


AUTHOR QUERY FORM

 ELSEVIER	Journal: DIFGEO Article Number: 993	Please e-mail or fax your responses and any corrections to: E-mail: corrections.esch@elsevier.vtex.it Fax: +1 61 9699 6735
--	--	--

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list. Note: if you opt to annotate the file with software other than Adobe Reader then please also highlight the appropriate place in the PDF file. To ensure fast publication of your paper please return your corrections within 48 hours.

For correction or revision of any artwork, please consult <http://www.elsevier.com/artworkinstructions>

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the '[Q](#)' link to go to the location in the proof.

Location in article	Query / Remark: click on the Q link to go Please insert your reply or correction at the corresponding line in the proof
Q1 Q2	<p>Please confirm that given names and surnames have been identified correctly and are presented in the desired order. (p. 1/ line 5)</p> <p>Please complete and update the reference(s) given here: [12]. (p. 11/ line 22)</p> <div data-bbox="568 1260 1110 1364" style="border: 1px solid black; padding: 10px; margin: 20px auto; width: fit-content;"> <p>Please check this box if you have no corrections to make to the PDF file</p> <input data-bbox="1010 1282 1086 1342" type="checkbox"/> </div>



Contents lists available at SciVerse ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo



Sprays metrizable by Finsler functions of constant flag curvature

Ioan Bucataru^{a,*}, Zoltán Muzsnay^b^a Faculty of Mathematics, Alexandru Ioan Cuza University, Iași, Romania^b Institute of Mathematics, University of Debrecen, Debrecen, Hungary

ARTICLE INFO

Article history:

Received 6 December 2012

Available online xxxx

Communicated by Z. Shen

MSC:

53C60

58B20

49N45

58E30

Keywords:

Isotropic sprays

Finsler metrizable

Flag curvature

ABSTRACT

In this paper we characterize sprays that are metrizable by Finsler functions of constant flag curvature. By solving a particular case of the Finsler metrizability problem, we provide the necessary and sufficient conditions that can be used to decide whether or not a given homogeneous system of second order ordinary differential equations represents the geodesic equations of a Finsler function of constant flag curvature. The conditions we provide are tensorial equations on the Jacobi endomorphism. We identify the class of homogeneous SODE where the Finsler metrizable is equivalent with the metrizable by a Finsler function of constant flag curvature.

© 2013 Published by Elsevier B.V.

1. Introduction

The inverse problem of Lagrangian mechanics can be formulated as follows: decide whether or not a given system of second order ordinary differential equations (SODE) coincides with the Euler–Lagrange equations of some Lagrangian, [2,6,11,15,17,19]. When the given system of SODE is homogeneous and the Lagrangian to search for is the square of a Finsler function, the problem is known as the Finsler metrizable problem, [8,14,18,23]. If the sought after Lagrangian is a Finsler function, the problem is known as the projective metrizable problem, or as the Finslerian version of Hilbert's fourth problem, [1,7–10,24].

In this paper we address the special case of the Finsler metrizable problem, where the Finsler function we seek for has constant curvature. When the spray has zero constant curvature, then there is no obstruction for the existence of a locally defined Finsler structure that metrizes the given spray, [7,9,18]. Therefore, in this work we will focus on the case when the curvature is non-zero.

In Theorem 4.1, we solve the above mentioned problem by providing a set of equations, which contains an algebraic equation A) and two tensorial differential equations D_1) and D_2) in (4.1), which have to be satisfied by the Jacobi endomorphism. One of these two tensorial equations restricts the class of homogeneous SODE (sprays), which we discuss, to the class of isotropic sprays. Therefore, we focus our attention on isotropic sprays and their relation with the Finsler

* Corresponding author.

E-mail address: bucataru@uaic.ro (I. Bucataru).

URLs: <http://www.math.uaic.ro/~bucataru/> (I. Bucataru), <http://www.math.klte.hu/~muzsnay/> (Z. Muzsnay).

metrizability problem. In [Theorem 4.2](#), we characterize the class of isotropic sprays S for which the following conditions are equivalent:

- S is Finsler metrizable,
- S is metrizable by a Finsler metric of scalar flag curvature;
- S is Finsler metrizable by an Einstein metric;
- S is metrizable by a Finsler metric of constant flag curvature;
- S is Ricci constant.

If the Finsler function is reducible to a Riemannian metric, the equivalence of the above conditions, on manifolds of dimension greater than or equal to three, is always true for affine isotropic sprays, and it is not true in the general Finslerian context. Therefore, [Theorem 4.2](#) identifies the class of sprays where this equivalence is still true. The last condition is a technical one, can be checked very easily, and it is common for both [Theorems 4.1 and 4.2](#). [Theorem 4.1](#) has the advantage of being true for spray spaces of dimension greater than or equal to two, and it has the disadvantage of disregarding Finsler metrizable sprays of non-constant flag curvature. The main advantage of [Theorem 4.2](#) is that it treats sprays for which the Finsler metrizability is equivalent with the metrizability by a Finsler metric of constant non-zero flag curvature. The key ingredient in proving [Theorem 4.2](#) is the Finslerian version of Schur's Lemma, [\[4, Lemma 3.10.2\]](#), which is true only for spray spaces of dimension greater than or equal to three.

Since any spray on a [two-dimensional](#) manifold is isotropic, one of the two equations [\(4.1\)](#) in [Theorem 4.1](#) simplify. In [Theorem 4.3](#) we provide necessary and sufficient conditions for the metrizability of a two-dimensional spray space by a Finsler function of constant (Gaussian) curvature.

To support our results, in the last section, we consider various examples of isotropic sprays that satisfy, or not, one or more of the necessary and sufficient conditions, which we provide, for Finsler metrizability.

2. Sprays and their geometric setting

The natural geometric framework for studying systems of second order ordinary differential equations is the tangent bundle of some configuration manifold.

In this work, M denotes a C^∞ -smooth, real, and n -dimensional manifold. We will denote by TM its tangent bundle and by $T_0M = TM \setminus \{0\}$ the tangent bundle with the zero section removed. Local coordinate charts $(U, (x^i))$ on M induce local coordinate charts $(\pi^{-1}(U), (x^i, y^i))$ on TM , where $\pi: TM \rightarrow M$ is the canonical submersion. We will assume that M is a connected manifold of dimension $n \geq 2$. Therefore, TM and T_0M are $2n$ -dimensional connected manifolds.

In this section we discuss the natural geometric setting determined by a spray S , which includes canonical nonlinear connection, dynamical covariant derivative and curvature tensors. This setting, as well as the proofs of our results in the next sections, are based on the Frölicher-Nijenhuis theory and the corresponding differential calculus that can be developed on TM , [\[11,13,22\]](#). There are two canonical structures on TM , which we will use to develop our setting. One is the tangent structure, J , and the other one is the Liouville vector field, \mathbb{C} , locally given by

$$J = \frac{\partial}{\partial y^i} \otimes dx^i, \quad \mathbb{C} = y^i \frac{\partial}{\partial y^i}.$$

A system of homogeneous second order ordinary differential equations on a manifold M , whose coefficients do not depend explicitly on time, can be identified with a special vector field on T_0M that is called a spray. A vector field $S \in \mathfrak{X}(T_0M)$ is called a *spray* if $JS = \mathbb{C}$ and $[\mathbb{C}, S] = S$. Locally, a spray S is given by

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}, \tag{2.1}$$

where functions $G^i(x, y)$ are smooth functions on domains of induced coordinates on T_0M and 2-homogeneous with respect to the y -variable.

It is well known that a spray induces a nonlinear connection, with the corresponding projectors h and v given by

$$h = \frac{1}{2}(\text{Id} - \mathcal{L}_S J), \quad v = \frac{1}{2}(\text{Id} + \mathcal{L}_S J).$$

For a spray S , we consider the map $\nabla: \mathfrak{X}(T_0M) \rightarrow \mathfrak{X}(T_0M)$, given by [\[6\]](#)

$$\nabla = \mathcal{L}_S + h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v. \tag{2.2}$$

We require that the action of ∇ on scalar functions is given by $\nabla f = S(f)$, for $f \in C^\infty(T_0M)$. Further requirements that ∇ satisfies the Leibnitz rule and commutes with contractions allow us to extend its action to arbitrary tensor fields on T_0M . We will refer to ∇ as to the *dynamical covariant derivative* induced by the spray S . Its action on semi-basic forms was called the *semi-basic derivation* and studied, in connection with the inverse problem of the calculus of variation, in [\[11\]](#).

An important geometric structure induced by a spray is the *Jacobi endomorphism*, which is the vector valued semi-basic 1-form given by

$$\Phi = v \circ \mathcal{L}_S h = \mathcal{L}_S h \circ h.$$

Locally, the Jacobi endomorphism, is given by

$$\Phi = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j = \left(2 \frac{\partial G^i}{\partial x^j} - S \left(\frac{\partial G^i}{\partial y^j} \right) - \frac{\partial G^i}{\partial y^r} \frac{\partial G^r}{\partial y^j} \right) \frac{\partial}{\partial y^i} \otimes dx^j. \quad (2.3)$$

We consider also the *Ricci curvature*, Ric , and the *Ricci scalar*, ρ , [5], [21, Definition 8.1.7], which are given by

$$\text{Ric} = (n-1)\rho = R^i_i = \text{Tr}(\Phi). \quad (2.4)$$

Definition 2.1. A spray S is called

- i) *Ricci-constant* if $d_h \rho = 0$,
- ii) *weakly Ricci-constant* if $S(\rho) = 0$.

We say that a spray S is *isotropic* if its Jacobi endomorphism has the form,

$$\Phi = \rho J - \alpha \otimes \mathbb{C}, \quad (2.5)$$

where α is a semi-basic 1-form on T_0M .

It is easy to see that if a spray S is Ricci constant, then it is also weakly Ricci constant. Indeed, for a Ricci constant spray S , it follows that the Ricci scalar satisfies $d_h \rho = 0$. Using the corresponding commutation formula, we obtain

$$0 = i_S d_h \rho = -d_h i_S \rho + \mathcal{L}_S i_S \rho + i_{[h,S]} \rho = S(\rho),$$

and hence the spray S is weakly Ricci constant.

For an isotropic spray S , due to the homogeneity condition, it follows that $0 = \Phi(S) = (\rho - i_S \alpha) \mathbb{C}$ on T_0M and hence $\rho = i_S \alpha$. Isotropic sprays can be characterized using the Weyl curvature, see Proposition 13.4.1 in [21]. The semi-basic vector valued 1-form Φ is 2-homogeneous, which means $\Phi = [\mathbb{C}, \Phi]$. For an isotropic spray S , we have

$$\Phi = [\mathbb{C}, \Phi] = [\mathbb{C}, \rho J] - [\mathbb{C}, \alpha \otimes \mathbb{C}] = (\mathbb{C}(\rho) - \rho) J - \mathcal{L}_\mathbb{C} \alpha \otimes \mathbb{C},$$

which implies $\mathbb{C}(\rho) = 2\rho$ and $\mathcal{L}_\mathbb{C} \alpha = \alpha$. Therefore, the Ricci curvature Ric and the Ricci scalar ρ are 2-homogeneous, while the 1-form α is 1-homogeneous.

Lemma 2.2. Consider S an isotropic spray, whose Jacobi endomorphism is given by formula (2.5). The following two conditions are equivalent

- i) $d_J \alpha = 0$,
- ii) $d_J \rho = 2\alpha$.

Proof. Using the fact that $\rho = i_S \alpha$ and the commutation rules, it follows

$$d_J \rho = d_J i_S \alpha = -i_S d_J \alpha + \mathcal{L}_J i_S \alpha + i_{[J,S]} \alpha = -i_S d_J \alpha + 2\alpha. \quad (2.6)$$

In the above equations, we have used that $[J, S] = h - v$ and since α is a semi-basic 1-form, it follows that $i_{[J,S]} \alpha = i_h \alpha = \alpha$. Therefore, the assumption $d_J \alpha = 0$ implies that $d_J \rho = 2\alpha$. The other implication is straightforward, using the fact that the tangent structure J is integrable, and hence $d_J^2 = 0$. \square

It is known that the projective deformations of a spray preserves the class of isotropic sprays. These deformations may preserve or not the condition $d_J \alpha = 0$. In Section 5.1 we provide examples of isotropic sprays such that $d_J \alpha \neq 0$. See formula (5.1) for a convenient choice of a 1-homogeneous function $P \in C^\infty(T_0M)$.

3. Finsler metrizable sprays

In this section we recall the notion of a Finsler space and its geodesic spray. We will focus our discussions on Finsler spaces of scalar flag curvature.

Definition 3.1. By a *Finsler function* we mean a continuous function $F : TM \rightarrow \mathbb{R}$ satisfying the following conditions:

- i) F is smooth and strictly positive on T_0M ;
- ii) F is positively homogeneous of order 1, which means that $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda \geq 0$ and $(x, y) \in TM$;
- iii) The *metric tensor* with components

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \quad \text{has rank } n \text{ on } T_0M. \quad (3.1)$$

The above conditions of Definition 3.1 imply that the metric tensor g_{ij} of a Finsler function is positive definite on T_0M , see [16]. Some relaxations of the above conditions, which lead to the notion of *conic pseudo-Finsler metric*, were proposed in [12]. See also [3, §1.1.2, §1.2.1] for more discussions about the regularity conditions and their relaxation for a Finsler function. In Subsection 5.4 we will discuss an example of a spray metrizable by such a conic pseudo-Finsler function.

A Finsler function is *reducible to a Riemannian metric* if the metric tensor g_{ij} in formula (3.1) does not depend on the fiber coordinates y . If $g_{ij}(x)$ is a Riemannian metric on M , then $F : TM \rightarrow \mathbb{R}$, $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$ is a Finsler function.

The regularity condition iii) of Definition 3.1 is equivalent to the fact that the Poincaré–Cartan 2-form of F^2 , $\omega_{F^2} = -dd_J F^2$, is non-degenerate and hence it is a symplectic structure. Therefore, the equation

$$i_S dd_J F^2 = -dF^2 \quad (3.2)$$

uniquely determine a vector field S on T_0M , which is called the *geodesic spray* of the Finsler function. In this work we will use more frequently, the equation $\mathcal{L}_S d_J F^2 = dF^2$, which is equivalent to Eq. (3.2).

Definition 3.2. A spray $S \in \mathfrak{X}(T_0M)$ is called *Finsler metrizable* if there exists a Finsler function F that satisfies Eq. (3.2).

Necessary and sufficient criteria for the Finsler metrizability problem for a spray S were formulated in [18] using the holonomy distribution \mathcal{H}_S . Also, such necessary and sufficient conditions were formulated in terms of a semi-basic 1-form in [6]. We will use these conditions in the next section to discuss the Finsler metrizability problem of a particular class of isotropic sprays.

Definition 3.3. Consider F a Finsler function and Φ the Jacobi endomorphism of its geodesic spray S .

- i) F is said to be of *scalar (constant) flag curvature* if there exists a scalar function (constant) κ on T_0M , such that

$$\Phi = \kappa (F^2 J - F d_J F \otimes \mathbb{C}). \quad (3.3)$$

- ii) F is called an *Einstein metric* if there exists a function $\lambda \in C^\infty(M)$ such that the Ricci scalar satisfies $\rho(x, y) = \lambda(x)F^2(x, y)$.

The notion of flag curvature extends to the Finslerian setting the concept of sectional curvature from the Riemannian setting.

Remark 3.4. If a Finsler function F is reducible to a Riemannian metric g on a manifold of dimension greater than or equal to three, and S is its geodesic spray, then the following conditions are equivalent, see [21, §13.4]:

- i) S is isotropic;
- ii) g is of scalar curvature;
- iii) g is of constant curvature.

In the general Finslerian context, conditions ii) and iii) above are not equivalent anymore. In the next section, we provide the necessary and sufficient condition for an isotropic geodesic spray such that this equivalence remains true. See the equivalence of the conditions of Theorem 4.2.

4. Finsler metrizable isotropic sprays

In this section we use the necessary and sufficient conditions, expressed in terms of a semi-basic 1-form, which were formulated in [6, Theorem 5.4], to discuss the Finsler metrizability problem for some classes of isotropic sprays.

4.1. Sprays metrizable by Finsler functions of constant curvature

In the next theorem we provide the necessary and sufficient conditions one has to check if we want to decide if a spray is metrizable by a Finsler function of non-zero constant curvature.

Theorem 4.1. Consider S a spray with non-vanishing Ricci curvature. The spray S is metrizable by a Finsler function of non-zero constant flag curvature if and only if its Jacobi endomorphism satisfies the following equations:

$$\begin{aligned} A) \quad & \text{rank } dd_J(\text{Tr } \Phi) = 2n; \\ D_1) \quad & 2(n-1)\Phi - 2(\text{Tr } \Phi)J + d_J(\text{Tr } \Phi) \otimes \mathbb{C} = 0; \\ D_2) \quad & d_h(\text{Tr } \Phi) = 0. \end{aligned} \quad (4.1)$$

Proof. Consider S a spray with non-vanishing Ricci curvature. We assume that its Jacobi endomorphism, Φ , satisfies the algebraic assumption A) as well as the two tensorial equations (4.1). Since Φ satisfies D_1), it follows that the Jacobi endomorphism is given by formula (2.5), where $2(n-1)\alpha = (n-1)d_J\rho = d_J(\text{Tr } \Phi)$. Therefore the spray S is isotropic and satisfies the condition $d_J\alpha = 0$.

Due to condition D_2), we have that S is Ricci constant and, as we have seen already, it follows that the spray S is weakly Ricci constant.

Using the fact that $2\alpha = d_J\rho$ we obtain

$$2\mathcal{L}_S\alpha = \mathcal{L}_S d_J\rho = d_{[S, J]}\rho + d_J\mathcal{L}_S\rho = d_V\rho = d\rho. \quad (4.2)$$

Within the assumption that the Ricci curvature does not vanish on T_0M , we may consider the function $F > 0$ such that $F^2 = \text{sign}(\rho)\rho > 0$ on T_0M . Since $dd_J(\text{Tr } \Phi) = (n-1)dd_J\rho = (n-1)dd_JF^2$, the assumption A) assures that F is a Finsler function. The condition $2\alpha = d_J\rho$ reads now $2\alpha = d_JF^2$ and using formula (4.2) we obtain $\mathcal{L}_S d_JF^2 = dF^2$, which means that S is the geodesic spray of the Finsler function F .

We replace $\text{Tr } \Phi = (n-1)F^2 = (n-1)\rho = (n-1)i_S\alpha$ and $d_J(\text{Tr } \Phi) = 2(n-1)Fd_JF = 2(n-1)\alpha = (n-1)d_J\rho$ in the expression for Φ and obtain formula (3.3) for $\kappa = \text{sign}(\rho)$. It follows that spray S is Finsler metrizable by the Finsler function F of constant curvature $\kappa = \text{sign}(\rho)$.

Conversely, if the spray S is Finsler metrizable by a Finsler function of non-zero constant flag curvature then its Jacobi endomorphism is given by formula (3.3). It is a straightforward computation to see that Φ satisfies all three conditions A), D_1) and D_2) in (4.1). \square

Some of the conditions of Theorem 4.1 are related to the conditions of Theorem 7.2 in [11] as follows. Both theorems use the assumption of non-vanishing Ricci curvature $\rho = i_S\alpha$. Condition D_1), which implies that the spray S is isotropic and $d_J\alpha = 0$, is stronger than condition 2. of Theorem 7.2 in [11]. Also condition D_2) implies that $\nabla\alpha = 0$ and therefore this implies condition 3. of Theorem 7.2 in [11]. Note that $\nabla\alpha = \alpha'$ is the semi-basic derivation in [11]. However, the conclusion in Theorem 4.1 is stronger, for a given spray, we seek for the metrizable by a Finsler function of constant flag curvature. Theorem 7.2 in [11] characterizes local variational non-flat isotropic typical sprays. The differentiability assumptions are different, while we use smoothness, in [11] the analyticity of all geometric structures is assumed.

The algebraic condition A) assures the regularity condition for the sought after Finsler function. Condition D_1) is equivalent to the fact that the spray S is isotropic, its Jacobi endomorphism is given by formula (2.5), where $2(n-1)\alpha = d_J(\text{Tr } \Phi)$ and hence satisfies the condition $d_J\alpha = 0$. Condition D_2) says that the spray S is Ricci constant.

Conditions D_1) and D_2) in (4.1) are very useful to decide whether or not a given spray is metrizable by a Finsler function of non-zero constant curvature. However, there are metrizable sprays by Finsler functions of non-constant flag curvature, and hence where conditions D_1) and D_2) cannot be used. See the example in Section 5.2. In Theorem 4.2 we will strengthen the result of Theorem 4.1 by limiting our discussion to the case where Finsler metrizable is equivalent to the metrizable by a Finsler function of constant curvature. In this discussion, we use the Finslerian version of Schur's Lemma, [4, Lemma 3.10.2]. Therefore, we will have to limit our considerations to the case where $\dim M \geq 3$. The case $\dim M = 2$ will be treated separately.

4.2. $\dim M \geq 3$

Consider S an isotropic spray, whose Jacobi endomorphism is given by formula (2.5). In the next theorem we will see that the condition $d_J\alpha = 0$ is the best we can require to make sure that the three equivalent conditions in Remark 3.4 are also true in the Finslerian context. We add two more conditions, the last one is condition D_2) in Theorem 4.1.

Theorem 4.2. Consider S a spray of non-vanishing Ricci curvature. Then, the spray is isotropic, satisfies the algebraic condition A), and the condition $d_J\alpha = 0$, if and only if the following five conditions are equivalent:

- i) S is Finsler metrizable;
- ii) S is metrizable by a Finsler metric of non-vanishing scalar flag curvature;
- iii) S is Finsler metrizable by an Einstein metric;
- iv) S is metrizable by a Finsler metric of non-zero constant flag curvature;
- v) S is Ricci constant.

Proof. We assume that the spray is isotropic, satisfies the algebraic condition A), and the condition $d_J\alpha = 0$. Within these assumptions, we will prove the following implications i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow v) \Rightarrow i). For some of these implications we will not need the condition $d_J\alpha = 0$. More exactly, we will use it for the implications ii) \Rightarrow iii), iii) \Rightarrow iv), and v) \Rightarrow i). Also, the assumption $\dim M \geq 3$ will be needed only for the implication iii) \Rightarrow iv).

In order to prove that condition i) implies condition ii) we assume that the spray S is Finsler metrizable. We will prove that the corresponding Finsler space has scalar flag curvature. This result coincides with Lemma 8.2.2 in [21], where the proof uses different (local) techniques. Our proof is based on the differential calculus on T_0M associated to a spray within the Frölicher–Nijenhuis formalism.

We consider F a Finsler function that metricizes the spray S . It follows that S is the unique solution of the equation $\mathcal{L}_S d_J F^2 = dF^2$. Therefore, the corresponding Helmholtz conditions are satisfied. One of these Helmholtz conditions involves the Jacobi endomorphism Φ , and can be expressed using the semi-basic 1-form $\theta = d_J F^2$ as follows, see [6, Theorem 4.1.],

$$0 = d\Phi\theta = d_{\rho J} - \alpha \otimes \mathbb{C}\theta = \rho d_J\theta - d_{\alpha \otimes \mathbb{C}}\theta = -\alpha \wedge \mathcal{L}_C\theta = -\alpha \wedge \theta. \quad (4.3)$$

In the above formula we used $d_J\theta = d_J^2 F^2 = 0$, since $d_J^2 = d_{[J, J]} = 0$, and $\mathcal{L}_C\theta = \theta$, since θ is a 1-homogeneous 1-form. Helmholtz condition (4.3) implies $\alpha \wedge d_J F^2 = 0$ and hence there exists a function $\kappa \in C^\infty(T_0M)$ such that $\alpha = \kappa d_J F^2 / 2 = \kappa F d_J F$. It follows that $\rho = i_S \alpha = \kappa F^2$ and the Jacobi endomorphism (2.5) can be written now as in formula (3.3), which shows that the Finsler space (M, F) has scalar flag curvature κ .

In order to prove that condition ii) implies condition iii) we make use of the differential assumption $d_J\alpha = 0$. According to Lemma 2.2 it follows $d_J\alpha = 0$ is equivalent to $2\alpha = d_J\rho$. Using the fact that $\rho = \kappa F^2$ and $2\alpha = \kappa d_J F^2$ it follows

$$2\alpha = d_J\rho = F^2 d_J\kappa + \kappa d_J F^2 = F^2 d_J\kappa + 2\alpha.$$

Above formulae imply that $d_J\kappa = 0$ and hence the scalar flag curvature κ does not depend on the flagpole y . It follows that $\rho(x, y) = \lambda(x)F^2(x, y)$, where $\lambda(x) = \kappa(x)$. Hence, the Finsler Function F is an Einstein metric that metricizes the spray S .

In order to prove that condition iii) implies condition iv) we will also make use of Schur's Lemma. We know that the spray S is Finsler metrizable by an Einstein metric F . Then, there exists a non-vanishing function $\lambda \in C^\infty(M)$ such that $\rho(x, y) = \lambda(x)F^2(x, y)$. Since S is isotropic, it follows that its Jacobi endomorphism is given by formula

$$\Phi = \lambda F^2 J - \alpha \otimes \mathbb{C}.$$

Now we use the assumption that $d_J\alpha = 0$, which by Lemma 2.2 implies that $2\alpha = d_J\rho = \lambda d_J F^2$. This implies that the Jacobi endomorphism is given by formula (3.3), where $\kappa(x) = \lambda(x)$ is the scalar flag curvature. Since $\dim M \geq 3$, we use Schur's Lemma and obtain that κ is a non-zero constant. This implies that the spray S is metrizable by the Finsler function F of constant flag curvature.

The implication iv) \Rightarrow v) is straightforward. Consider S the geodesic spray of Finsler metric F of non-zero constant flag curvature κ . It follows that the Ricci scalar is given by $\rho = \kappa F^2$, where κ is a constant. Therefore, $d_h\rho = \kappa d_h F^2 = 0$ since $d_h F^2 = 0$.

For the last implication v) \Rightarrow i), we use the first implication of Theorem 4.1. Since S is isotropic and satisfies $d_J\alpha = 0$ it follows that the Jacobi endomorphism Φ satisfies equation D_1) in (4.1). The fact that S is Ricci constant means that Φ satisfies also equation D_2) in (4.1). By Theorem 4.1 we obtain that the spray S is Finsler metrizable.

To conclude the proof, we have to show that if a spray S satisfies one of the five equivalent conditions of the theorem, then necessarily S is isotropic, satisfies the algebraic condition A), as well as the condition $d_J\alpha = 0$. We assume that condition iv) is satisfied and hence the spray S is Finsler metrizable by a Finsler metric F of non-zero constant curvature κ . It follows that its Jacobi endomorphism is given by formula (3.3) and hence $\alpha = \kappa F d_J F = \kappa d_J F^2$. Since F is a Finsler function we obtain that $\text{rank} dd_J F^2 = 2n$ and hence the algebraic condition A) is satisfied. The condition $d_J\alpha = 0$ is also satisfied. \square

If a Finsler function is reducible to a Riemannian metric, and S is its isotropic geodesic spray, it follows that $\alpha = \kappa d_J F^2$, where $d_J\kappa = 0$, and hence we always have $d_J\alpha = 0$. This shows that the equivalence of conditions i), ii) and iv) of Theorem 4.2 generalize to the Finslerian context the equivalence of the three conditions of Remark 3.4.

In Theorem 4.2, the condition $\dim M \geq 3$ was very important, since it allowed us to use the Finslerian version of Schur's Lemma.

4.3. $\dim M = 2$

In this subsection we pay attention to the Finsler metrizable problem on 2-dimensional manifolds. It is known that any spray on a 2-dimensional manifold is isotropic. This result allows us to simplify the two conditions (4.1) of Theorem 4.1.

Theorem 4.3. Consider S a spray of non-vanishing Ricci curvature that satisfies the algebraic condition A), for $n = 2$.

- i) The spray S is isotropic, which means that its Jacobi endomorphism is given by formula (2.5), where the semi-basic 1-form $\alpha = \alpha_i dx^i$ has the components

$$\alpha_1 = \frac{R_2^2}{y^1} = \frac{-R_1^2}{y^2}, \quad \alpha_2 = \frac{-R_2^1}{y^1} = \frac{R_1^1}{y^2}. \quad (4.4)$$

- ii) The spray S is metrizable by a Finsler function of non-zero constant flag curvature if and only if it satisfies the following two conditions

$$d_J \alpha = 0, \quad d_h \rho = 0. \quad (4.5)$$

Proof. i) Due to the homogeneity conditions of the spray we have that $\Phi(S) = 0$ and by formula (2.3) we obtain $R_j^i y^j = 0$. It follows that Φ can be written as in formula (2.5), where $\rho = R_1^1 + R_2^2$ and the semi-basic 1-form $\alpha = \alpha_i dx^i$ has the components (4.4).

Since any spray S is isotropic we have that the Jacobi endomorphism Φ satisfies Eq. D_1) in (4.1) if and only if it satisfies first equation in (4.5). This part is then a consequence of Theorem 4.1. \square

The first part of Theorem 4.3 coincides with Lemma 8.1.10 in [21] and formulae (4.4) coincide with formulae (8.37) and (8.38) in [21].

The two conditions (4.5) can be written as follows

$$\begin{aligned} d_J \alpha = 0 &\Leftrightarrow 2\alpha = d_J \rho \Leftrightarrow \frac{\partial \alpha_1}{\partial y^2} = \frac{\partial \alpha_2}{\partial y^1} \Leftrightarrow 2\alpha_1 = \frac{\partial \rho}{\partial y^1} \quad \text{and} \quad 2\alpha_2 = \frac{\partial \rho}{\partial y^2}, \\ d_h \rho = 0 &\Leftrightarrow \frac{\delta \rho}{\delta x^1} = \frac{\delta \rho}{\delta x^2} = 0, \quad \text{where} \quad \frac{\delta}{\delta x^i} = h \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} - \frac{\partial G^j}{\partial y^i} \frac{\partial}{\partial y^j}. \end{aligned} \quad (4.6)$$

We will use the above conditions in the next section to test whether or not some sprays on a two-dimensional manifold are metrizable by Finsler functions of constant curvature.

5. Examples

In this section we provide examples to show the consistency of the conditions we discussed so far.

5.1. The case $d_J \alpha = 0$

It is well known that the class of isotropic sprays is invariant under projective deformations of sprays. We start with a Finsler metrizable isotropic spray, S_0 , which satisfies the condition $d_J \alpha = 0$. We will study projective deformations of S_0 , which also satisfy the condition $d_J \alpha = 0$. Within this projective deformations, we will seek for those which do not preserve the condition of being Ricci constant. Using Theorem 4.2, this will lead us to examples of non-Finsler metrizable isotropic sprays.

Let S_0 be the geodesic spray of a Finsler function F_0 , which has constant flag curvature κ_0 . Denote by h_0 the horizontal projector induced by the spray S_0 and ∇_0 the corresponding dynamical covariant derivative. The spray S_0 is isotropic, its Jacobi endomorphism is given by

$$\Phi_0 = \kappa_0 (F_0^2 J - F_0 d_J F_0 \otimes \mathbb{C}).$$

Consider the projectively equivalent spray $S = S_0 - 2P\mathbb{C}$, where $P \in C^\infty(T_0M)$ is a 1-homogeneous function. According to formulae (4.8) in [8, Proposition 4.4], the horizontal projector and the Jacobi endomorphism of the spray S are given by

$$\begin{aligned} h &= h_0 - 2(PJ + d_J P \otimes \mathbb{C}), \\ \Phi &= \Phi_0 + (P^2 - S_0(P))J + (2d_{h_0} P - Pd_J P - \nabla_0 d_J P) \otimes \mathbb{C}. \end{aligned}$$

The spray S is also isotropic. We will study now when a projective deformation preserves the condition $d_J\alpha = 0$, where the 1-form α is given by $\alpha = \kappa_0 F_0 d_J F_0 - 2d_{h_0}P + Pd_JP + \nabla_0 d_JP$. The 2-form $d_J\alpha$ has been computed in the proof of Proposition 4.4 in [8] and it is given by

$$d_J\alpha = 3d_Jd_{h_0}P = -3d_{h_0}d_JP. \quad (5.1)$$

Therefore $d_J\alpha = 0$ if and only if $d_{h_0}P = d_Jg$, for some function $g \in C^\infty(T_0M)$. For $g = P^2/2$ the function P is called a *Funk function*, see [21, Definition 12.1.4]. It has been shown in [21, Proposition 12.1.3] that projective deformations by a Funk function preserve the Jacobi endomorphism.

In order to simplify some of the calculations we assume that the function P satisfies $d_{h_0}P = 0$ and hence $d_J\alpha = 0$. It follows that $S_0(P) = 0$ and using the commutation formula (4.11) in [8] we have $\nabla_0 d_JP = d_J\nabla_0P - d_{h_0}P = 0$. Therefore, the Jacobi endomorphism Φ of the spray S is given by

$$\Phi = (\kappa_0 F_0^2 + P^2)J - (\kappa_0 F_0 d_J F_0 + Pd_JP) \otimes \mathbb{C}. \quad (5.2)$$

It follows that the Ricci scalar is given by

$$\rho = \kappa_0 F_0^2 + P^2.$$

We check now the last condition of Theorem 4.2. Using the assumption that $d_{h_0}P = 0$ it follows that $d_{h_0}\rho = 0$ and hence we have

$$d_h\rho = -2d_{P_J+d_JP\otimes\mathbb{C}}\rho = -2(Pd_J\rho + 2\rho d_JP) = -\frac{2}{P}d_J(\rho P^2). \quad (5.3)$$

From the above formula, we see that we can choose a function P such that $d_h\rho \neq 0$, which will imply that the spray S is not Ricci constant. According to Theorem 4.2 we conclude that the spray S is not Finsler metrizable. Indeed, we can take $P = \lambda F_0$, where λ is a constant. If we replace this in formula (5.3), we obtain

$$d_h\rho = -4\lambda(\kappa_0 + \lambda^2)d_JF_0^2. \quad (5.4)$$

In this case we have that the spray S is Ricci constant, and hence it is Finsler metrizable, if and only if either $\lambda = 0$ or $\lambda^2 + \kappa_0 = 0$. See also Theorem 5.1 in [8]. When the spray S_0 is projectively flat, we can view this as an alternative proof of Theorem 1.2 in [24].

5.2. The case $d_J\alpha \neq 0$.

We present now an example of a Finsler metrizable, isotropic spray, which does not satisfy the condition $d_J\alpha = 0$. This also shows that the assumption $d_J\alpha = 0$, which we made in Theorem 4.2, is the best assumption we could consider in order to have the equivalence of the five conditions.

We will use the following Randers metric studied by Shen, see Example 11.2 in [20]. Consider a domain $M \subset \mathbb{R}^n$, where $\Delta(x) = 1 - |a|^2|x|^4 > 0$. Denote by $\beta(x, y) = 2\langle a, x \rangle \langle x, y \rangle - |x|^2 \langle a, y \rangle$. The Finsler function $F : M \times \mathbb{R}^n \rightarrow \mathbb{R}$, given by

$$F(x, y) = \frac{\sqrt{\beta^2(x, y) + \Delta(x)|y|^2} + \beta(x, y)}{\Delta(x)}$$

has scalar flag curvature given by

$$\kappa(x, y) = 3\frac{\langle a, y \rangle}{F} + 3\langle a, x \rangle^2 - 2|a|^2|x|^2.$$

The geodesic spray S of the Finsler function F is isotropic and the 1-form α , in formula (2.5), is given by $\alpha = \kappa F d_JF$ and hence

$$d_J\alpha = Fd_J\kappa \wedge d_JF.$$

The scalar flag curvature κ is 0-homogeneous and therefore $0 = \mathbb{C}(\kappa) = i_S d_J\kappa$. Moreover the flag curvature κ depends on the flagpole y , which means that $d_J\kappa \neq 0$. Therefore,

$$i_S d_J\alpha = Fi_S d_J\kappa d_JF - Fi_S d_JF d_J\kappa = -F^2 d_J\kappa \neq 0,$$

and this implies that $d_J\alpha \neq 0$. Therefore, the spray S is Finsler metrizable and isotropic. However, the five conditions in Theorem 4.2 are not equivalent and this is due to the fact that $d_J\alpha \neq 0$.

5.3. Two-dimensional examples

We consider now some examples of sprays on a two-dimensional manifold and use the conditions (4.5) to test if they are metrizable by a Finsler function of constant curvature.

5.3.1. The Poincaré model and the Finslerian Poincaré disk

Consider the geodesic equations of the Poincaré half plane $M = \{(x^1, x^2) \in \mathbb{R}^2, x^2 > 0\}$:

$$\begin{aligned} \frac{d^2 x^1}{dt^2} - \frac{2}{x^2} \frac{dx^1}{dt} \frac{dx^2}{dt} &= 0, \\ \frac{d^2 x^2}{dt^2} + \frac{1}{x^2} \left(\left(\frac{dx^1}{dt} \right)^2 - \left(\frac{dx^2}{dt} \right)^2 \right) &= 0. \end{aligned}$$

The above system of second order ordinary differential equations determines a spray $S \in \mathfrak{X}(TM)$. For this spray S , the local components (2.3) of the Jacobi endomorphism are given by

$$R_1^1 = -\frac{(y^2)^2}{(x^2)^2}, \quad R_2^1 = R_1^2 = \frac{y^1 y^2}{(x^2)^2}, \quad R_2^2 = -\frac{(y^1)^2}{(x^2)^2}.$$

According to first part of Theorem 4.3, it follows that the spray S is isotropic, with the two components of the semi-basic 1-form α given by formula (4.4):

$$\alpha_1 = \frac{R_2^1}{y^1} = -\frac{y^1}{(x^2)^2}, \quad \alpha_2 = \frac{R_1^1}{y^2} = -\frac{y^2}{(x^2)^2}.$$

It is very easy to check that $d_J \alpha = 0$ and hence the first condition (4.5) is satisfied. The Ricci scalar of the spray is given by

$$\rho = R_1^1 + R_2^2 = -\frac{1}{(x^2)^2} \{(y^1)^2 + (y^2)^2\}. \quad (5.5)$$

It follows that $d_h \rho = 0$ and hence the second condition (4.5) is satisfied. By the second part of Theorem 4.3 we obtain that S is metrizable by a Finsler function of constant negative sectional curvature. From formula (5.5) it follows that, up to a multiplicative constant, the Finsler function F is given by

$$F(x, y) = \frac{1}{x^2} \sqrt{(y^1)^2 + (y^2)^2}.$$

The above Finsler function is reducible to a Riemannian metric, which we can recognize to be the Poincaré metric of the upper half plane.

Although the previous example is Riemannian, one can modify it to obtain a Finslerian one. Consider the disk $M = \{(x^1, x^2) \in \mathbb{R}^2, (x^1)^2 + (x^2)^2 < 4\}$ with the following Finsler function $F: TM \rightarrow [0, +\infty)$, known as the Finslerian Poincaré disk [4,5],

$$F = 4 \frac{\sqrt{(y^1)^2 + (y^2)^2}}{4 - r^2} + 16 \frac{x^1 y^2 + x^2 y^1}{(4 - r^2)(4 + r^2)},$$

where $r^2 = (x^1)^2 + (x^2)^2$. In the first term of the right-hand side of the above formula we can recognize the Poincaré metric on the disk M . The above Finsler function has constant flag curvature $\kappa = -1/4$, and its geodesic spray satisfies all the assumptions of Theorem 4.3.

5.3.2. Non-constant Ricci scalar

We consider an example proposed by Bao and Robles in [5] of a two-dimensional spray, which is metrizable by a Finsler function of non-constant Ricci scalar. For this example, the first condition (4.5) is satisfied, but the second condition (4.5) it is not.

Consider the portion of the elliptic paraboloid $M = \{(x^1, x^1, x^3) \in \mathbb{R}^3, x^3 = (x^1)^2 + (x^2)^2, (x^1)^2 + (x^2)^2 < 1\}$. We denote $\Delta(x^1, x^1) = 1 - (x^1)^2 - (x^2)^2 > 0$. The Randers metric $F: TM \rightarrow [0, +\infty)$, $F = \alpha + \beta$, where

$$\begin{aligned} \alpha &= \frac{\sqrt{(x^1 y^2 - x^2 y^1)^2 + ((1 + 4(x^1)^2)(y^1)^2 + 8x^1 x^2 y^1 y^2 + (1 + 4(x^2)^2)(y^2)^2) \Delta}}{\Delta}, \\ \beta &= \frac{x^2 y^1 - x^1 y^2}{\Delta(x^1, x^2)}, \end{aligned}$$

has scalar flag curvature

$$\kappa(x^1, x^2) = \frac{4}{(1 + 4(x^1)^2 + 4(x^2)^2)^2}.$$

Let S be the geodesic spray of the Finsler function F and h the horizontal projector. It follows that S is isotropic, its Jacobi endomorphism is given by formula (2.5), where

$$\rho = \kappa F^2, \quad \alpha = \kappa F d_J F.$$

For the two-dimensional spray S , we check now the conditions of Theorem 4.3. The Ricci scalar ρ does not vanish in T_0M . Since $d_J\kappa = 0$ it follows that $d_J\alpha = 0$ and hence first condition (4.5) is satisfied. However, since $d_h\kappa \neq 0$ and $d_hF = 0$ it follows that $d_h\rho \neq 0$, which shows that the spray S is not Ricci constant, which means that second condition (4.5) is not satisfied.

5.4. The non-vanishing condition of the Ricci curvature

In this subsection we discuss the non-vanishing condition of the Ricci curvature, which we assumed in our results. We will explain that although the Ricci curvature might vanish, if it is not identically zero, then the conditions of metrizability in Theorems 4.1, 4.2 and 4.3 are still necessary and sufficient, but the metrizability to consider has to be with respect to a conic pseudo-Finsler function, as defined in [12].

Consider the following affine spray on some open domain $M \subset \mathbb{R}^2$

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - \phi(x^1, x^2)(y^1)^2 \frac{\partial}{\partial y^1} - \psi(x^1, x^2)(y^2)^2 \frac{\partial}{\partial y^2}. \quad (5.6)$$

For this spray S , the local components (2.3) of the Jacobi endomorphism are given by

$$R_1^1 = -\phi_{x^2} y^1 y^2, \quad R_2^1 = \phi_{x^2} (y^1)^2, \quad R_1^2 = \psi_{x^1} (y^2)^2, \quad R_2^2 = -\psi_{x^1} y^1 y^2.$$

The spray S is isotropic and the two components of the semi-basic 1-form α are given by formula (4.4):

$$\alpha_1 = \frac{R_2^2}{y^1} = -\psi_{x^1} y^2, \quad \alpha_2 = \frac{R_1^1}{y^2} = -\phi_{x^2} y^1.$$

In view of the local formulae (4.6), the condition $d_J\alpha = 0$ is satisfied if and only if

$$\frac{\partial \alpha_1}{\partial y^2} = \frac{\partial \alpha_2}{\partial y^1}, \quad \text{which is equivalent to: } \psi_{x^1} = \phi_{x^2}.$$

Next, we assume that the above condition is satisfied. Within this assumption, it follows that the Ricci scalar is given by

$$\rho = -y^1 y^2 (\phi_{x^2} + \psi_{x^1}) = -2y^1 y^2 \phi_{x^2} = -2y^1 y^2 \psi_{x^1}.$$

To test the second condition (4.5), we use the local expression (4.6):

$$\frac{\delta \rho}{\delta x^1} = 2y^1 y^2 (\phi \phi_{x^2} - \phi_{x^1 x^2}), \quad \frac{\delta \rho}{\delta x^2} = 2y^1 y^2 (\psi \psi_{x^1} - \psi_{x^1 x^2}).$$

For the functions $\phi, \psi \in C^\infty(M)$ we consider the following system of partial differential equations:

$$\phi_{x^2} = \psi_{x^1}, \quad \phi_{x^1 x^2} - \phi \phi_{x^2} = 0, \quad \psi_{x^1 x^2} - \psi \psi_{x^1} = 0. \quad (5.7)$$

For example on $M = \{(x^1, x^2) \in \mathbb{R}^2, x^1 + x^2 > 0\}$ we can consider the functions

$$\phi(x^1, x^2) = \psi(x^1, x^2) = -\frac{2}{x^1 + x^2}, \quad (5.8)$$

which satisfy the above system of partial differential equations. For this choice of functions, the spray

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + \frac{2(y^1)^2}{x^1 + x^2} \frac{\partial}{\partial y^1} + \frac{2(y^2)^2}{x^1 + x^2} \frac{\partial}{\partial y^2}$$

satisfies the two conditions (4.5) of Theorem 4.3. Since the Ricci curvature of the spray S is given by

$$\rho = -\frac{4y^1 y^2}{(x^1 + x^2)^2},$$

it follows that the spray S is metrizable by a conic pseudo-Finsler function, see [12], $F : A = \{(x^1, x^2, y^1, y^2) \in TM, y^1 y^2 > 0\} \subset TM \rightarrow [0, +\infty)$, where

$$F^2(x, y) = \frac{4y^1 y^2}{(x^1 + x^2)^2},$$

has constant sectional curvature $\kappa = -1$.

The case when the Ricci scalar ρ is identically zero, which corresponds to $\phi_{x^2} = -\psi_{x^1}$, has been studied in [21, Example 8.2.4], where it has been shown that the spray S , given by (5.6) is not Finsler metrizable.

Acknowledgements

The work of I.B. has been supported by the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2012-4-0131. The work of Z.M. has been supported by the Hungarian Scientific Research Fund (OTKA) Grant K67617.

References

- [1] J.C. Álvarez Paiva, Symplectic geometry and Hilbert's fourth problem, *J. Differential Geom.* 69 (2) (2005) 353–378.
- [2] I. Anderson, G. Thompson, The inverse problem of the calculus of variations for ordinary differential equations, *Mem. Amer. Math. Soc.* 98 (473) (1992) 1–110.
- [3] P.L. Antonelli, R.S. Ingarden, M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer, 1993.
- [4] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann–Finsler Geometry*, Springer, 2000.
- [5] D. Bao, C. Robles, Ricci and flag curvatures in Finsler geometry, in: D. Bao, et al. (Eds.), *A Sampler of Riemann–Finsler Geometry*, in: *Math. Sci. Res. Inst. Publ.*, vol. 50, Cambridge Univ. Press, Cambridge, 2004, pp. 197–259.
- [6] I. Bucataru, M.F. Dahl, Semi-basic 1-forms and Helmholtz conditions for the inverse problem of the calculus of variations, *J. Geom. Mech.* 1 (2) (2009) 159–180.
- [7] I. Bucataru, Z. Muzsnay, Projective metrizable and formal integrability, *SIGMA* 7 (2011) 114, 22 p.
- [8] I. Bucataru, Z. Muzsnay, Projective and Finsler metrizable: parametrization rigidity of geodesics, *Int. J. Math.* 23 (6) (2012) 1250099, 15 p.
- [9] M. Crampin, Isotropic and R-flat sprays, *Houston J. Math.* 33 (2007) 451–459.
- [10] M. Crampin, T. Mestdag, D.J. Saunders, The multiplier approach to the projective Finsler metrizable problem, *Differential Geom. Appl.* 30 (6) (2012) 604–621.
- [11] J. Grifone, Z. Muzsnay, *Variational Principles for Second-Order Differential Equations*, World-Scientific, 2000.
- [12] M.A. Javaloyes, M. Sánchez, On the definition and examples of Finsler metrics, *Ann. Sc. Norm. Sup. Pisa*, <http://dx.doi.org/10.2422/2036-2145.201203-002>, 40 p.
- [13] I. Kolar, P.W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.
- [14] D. Krupka, A.E. Sattarov, The inverse problem of the calculus of variations for Finsler structures, *Math. Slovaca* 35 (1985) 217–222.
- [15] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Mathematics, vol. 1678, Springer-Verlag, Berlin, 1997.
- [16] R.L. Lovas, A note on Finsler–Minkowski norms, *Houston J. Math.* 33 (2007) 701–707.
- [17] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo, C. Rubano, The inverse problem in the calculus of variations and the geometry of the tangent bundle, *Phys. Rep.* 188 (3–4) (1990).
- [18] Z. Muzsnay, The Euler–Lagrange PDE and Finsler metrizable, *Houston J. Math.* 32 (2006) 79–98.
- [19] W. Sarlet, The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics, *J. Phys. A: Math. Gen.* 15 (1982) 1503–1517.
- [20] Z. Shen, Landsberg curvature, S-curvature and Riemann curvature, in: D. Bao, et al. (Eds.), *A Sampler of Riemann–Finsler Geometry*, in: *Math. Sci. Res. Inst. Publ.*, vol. 50, Cambridge Univ. Press, Cambridge, 2004, pp. 305–356.
- [21] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Springer, 2001.
- [22] J. Szilasi, A setting for spray and Finsler geometry, in: P.L. Antonelli (Ed.), *Handbook of Finsler Geometry*, vol. 2, Kluwer Acad. Publ., Dordrecht, 2003, pp. 1183–1426.
- [23] J. Szilasi, S. Vattamány, On the Finsler-metrizabilities of spray manifolds, *Period. Math. Hungar.* 44 (2002) 81–100.
- [24] G. Yang, Some classes of sprays in projective spray geometry, *Differential Geom. Appl.* 29 (4) (2011) 606–614.