

What can we learn from Tamás Varga's work regarding the arithmetic-algebra transition?

GYÖNGYI SZANYI AND ZSUZSANNA JÁNVÁRI

Abstract. Tamás Varga's Complex Mathematics Education program plays an important role in Hungarian mathematics education. In this program, attention is given to the continuous "movement" between concrete and abstract levels. In the process of transition from arithmetic to algebra, the learner moves from a concrete level to a more abstract level. In our research, we aim to track the transition process from arithmetic to algebra by studying the 5-8-grader textbooks and teacher manuals edited under Tamás Varga's supervision. For this, we use the appearance of "working backward" and "use an equation" heuristic strategies in the examined textbooks and manuals, which play a central role in the mentioned process.

Key words and phrases: Tamás Varga's textbooks, heuristics strategies, arithmetic-algebra transition.

MSC Subject Classification: 97-01, 97-03, 97D50.

Introduction

Tamás Varga's figure is known for his work in mathematics education in Hungary, he was one of the contributors in the renewal of the 20th century school mathematics. He was the leader of the "Complex Mathematics Education Reform Program" in the 1960s-70s,

put the focus on problem solving and the “discovery” of mathematics in order to develop mathematical thinking, understanding better the curriculum and the motivation of students (Klein, 1980).

The related documents (curriculum, textbook, teacher manuals, etc.) and some of Tamás Varga's studies provide insight into the concepts of the mentioned reform program, but they only present Tamás Varga's approach through specific examples and only indirect access to his didactic concept (Gosztonyi et al, 2018).

In our research, we aim to track Tamás Varga's didactic concept regarding the transition process from arithmetic to algebra by the detailed examination of the solutions of the tasks of 5-8-grader textbooks edited under his supervision which can be found in teacher's manuals. We intend to achieve this by analysing the presence of two heuristic strategies in textbooks that play a central role in the above-mentioned transition process. The research of Tamás Varga's oeuvre and the revitalization of his ideas about teaching mathematics were one of the goals of the MTA-ELTE Complex Mathematics Education Research Group. As part of this project, other heuristic strategies were studied in school settings (Kónya & Kovács, 2018).

Literature review and framework

Although Tamás Varga's reform movement was inspired by the New Math movement, according to his colleagues, it is a special Hungarian concept in which the teaching of mathematics through discovery is given special attention by fitting it into local traditions. Tamás Varga kept contact with a number of prominent mathematicians of this tradition (L. Kalmár, R. Péter, etc.) who – together with the well-known thinkers (Gy. Pólya, I. Lakatos) – represented a fairly coherent “heuristic” epistemology of mathematics (Gosztonyi et al., 2018).

In addition to international influences, these personal relationships also had an influence on giving birth to the different concepts of the attempt to teach complex mathematics. One of these principles is that mathematics should be taught using problem situations. The problem-solving process is a mental operation that involves a series of procedures and heuristic strategies to find a solution (Mayer, 1985).

In Tamás Varga's work on the teaching of mathematics (Varga, 1969), among other things, we find the principles of two heuristic strategies that are closely related and play

an important role in the transition from arithmetic (concrete level) to algebra (abstract level). These strategies are: (1) „*working backward*” strategy (in the problem we describe a procedure i.e. describe an algorithm in words and require the initial state based on the final state, or the initial state and the final state are known and a procedure is requested by which we can get from the initial state to the final state); (2) „*use an equation*” strategy (in the problem, a letter is used to represent the unknown quantity as a variable). The (1) is substantiated and prepares the engagement of the (2).

In moving from the concrete to the abstract, Tamás Varga considered it important to continuously “mature” the concepts in such a way that there is a constant back-and-forth movement between the concrete and the abstract: “We have to go through all the stages of abstraction again and again in order to be able to advance from concrete experiences to more and more abstract concepts” (Varga, 1969, p.7). We find a reference to the central role of the above-mentioned two heuristic strategies in this process in Tamás Varga's work on solving problems with and / or without equations: after solving word problems without equations it is worth to solve it using equations, or reversely, after solving tasks with equations we should look for a simpler, equation-free solution if possible (Varga, 1969).

After Tamás Varga's attempt to teach complex mathematics, we can meet the findings supporting the concept above in researches on the transition from arithmetic (concrete) to algebra (abstract). For example, Carraher et al. (2006) and Kaput (2000) claimed that the separation between arithmetic and algebra accentuate and prolong the students' difficulties regarding the understanding of the concepts of algebra. When students converse from arithmetic to algebra it is inevitable to be aware of the coherence of numerical data of a problem; to be able to represent the problem in an informal language then to be able to replace a quantity with symbol (letter) by examining arithmetic operations and characterization of the problem with letters – build and solve equations (Herscovics & Linchevski, 1994, Warren, 2003). This procedure presumes that a problem is being solved with an “arithmetic equation” (the algebraic model of which for example $Ax + B = C$), i.e. using “working backward” strategy, then working with “algebraic equation” (i.e. using “use an equation” strategy). Zsolt Fülöp (2017) assessed that it is important in the primary school education to show the students arithmetic strategies (i.e. “working backward”) and algebraic also (i.e. “use an equation”) during the finding of the solution of the problems in a relevant exercise. Dettori et al. (2001) emphasize that getting acquainted with the nature of algebra could be supported by comparing different problem-solving methods, as well as arithmetic and algebraic.

To summarize, following Tamás Varga's reform program, several studies have reported that the return to arithmetic “roots” plays a significant role in the transition from arithmetic to algebra. It also means that in relevant cases, both applications of the examined strategies can support learning algebraic concepts.

Research Questions

- RQ1. How the "dynamic transition from arithmetic to algebra" principle appears in textbooks edited under Varga's supervision?
- RQ2. What is the role of the different mathematical topics in the transition, in particular, in what proportion do the “working backward” and “use an equation” heuristic strategies appear?

Methodology

In our research, the textbooks and the related teacher's manuals of Tamás Varga (from grade 5 to grade 8) (Eglesz et al, 1979, 1981, Imrecze et al., 1981, Kovács et al., 1980) were analysed by content analysis (Berelson, 1952, Meyers, 1973, as cited in Dárdai, 2002) in such a way that based on quantitative data we prepared a qualitative analysis, by which we aimed at catching the realisation of the transition from arithmetic to algebra.

For this, we coded each textbook task based on the following aspects, taking into account its solution given in the manual:

1. Which of the following 5 main topics appears the most in the task:
(1) thinking methods; (2) arithmetic, algebra; (3) functions; (4) geometry; (5) probability, statistics. (Organising the curriculum around similar topics was initiated by Tamás Varga.)
2. The task belongs to which competency clusters defined by PISA (2001, p. 23):
 - reproduction – consists of simple computations or definitions of the type most familiar in conventional assessments of mathematics;
 - connection – requires bringing together the mathematical ideas and procedures to solve straightforward and somewhat familiar problems;

- reflection – consists of mathematical thinking, generalization and insight, and requires students to engage in analysis, to identify the mathematical element in a situation and to pose their own problems.

Thereafter, tasks belonging to the reproduction cluster (i.e. non-routine tasks) were not further coded.

3. Tasks belonging to the connection or reflection cluster, i.e. *problems* were further coded regarding the (maximum of 3) *heuristic strategies* that appear in the solution of the problem reported in the textbook and the manual: iconic representation, working backward, look for a pattern, algorithmic thinking, use an equation, restate the problem, simplify the problem.

So, each task was given 1 code for the topic, 1 code for the competency class, and a maximum of 3 codes based on its inherent heuristic strategies (see Figure 1).

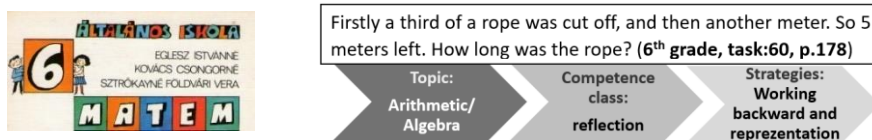


Figure 1. An example of the Methodology

Using this methodology, 797 tasks in the 5th grade, 876 tasks in the 6th grade, 741 tasks in the 7th grade and 1320 tasks in the 8th grade textbooks were analysed.

The analysis of each textbook was performed by teachers of mathematics with many years of practical experience in the given age groups, including the authors, too. Teachers received detailed, exemplified guidance for the analysis from the researchers of the MTA-research group mentioned in the introduction, including the authors. In the questionable cases, textbook analysts consulted with the researchers.

Results

Among all tasks which were analysed there were 380 problems in 5th grade, 406 problems in 6th grade, 423 problems in 7th grade and 865 problems in 8th grade.

Since the appearance of the “working backward” and “use an equation” strategies in textbooks are the most relevant regarding to our research questions, in this chapter we only deal with the problems in whose solutions presented in the manual the one or both of two

mentioned strategies appear. Furthermore, although the geometrical constructing problems require the use of the “working backward” strategy, they do not play a major role in the process of transition from arithmetic to algebra, so we did not work with this type of problems in the data processing. We also point out that the problems with the code “use an equation” in the vast majority were problems in which it was not necessary to solve a given equation but to build the equation to solve the problem. Based on these, there was a total of $N = 692$ problems in the studied age groups in which were used (1) the “working backward” strategy (maybe another one also, but not the “use an equation”), (2) the “use an equation” strategy (maybe another one also, but not the “working backward”) or (3) the two examined strategies (maybe another one) together. The relative frequency of these strategies by grade is shown in Figure 2.

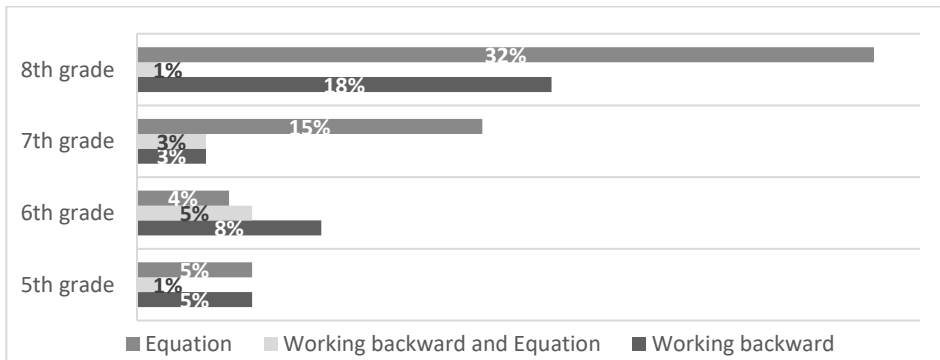


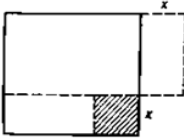
Figure 2. Relative frequency of the analysed strategies in the textbooks ($N=692$)

As it is shown by Figure 2, in the 5th grade the two examined strategies are recommended by the handbook in the same proportions to solve problems and two strategies together within a problem are shown in a much smaller proportion. This data can be explained by the fact that at the beginning of the textbook we can see the application of the “working backward” strategy, and then as we move forward in the textbook, the textbook leads the students to the equations and their solution using the “working backward” strategy and then without it. (Note here that the “working backward” and “use an equation” strategies are often associated with the iconic representation strategy, but we will not elaborate on this in the present study.) The appearance of the “working backward” strategy is increasing from grade 5 to grade 6. In grade 6, applying the two strategies in the same problems appears more often than in grade 5. The single application of the “use an equation” strategy for the finding of the solution of the problems appears in fewer problems. Based on these, we see the continuous “maturation” of the abstract concept of

the equation realized in the textbooks according to the idea of returning to arithmetic (concrete) (see the example below which is a translation of the problem).

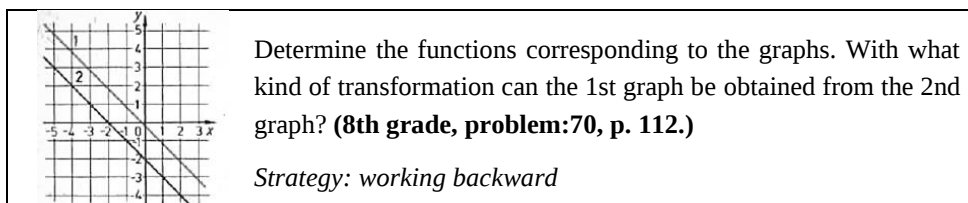
Problem: 33, p.151., 5th grade	The solution of the problem in the teacher's manual
The sum of three consecutive integers in the sequence of natural numbers is 645. What are these numbers?	(1) The smaller number is 1 less, the bigger number is 1 more than the middle one. So the sum of these three numbers is three times the middle number. (strategy: <i>working backward</i>) (2) The middle number: x, the number before the middle one: $x-1$, the number after the middle one: $x+1$ The sum of these three numbers: $x+x-1+x+1=645$, $x=215$ The numbers: 214, 215, 216. (strategy: <i>use an equation</i>)

According to Figure 2, in grade 7, in some problems the “working backward” strategy appears but the period of transition to algebra is marked by a significant increasing of the application of the “use an equation” strategy. Mathematical knowledge is expanding more and more in this age group, and the textbook further deepens the concept of the variable by discussing – among others – the first-order functions in detail and also covers solving equations by graphs. In addition, the movement between the concrete and the abstract can be found in this age group, as well (see the example below, where the “working backward” strategy appears so that if from the area of the given square we subtract the area of the rectangle the result will be the area of the painted square whose side length is x):

Problem: 69, p. 62., 7th grade	The solution of the problem in the teacher's manual
The length of the side of a square is 30 cm. Reduce one side by a certain length and increase another side with the same length. The area of the obtained rectangle is 800 cm ² . By what length did we reduce (and increase) the length of the side of the original square?	69. The solution can be found from the figure  $x^2 = 100$, $x = 10$ (cm). Or we can write an equation $(30-x) \cdot (30+x) = 800$, $900 - x^2 = 800$, $x^2 = 100$, $x = 10$ (cm). Strategies: (1) Working backward; (2) Use an equation (3) Iconic representation

It can be considered an expected result that in grade 8 the textbook contains a significant proportion of problems in the solution of which the “use an equation” strategy appears. We see the reason for this in the fact that in this age group the textbook deals with solving word problems by using equations in a separate chapter. But, as Figure 2 shows,

the return to arithmetic, that is, the moving between the abstract and the concrete level can be observed in grade 8. This is reflected in an increase in the use of the “working backward” strategy compared to the 7th grade data. All this can be explained by the introduction of the function concept and the expansion of the functional knowledge. This is because in several function-related problems, using the “working backward” strategy it is necessary to represent the functions with the formula based on their given graphs (see the example below).



The “working backward” strategy also appears in a number of geometric problems in whose background there is some sort of geometric formula.

We examined the proportion of “working backward” and “use an equation” strategies appearing in the examined 692 textbook problems by age group and topic also. Thus, out of the 692 problems used for the analyses, we examined the proportion of the two strategies appearing in the given age groups in each topic, in the 5th grade having the case of $N_5 = 76$, in the 6th grade having $N_6 = 120$, in the 7th grade having $N_7 = 146$, in the 8th grade having $N_8 = 350$ (Table 1).

Because of that, the examined strategies have an important role in the arithmetic-algebra transition considered as an expected result that the examined strategies appear mostly in the topic of Arithmetic / Algebra. For example, in the 5th grade, 25% of the 76 problems were related to Arithmetic/Algebra for which the “working backward” strategy was applied in the textbook or the manual. However, the concept of the textbook that implementing the “maturation” of abstract concepts by moving back and forth between the concrete (using the “working backward” strategy) and the abstract (using the “use an equation” strategy) in the relevant problems of new topics in each age group should also be emphasized.

In the 8th grade, the textbook does a lot to discuss the new concept, the function. Therefore, in the Functions topic, the “working backward” strategy is used to a large extent to solve problems (in 17% of the 350 examined problems).

Strategies Topics	Working Backward (%)				Working Backward and Equation (%)				Equation (%)			
	Grades				Grades				Grades			
	5	6	7	8	5	6	7	8	5	6	7	8
Functions	5	0	3	16,9	0	0	6	0	0	0	3	0
Geometry	11	13	8	13,7	0	6	2	1,4	4	6	33	4
Arithmetic/Algebra	25	34	5	4,3	10	23	4	0,3	45	18	36	59,4

Table 1. The proportion of the analysed strategies in different topics of the textbooks

Conclusions

In this study, we analysed the textbooks and the related teacher's handbooks from grade 5 to 8 grade for Tamás Varga's reform program. Our aim with the analyses was to get to know Tamás Varga's didactic concept of the transition from arithmetic to algebra. To this end, we examined the emergence of two heuristic strategies in textbook problems that play a central role in this process. We summarize our experiences based on the research questions.

RQ1. There is a dynamic transition from arithmetic to algebra in Tamás Varga's textbooks. According to the results, this transition takes place from the 6th to the 7th grade. From the 5th grade the textbooks gradually lead to abstract algebraic concepts based on arithmetic "roots". In this transition, the principle of continuous "maturation" could be discovered with the emergence of movement between the concrete and the abstract in the relevant problems. The follow-up of this movement was realized in our study by applying the "working backward" and "use an equation" strategies in the problem solution presented in the manual.

RQ2. The "working backward" and "use an equation" strategies that play a central role in the transition from arithmetic to algebra appeared mostly in the problems related to Arithmetic/Algebra, Geometry and Functions. It should be emphasized that during the transition from the concrete level to the abstract level in the textbook, these strategies were

typically included in the problems related to the new topics appearing in the given age group. The 7th grade textbook deals with the equations and their different solving methods. Therefore the textbook uses the “use an equation” strategy to a high extent. In addition to equations, the discussion of functions also plays a central role in grade 8. The application of the examined strategies appears in the solution of the problems related to these topics.

Summarizing the results, it can be said that the textbooks prepared for Tamás Varga's reform program contain a conscious structure of the transition from arithmetic to algebra, which aspect is highly relevant in teaching mathematics nowadays, too. The investigated textbooks serve a good example with respect to the analysed field.

Acknowledgement

This article was written in the frame of the “MTA-ELTE Complex Mathematics Education in the 21st Century” project supported by the Hungarian Academy of Sciences (ID number 471028).

References

- Berelson, B. (1952). *Content analysis in communication research*. Free Press.
- Carraher, D., Schliemann, A. D., Brizuela B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education* 37(2), 87-115.
- Dettori G., Garuti R., Lemut E. (2001) From Arithmetic to Algebraic Thinking by Using a Spreadsheet. In: Sutherland R., Rojano T., Bell A., Lins R. (eds), *Perspectives on School Algebra*. Mathematics Education Library, vol 22. Springer, Dordrecht.
- Dárdai, Á. (2002). *A tankönyvkutatás alapjai*. Dialóg Campus Kiadó, Budapest.
- Eglesz, I., Kovács, Cs., & Sz. Földvári, V. (1979). *Matematika, 5. osztály*. Tankönyvkiadó, Budapest.
- Eglesz, I., Kovács, Cs., & Sz. Földvári, V. (1981). *Matematika, 6. osztály*. Tankönyvkiadó, Budapest.

- Fülöp, Zs. (2017). *Feladat megoldási módszerek összehasonlító vizsgálata a pedagógus illetve a diák rendelkezésére álló ismeretek birtokában*. PhD értekezés, Szegedi Tudományegyetem.
- Herscovics, N. and Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27(1), 59-78.
- Imrecze, Z., Kovács, Cs., Sz. Földvári, V., & Szeredi, É. (1981). *Matematika 8. osztály*. Tankönyvkiadó, Budapest.
- Gosztonyi, K., Kosztolányi J., Pintér, K., Vancsó, Ö., & Varga, E. (2018) Varga's «complex mathematics education» reform: at the crossroad of the New Math and Hungarian traditions. In Y. Shimizu & R. Vithal (Eds.) *ICMI Study 24: School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities*. Conference proceedings (pp 133-140).
- Kaput, J., (2000). *Transforming Algebra from an Engine of Inequity to an Engine of Mathematical Power By "Algebrafying" the K-12 Curriculum*. Paper from National Center for Improving Student Learning and Achievement in Mathematics and Science, Dartmouth, MA. (ERIC Document Reproduction Service No. ED 441 664).
- Klein, S. (1980). *A komplex matematikatanítási módszer pszichológiai hatásvizsgálata*. Akadémiai Kiadó, Budapest.
- Kónya, E., & Kovács, Z. (2018). Let's explore the solution: Look for pattern! In B. Maj-Tatsis, K. Tatsis & E. Swoboda (Eds.), *Mathematics In The Real World* (pp. 136-147). Rzeszów, Poland: Wydawnictwo Uniwersytetu Rzeszowskiego.
- Kovács, Cs., Sz. Földvári, V., & Szeredi, É. (1980). *Matematika 7. osztály*, Tankönyvkiadó, Budapest.
- Mayer, R. E. (1985). Implications of cognitive psychology for instruction in mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 123-138). NJ: Lawrence Erlbaum.
- PISA Knowledge and Skills for Life. First Results from PISA 2000. OECD, 2001. Retrieved from <https://www.oecd-ilibrary.org/docserver/9789264195905-en.pdf?expires=1596466182&id=id&accname=guest&checksum=49C3C0417459BB68D4849B54F6B3F2EE>
- Varga, T. (1969). *A matematika tanítása*. Tankönyvkiadó, Budapest.
- Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, 15(2), 122-137.

GYÖNGYI SZANYI

UNIVERSITY OF DEBRECEN, FACULTY OF ENGINEERING

4028 DEBRECEN, ÓTEMETŐ STR. 2-4.

E-mail: szanyi.gyongyi@eng.unideb.hu

ZSUZSANNA JÁNVÁRI

SZERB ANTAL SECONDARY GRAMMAR SCHOOL, BUDAPEST, HUNGARY

E-mail: zsjanvari@gmail.com