General Bahr-Esseen inequalities and their applications

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Abstract

The Bahr-Esseen inequality is studied. It is shown that the Bahr-Esseen inequality holds with exponent p, if it holds with exponent q ($q > p$) for the truncated and centered random variables. The Bahr-Esseen inequality is also true if the truncated random variables are acceptable. Then the results are applied to obtain weak and strong laws of large numbers and complete convergence.

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1 Introduction

First we recall the well-known Bahr-Esseen inequality. Let $1 \le p \le 2$ and let X_n, $n = 1, 2, \dots$ be a sequence of independent random variables (r.v.'s) with finite p-th moment and mean zero ($\mathbb{E}|X_n|^p < \infty$, $\mathbb{E}X_n = 0$ for all $n = 1, 2, \dots$). Then

$$\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p \le c_{p,n} \sum_{k=1}^{n} \mathbb{E}|X_k|^p, \tag{1.1}$$

for all $n = 1, 2, \dots$, where $c_{p,n} \le 2 - n^{-1}$ (von Bahr and Esseen, [15]). Inequality (1.1) is the p-th von Bahr-Esseen moment inequality.

We remark that the p-th von Bahr-Esseen moment inequality is obviously true for $0 < p \le 1$, that is $\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p \le \sum_{k=1}^{n} \mathbb{E}|X_k|^p$, if $0 < p \le 1$ for any sequence X_n, $n = 1, 2, \dots$ of random variables with finite p-th moment.

Dharmadhikari and Jogdeo [7] proved the following inequality, which can be considered as an extension of the Bahr-Esseen inequality to the case of $p > 2$. Let $p \ge 2$ and let X_n, $n = 1, 2, \dots$ be a sequence of independent random variables with finite p-th moment and mean zero. Then (1.1) is satisfied with

$$c_{p,n} = n^{p/2-1} \frac{p(p-1)}{2} \max\{1, 2^{p-3}\} \left[1 + 2p^{-1} D_{2m}^{(p-2)/2m}\right],$$

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where the integer m satisfies $2m \le p < 2m+2$ and the constant D_{2m} is defined as

$$D_{2m} = \sum_{k=1}^{m} k^{2m-1}/(k-1)!\,.$$

In [4] the p-th von Bahr-Esseen moment inequality was obtained for pairwise independent random variables if $1 < p < 2$. The 2nd von Bahr-Esseen moment inequality is obvious for pairwise independent zero mean random variables, and in [4] this fact is applied to prove the p-th ($1 < p < 2$) von Bahr-Esseen moment inequality. Analysing the proof in [4], we can obtain the following result. If $1 < p < 2$ and X_n, $n = 1, 2, \dots$ is a sequence of arbitrary random variables with finite p-th moment and mean zero and the 2nd von Bahr-Esseen moment inequality holds for the truncated and centered variables $X_k \mathbb{I}\left(|X_k| \le x\right) - \mathbb{E} X_k \mathbb{I}\left(|X_k| \le x\right)$, $k = 1, 2, \dots, n$, for any $x > 0$, then the p-th ($1 < p < 2$) von Bahr-Esseen moment inequality is true for the random variables X_n, $n = 1, 2, \dots$ themselves. (Here and in what follows $\mathbb{I}$ denotes the indicator function of a set.) Moreover, we can generalize the previous result using q instead of 2. That is if $1 < p < q$ and the q-th von Bahr-Esseen moment inequality holds for the truncated and centered variables, then the p-th von Bahr-Esseen moment inequality is true for the original random variables themselves.

However, there is an other version of truncation. Given a r.v. X and a positive number t, we can use the following truncated r.v.

$$^{(-t)}X^{(t)} = -t\mathbb{I}\left\{X < -t\right\} + X\mathbb{I}\left\{|X| \le t\right\} + t\mathbb{I}\left\{X > t\right\}. \tag{1.2}$$

The advantage of this truncation is that $^{(-t)}X^{(t)} = h(X)$ with an increasing real function h. And we know that certain dependence conditions are inherited if the random variables are inserted into increasing functions. Therefore it is more important to know that the qth von Bahr-Esseen moment inequality for the truncated and centered variables $^{(-x)}X_k^{(x)} - \mathbb{E}\,{}^{(-x)}X_k^{(x)}$ implies the p-th von Bahr-Esseen moment inequality for the original random variables X_k themselves ($1 < p < q$). This fact is proved in our Theorem 2.1. We underline that in our Theorem 2.1 we do not assume any weak dependence condition for the random variables. We also emphasize that throughout the paper we use versions of truncation given in (1.2).

It is well-known that certain exponential relations play fundamental role in the proofs of asymptotic results for independent and weakly dependent random variables. A general shape of such kind of relations is included in the definition of acceptability. The r.v.'s $X_1, X_2, \dots, X_n$ are called acceptable if

$$\mathbb{E} e^{\sum_{i=1}^{n} \lambda X_i} \le \prod_{i=1}^{n} \mathbb{E} e^{\lambda X_i} \tag{1.3}$$

for any real number λ, see [1]. In Subsection 2.3 of this paper we shall show that a version of inequality (1.3) implies an exponential inequality, see Proposition 2.1. Then, using the exponential inequality, we obtain a Rosenthal's inequality (Proposition 2.2). Finally, we shall see that a version of inequality (1.3) implies the p-th von Bahr-Esseen moment inequality, see Theorem 2.2. Applying Theorem 2.2, we obtain von Bahr-Esseen's moment inequality for WOD sequences (Theorem 2.3).

Important applications of moment inequalities are convergence theorems. In Subsection 2.4 of this paper we shall present laws of large numbers and complete convergence as consequences of our inequalities. According to the well-known Etemadi's strong law of large numbers (SLLN), if $X_1, X_2, \ldots$ are pairwise independent and identically distributed random variables with finite first moment, then

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mathbb{E}X_1$$

almost surely, see [8]. Our Theorem 2.4 is an Etemadi style SLLN. In our theorem, instead of pairwise independence, we assume either (1.3) or (1.1) for the truncated r.v.'s. Also weell-known SLLN is the result of Csörgő, Tandori and Totik, see [6]. There pairwise independent but not identically distributed r.v.'s were considered. Our Theorem 2.5 is a new version of the Csörgő, Tandori and Totik SLLN. In our theorem, we replace pairwise independence with appropriate versions of (1.3) or (1.1). We also present a weak law of large numbers (WLLN, see Theorem 2.6).

The rate of convergence in laws of large numbers can be described by so-called complete convergence theorems. Classical complete convergence results are due to Hsu, Robbins, Erdős, Baum and Katz, see [3]. First complete convergence results concerned probabilities, later such results were proved for moments as well. The general form of the complete moment convergence of the random variables $Y_1, Y_2, \ldots$ is

$$\sum_{n}^{\infty} a_n \mathbb{E}\left(|Y_n|/b_n - \varepsilon\right)_+^q < \infty$$

for all $\varepsilon > 0$, where $(\cdot)_+$ denotes the positive part. Here Y_n is usually the partial sum of r.v.'s. The classical paper dealing with complete moment convergence for independent r.v.'s is [5]. Then several papers were devoted to the topic. In [14] it is shown that when certain moment inequalities are satisfied for the truncated r.v.'s, then complete moment convergence holds. In our paper we prove complete moment convergence if (1.3) is true for the truncated r.v.'s (Theorem 2.7).

2 Results and discussion

2.1 Methods

In this paper we apply truncations of random variables then approximations of probabilities and moments. The combination of these methods enables us to obtain general versions of moment inequalities and convergence theorems.

2.2 The von Bahr-Esseen moment inequality

In this subsection, we will prove the following general theorem. If the von Bahr-Esseen moment inequality holds for q for the truncated and centered random variables, then it holds for the random variables themselves for any p with $1 < p < q$. We emphasize, that there is no additional assumption on the dependence structure of the random variables. We mention that Theorem 2.1 in [4] is a Bahr-Esseen's inequality for pairwise independent random variables. In our paper we apply the method of the proof presented in [4]. However, as we use truncation (1.2) instead of $X_k \mathbb{I}\left(|X_k| \le x\right)$, our proof is shorter than the one in [4].

Theorem 2.1. *Let $1 < p < q$. Let $X_n, n = 1, 2, \dots$ be a sequence of random variables with $\mathbb{E}|X_n|^p < \infty$, $\mathbb{E}X_n = 0$ for all $n = 1, 2, \dots$. Assume that for any $x > 0$*

$$\mathbb{E}\left|\sum_{k=1}^{n}\left({}^{(-x)}X_k^{(x)} - \mathbb{E}\,{}^{(-x)}X_k^{(x)}\right)\right|^q \le g_q(n)\sum_{k=1}^{n}\mathbb{E}\left|{}^{(-x)}X_k^{(x)} - \mathbb{E}\,{}^{(-x)}X_k^{(x)}\right|^q. \tag{2.1}$$

Then

$$\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p \le f_{p,q}(n)\sum_{k=1}^{n}\mathbb{E}|X_k|^p, \tag{2.2}$$

where $f_{p,q}(n)$ depends only on $g_q(n)$, p and q (a possible choice is $f_{p,q}(n) = 5 + 2c_q g_q(n) 2^q \left(\frac{q}{q-p}\right)^2$, where $c_q = 2^{q-1}$).

Proof. Let $V = \sum\limits_{k=1}^{n}\mathbb{E}|X_k|^p$. If $V = 0$, then $X_k = 0$ a.s. for all $k = 1, 2, \dots, n$, so we may assume that $V \ne 0$. For simplicity, Z_i denotes the truncated random variable, that is $Z_i = {}^{(-x^{1/p})}X_i^{(x^{1/p})}$, where x is an arbitrary positive number. For any $\varepsilon > 1$,

$$\begin{aligned}
\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p &= \int_0^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n} X_k\right|^p > x\right\}\mathrm{d}x \le \\
&\le (1+\varepsilon)V + \int_{(1+\varepsilon)V}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n} X_k\right| > x^{1/p}\right\}\mathrm{d}x \le \\
&\le (1+\varepsilon)V + \int_{(1+\varepsilon)V}^{\infty}\sum_{k=1}^{n}\mathbb{P}\left\{|X_k| > x^{1/p}\right\}\mathrm{d}x + \\
&+ \int_{(1+\varepsilon)V}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n} Z_k\right| > x^{1/p}\right\}\mathrm{d}x = \\
&= (1+\varepsilon)V + I_1 + I_2.
\end{aligned} \tag{2.3}$$

We see that

$$I_1 \le \sum_{k=1}^{n} \int_{0}^{\infty} \mathbb{P}\left\{|X_k| > x^{1/p}\right\} \mathrm{d}x = \sum_{k=1}^{n} \mathbb{E}|X_k|^p = V. \tag{2.4}$$

Using that $\mathbb{E}X_k = 0$, we have that $\mathbb{E}X_k\mathbb{I}\left(|X_k| \le x^{1/p}\right) = -\mathbb{E}X_k\mathbb{I}\left(|X_k| > x^{1/p}\right)$, so we obtain

$$\begin{aligned}\sup_{x\ge(1+\varepsilon)V} x^{-1/p}\left|\sum_{k=1}^{n}\mathbb{E}Z_k\right| &= \sup_{x\ge(1+\varepsilon)V} x^{-1/p}\Big|\sum_{k=1}^{n} -\mathbb{E}X_k\mathbb{I}\left(|X_k| > x^{1/p}\right) + \\ &\quad + x^{1/p}\mathbb{P}\left(X_k > x^{1/p}\right) - x^{1/p}\mathbb{P}\left(X_k < -x^{1/p}\right)\Big| \le \\ &\le \sup_{x\ge(1+\varepsilon)V} x^{-1/p}\left|\sum_{k=1}^{n}\mathbb{E}\left|X_k\right|\mathbb{I}\left(|X_k| > x^{1/p}\right) + x^{1/p}\mathbb{P}\left(|X_k| > x^{1/p}\right)\right| \le \\ &\le 2\sup_{x\ge(1+\varepsilon)V} x^{-1/p}\sum_{k=1}^{n}\mathbb{E}\left|X_k\right|\mathbb{I}\left(|X_k| > x^{1/p}\right) \le \\ &\le 2\sup_{x\ge(1+\varepsilon)V} x^{-1/p}\cdot x^{1/p-1}\sum_{k=1}^{n}\mathbb{E}\left|X_k\right|^p\mathbb{I}\left(|X_k| > x^{1/p}\right) \le \\ &\le 2(1+\varepsilon)^{-1}V^{-1}\cdot V = 2(1+\varepsilon)^{-1}.\end{aligned} \tag{2.5}$$

Now we apply (2.5) and then, as $\varepsilon > 1$, we can use Markov's inequality, so we obtain

$$\begin{aligned} I_2 &= \int_{(1+\varepsilon)V}^{\infty} \mathbb{P}\left\{\left|\sum_{k=1}^{n} Z_k\right| > x^{1/p}\right\}\mathrm{d}x \le \\ &\le \int_{(1+\varepsilon)V}^{\infty} \mathbb{P}\left\{\left|\sum_{k=1}^{n} Z_k - \sum_{k=1}^{n}\mathbb{E}Z_k\right| > x^{1/p} - \left|\sum_{k=1}^{n}\mathbb{E}Z_k\right|\right\}\mathrm{d}x \le \\ &\le \int_{(1+\varepsilon)V}^{\infty} \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left[Z_k - \mathbb{E}Z_k\right]\right| > \left[1 - 2(1+\varepsilon)^{-1}\right]x^{1/p}\right\}\mathrm{d}x \le \\ &\le \left[1 - 2(1+\varepsilon)^{-1}\right]^{-q}\int_{(1+\varepsilon)V}^{\infty} x^{-q/p}\mathbb{E}\left|\sum_{k=1}^{n}\left[Z_k - \mathbb{E}Z_k\right]\right|^q\mathrm{d}x \le \\ &\le 2c_q g_q(n)\left[1 - 2(1+\varepsilon)^{-1}\right]^{-q}\sum_{k=1}^{n}\int_{(1+\varepsilon)V}^{\infty} x^{-q/p}\mathbb{E}\left|Z_k\right|^q\mathrm{d}x = \\ &= 2c_q g_q(n)\left[1 - 2(1+\varepsilon)^{-1}\right]^{-q}\sum_{k=1}^{n} I_{2k}.\end{aligned} \tag{2.6}$$

Above, in the last step we applied (2.1) and the c_q-inequality. Then for a fixed k, $1 \leq k \leq n$, we have

$$\begin{aligned} I_{2k} = \int\limits_{(1+\varepsilon)V}^{\infty} x^{-q/p}\mathbb{E}\left|Z_k\right|^q \mathrm{d}x &= \int\limits_{(1+\varepsilon)V}^{\infty} x^{-q/p} \int\limits_{0}^{x^{q/p}} \mathbb{P}\left\{|X_k|^q > y\right\} \mathrm{d}y\mathrm{d}x = \\ &= \int\limits_{(1+\varepsilon)V}^{\infty} x^{-q/p} \int\limits_{0}^{(1+\varepsilon)^{q/p}V^{q/p}} \mathbb{P}\left\{|X_k| > y^{1/q}\right\} \mathrm{d}y\mathrm{d}x+ \\ &+ \int\limits_{(1+\varepsilon)V}^{\infty} x^{-q/p} \int\limits_{(1+\varepsilon)^{q/p}V^{q/p}}^{x^{q/p}} \mathbb{P}\left\{|X_k| > y^{1/q}\right\} \mathrm{d}y\mathrm{d}x = \\ &= I_{21k} + I_{22k}. \end{aligned} \tag{2.7}$$

Again using Markov's inequality

$$\begin{aligned} I_{21k} &= \frac{p}{q-p}(1+\varepsilon)^{1-q/p}V^{1-q/p} \int\limits_{0}^{(1+\varepsilon)^{q/p}V^{q/p}} \mathbb{P}\left\{|X_k| > y^{1/q}\right\} \mathrm{d}y \leq \\ &\leq \frac{p}{q-p}(1+\varepsilon)^{1-q/p}V^{1-q/p} \int\limits_{0}^{(1+\varepsilon)^{q/p}V^{q/p}} \mathbb{E}|X_k|^p \cdot y^{-p/q}\mathrm{d}y = \\ &= \frac{qp}{(q-p)^2}\mathbb{E}|X_k|^p. \end{aligned} \tag{2.8}$$

For I_{22k}, we also get

$$\begin{aligned} I_{22k} &= \int\limits_{(1+\varepsilon)^{q/p}V^{q/p}}^{\infty} \mathbb{P}\left\{|X_k| > y^{1/q}\right\} \int\limits_{y^{p/q}}^{\infty} x^{-q/p}\mathrm{d}x\mathrm{d}y = \\ &= \frac{p}{q-p} \int\limits_{(1+\varepsilon)^{q/p}V^{q/p}}^{\infty} y^{p/q-1}\mathbb{P}\left\{|X_k| > y^{1/q}\right\} \mathrm{d}y \leq \\ &\leq \frac{p}{q-p} \int\limits_{0}^{\infty} y^{p/q-1}\mathbb{P}\left\{|X_k| > y^{1/q}\right\} \mathrm{d}y = \\ &= \frac{q}{q-p}\mathbb{E}|X_k|^p. \end{aligned} \tag{2.9}$$

Using relations (2.6)-(2.9), we get

$$\begin{aligned} I_2 &\le 2c_q g_q(n)[1-2(1+\varepsilon)^{-1}]^{-q}\left[\frac{qp}{(q-p)^2}+\frac{q}{q-p}\right]V = \\ &= 2c_q g_q(n)[1-2(1+\varepsilon)^{-1}]^{-q}\left(\frac{q}{q-p}\right)^2 V. \end{aligned} \tag{2.10}$$

Summarizing (2.3), (2.4) and (2.10), we obtain

$$\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p \le \left\{2+\varepsilon+2c_q g_q(n)[1-2(1+\varepsilon)^{-1}]^{-q}\left(\frac{q}{q-p}\right)^2\right\}V.$$

One can see that the function

$$f(\varepsilon)=2+\varepsilon+2c_q g_q(n)[1-2(1+\varepsilon)^{-1}]^{-q}\left(\frac{q}{q-p}\right)^2$$

is positive and continuous on the interval $(1,\infty)$, and $\lim_{\varepsilon\to1^+} f(\varepsilon)=\lim_{\varepsilon\to\infty} f(\varepsilon)=\infty$. Therefore $f(\varepsilon)$ has a minimum on $(1,\infty)$. Let $f_{p,q}(n)=\inf_{1<\varepsilon<\infty} f(\varepsilon)$. One can see that $f_{p,q}(n)>3$, it depends only on $g_q(n)$, p and q, so (2.2) is proved. □

2.3 Exponential inequalities and their consequences

In this subsection we shall see that if we assume that the exponential relation (1.3) is true for the truncated random variables, then we obtain an exponential inequality (Proposition 2.1), which implies Rosenthal's inequality (Proposition 2.2) and von Bahr-Esseen's moment inequality (Theorem 2.2).

Let $\eta_1,\eta_2,\dots,\eta_n$ be a sequence of r.v.'s. Consider the condition

$$\mathbb{E}e^{\sum_{i=1}^n \lambda\eta_i} \le g(n)\prod_{i=1}^{n}\mathbb{E}e^{\lambda\eta_i}. \tag{2.11}$$

If condition (2.11) is satisfied for $g(n)=1$ and for all $\lambda\in\mathbb{R}$, then $\eta_1,\eta_2,\dots,\eta_n$ are called acceptable. It is easy to see that, if (2.11) is true for $\eta_1,\eta_2,\dots,\eta_n$, then it is true for $\eta_1-a_1,\eta_2-a_2,\dots,\eta_n-a_n$ for any real numbers $a_1,\dots,a_n$, in particular it is true for $\eta_1-\mathbb{E}\eta_1,\eta_2-\mathbb{E}\eta_2,\dots,\eta_n-\mathbb{E}\eta_n$.

Given a r.v. X and numbers $a<b$, we define the following (asymmetrically) truncated r.v.

$${}^{(a)}X^{(b)} = a\mathbb{I}\{X<a\}+X\mathbb{I}\{a\le X\le b\}+b\mathbb{I}\{X>b\}. \tag{2.12}$$

This truncation ${}^{(a)}X^{(b)}$ is an increasing function of X.

Proposition 2.1. *Let* $X_1, X_2, \dots, X_n$ *be a sequence of r.v.'s. Assume that* (2.11) *is satisfied for any* $\lambda \in \mathbb{R}$ *and for* $\eta_i = {}^{(a_i)}X_i^{(b_i)}$ *for any* $a_i < b_i$, $i = 1, 2, \dots, n$. *Let* $d > 0$ *be fixed and let* $Y_i = {}^{(-d)}X_i^{(d)} - \mathbb{E}\,{}^{(-d)}X_i^{(d)}$, $i = 1, 2, \dots, n$, *be the truncated and centered r.v.'s. Let* $S_n = \sum_{i=1}^n Y_i$ *be the sum and let* $B_n = \sum_{i=1}^n \mathbb{E}Y_i^2$ *be the sum of variances. Then for any* $x > 0$ *and* $t > 0$, *we have*

$$\mathbb{P}\left(|S_n| > x\right) \le \mathbb{P}\left(\max_{1\le i\le n} |Y_i| > t\right) + 2g(n)\exp\left(\frac{x}{t} - \frac{x}{t}\ln\left(1 + \frac{xt}{B_n}\right)\right). \tag{2.13}$$

Proof. One can follow the classical ideas of [11] (see also [12] and [10]). For a real number $t > 0$ and a r.v. ξ

$$\xi^{(t)} = \min\{\xi, t\}$$

will denote the r.v. truncated from above. Let $\eta_i = Y_i^{(t)}$, $i = 1, 2, \dots, n$, denote our truncated r.v.'s. Then η_i is of the form ${}^{(a_i)}X_i^{(b_i)} - m_i$ for some $a_i < b_i$ and m_i, $i = 1, 2, \dots, n$. Therefore (2.11) is satisfied for $\eta_i = Y_i^{(t)}$. So usual argument (see [10]) gives

$$\mathbb{P}\left(\sum_{i=1}^n Y_i^{(t)} > x\right) \le g(n)\exp\left(\frac{x}{t} - \frac{x}{t}\ln\left(1 + \frac{xt}{B_n}\right)\right). \tag{2.14}$$

Inequality (2.11) is true for $\eta_i = (-Y_i)^{(t)}$, $i = 1, 2, \dots, n$, so (2.14) is true for the r.v.'s $-Y_1, -Y_2, \dots, -Y_n$, too. Applying (2.14) both for the r.v.'s $Y_1, Y_2, \dots, Y_n$ and the r.v.'s $-Y_1, -Y_2, \dots, -Y_n$, we get (2.13). ☐

Now we turn to Rosenthal's inequality.

Proposition 2.2. *Let* $X_1, X_2, \dots, X_n$ *be a sequence of r.v.'s. Assume that* (2.11) *is satisfied for any* $\lambda \in \mathbb{R}$ *and for* $\eta_i = {}^{(a_i)}X_i^{(b_i)}$ *for any* $a_i < b_i$, $i = 1, 2, \dots, n$. *Let* $d > 0$ *be fixed and let* $Y_i = {}^{(-d)}X_i^{(d)} - \mathbb{E}\,{}^{(-d)}X_i^{(d)}$, $i = 1, 2, \dots, n$, *be the truncated and centered r.v.'s. Let* $S_n = \sum_{i=1}^n Y_i$ *be their sum and* $B_n = \sum_{i=1}^n \mathbb{E}Y_i^2$ *be the sum of variances. Then*

$$\mathbb{E}|S_n|^p \le C_1 \mathbb{E}\max_{1\le i\le n} |Y_i|^p + 2C_2 g(n) B_n^{p/2}, \tag{2.15}$$

where $p > 0$ *and* C_1, C_2 *depend only on* p.

Proof. It is known, that the exponential inequality implies Rosenthal's inequality, see e.g. Theorem 3.1 in [10]. Therefore (2.13) implies (2.15). ☐

Now, we obtain the von Bahr-Esseen inequality.

Theorem 2.2. *Let* $1 < p \le 2$. *Let* $X_n, n = 1, 2, \dots$ *be a sequence of random variables with* $\mathbb{E}|X_n|^p < \infty$, $\mathbb{E}X_n = 0$ *for all* $n = 1, 2, \dots$. *Assume that* (2.11) *is satisfied for any* $\lambda \in \mathbb{R}$ *and for* $\eta_i = {}^{(a_i)}X_i^{(b_i)}$ *for any* $a_i < b_i$, $i = 1, 2, \dots, n$. *Then*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^p \le f_p(n) \sum_{k=1}^n \mathbb{E}|X_k|^p, \tag{2.16}$$

where $f_p(n)$ *depends only on* $g(n)$ *and* p.

Proof. Let $d > 0$ be fixed and let $Y_i = {}^{(-d)}X_i^{(d)} - \mathbb{E}\,{}^{(-d)}X_i^{(d)}$, $i = 1, 2, \dots, n$ be the truncated and centered r.v.'s. Let $S_n = \sum_{i=1}^n Y_i$ be their sum and $B_n = \sum_{i=1}^n \mathbb{E}Y_i^2$ be the sum of variances. Then, by Proposition 2.2 with exponent 2, we have

$$\mathbb{E}\left(\sum_{i=1}^n Y_i\right)^2 = \mathbb{E}|S_n|^2 \le Cg(n)B_n = \sum_{i=1}^n Cg(n)\mathbb{E}Y_i^2. \tag{2.17}$$

So we obtained that the von Bahr-Esseen moment inequality holds for exponent 2 for the truncated and centered random variables. Therefore, by Theorem 2.1, it holds for the random variables themselves for any exponent p with $1 < p < 2$. So (2.16) is proved for $1 < p < 2$. For $p = 2$ we use $d \uparrow \infty$ in (2.17). Then the dominated convergence theorem implies (2.16) if $p = 2$. □

Now, we apply our results to widely orthant dependent sequences. The sequence of r.v.'s $X_1, X_2, \dots$ is said to be widely orthant dependent (WOD) if for any positive integer n there exists a finite $g(n)$ so that for any real numbers $x_1, \dots, x_n$ we have

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \le g(n) \prod_{i=1}^n \mathbb{P}(X_i > x_i) \tag{2.18}$$

and

$$\mathbb{P}(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n) \le g(n) \prod_{i=1}^n \mathbb{P}(X_i \le x_i), \tag{2.19}$$

see [17]. It is known that extended negatively orthant dependent sequences, negatively orthant dependent sequences, negatively superadditive dependent sequences, negatively associated and independent sequences are WOD, see [16]. We list a few known facts on WOD sequences.

If $X_1, X_2, \dots$ is a WOD sequence and the real functions $f_1, f_2, \dots$ are either all non-decreasing of all non-increasing, then the sequence $f_1(X_1), f_2(X_2), \dots$ is also WOD. In particular, the truncated sequence ${}^{(a_i)}X_i^{(b_i)}$, $i = 1, 2, \dots$ is WOD. Moreover

$$\mathbb{E}e^{\sum_{i=1}^n \lambda X_i} \le g(n) \prod_{i=1}^n \mathbb{E}e^{\lambda X_i} \tag{2.20}$$

for any real number λ and with $g(n)$ in (2.18)-(2.19). Now, we obtain the von Bahr-Esseen inequality for WOD sequences. We remark that the following theorem was obtained using a different set-up in [16] (see Corollary 2.3 of [16]).

Theorem 2.3. *Let $1 < p \le 2$. Let $X_n, n = 1, 2, \dots$ be a WOD sequence of random variables satisfying* (2.18) *and* (2.19). *Assume $\mathbb{E}|X_n|^p < \infty$, $\mathbb{E}X_n = 0$ for all $n = 1, 2, \dots$. Then*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^p \le f_p(n) \sum_{k=1}^n \mathbb{E}|X_k|^p, \tag{2.21}$$

where $f_p(n)$ depends only on p and $g(n)$ from inequalities (2.18)-(2.19).

Proof. Because of the above mentioned properties of WOD sequences we can apply Theorem 2.2. □

2.4 Convergence theorems

In this subsection we shall prove general convergence theorems. We shall show that when the acceptability relation (2.11) is satisfied for the truncated random variables, then weak and strong laws of large numbers (WLLN, SLLN) and complete convergence hold without any further weak dependence assumption. As the proofs go through the Bahr-Esseen inequality, we can see that the validity of (2.16) for the truncated and centered random variables implies the above mentioned asymptotic results.

We start with an Etemadi style SLLN.

Theorem 2.4. *Let* $X_n, n = 1, 2, \dots$ *be a sequence of identically distributed r.v.'s satisfying* $\mathbb{E}X_1^2 < \infty$ *and* $\mathbb{E}X_1 = 0$.

(1) Assume that (2.11) *is satisfied with* $g(n) = C$ *for any* $\lambda \in \mathbb{R}$ *and for* $\eta_i = {}^{(a_i)}X_i^{(b_i)}$ *with any* $a_i < b_i$, $i = 1, 2, \dots$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = 0 \tag{2.22}$$

with probability 1.

(2) If, instead of (2.11), *the Bahr-Esseen inequality is satisfied for the truncated and centered r.v.'s, that is if*

$$\mathbb{E}\left(\sum_{i=1}^{n}\left({}^{(a_i)}X_i^{(b_i)} - \mathbb{E}\,{}^{(a_i)}X_i^{(b_i)}\right)\right)^2 \le C\sum_{i=1}^{n}\mathbb{E}\left({}^{(a_i)}X_i^{(b_i)} - \mathbb{E}\,{}^{(a_i)}X_i^{(b_i)}\right)^2, \tag{2.23}$$

then (2.22) *is satisfied.*

Proof. First we remark that, by Theorem 2.2, inequality (2.23) is always satisfied under the conditions of our theorem. We know that the original Etemadi's SLLN is satisfied for pairwise independent r.v.'s. However, analysing the proof (see [8] or [2]) the only step, where pairwise independence is applied is the use of inequality (2.23) with $a_i = 0$, $b_i = i$ and with $a_i = -i$, $b_i = 0$. □

A well-known SLLN for pairwise independent r.v.'s is the result of Csörgő, Tandori and Totik [6]. We show that Theorem 1 in [6] is valid if pairwise independence is replaced by an acceptability condition. We mention that in our theorem p is arbitrary with $1 < p < 2$, while in [6] $p = 2$.

Theorem 2.5. *Let* $1 < p < 2$. *Let* $X_n, n = 1, 2, \dots$ *be a sequence of r.v.'s. Assume that*

$$\sum_{m=1}^{\infty} \mathbb{E}\frac{|X_m - \mathbb{E}X_m|^p}{m^p} < \infty \tag{2.24}$$

and

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left|X_m-\mathbb{E}X_m\right| \qquad \textit{is bounded.} \tag{2.25}$$

(1) Assume that (2.11) *is satisfied with* $g(n)=C$ *for any* $\lambda\in\mathbb{R}$ *and for* $\eta_i={}^{(a_i)}X_i^{(b_i)}$ *with any* $a_i<b_i$, $i=1,2,\dots$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}(X_m-\mathbb{E}X_m)=0 \tag{2.26}$$

with probability 1.

(2) If, instead of (2.11), *the Bahr-Esseen inequality is satisfied for the truncated and centered r.v.'s, that is if*

$$\mathbb{E}\left|\sum_{i=1}^{n}\left({}^{(a_i)}X_i^{(b_i)}-\mathbb{E}\,{}^{(a_i)}X_i^{(b_i)}\right)\right|^p\le C\sum_{i=1}^{n}\mathbb{E}\left|{}^{(a_i)}X_i^{(b_i)}-\mathbb{E}\,{}^{(a_i)}X_i^{(b_i)}\right|^p, \tag{2.27}$$

then (2.26) *is satisfied.*

Proof. By Theorem 2.2, inequality (2.27) is always satisfied under the conditions of our theorem. In the original proof (see [6]) the only step, where pairwise independence is applied is the use of inequality (2.27) with with $a_i=0$, $b_i=\infty$ and with $a_i=-\infty$, $b_i=0$. □

It is known, that in the case of non-identically distributed random variables certain regularity conditions should be imposed for the moments or for the distributions (e.g. conditions (2.24), (2.25)). Such a condition is the weak mean domination.

The sequence of r.v.'s Y_n, $i=1,2,\dots$ is called weakly mean dominated (wmd) by the r.v. Y, if

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}\left(|Y_i|>t\right)\le C\mathbb{P}\left(|Y|>t\right) \tag{2.28}$$

for all $t\ge 0$ and $n=1,2,\dots$ (see Gut [13]).

We shall often use the following lemma (see [9]).

Lemma 2.1. *Let the sequence* Y_n, $i=1,2,\dots$ *be weakly mean dominated by the r.v.* Y. *Let* $t>0$ *be fixed. Let* $f:[0,\infty)\to[0,\infty)$ *be a strictly increasing unbounded function with* $f(0)=0$. *Then*
(a)

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}|Y_i|\le C\mathbb{E}|Y|; \tag{2.29}$$

(b) the sequence $f(|Y_n|)$, $i=1,2,\dots$ *is weakly mean dominated by the r.v.* $f(|Y|)$;
(c) the truncated sequence ${}^{(-t)}Y_n^{(t)}$, $i=1,2,\dots$ *is weakly mean dominated by the truncated*

r.v. ${}^{(-t)}Y^{(t)}$;
(d)

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}|Y_i|\mathbb{I}\left\{|Y_i|>t\right\}\le C\mathbb{E}|Y|\mathbb{I}\left\{|Y|>t\right\}. \tag{2.30}$$

The following theorem contains a WLLN and L_p-convergence.

Theorem 2.6. *Let* $1<p<2$. *Let the sequence* $X_n, n=1,2,\dots$ *be weakly mean dominated by the r.v.* X *with* $\mathbb{E}|X|^p<\infty$. *Assume that* $\mathbb{E}X_n=0$ *for all* $n=1,2,\dots$. *Assume that* (2.11) *is satisfied with* $g(n)=C$ *for any* $\lambda\in\mathbb{R}$ *and for* $\eta_i={}^{(a_i)}X_i^{(b_i)}$ *with any* $a_i<b_i$, $i=1,2,\dots$. *Then*

$$\lim_{n\to\infty}\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}X_k\right|^p=0. \tag{2.31}$$

Moreover,

$$\lim_{n\to\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}X_k=0 \tag{2.32}$$

in probability.

Proof. Let $t>0$. Define

$$ {}^{(-\infty)}Z^{(-t)}=\min\{-t,Z\},\quad {}^{(t)}Z^{(\infty)}=\max\{t,Z\}.$$

As $\mathbb{E}X_k=0$, so we have

$$\begin{aligned}\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}X_k\right|^p &\le c\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(-\infty)}X_k^{(-t)}-\mathbb{E}\,{}^{(-\infty)}X_k^{(-t)}\right)\right|^p+\\ &+c\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(-t)}X_k^{(t)}-\mathbb{E}\,{}^{(-t)}X_k^{(t)}\right)\right|^p+\\ &+c\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(t)}X_k^{(\infty)}-\mathbb{E}\,{}^{(t)}X_k^{(\infty)}\}\right)\right|^p=\\ &=T_1+T_2+T_3,\end{aligned} \tag{2.33}$$

say. Applying Theorem 2.2, we obtain

$$\begin{aligned}
T_3 &\le \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}\left|{}^{(t)}X_k^{(\infty)} - \mathbb{E}\,{}^{(t)}X_k^{(\infty)}\right|^p \le \\
&\le \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}\big|t\mathbb{I}\{X_k > t\} - \mathbb{E}t\mathbb{I}\{X_k > t\}\big|^p + \\
&+ \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}\left|X_k\mathbb{I}\{X_k > t\} - \mathbb{E}X_k\mathbb{I}\{X_k > t\}\right|^p \le \\
&\le \frac{c}{n}\sum_{k=1}^{n} t^p\mathbb{P}\{X_k > t\} + \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}X_k^p\mathbb{I}\{X_k > t\} \le \\
&\le \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}X_k^p\mathbb{I}\{X_k > t\}.
\end{aligned}$$

Similarly

$$T_1 \le \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}|X_k|^p\mathbb{I}\{X_k < -t\}.$$

Therefore, by (2.30),

$$T_1 + T_3 \le \frac{c}{n}\sum_{k=1}^{n}\mathbb{E}|X_k|^p\mathbb{I}\{|X_k| > t\} \le c\mathbb{E}|X|^p\mathbb{I}\{|X| > t\} \le \frac{\varepsilon}{2}$$

for any fixed $\varepsilon > 0$, if t is large enough, that is $t \ge t_\varepsilon$, say. Now, applying Theorem 2.2 with exponent 2, we obtain

$$\begin{aligned}
T_2 &\le \frac{c}{n}\left(\sum_{k=1}^{n}\mathbb{E}\left({}^{(-t)}X_k^{(t)} - \mathbb{E}\,{}^{(-t)}X_k^{(t)}\right)^2\right)^{p/2} \le \\
&\le \frac{c}{n}\left(n(2t)^2\right)^{p/2} = ct^p n^{p/2}/n.
\end{aligned} \tag{2.34}$$

Let $t = t_\varepsilon$ and choose n large enough so that $ct_\varepsilon^p n^{p/2}/n \le \frac{\varepsilon}{2}$. Then $T_2 \le \varepsilon/2$. □

Remark 2.1. Our Theorem 2.6 is similar to Theorem 3.1 of [4], where pairwise independent r.v.'s were considered. We can see that in our theorem the weak mean domination assumption can be replaced by the p-th uniform integrability assumption used in Theorem 3.1 of [4].

In the following theorem we shall see that if the acceptability condition, that is (2.11) with $g(n) = C$, holds for the truncated random variables, then complete (moment) convergence results can be obtained. In particular, if the Bahr-Esseen inequality holds for the truncated and centered random variables, then complete (moment) convergence holds.

Theorem 2.7. *Let* $0 < p < 2$, $1 \le r < 2$, *and* $0 < \alpha < 2$. *Let the sequence* X_n, $n = 1, 2, \dots$ *be weakly mean dominated by the r.v.* X. *Assume that* $\mathbb{E}X_n = 0$ *for all* $n = 1, 2, \dots$. *Assume that* (2.11) *is satisfied with* $g(n) = C$ *for any* $\lambda \in \mathbb{R}$ *and for* $\eta_i = {}^{(a_i)}X_i^{(b_i)}$ *with any* $a_i < b_i$, $i = 1, 2, \dots$. *(i) If* $r < \alpha$, *then assume* $\mathbb{E}|X|^\alpha < \infty$. *(ii) If* $r = \alpha$, *then assume* $\mathbb{E}|X|^r \log(1 + |X|) < \infty$. *(iii) If* $r > \alpha$, *then assume* $\mathbb{E}|X|^r < \infty$. *Then*

$$\sum_{n=1}^{\infty} n^{\alpha/p-2}\mathbb{E}\left\{\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_k\right| - \varepsilon\right\}_+^r < \infty \tag{2.35}$$

for any $\varepsilon > 0$.

Proof. Let $t = n^{1/p}$. As $\mathbb{E}X_k = 0$, so we have

$$\begin{aligned}
B &\overset{\text{def}}{=} \sum_{n=1}^{\infty} n^{\alpha/p-2}\mathbb{E}\left\{\frac{1}{n^{1/p}}\left|\sum_{k=1}^{n} X_k\right| - \varepsilon\right\}_+^r = \\
&= \sum_{n=1}^{\infty} n^{\alpha/p-2}\mathbb{E}\left\{\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(-t)}X_k^{(t)} - \mathbb{E}\,{}^{(-t)}X_k^{(t)}\right) + \right.\right. \\
&\quad + \sum_{k=1}^{n}\left({}^{(-\infty)}X_k^{(-t)} - \mathbb{E}\,{}^{(-\infty)}X_k^{(-t)}\right) + \\
&\quad \left.\left. + \sum_{k=1}^{n}\left({}^{(t)}X_k^{(\infty)} - \mathbb{E}\,{}^{(t)}X_k^{(\infty)}\}\right)\right| - \varepsilon\right\}_+^r \le \\
&\le c\sum_{n=1}^{\infty} n^{\alpha/p-2}\mathbb{E}\left(\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(-t)}X_k^{(t)} - \mathbb{E}\,{}^{(-t)}X_k^{(t)}\right)\right)^2 + \\
&\quad + c\sum_{n=1}^{\infty} n^{\alpha/p-2}\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(-\infty)}X_k^{(-t)} - \mathbb{E}\,{}^{(-\infty)}X_k^{(-t)}\right)\right|^r + \\
&\quad + c\sum_{n=1}^{\infty} n^{\alpha/p-2}\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left({}^{(t)}X_k^{(\infty)} - \mathbb{E}\,{}^{(t)}X_k^{(\infty)}\}\right)\right|^r,
\end{aligned} \tag{2.36}$$

where we applied Lemma 3.1 of [4]. Now, as $g(n) = C$, by Theorem 2.2, we obtain

$$\begin{aligned}
B &\le c\sum_{n=1}^{\infty} n^{\alpha/p-2}\frac{1}{n^{2/p}}\sum_{k=1}^{n}\mathbb{E}\left({}^{(-t)}X_k^{(t)} - \mathbb{E}\,{}^{(-t)}X_k^{(t)}\right)^2 + \\
&\quad + c\sum_{n=1}^{\infty} n^{\alpha/p-2}\frac{1}{n^{r/p}}\sum_{k=1}^{n}\mathbb{E}\left|{}^{(-\infty)}X_k^{(-t)} - \mathbb{E}\,{}^{(-\infty)}X_k^{(-t)}\right|^r + \\
&\quad + c\sum_{n=1}^{\infty} n^{\alpha/p-2}\frac{1}{n^{r/p}}\sum_{k=1}^{n}\mathbb{E}\left|{}^{(t)}X_k^{(\infty)} - \mathbb{E}\,{}^{(t)}X_k^{(\infty)}\right|^r = \\
&= T_2 + T_1 + T_3,
\end{aligned} \tag{2.37}$$

say.

First consider T_2. Applying Lemma 2.1, and using that $t = n^{1/p}$, we obtain

$$\begin{aligned} T_2 &\leq c\sum_{n=1}^{\infty} n^{\alpha/p-2/p-2} \sum_{k=1}^{n} \mathbb{E}\left({}^{(-t)}X_k^{(t)}\right)^2 \leq \\ &\leq c\sum_{n=1}^{\infty} n^{\alpha/p-2/p-1} \mathbb{E}\left({}^{(-t)}X^{(t)}\right)^2 = \\ &= c\sum_{n=1}^{\infty} n^{\alpha/p-2/p-1} \mathbb{E}|X|^2 \mathbb{I}\{|X| \leq n^{1/p}\} + c\sum_{n=1}^{\infty} n^{\alpha/p-2/p-1} \left(n^{1/p}\right)^2 \mathbb{P}\{|X| > n^{1/p}\} = \\ &= T_{21} + T_{22}. \end{aligned}$$

Now we have

$$\begin{aligned} T_{22} &\leq c\sum_{n=1}^{\infty} n^{\alpha/p-1} \sum_{k=n}^{\infty} \mathbb{P}\{k^{1/p} < |X| \leq (k+1)^{1/p}\} = \\ &= c\sum_{k=1}^{\infty} \mathbb{P}\{k^{1/p} < |X| \leq (k+1)^{1/p}\} \sum_{n=1}^{k} n^{\alpha/p-1} \leq c\mathbb{E}|X|^{\alpha}. \end{aligned}$$

Furthermore

$$\begin{aligned} T_{21} &\leq c\sum_{n=1}^{\infty} n^{\alpha/p-2/p-1} \sum_{k=1}^{n} \mathbb{E}|X|^2 \mathbb{I}\{(k-1)^{1/p} < |X| \leq k^{1/p}\} = \\ &= c\sum_{k=1}^{\infty} \mathbb{E}|X|^2 \mathbb{I}\{(k-1)^{1/p} < |X| \leq k^{1/p}\} \sum_{n=k}^{\infty} n^{\alpha/p-2/p-1} \leq c\mathbb{E}|X|^{\alpha}. \end{aligned}$$

Therefore we see that $T_2 < \infty$.

Now, we turn to T_1 and T_3. Like in the proof of Theorem 2.6, as $t = n^{1/p}$, we obtain

$$T_3 \leq c\sum_{n=1}^{\infty} n^{\alpha/p-r/p-2} \sum_{k=1}^{n} \mathbb{E}X_k^r \mathbb{I}\{X_k > n^{1/p}\}.$$

Similarly,

$$T_1 \leq c\sum_{n=1}^{\infty} n^{\alpha/p-r/p-2} \sum_{k=1}^{n} \mathbb{E}|X_k|^r \mathbb{I}\{X_k < -n^{1/p}\}.$$

Therefore, by Lemma 2.1,

$$\begin{aligned} T_1 + T_3 &\le c \sum_{n=1}^{\infty} n^{\alpha/p - r/p - 2} \sum_{k=1}^{n} \mathbb{E}|X_k|^r \mathbb{I}\{|X_k| > n^{1/p}\} \le \\ &\le c \sum_{n=1}^{\infty} n^{\alpha/p - r/p - 1} \mathbb{E}|X|^r \mathbb{I}\{|X| > n^{1/p}\} \le \\ &\le c \sum_{n=1}^{\infty} n^{\alpha/p - r/p - 1} \sum_{k=n}^{\infty} \mathbb{E}|X|^r \mathbb{I}\{k^{1/p} < |X| \le (k+1)^{1/p}\} \le \\ &\le c \sum_{k=1}^{\infty} \mathbb{E}|X|^r \mathbb{I}\{k^{1/p} < |X| \le (k+1)^{1/p}\} \sum_{n=1}^{k} n^{\alpha/p - r/p - 1}. \end{aligned}$$

Now, we see the following.

(i) If $r < \alpha$, then $T_1 + T_3 \le c\mathbb{E}|X|^\alpha < \infty$.

(ii) If $r = \alpha$, then $T_1 + T_3 \le c\mathbb{E}|X|^r \log(1 + |X|) < \infty$.

(iii) If $r > \alpha$, then $T_1 + T_3 \le c\mathbb{E}|X|^r < \infty$.

Therefore we see that $B < \infty$ in all cases. □

Remark 2.2. For pairwise independent and identically distributed random variables, Theorem 3.7 in [4] states the same assertion as our Theorem 2.7. By our proof we can see, that Theorem 3.7 in [4] can be extended to weakly mean dominated pairwise independent random variables. We also see that our Theorem 2.7 implies complete convergence for WOD random variables if in (2.18) and (2.19) $g(n) = C$.

Remark 2.3. Under the conditions of Theorem 2.7 we have

$$\sum_{n=1}^{\infty} n^{\alpha/p-2} \mathbb{P}\left\{ \left| \frac{1}{n^{1/p}} \sum_{k=1}^{n} X_k \right| > \varepsilon \right\} < \infty \tag{2.38}$$

for any $\varepsilon > 0$. It can be proved by usual calculation, see, e.g. [14], Remark 2.6.

3 Conclusions

We have obtained general versions of the von Bahr-Esseen moment inequality, the exponential inequality and convergence theorems. Our results can be applied to prove new limit theorems for weakly dependent sequences.

4 Declaration

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References

[1] R. G. Antonini, Y. Kozachenko, A. Volodin: Convergence of series of dependent φ-sub-Gaussian random variables. *J. Math. Anal. Appl.* **338** (2008), no. 2, 1188–1203.

[2] Heinz Bauer: *Probability theory.* Walter de Gruyter & Co., Berlin, 1996.

[3] L. E. Baum, M. Katz: Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** (1965), 108–123.

[4] Pingyan Chen, Peng Bai, Soo Hak Sung: The von Bahr-Esseen moment inequality for pairwise independent random variables and applications. *J. Math. Anal. Appl.* **419** (2014), no.2, 1290–1302.

[5] Y. S. Chow: On the rate of moment convergence of sample sums and extremes. *Bull. Inst. Math. Acad. Sinica*, **6** (1988), no. 3, 177–201.

[6] S. Csörgő, K. Tandori, V. Totik: On the strong law of large numbers for pairwise independent random variables. *Acta Math. Hungar.* **42** (1983), no. 3-4, 319–330.

[7] S. W. Dharmadhikari, Kumar Jogdeo: Bounds on moments of certain random variables. *Ann. Math. Statist.* **40** (1969), no.4, 1506–1509.

[8] N. Etemadi: An elementary proof of the strong law of large numbers. *Z. Wahrsch. Verw. Gebiete* **55** (1981), no. 1, 119–122.

[9] I. Fazekas: Convergence rates in the law of large numbers for arrays. *Publ. Math. Debrecen*, **41** (1992), no. 1-2, 53–71.

[10] I. Fazekas, S. Pecsora, B. Porvázsnyik: General theorems on exponential and Rosenthal's inequalities and on complete convergence. Manuscript, 2016.

[11] D. H. Fuk, S. V. Nagaev: Probabilistic inequalities for sums of independent random variables. *Teor. Verojatnost. i Primenen.* **16** (1971), 660–675.

[12] Shixin Gan, Pingyan Chen, Dehua Qiu: Rosenthal inequality for NOD sequences and its applications. *Wuhan Univ. J. Nat. Sci.* **16** (2011), no. 3, 185–189.

[13] A. Gut: Complete convergence for arrays. *Period. Math. Hungar.* **25** (1992), no. 1, 51–75.

[14] Soo Hak Sung: Moment inequalities and complete moment convergence. *J. Inequal. Appl.* (2009), Art. ID 271265, 14 pp.

[15] B. von Bahr, C. G. Esseen: Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist.* **36** (1965), no.1, 299–303.

[16] Xuejun Wang, Chen Xu, Tien-Chung Hu, Andrei Volodin, Shuhe Hu: On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models. *TEST* **23** (2014), no. 3, 607–629.

[17] Kaiyong Wang, Yuebao Wang, Qingwu Gao: Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate. *Methodol. Comput. Appl. Probab.* **15** (2013), no. 1, 109–124.