work, we developed a mathematical model based on a system of nine ordinary differential equations. The main objective of this study is to develop a new mathematical model for Waquoit Bay and test whether the findings are reasonably close to the actual data and what can be suggested to improve or control the pollutant situation of Waquoit Bay. Studies conducted physically requires a lot of effort, time and funds to produce the results of changes occurring in Waquoit Bay whereas a mathematical model, if provided with accurate data can predict the results very accurately. Therefore, in this work, optimal control strategy will be applied to study the impact of different controls on the increasing situation of nutrient pollution in Waquoit Bay. Now let's form the mathematical model for water pollution in Waquoit Bay.

2. Mathematical formulation

In this section, we are going to develop a mathematical model of Waquoit Bay water pollution exactly as suggested by the authors of [9], [7], [8]. Waquoit bay, shown in Fig. 1, has eight other water bodies connected through channels of different widths. The red dot denotes pollutants being introduced from the points or water bodies in Fig. 1 and in mathematical equations written as p(t). These red dots basis on the USA national center for Environmental Assessment Agency [10]. $p(t)$ is considered as the nutrient enrichment from x_2, x_3, and x_5 in the form of fertilizers from the garden, agricultural lands, lawns, waste treatment, industrial waste, and atmospheric deposition (Table 1).

To begin with assume C: concentration of the pollutant, Q: the volumetric flow rate through the Bay, v: volume of the Bay, M: mass of the Pollutant.

Few assumptions have been made for this model, i.e., v will remain constant, the flow rate remains constant and denoted by F_{ij} which shows

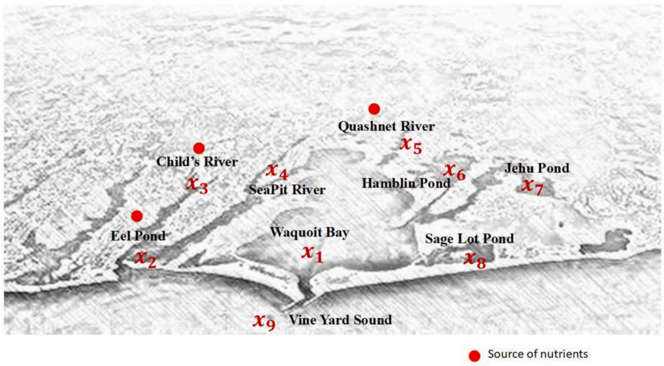


Fig. 1. Picture of interconnected water bodies to Waquoit Bay, i.e., Child's River, Quashnet River, and Seapit River, they flow towards Waquoit Bay. Whereas Eel Pond, Hamblin Pond, Jehu Pond, and Sage lot Pond are smaller water bodies and flow inwards with smaller volume to Waquoit Bay. Vine Yard Sound is linked with Waquoit Bay, and therefore contamination in Waquoit Bay contaminates the Vine Yard Sound. Red circle marks the major source of pollution of Waquoit Bay.

Table 1
The water bodies (i.e., rivers or ponds, watersheds, etc.) around Waquoit Bay, named as variables for the calculation of the Waquoit Bay model.

Variable	Water Bodies
x_1	Waquoit Bay
x_2	Eel pond
x_3	Child's River
x_4	Seapit River
x_5	Quashnet River
x_6	Hamblin pond
x_7	Jehu pond
x_8	Sagelot pond
x_9	Vineyard Sound

the flow of water from water body i to j, reaction rate remains constant, the lake is well mixed. The movement of nutrients is in the same direction as the fluid flow known as the advective flow will be assumed in this model.
As it is proved that the rate of change of nutrients in water bodies = Input rate − output rate. Hence the basic mixture model introduced in [8] is

$$mass \times time^{-1} = rate\ in\ -\ rate\ out, \tag{1}$$

Now, for this case, nine water bodies are interconnected with each other see Fig. 1. The nutrient source is considered from the highly populated areas only, i.e., Eel pond, Child's river, and Quashnet river. The channels connecting through have no backward flow. Flow from Child's river and Sea pit river is downward, entering the eel pond and then the Vineyard sound, so this can be written as F_{34}, F_{29}, F_{41}. Therefore, let the amount of pollution in each water body is $p_i(t)$ and the volume of each water body is v_i, then the concentration of the pollutant in each water body be $C_i(t) = \frac{x_i(t)}{v_i}$. Initially, each water body is considered to be pollution free, so $x_i(0) = 0$. Since flux of pollutant will occur with the flow of water, the flow of flux of the pollutant from water body i to water body j denoted by $m_{ij}(t)$ is given in gal/min as $m_{ij}(t) = F_{ij}$ and therefore $C_i(t) = \frac{F_{ij} x_i(t)}{v_i}\ kg/min$. Applying this principle to each water body results in the following system of first-order equations modeling the dynamic behavior of the lake system:

$$\begin{aligned}
\frac{dx_1}{dt} =& \frac{F_{41}}{v_4}x_4(t) + \frac{F_{51}}{v_5}x_5(t) - \frac{F_{19}}{v_1}x_1(t) \\
&- \frac{F_{16}}{v_1}x_1(t) - \frac{F_{17}}{v_1}x_1(t) - \frac{F_{18}}{v_1}x_1(t) \\
&+ \frac{F_{91}}{v_9}x_9(t) - \frac{F_{15}}{v_1}x_1(t) + \frac{F_{61}}{v_6}x_6(t) + \frac{F_{71}}{v_7}x_7(t) \\
&+ \frac{F_{81}}{v_8}x_8(t) - \frac{F_{41}}{v_1}x_1(t), \\
\frac{dx_2}{dt} =& p(t) - \frac{F_{29}}{v_2}x_2(t) + \frac{F_{42}}{v_4}x_4(t) + \frac{F_{32}}{v_3}x_3(t) \\
&+ \frac{F_{92}}{v_9}x_9(t) - \frac{F_{23}}{v_2}x_2(t), \\
\frac{dx_3}{dt} =& p(t) - \frac{F_{32}}{v_3}x_3(t) + \frac{F_{23}}{v_2}x_2(t), \\
\frac{dx_4}{dt} =& \frac{F_{41}}{v_1}x_1(t) - \frac{F_{14}}{v_4}x_4(t) - \frac{F_{42}}{v_4}x_4(t), \\
\frac{dx_5}{dt} =& p(t) - \frac{F_{51}}{v_5}x_5(t) + \frac{F_{15}}{v_1}x_1(t), \\
\frac{dx_6}{dt} =& \frac{F_{16}}{v_1}x_1(t) - \frac{F_{61}}{v_6}x_6(t), \\
\frac{dx_7}{dt} =& \frac{F_{17}}{v_1}x_1(t) - \frac{F_{71}}{v_7}x_7(t), \\
\frac{dx_8}{dt} =& \frac{F_{18}}{v_1}x_1(t) - \frac{F_{81}}{v_8}x_8(t), \\
\frac{dx_9}{dt} =& \frac{F_{19}}{v_1}x_1(t) + \frac{F_{29}}{v_2}x_2(t) - \frac{F_{92}}{v_9}x_9(t) - \frac{F_{91}}{v_9}x_9(t).
\end{aligned} \tag{2}$$

In order for the volume of each lake to remain constant, the flow rate into each lake must balance the flow out of the lake. Thus we assume the following conditions:

$$\begin{aligned}
Waquoit Bay :& F_{41} + F_{51} + F_{91} + F_{61} + F_{71} + F_{81} = \\
& F_{19} + F_{16} + F_{17} + F_{18} + F_{15} + F_{41}, \\
Eelpond :& F_{42} + F_{32} + F_{92} = F_{23} + F_{29}, \\
Child`sRiver :& F_{23} = F_{32}, \\
SeapitRiver :& F_{41} = F_{14} + F_{42}, \\
QuashnetRiver :& F_{15} = F_{51}, \\
Hamblinpond :& F_{16} = F_{61}, \\
Jehupond :& F_{17} = F_{71}, \\
Sagelotpond :& F_{18} = F_{81}, \\
VineyardSound :& F_{19} + F_{29} = F_{92} + F_{91},
\end{aligned} \tag{3}$$

with the initial condition

$$x_1(0) \geq 0, x_2(0) \geq 0, x_3(0) \geq 0, x_4(0) \geq 0,$$

$$x_5(0) \geq 0, x_6(0) \geq 0, x_7(0) \geq 0, x_8(0) \geq 0, x_9(0) \geq 0.$$

Now, if we notice, this model can further be reduced into a system of five ordinary differential equations since we are concerned with pollutant intake of Waquoit Bay, so to study that, we can remove the differential equation for $x_9(t)$ vineyard. So if we remove this equation, Eel pond and Child's river do not directly link to Waquoit Bay, so we can remove equations for $x_2(t)$ and $x_3(t)$. Also, the minute amount interfering in Waquoit Bay of these water bodies can be considered a constant in the first equation. Let this constant be known as c. Now the remaining system has five differential equations, such as

$$\begin{aligned}
\frac{dx_1}{dt} =& c+\frac{F_{51}}{v_5}x_5(t)-\frac{F_{16}}{v_1}x_1(t)-\frac{F_{17}}{v_1}x_1(t)-\frac{F_{18}}{v_1}x_1(t)-\frac{F_{15}}{v_1}x_1(t)+\\
&\frac{F_{61}}{v_6}x_6(t)+\frac{F_{71}}{v_7}x_7(t)+\frac{F_{81}}{v_8}x_8(t)-\frac{F_{41}}{v_1}x_1(t),\\
\frac{dx_4}{dt} =& \frac{F_{41}}{v_1}x_1(t)-\frac{F_{14}}{v_4}x_4(t)-\frac{F_{42}}{v_4}x_4(t),\\
\frac{dx_5}{dt} =& p(t)-\frac{F_{51}}{v_5}x_5(t)+\frac{F_{15}}{v_1}x_1(t),\\
\frac{dx_6}{dt} =& \frac{F_{16}}{v_1}x_1(t)-\frac{F_{61}}{v_6}x_6(t),\\
\frac{dx_7}{dt} =& \frac{F_{17}}{v_1}x_1(t)-\frac{F_{71}}{v_7}x_7(t),\\
\frac{dx_8}{dt} =& \frac{F_{18}}{v_1}x_1(t)-\frac{F_{81}}{v_8}x_8(t). \qquad (4)
\end{aligned}$$

3. Optimal control model analysis

In this section, we will formulate an optimal control problem to minimize the impact of nutrient pollution in Waquoit Bay with two optimal control variables $u_1(t)$ and $u_2(t)$. These control variables are efforts to reduce the pollutant amount going into Waquoit Bay from two primary sources, $p(t)$ from Quashnet river and the remaining water bodies collectively contributing in Waquoit Bay as c. These efforts are to keep the nutrients under control according to the threshold provided by Massachusetts Estuaries Project [14] by controlling the septic systems that cause the 75% of the nutrient pollution. Also, we suggest using Bio-dredging in Waquoit Bay. As indicated by [15], bio-dredging effectively reduces nutrients from a Bay compared to dredging. But it can cause an initial decline in the regrowth of Eel grass, but it can increase after the two seasons. Main reason for suggesting only these two optimal controls, i.e. control nutrient input and bio dredging is because these two options can be started at once and does not require hefty funds. Simple dredging means to pick the sludge and try to clean the water bodies from this sludge which is the host of several water borne pathogens and produce foul smell. Whereas in bio dredging a nano scale mixture is added in the water and it digests the organic waste with time and it does not effect the environment or marine life present in water bodies.

$$\begin{aligned}
\frac{dx_1}{dt} =& c+u_2\frac{F_{51}}{v_5}x_5(t)-\frac{F_{16}}{v_1}x_1(t)-\frac{F_{17}}{v_1}x_1(t)-\frac{F_{18}}{v_1}x_1(t)\\
&-u_1\frac{F_{15}}{v_1}x_1(t)+\frac{F_{61}}{v_6}x_6(t)+\frac{F_{71}}{v_7}x_7(t)+\frac{F_{81}}{v_8}x_8(t)-\frac{F_{41}}{v_1}x_1(t),\\
\frac{dx_4}{dt} =& \frac{F_{41}}{v_1}x_1(t)-\frac{F_{14}}{v_4}x_4(t)-\frac{F_{42}}{v_4}x_4(t),\\
\frac{dx_5}{dt} =& p(t)-u_2\frac{F_{51}}{v_5}x_5(t)+u_1\frac{F_{15}}{v_1}x_1(t),\\
\frac{dx_6}{dt} =& \frac{F_{16}}{v_1}x_1(t)-\frac{F_{61}}{v_6}x_6(t),\\
\frac{dx_7}{dt} =& \frac{F_{17}}{v_1}x_1(t)-\frac{F_{71}}{v_7}x_7(t),\\
\frac{dx_8}{dt} =& \frac{F_{18}}{v_1}x_1(t)-\frac{F_{81}}{v_8}x_8(t). \qquad (5)
\end{aligned}$$

Eq. (7) minimizes the impact of Eq. (5) by optimal control variables u_1 and u_2 therefore written as:

$$I=\int_0^t \left[w_1x_1(t)+w_2x_5(t)+\frac{1}{2}(w_3(u_1(t))^2+w_4(u_2(t))^2)\right]dt. \qquad (6)$$

In Eq. (6) t denotes the final time, w_1, w_2, w_3 and w_4 are the weight constants of x_1, x_5, u_1 and u_2 respectively. The purpose of using quadratic cost function is because cost by nature is not linear [16].
Let us prove its existence using the concepts given in [17]. Let us consider a closed set $U=\{(u_1,u_2)|u_j(t), 0\le u_j(t)\le 1, j=1,2\}$, where $u_j(t)$ is measurable on $[0,t]$ then there is $u*=(u_1^*,u_2^*\in U)$ an optimal control pair based on Eq. (5) with its initial conditions for which $I(u_1^*,u_2^*)=min\{I(u_1,u_2)|u_1(t),u_2(t)\in U\}$

Theorem 3.1. *For the proof of the existence of optimal control problem in Eq. (5), we need to show*

- $\{u_1,u_2\}>0$ *also* $x_1,x_4,x_5,x_6,x_7,x_8>0$.
- U *defined above is closed and convex.*
- *Eq. (5) should be bounded in an equilibrium state by a linear function.*
- *Eq. (6) is concave in U.*
- *Integrand* $M(t,u_1,u_2)\triangleq x_1(t)+x_5(t)+\frac{1}{2}u_1^2(t)\frac{1}{2}u_2^2(t)$ *satisfy* $M(t,u_1,u_2)\ge a_1\left(|u_1|^2+|u_2|^2\right)^2-a_2$ *for constants positive constants* a_1,a_2.

Proof. As mentioned above that, both state and control variables are positive. Also, the set U is closed and convex. Also, I the integrand is a convex function on U with respect to (u_1,u_2). Since the system takes the form

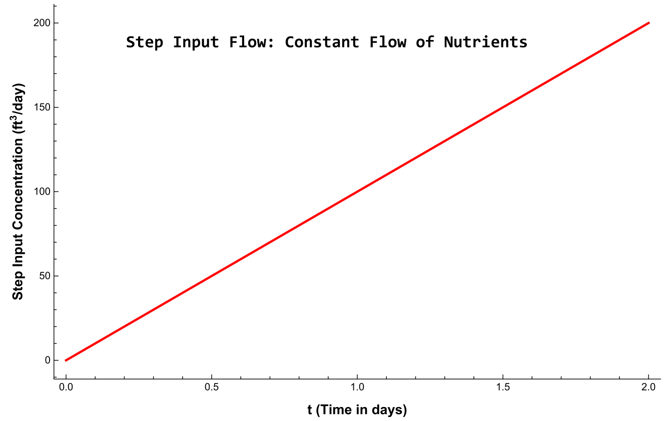
$$\begin{aligned}
\frac{dx_1}{dt} \le& c+u_2\frac{F_{51}}{v_5}x_5(t)+\frac{F_{61}}{v_6}x_6(t)+\frac{F_{71}}{v_7}x_7(t)+\frac{F_{81}}{v_8}x_8(t),\\
\frac{dx_4}{dt} \le& \frac{F_{41}}{v_1}x_1(t),\\
\frac{dx_5}{dt} \le& p(t)+u_1\frac{F_{15}}{v_1}x_1(t),\\
\frac{dx_6}{dt} \le& \frac{F_{16}}{v_1}x_1(t),\\
\frac{dx_7}{dt} \le& \frac{F_{17}}{v_1}x_1(t),\\
\frac{dx_8}{dt} \le& \frac{F_{18}}{v_1}x_1(t). \qquad (7)
\end{aligned}$$

From this, the Lipschitz property is satisfied for the state system based on state variables as they are bounded.
Now if $a_1=min\left\{\frac{w_1}{2},\frac{w_2}{2}\right\}$ is chosen then $M(t,u_1,u_2)\ge a_1\left(|u_1|^2+|u_2|^2\right)^2-a_2$ proved to be true $\forall a_2\in\Re^+$. Hence the proof is complete. □

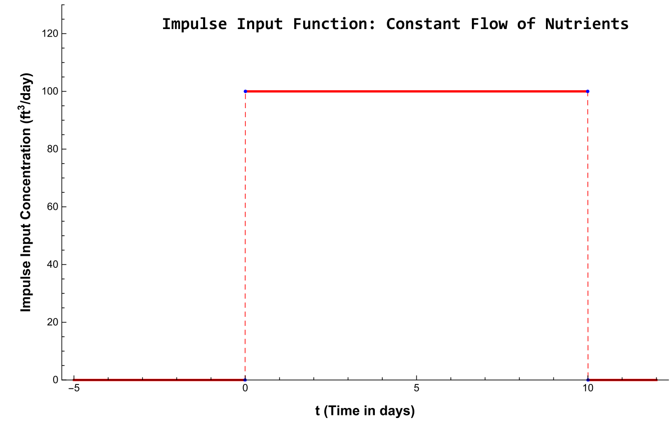
4. Hamiltonian system and optimal control

[18] suggests that the best way to describe control systems is through the trajectory of the Hamiltonian system, therefore the Hamiltonian obtained for this system is given as

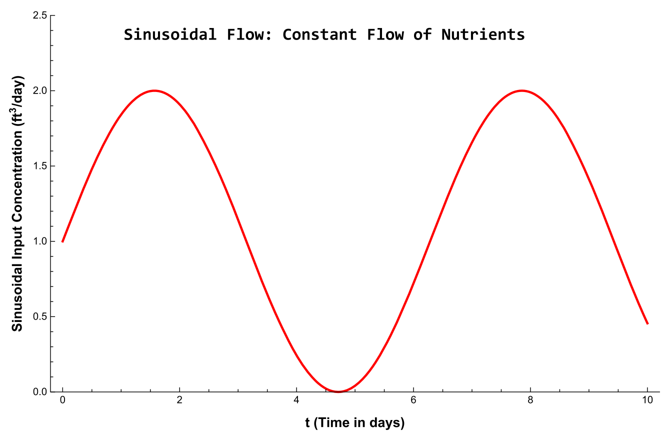
$$\begin{aligned}
\mathcal{H} =& w_1x_1+w_2x_2+\frac{1}{2}(w_3u_1^2+w_4u_2^2)\\
&+\lambda_1\left(c+u_2\frac{F_{51}}{v_5}x_5(t)-\frac{F_{16}}{v_1}x_1(t)\right.\\
&-\frac{F_{17}}{v_1}x_1(t)-\frac{F_{18}}{v_1}x_1(t)-u_1\frac{F_{15}}{v_1}x_1(t)\\
&+\frac{F_{61}}{v_6}x_6(t)+\frac{F_{71}}{v_7}x_7(t)+\frac{F_{81}}{v_8}x_8(t)\\
&\left.-\frac{F_{41}}{v_1}x_1(t)\right)+\lambda_2\left(\frac{F_{41}}{v_1}x_1(t)-\frac{F_{14}}{v_4}x_4(t)-\frac{F_{42}}{v_4}x_4(t)\right)\\
&+\lambda_3\left(p(t)-u_2\frac{F_{51}}{v_5}x_5(t)+u_1\frac{F_{15}}{v_1}x_1(t)\right)\\
&+\lambda_4\left(\frac{F_{16}}{v_1}x_1(t)-\frac{F_{61}}{v_6}x_6(t)\right)\\
&+\lambda_5\left(\frac{F_{17}}{v_1}x_1(t)-\frac{F_{71}}{v_7}x_7(t)\right)
\end{aligned}$$



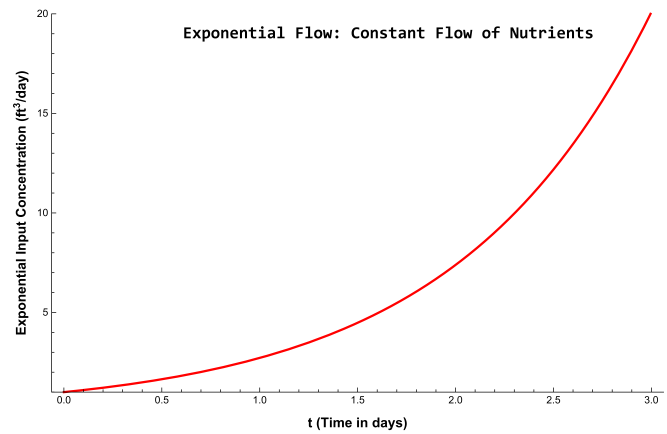
(a) Step input behavior is shown, i.e., after t_0, it abruptly moves to 100 and remains constant afterward.



(b) Impulse input behavior is shown, i.e., a sudden huge amount of nutrients is added in the water body for continuous 10 days whereas before and after, the amount is zero. So initially, it abruptly moves to 100 and remains constant afterward.



(c) Sinusoidal input is shown here. The input pattern is a waveform pattern that represents the oscillated pattern for nutrient input.



(d) Exponential input is shown here. The input pattern is increasing continuously with respect to time.

Fig. 2. Graphical interpretation of four different patterns of entering pollutants in the water bodies.

$$+\lambda_6\left(\frac{F_{18}}{v_1}x_1(t)-\frac{F_{81}}{v_8}x_8(t)\right), \tag{8}$$

where the adjoint variable functions $\lambda_j, j=1,\cdots,5$ are yet to be determined.

Theorem 4.1. *Consider the equations that satisfy the adjoint variables $\lambda_j, j=1,\cdots,5$ as*

$$\begin{aligned}
\frac{d\lambda_1}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_1}(t), & \frac{d\lambda_2}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_4}(t),\\
\frac{d\lambda_3}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_5}(t), & \frac{d\lambda_4}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_6}(t),\\
\frac{d\lambda_5}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_7}(t), & \frac{d\lambda_6}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_8}(t),
\end{aligned} \tag{9}$$

having conditions $\lambda_j(t)=0, j=1,\cdots,5$ and with the control set as

$$\begin{aligned}
u_1^* &= max\left\{0, min\left(1, \frac{F_{15}}{v_1 w_3}x_1(t)\left(\lambda_1-\lambda_3\right)\right)\right\},\\
u_2^* &= max\left\{0, min\left(1, \frac{F_{51}}{v_5 w_4}x_5(t)\left(\lambda_3-\lambda_1\right)\right)\right\}.
\end{aligned} \tag{10}$$

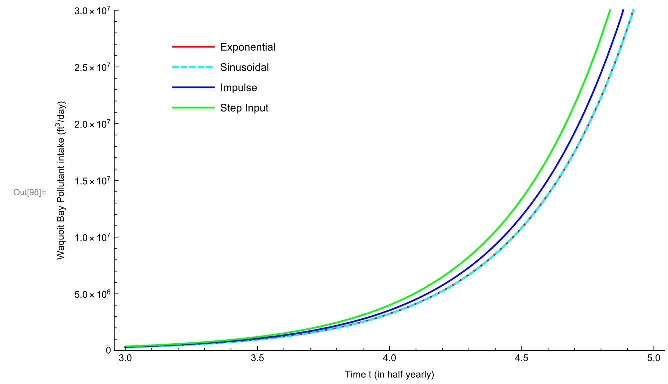
Proof. According to [18], the adjoint system can be obtained by differentiating the Hamiltonian as

$$\begin{aligned}
\frac{d\lambda_1}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_1}(t)=-w_1-\left(\lambda_4-\lambda_1\right)\frac{F_{16}}{v_1}-\left(\lambda_5-\lambda_1\right)\frac{F_{17}}{v_1}\\
&\quad-\left(\lambda_6-\lambda_1\right)\frac{F_{18}}{v_1}-\left(\lambda_3-\lambda_1\right)u_1\frac{F_{15}}{v_1}-\left(\lambda_2-\lambda_1\right)\frac{F_{41}}{v_1},\\
\frac{d\lambda_2}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_4}(t)=-\lambda_2\left(-\frac{F_{14}}{v_4}-\frac{F_{42}}{v_4}\right),\\
\frac{d\lambda_3}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_5}(t)=-\left(\lambda_1-\lambda_3\right)u_2\frac{F_{51}}{v_5},\\
\frac{d\lambda_4}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_6}(t)=-\left(\lambda_1-\lambda_4\right)\frac{F_{61}}{v_6},\\
\frac{d\lambda_5}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_7}(t)=-\left(\lambda_1+\lambda_5\right)\frac{F_{71}}{v_7},\\
\frac{d\lambda_6}{dt}&=-\frac{\partial\mathcal{H}}{\partial x_8}(t)=-\left(\lambda_1+\lambda_6\right)\frac{F_{81}}{v_8},
\end{aligned} \tag{11}$$

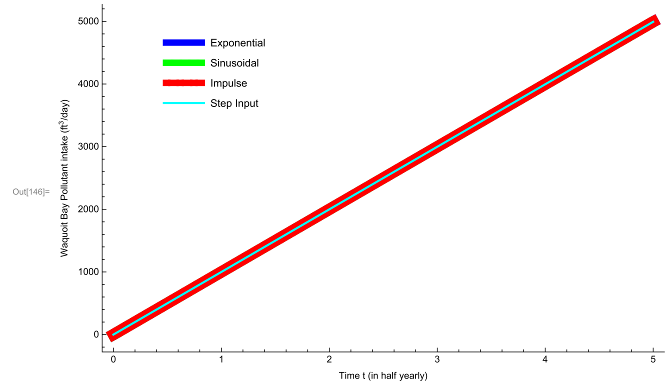
by applying the transversally conditions, i.e., $\lambda_j(t)=0 \quad \forall \quad j$. It can also be observed that (u_1^*, u_2^*) is satisfying the condition very clearly i.e.

$$\begin{aligned}
\frac{\partial\mathcal{H}}{\partial u_1}&=0,\\
\frac{\partial\mathcal{H}}{\partial u_2}&=0.
\end{aligned} \tag{12}$$

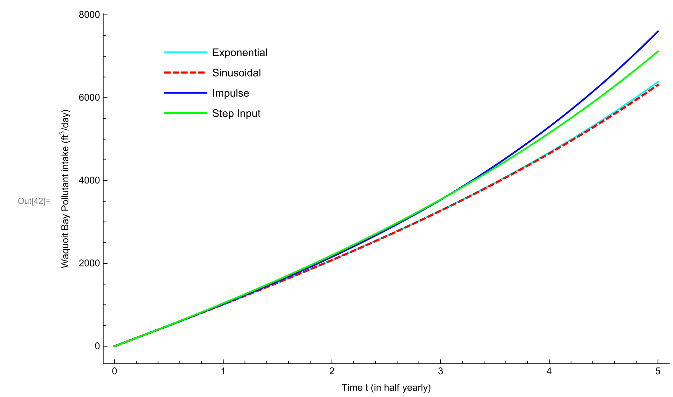
Upon solving Eq. (12) the theorem is proved. □



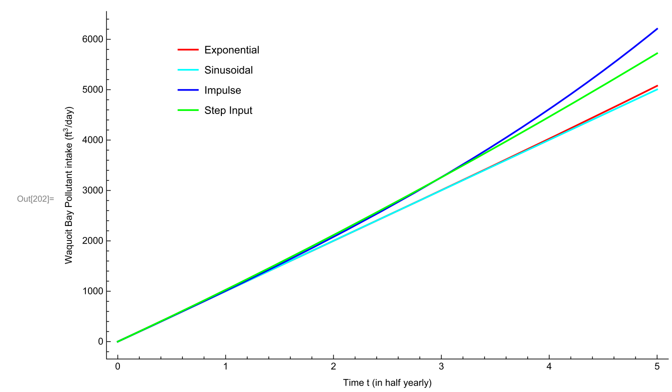
(a) Numerical Solution of Eq.2.Pollutants entering Waquoit Bay according to the model Eq.2 and impact of pollutants with no control i.e. $u_1 = 0, u_2 = 0$. A graphical representation of no control with four different types of input for Waqoit Bay.



(b) A graphical representation of control parameters $u_1 \neq 0, u_2 = 0$ on nutrient pollutants entering Waquoit Bay.



(c) A graphical representation of control parameters $u_1 \neq 0, u_2 \neq 0$ on nutrient pollutants entering Waquoit Bay.



(d) A graphical representation of control parameters $u_1 = 0, u_2 \neq 0$ on nutrient pollutants entering Waquoit Bay.$u_1 = 0, u_2 \neq 0$.

Fig. 3. Graphical results of numerical solution of Waquoit Bay showing the nutrient pollution input and behavior with control and no control parameters.

5. Numerical approximation of Waquoit Bay model

Mathematical model is just a set of equation until it is proved to be true numerically as well as theoretically. After proving it correct theoretically now let us prove it accurate numerically. Nutrients enter the water bodies in four different ways, i.e. day and night continuous constant flow of nutrients in water bodies mathematically this is called Step input. Sudden dumping of contaminated water in water bodies for a certain time period and no dumping before and after this time frame, mathematically this is called impulse input. Sinusoidal input is the day and night dumping of contaminated water in Waquoit Bay and its water bodies. Exponential input is the type of dumping which increases with time. These input models enable the pollution equation to simulate nutrient channels closer than standard equations to the real world. These input behavior are shown in Fig. 2. The first step after the modeling is verifying the model's accuracy with the data from authorized officials. Data from the official Cape Cod mission reports were collected to verify the accuracy of this model see [19], [20], [21]. To compare the simulated data with the original values following formulas have been implemented. Volume = average water body depth (ft) × Area (ft^2) and Flow rate (ft^3/day)= Area (ft^2) × Velocity (ft/day).

5.1. Step input model

The first case is the step-input model. The step input model is used for nutrients entering the water bodies at a constant rate and frequency and then proceeding indefinitely in the same way. The nutrients penetrated the system at $t_0 = 0$, and there was nothing in the sample until time zero. The important points for this case are that the input increases abruptly at $t_0 = 0$ and that after zero time, the input remains relatively constant see Fig. 2a. So put $p(t) = 100 \times t$ in Eq. (2).

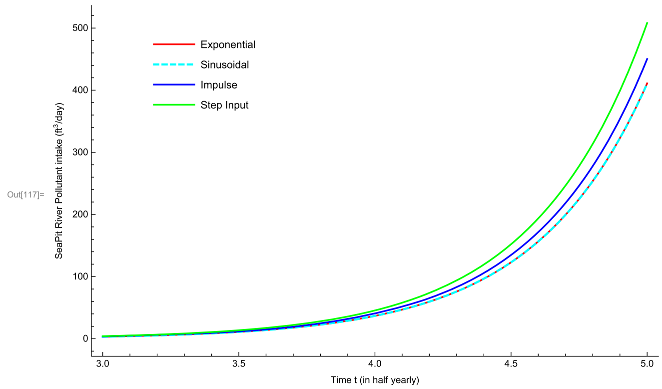
5.2. Impulse input model

$$p(t) = \begin{cases} 0 & t < 0 \\ 100 & 0 \le t \le 10 \\ 0 & t > 10 \end{cases}$$

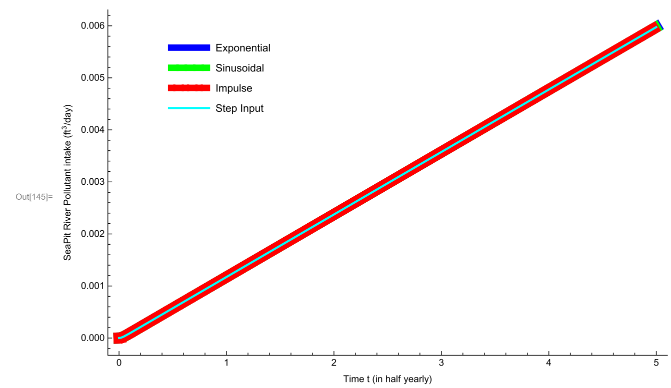
The impulse input model is used for immediately releasing pollutants into the lake. Impulse input functions have a pulse, and the variable is zero everywhere else. The steep rise is the time the contaminant was dumped at. The model shows the level of water pollution after some time has passed, provided the impulse has to be at time zero. The significant points are that the lake continues with an initial pollutant concentration, and no pollutant enters the lake once the model has appropriately started. It can be a case of sudden dumping of something containing a higher amount of nutrients. The behavior of such $p(t)$ is shown in Fig. 2b.

5.3. Sinusoidal input model

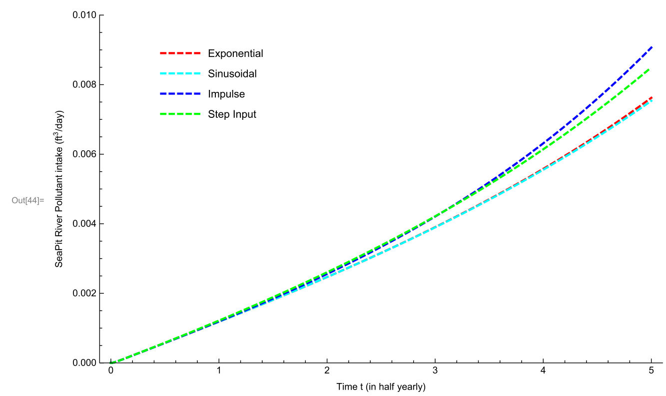
The sinusoidal interface model will be used for contaminants periodically added to the Bay. The pollution inhabits the system with an average concentration and periodically fluctuates around that average. Sinusoidal interface converts p(t) towards something more practical i.e.



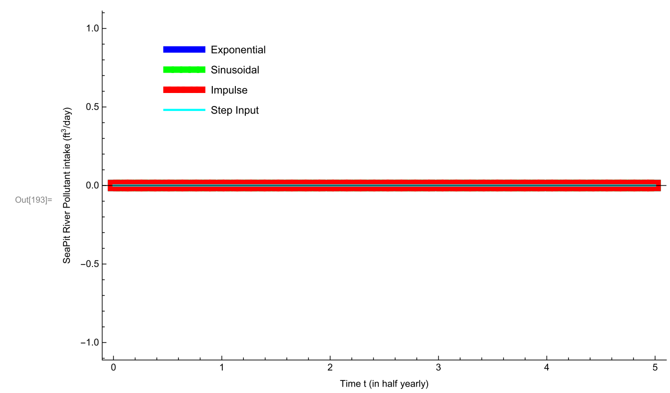
(a) Numerical Solution of Eq.2.Pollutants entering Sea Pit river according to the model Eq.2 and impact of pollutants with no control i.e. $u_1 = 0, u_2 = 0$. A graphical representation of no control with four different types of input for Sea Pit river.



(b) A graphical representation of control parameters $u_1 \neq 0, u_2 = 0$ on nutrient pollutants entering Sea Pit river.



(c) A graphical representation of control parameters $u_1 \neq 0, u_2 \neq 0$ on nutrient pollutants entering Sea Pit river.



(d) A graphical representation of control parameters $u_1 = 0, u_2 \neq 0$ on nutrient pollutants entering Sea Pit river.

Fig. 4. Graphical results of numerical solution of Sea Pit river showing the nutrient pollution input and behavior with control and no control parameters.

$p(t) = n_i(1 + \lambda sin(\frac{2\pi t}{T}))$, where T Time variational period. λ Amplitude between 0 and 1. n_i The average number of nutrient input concentrations. For simplicity, assume $p(t) = 1 + sin(t)$ see Fig. 2c

5.4. Exponential input model

The last case is the exponential input model here, $p(t) = exp(t)$ will be assumed see Fig. 2d. This explains the situation where the nutrients are exponentially increasing with time in the pollutant areas of ponds and rivers. Due to this, the main Water body, Waquoit Bay, is affected.

6. Results & discussion

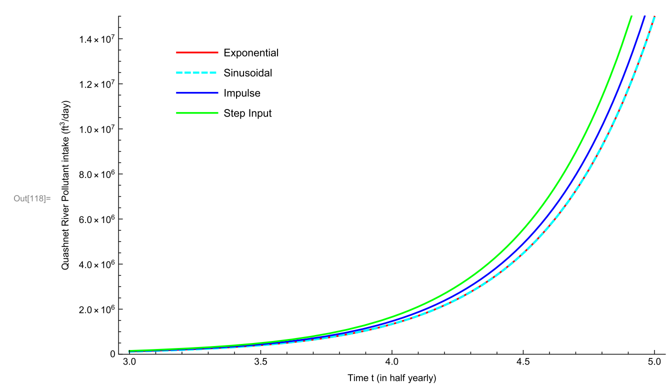
Now let us explain the behavior of each $p(t)$ on our model. Graphically the results of each water body with no control have been shown in Fig. 3a, 4a, 5a, 6a, 7a, 8a.

In Table 2 and Fig. 3a, pollutants entering our model's first water body, i.e., Waquoit Bay, are shown. It has also been compared with the actual data, and another model is called NLM model. In the first year, the actual pollutant that entered Waquoit Bay was approximately 27000, but NLM model predicted 22532, which is inaccurate. Whereas our four pollutant flow types are very close to the actual value. The closest value is the Step input, which depicts that the water bodies connected to Waquoit Bay constantly pass nutrients into the Bay as time passes. Fig. 3a also shows that all these inputs depict a very closer value to each other.

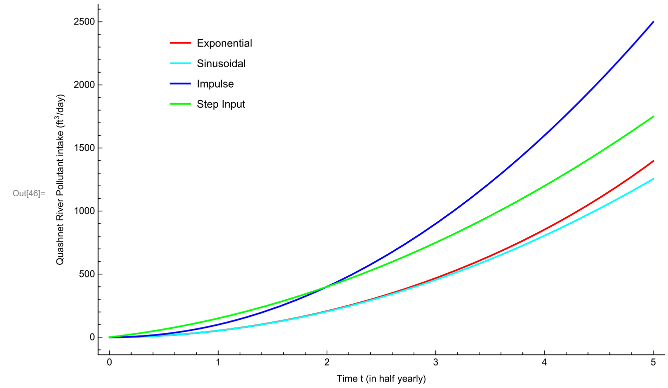
There are two control parameters that have been applied on Eq. (2) for suggesting the control on nutrient pollution. First one is to control the dumping of nutrients from nearby areas through sewerage whether its the residential area, industrial or tourist destinations i.e. u_1. Second control parameter is u_2 that describes the bio dredging and through this the amount of nutrient population can be controlled as well.

The threshold for Nitrogen in Waquoit Bay is $14.04 TMDL^3$ per day, i.e., Total Maximum Daily Load, and Quashnet river is $11.25 TMDL^3$ per day. Therefore by applying our control variables, the value is lower than the threshold see Table 3, 4 and 5. Results in these tables graphically show that if these strategies are applied, then we can reduce the number of nutrients entering Waquoit Bay.

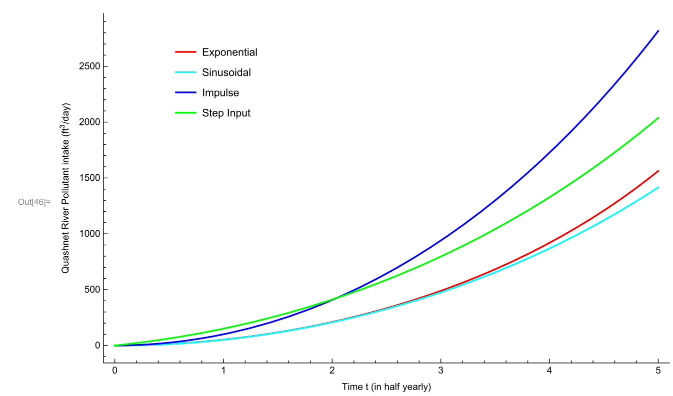
First case of this simulation is shown in Fig. 3 for Waquoit Bay. In these graphical results, the first graph Fig. 3a shows the pollutants entering the water streams and Bay with four different input behavior such as impulse input, step input, exponential input or sinusoidal input but with no controls. As we can observe in Fig. 3a amount of pollutants entering in Waquoit Bay is very high and increasing exponentially since this is the case of no control where $u_1 = 0, u_2 = 0$. Therefore, when we apply the first control i.e. when $u_1 \neq 0$ and $u_2 = 0$, we can observe a drastic reduction in the nutrient addition in Waquoit Bay just by controlling the nutrient factor from water pumps and sewerage system near Waquoit Bay see Fig. 3c. Now for the second case, considering $u_1 \neq 0, u_2 \neq 0$ means that with controlling the nutrient input, bio dredging is also being applied and as it can be observed in Fig. 3b, the amount of pollutants is reduced drastically. Third case under discussion is where only bio dredging is being applied in Fig. 3d i.e. $u_1 = 0, u_2 \neq 0$ and it is also effective as the other control parameter was effective. By observing these four cases the most effective strategy seems to be $u_1 \neq 0, u_2 = 0$ for Waquoit Bay i.e. just by controlling the amount being dumped in



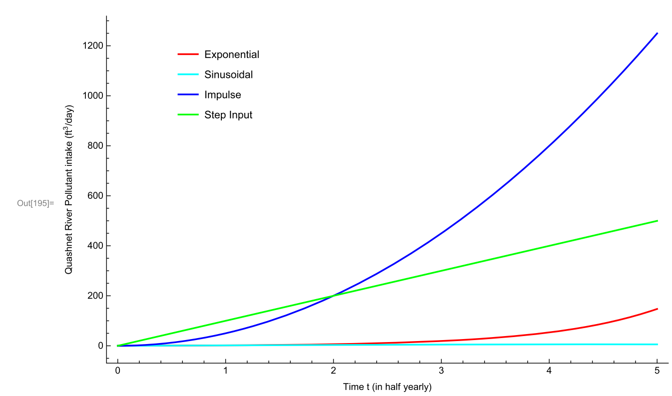
(a) Numerical Solution of Eq.2.Pollutants entering Quashnet river according to the model Eq.2 and impact of pollutants with no control i.e. $u_1 = 0, u_2 = 0$. A graphical representation of no control with four different types of input for Quashnet river.



(b) A graphical representation of control parameters $u_1 \neq 0, u_2 = 0$ on nutrient pollutants entering Quashnet river.



(c) A graphical representation of control parameters $u_1 \neq 0, u_2 \neq 0$ on nutrient pollutants entering Quashnet river.



(d) A graphical representation of control parameters $u_1 = 0, u_2 \neq 0$ on nutrient pollutants entering Quashnet river.

Fig. 5. Graphical results of numerical solution of Quashnet river showing the nutrient pollution input and behavior with control and no control parameters.

Bay through sewerage.

Now for the Sea Pit river, all four input cases studied with different control parameters and the results imply the same as in Waquoit Bay. In Fig. 4a there is no control, hence the amount of pollution is very high in any of the input patterns but as we apply control in Fig. 4c, Fig. 4b and Fig. 4d, a sudden reduction in pollutants can be observed in graphs. Among all four cases with no control, just nutrient control, nutrient control and bio dredging and only bio dredging, the better results for Sea Pit river are for the case where both controls have been applied i.e. $u_1 \neq 0$ and $u_2 \neq 0$.

Fig. 5a represents the no control $u_1 = 0, u_2 = 0$ simulation for Quashnet river and as expected the amount of pollutants in very high. For the control theory Fig. 5c shows the first case in which only pollutant control have been applied i.e. $u_1 \neq 0, u_2 = 0$, it reduces the amount of pollutants drastically. Similar is the case with other two controls i.e. $u_1 \neq 0, u_2 \neq 0$ and $u_1 = 0, u_2 \neq 0$, the pollutants reduce very quickly. Among these cases for Quashnet river, the control that gives better result is Fig. 5d where $u_1 = 0, u_2 \neq 0$ i.e. only bio dredging is being applied.

Fig. 6a shows the no control numerical solution with four inputs of Hamblin pond, which is very high. When the control is being applied on Hamblin pond as shown in Fig. 6c, Fig. 6b and Fig. 6d, the best solution to control nutrient pollution in Hamblin pond is $u_1 = 0, u_2 \neq 0$ that is shown in Fig. 6d.

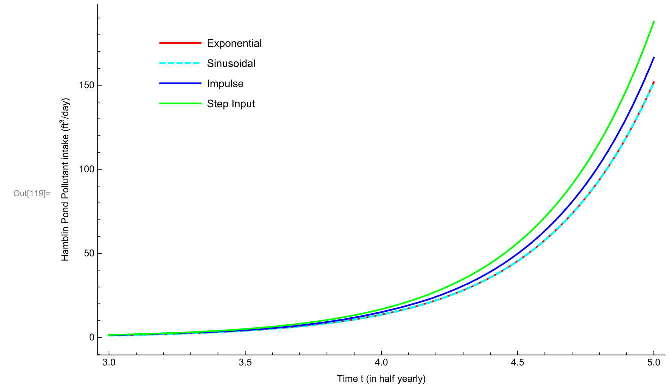
Fig. 7a shows the no control numerical solution with four inputs of Jehu pond, which is very high. When the control is being applied on Jehu pond as shown in Fig. 7c, Fig. 7b and Fig. 7d, the best solution to control nutrient pollution in Jehu pond is $u_1 = 0, u_2 \neq 0$ that is shown in Fig. 7d.

Fig. 8a shows the no control numerical solution with four inputs of Sage Lot pond, which is very high. When the control is being applied on Sage Lot pond as shown in Fig. 6c, Fig. 8b and Fig. 8d, the best solution to control nutrient pollution in Sage Lot pond is $u_1 \neq 0, u_2 = 0$ that is shown in Fig. 8b. Table values prove the closest result among all inputs to the actual data is sinusoidal. The most relevant result that suggests fewer nutrient pollutants is sinusoidal input, even when the control parameters are applied.

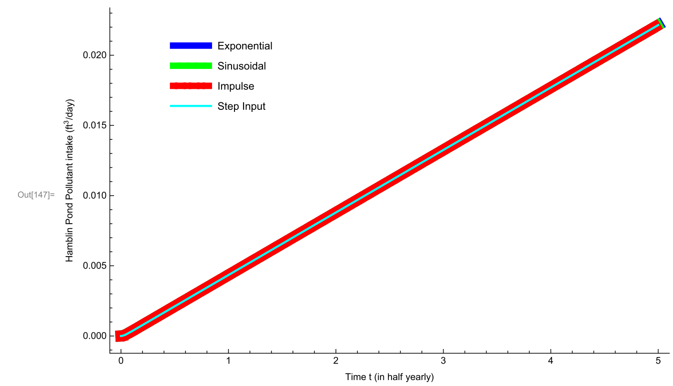
7. Conclusion

In this paper, the lake model developed for estimating pollutants in interconnected lakes by channels is tested for a real-life case of Waquoit Bay. Waquoit Bay is connected to different smaller and large water bodies in the form of ponds and lakes. Therefore it was the best scenario to develop the Waquoit Bay model due to two reasons. Firstly, it is linked with ponds and rivers; secondly, studies conducted on this Bay and the increment in the number of nutrients were studied by U.S. Government officials; hence, a fair amount of data is easily available for comparing these results. In this work, a mathematical model is developed to calculate the change in the nutrient pollution being added in Waquoit Bay and vineyard sound by rivers and ponds. To check the validity of this model, values have been compared by the NLM model and with actual data.

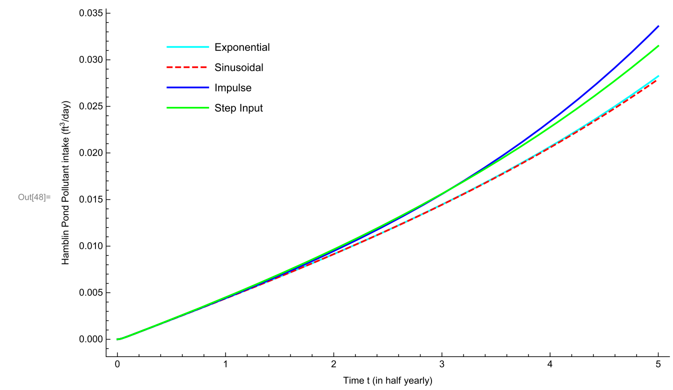
Four different pollutant input functions have been considered, such as the exponential function, impulse function, sinusoidal function, and step function. These inputs describe the water dumping and usage in coastal areas i.e. when there are tourists and the parks and Bay are open for visiting hours, it may have impulse input or step input. For the



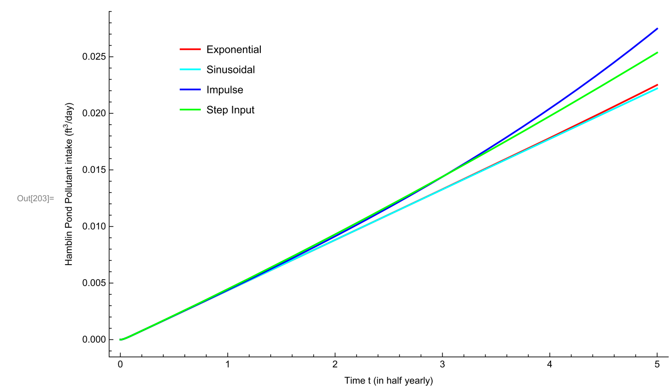
(a) Numerical Solution of Eq.2. Pollutants entering Hamblin pond according to the model Eq.2 and impact of pollutants with no control i.e. $u_1 = 0, u_2 = 0$. A graphical representation of no control with four different types of input for Hamblin pond.



(b) A graphical representation of control parameters $u_1 \neq 0, u_2 = 0$ on nutrient pollutants entering Hamblin pond.



(c) A graphical representation of control parameters $u_1 \neq 0, u_2 \neq 0$ on nutrient pollutants entering Hamblin pond.



(d) A graphical representation of control parameters $u_1 = 0, u_2 \neq 0$ on nutrient pollutants entering Hamblin pond.

Fig. 6. Graphical results of numerical solution of Hamblin pond showing the nutrient pollution input and behavior with control and no control parameters.

Table 2
Flow of nutrients into the water bodies calculated in this work is compared with the actual values and NLM Model. Four types of flows are considered for comparison. Volume is calculated in ft^3 and flow rate in ft^3/day.

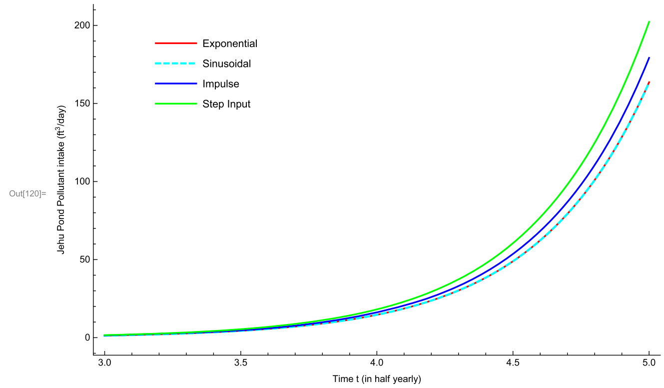
Water Bodies	Actual Data	NLM Model	Step Input	Sinusoidal Input	Exponential Input	Impulse Input
Waquoit Bay	27214 ± 16.3	22532	28243.6	25939.6	25954.6	32000.1
Quashnet River	9879 ± 11.0	8406	11587	10568.3	10575.8	13117.2

Table 3
Flow of nutrients after applying the control parameters $u_1 = 0.1$, $u_2 = 0$. Volume is calculated in ft^3 and flow rate in ft^3/day.

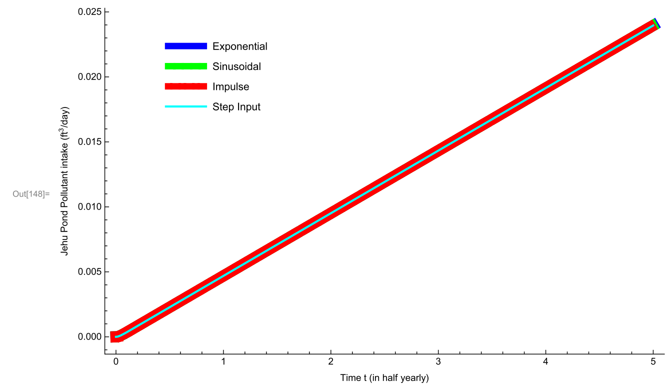
Water Bodies	Actual Data	NLM Model	Step Input	Sinusoidal Input	Exponential Input	Impulse Input
Waquoit Bay	27214+16.3	22532	2076.72	2000.79	2001.55	2115.57
Quashnet River	9879+11.0	8406	199.965	3.41534	6.38791	199.948

Table 4
Flow of nutrients after applying the control parameters $u_1 = u_2 = 0.1$. Volume is calculated in ft^3 and flow rate in ft^3/day.

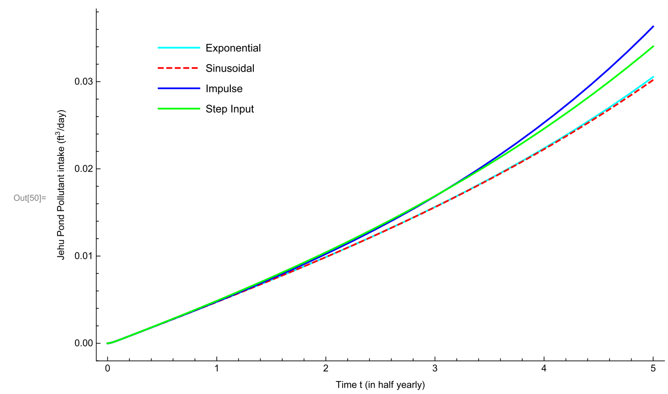
Water Bodies	Actual Data	NLM Model	Step Input	Sinusoidal Input	Exponential Input	Impulse Input
Waquoit Bay	27214+16.3	22532	1998.9	1998.9	1998.9	1998.9
Quashnet River	9879+11.0	8406	399.927	203.344	206.316	399.927



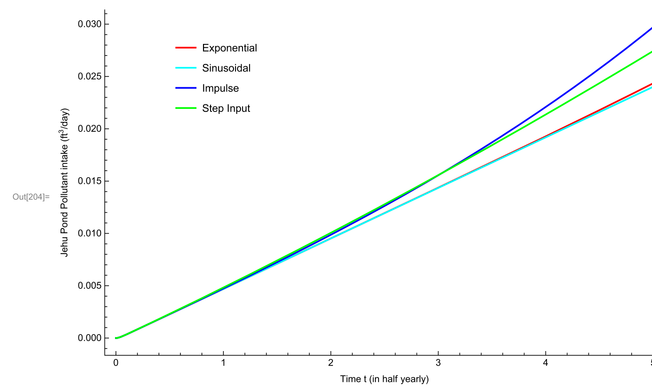
(a) Numerical Solution of Eq.2. Pollutants entering Jehu pond according to the model Eq.2 and impact of pollutants with no control i.e. $u_1 = 0, u_2 = 0$. A graphical representation of no control with four different types of input for Jehu pond.



(b) A graphical representation of control parameters $u_1 \neq 0, u_2 = 0$ on nutrient pollutants entering Jehu pond.



(c) A graphical representation of control parameters $u_1 \neq 0, u_2 \neq 0$ on nutrient pollutants entering Jehu pond.



(d) A graphical representation of control parameters $u_1 = 0, u_2 \neq 0$ on nutrient pollutants entering Jehu pond.

Fig. 7. Graphical results of numerical solution of Jehu pond showing the nutrient pollution input and behavior with control and no control parameters.

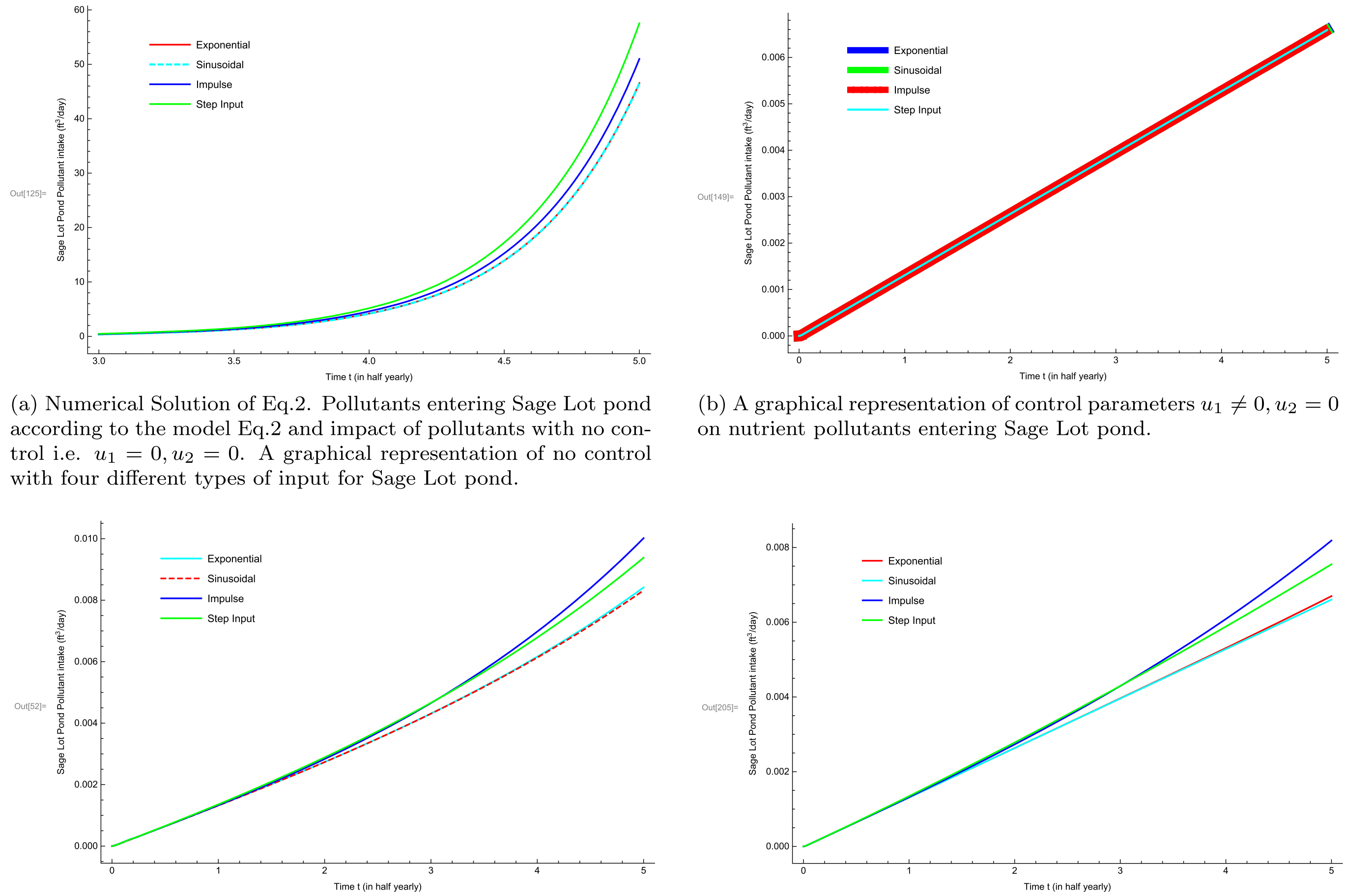
Table 5
Flow of nutrients after applying the control parameters $u_1 = 0$, $u_2 = 0.1$. Volume is calculated in ft^3 and flow rate in ft^3/day.

Water Bodies	Actual Data	NLM Model	Step Input	Sinusoidal Input	Exponential Input	Impulse Input
Waquoit Bay	27214+16.3	22532	2156.16	2079.35	2080.11	2196.38
Quashnet River	9879+11.0	8406	407.69	207.337	210.337	411.619

household water usage and this factor contributing in increase in nutrient pollution in lakes and Bay is described mathematically as sinusoidal input. Because household water usage is maximum in day time and very less and night time. Also in those areas there is industrial wastage that shows the exponential input of nutrient pollution. In our model, four cases have been simulated in this work as the mixing of nutrients in water bodies as they can adapt any of these patterns from the coastal area for water dumping. Two control parameters have been applied, Nutrient control from residential areas and bio dredging.
Let us consider the bio-dredging first in Waquoit Bay; then it means we have nonzero u_1 and $u_2 = 0$. Then, whatever the flow of input is by bio-dredging, the impact is similar on Waquoit Bay, and it reduces up to 92% even below the threshold see Table 3, Fig. 3b, Fig. 4b, Fig. 5b, Fig. 6b, Fig. 7b, Fig. 8b.
If we consider bio-dredging and household control with awareness in public in water bodies, then we have nonzero u_1 and u_2. Then the impact on all water bodies shows it reduces up to 93% even below the threshold see Table 4, Fig. 3c, Fig. 4c, Fig. 5c, Fig. 6c, Fig. 7c, Fig. 8c.
The last case is considered just the household control with awareness in public in Quashnet Bay, meaning we have nonzero u_2 and $u_1 = 0$. Then the impact on all water bodies shows it reduces up to 92% even below the threshold see Table 5, Fig. 3d, Fig. 4d, Fig. 5d, Fig. 6d, Fig. 7d, Fig. 8d. Hence in this work each water body have been studied in detail with two control parameters and if we apply these parameters, it can save the eel grass in Waquoit Bay, prevent the destruction of marine life in Waquoit Bay and improve the environment for human population as well, so that it can be an attractive tourist destination for a very long time.

CRediT authorship contribution statement

Conceptualization; FSK & MK & AHA. Writing - original draft preparation; FSK & MK & AHA. Methodology; FSK & MK & AHA. Software; FSK & AHA. Visualization; FSK & AHA. Formal analysis and investigation; OB & FG. Writing - review and editing; OB & FG. Resources; OB & FG. Supervision; MK & AHA.



(a) Numerical Solution of Eq.2. Pollutants entering Sage Lot pond according to the model Eq.2 and impact of pollutants with no control i.e. $u_1 = 0, u_2 = 0$. A graphical representation of no control with four different types of input for Sage Lot pond.

(b) A graphical representation of control parameters $u_1 \neq 0, u_2 = 0$ on nutrient pollutants entering Sage Lot pond.

(c) A graphical representation of control parameters $u_1 \neq 0, u_2 \neq 0$ on nutrient pollutants entering Sage Lot pond.

(d) A graphical representation of control parameters $u_1 = 0, u_2 \neq 0$ on nutrient pollutants entering Sage Lot pond.

Fig. 8. Graphical results of numerical solution of Sage Lot pond showing the nutrient pollution input and behavior with control and no control parameters.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] D.F. Boesch (Ed.), Proceedings of the Conference on Coastal Erosion and Wetland Modification in Louisiana: Causes, Consequences, and Options (FWS/OBS-82/59), U.S. Fish and Wildlife Service, Biological Services Program, Louisiana, USA, 1982.
[2] S.W. Nixon, C.A. Oviatt, J. Frithsen, B. Sullivan, Nutrients and the productivity of estuarine and coastal marine ecosystems, J. Limnol. Soc. South Africa 12 (1986) 43–71.
[3] V. Ivan, Couplings of watersheds and coastal waters: sources and consequences of nutrient enrichment in Waquoit Bay, Estuaries Coasts 15 (4) (1992) 443–457.
[4] H. Mortazavy Beni, H. Mortazavi, Mathematical modeling of the solar regenerative heat exchanger under turbulent oscillating flow: applications of renewable and sustainable energy and artificial heart, Results Eng. (ISSN 2590-1230) 13 (2022) 100321.
[5] I. Valiela, K. Foreman, M. Lamontagne, D. Hersh, J. Costa, P. Peckol, B.D. Andreson, C. Dávanzo, M. Babione, Couplings of watersheds and coastal waters: sources and consequences of nutrient enrichment in Waquoit Bay, Massachusetts, Estuaries 15 (4) (1992) 443–457.
[6] S.R. Pritchard, R.W. Golledge, J.G. Haas, Quashnet river, hamblin pond, little river, jehu pond, and great river in the Waquoit Bay system total maximum daily loads for total nitrogen, Tech. Rep., Commonwealth of Massachusetts, Cape Cod Commission, Town of Mashpee, 2006, data available from https://www.sandwichmass.org/DocumentCenter/View/4212/Waquoit_Quashnet_TMDL-2006-Jan.
[7] J. Aguirre, D. Tully, Lake Pollution Model, May 1999, 10 pages.
[8] J. Hoggard, Lake pollution modeling, Tech. Rep., Virginia Tech. Available at https://www.math.vt.deu/people/hoggard/links/new/main.html, 2007.
[9] J. Biazar, L. Farrokhi, M.R. Islam, Modeling the pollution of a system of lakes, Appl. Math. Comput. 178 (2006) 423–430.
[10] EPA, Waquoit Bay watershed ecological risk assessment, Tech. Rep., National Technical Information Service, Springfield, VA, 2002; PB2003–102013, Available from https://www.epa.gov/ncea.
[11] U.S. Environmental Protection Agency, Waquoit Bay watershed ecological risk assessment: the effect of land-derived nitrogen loads on estuarine eutrophication, Tech. Rep., EPA/600/R-02/079, U.S. Environmental Protection Agency, Office of Research and Development, National Center for Environmental Assessment, Washington Office, Washington, DC, 2002.
[12] M.A. Charette, K.O. Buesseler, E. John, Utility of radium isotopes for evaluating the input and transport of groundwater-derived nitrogen to a Cape Cod estuary, Limnol. Oceanogr. 46 (2) (2001) 465–470.
[13] B. Howes, N. Millham, S. Kelley, J.S. Ramsey, R. Samimy, D. Schlezinger, E. Eichner, Linked Watershed-EmBayment Model to Determine Critical Nitrogen Loading Thresholds for the Slocum's and Little River Estuaries, Dartmouth, Massachusetts. Tech. Rep., SMAST/DEP Massachusetts Estuaries Project, Massachusetts Department of Environmental Protection, Boston, MA, 2007.
[14] B. Howes, S. Kelley, E. Eichner, R. Samimy, J.S. Ramsey, D. Schlezinger, P. Detjens, Linked Watershed-EmBayment Approach to Determine Critical Nitrogen Loading Thresholds for the Waquoit Bay and Eel Pond EmBayment System, Tech. Rep., Massachusetts Department of Environmental Protection, Boston, MA, 2011.

[15] S. Bruce, S. Deborah, L. Elizabeth, Dredging effects on eelgrass (Zostera marina) in a New England small boat harbor, J. Marine Environ. Eng. 7 (2005) 1–26.

[16] T.R. Haileyesus, D.M. Oluwole, M.T. David, Optimal control model and cost effectiveness analysis of maize streak virus pathogen interaction with pest invasion in maize plant, Egypt. J. Basic Appl. Sci. 7 (1) (2020) 180–193.

[17] W. Fleming, R. Rishel, Deterministic and Stochastic Optimal Control, 1st edition, Springer Verlag, Berlin Heidelberg New York, 1975.

[18] A. Agrachev, Hamiltonian systems and optimal control, NATO Sci. Peace Secur. Ser. (2008) 143–156.

[19] CAPE COD, Implementation report: watershed report, Data retrieved from http://www.capecodcommission.org, 2000.

[20] NOAA, United States Coast and Geodetic Survey, Atlantic Coast: Cape Cod to Sandy Hook, 25th edition, Government Printing Office, Washington, DC., USA, 1991, Available from https://www.books.google.com.pk/books?id=cwrP1jIQPc8C.

[21] C.M. Taube, Manual of fisheries survey methods II: three methods for computing the volume of a lake, in: State of Michigan, Department of Natural Resources, Fisheries Special Report 25, Ann Arbor., Michigan DNR, Lansing, MI, 2000, Ch. 12.

[22] B. Pimpunchat, W.L. Sweatman, G.C. Wake, W. Triampo, A. Parshotam, A mathematical model for pollution in a river and its remediation by aeration, Appl. Math. Lett. 22 (3) (2009) 304–308.

[23] V.Y. Filimonov, A.T. Zinoviev, O.V. Lovtskaya, Method for estimating the pollutant flow rate under diffuse pollution of small lowland rivers, Math. Comput. Simul. 182 (2021) 807–818.

[24] S. Shojaei, S. Shojaei, M. Sasani, The efficiency of eliminating Direct Red 81 by Zero-valent Iron nanoparticles from aqueous solutions using response surface Model (RSM), Model. Earth Syst. Environ. 3 (2017) 27.

[25] S. Shojaei, S. Shojaei, Optimization of process variables by the application of response surface methodology for dye removal using nanoscale zero-valent iron, Int. J. Environ. Sci. Technol. 16 (2019) 4601–4610.

[26] S. Shojaei, S. Shojaei, Optimization of process conditions in wastewater degradation process, in: Soft Computing Techniques in Solid Waste and Wastewater Management, Elsevier, 2021, pp. 381–392, Chapter 23.

[27] J. Yang, S. Shojaei, S. Shojaei, Removal of drug and dye from aqueous solutions by graphene oxide: adsorption studies and chemometrics methods, npj Clean Water 5 (2022) 1–10.

[28] A. Pourabadeh, L. Baharinikoo, A. Nouri, B. Mehdizadeh, S. Shojaei, The optimisation of operating parameters of dye removal: application of designs of experiments, Int. J. Environ. Anal. Chem. 101 (9) (2021) 1320–1329.

[29] R. Kaushik, Method for estimating the pollutant flow rate under diffuse pollution of small lowland rivers, Adv. Appl. Sci. Res. 6 (7) (2015) 57–64.

[30] E. Bonyaha, P. Agbekpornub, C. Unlu, Mathematical modeling of transmission of water pollution, J. Prime Res. Math. 17 (2) (2021) 20–38.

[31] A. Ghanbari, K. Mazaheri, T. Saneei, S. Keshavarzian, Development of a mathematical model to simulate a natural gas compression system and its anti-surge control to improve performance: a case study to reduce the energy consumption, and the environmental impact, Results Eng. (ISSN 2590-1230) 17 (2023) 100933.

[32] S. Narayan S, S. Saha, A. Bhattacharjee, A 2D FSI mathematical model of blood flow to analyze the hyper-viscous effects in atherosclerotic COVID patients, Results Eng. (ISSN 2590-1230) 12 (2021) 100275.

[33] S.H. Menon, M.G. Samjith, J.P. Thomas, J. Mathew, J. Jayaprakash, Mathematical modelling of electrodic heat source in a diffusion controlled electrochemical cell, Results Eng. (ISSN 2590-1230) 11 (2021) 100239.

[34] F. Ishola, V. Oladokun, O. Petinrin, O. Olatunji, S. Akinlabi, A mathematical model and application for fire risk management in commercial complexes in South Africa, Results Eng. (ISSN 2590-1230) 7 (2020) 100145.