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# Optimizing graphical network design using fuzzy hypersoft topologies and multi-criteria VIKOR decision algorithms

Shabana Anwar [a], Muhammad Kamran Jamil [a], Hothefa Shaker Jassim [b,*], Muhammad Azeem [a], Bandar Almohsen [c], Husam A. Neamah [d,e,*]

[a] Department of Mathematics, Riphah International University, Lahore, Pakistan
[b] Modern college of business and Science, Muscat, Oman
[c] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[d] Department of Electrical Engineering and Mechatronics, Faculty of Engineering, University of Debrecen, Òtemetö Utca 2-4, 4028 Debrecen, Hungary
[e] echnical Engineering College, Al-Ayen University, 64001 Thi-Qar, Iraq

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ABSTRACT

The accumulated data is displayed in the form of a figure called a graph. The notion of fuzziness was developed in graph theory to address issues not resolved by crisp graph theory. However, fuzzy soft ideas in graph theory, which is a parameterized family and provides more exact and generalized solutions to answer vague and ambiguous problems, can yield more generalized conclusions. Fuzzy hypersoft is another broader and more generic soft set, which is the Cartesian product of disjoint attribute-valued sets corresponding to distinct attributes, in which mapping is transformed into a multi-attribute. The number of fuzzy hypersoft subgraphs is equal to the cardinality of the set obtained after taking the Cartesian product of disjoint attributed valued sets, which could be applied in graph theory to get surprising results. The novelty of this manuscript is defining topological numbers for the first time in a fuzzy hypersoft environment. We have also initiated the definitions of path graph, cycle graph, complete graph, wheel graph, and star graph, and calculated the degrees of nodes of each graph in a fuzzy hypersoft environment. Then, we have derived some generalized results for these fuzzy hypersoft families of graphs for the first Zagreb number, the second Zagreb number, and the Randic number. This is also a novel approach that we have converted the decision-making algorithm, vizierkriterijumsko kompromisno rangiranje (VIKOR), into a fuzzy hypersoft framework. Our objectives are to lessen the complexity in the approaches and to establish a solid relationship with the multi-criteria decision-making procedures. It is intriguing that the decision-making issue can be addressed with hypersoft theory without the restrictions on the decision-maker's choice of values. In the end, we have presented an application to highlight the generic nature of the fuzzy hypersoft environment. We also have examined the best graphical network to maximize profit for a four-partner corporation, taking into account fuzzy hypersoft topological numbers as decision-makers. Then, VIKOR in the fuzzy hypersoft framework is used for the same purpose. The acquired results are enough to prove that the graphical network for yielding maximum profit is the same by applying MCDM as well as by applying the formulas of different topological numbers. Furthermore, we have presented the AHP method's methodology and eventually employed it to determine the top-ranked graphical network among commercial enterprises. Furthermore, a comparison has been made between fuzzy soft topological numbers and fuzzy hypersoft topological numbers.

## 1. Introduction

Strong and geodetic dominion integrity sets were examined in [1], along with the effects of node removal on these sets. A fuzzy set is a class of things that have a range of membership grades. Fuzzy sets are membership functions that assign a membership grade of zero to one for each item [2]. Research trends and publication channels related to the application of fuzzy set theory in production and operations management [3].

* Corresponding authors.
*E-mail addresses:* 30191@students.riphah.edu.pk (S. Anwar), kamran.jamil@riphah.edu.pk (M.K. Jamil), hothefa.shaker@mcbs.edu.om (H.S. Jassim), muhammad.azeem@riphah.edu.pk (M. Azeem), balmohsen@ksu.edu.sa (B. Almohsen), husam@eng.unideb.hu (H.A. Neamah).

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Fuzzy equivalents of several basic graph-theoretic concepts, such as trees and bridges, had been developed, and some of their properties were established in [4]. In [5], the concept of fuzzy soft topology was introduced, and several structural characteristics were analyzed as the vicinity of a different fuzzy soft environment's topology. The possibility fuzzy soft set notion was presented by the authors of [6], who also examined some of its characteristics. By applying graph theory to single-valued neutrosophic sets, the findings linking crisp and fuzzy graphs were expanded upon, and the single-valued neutrosophic graph, a novel kind of graph structure, was investigated [7]. An innovative use of establishing a chain of well-known brand schools in a certain area, including the chain's cluster center, and evaluating their performance using intuitionistic fuzzy-based topological indices was demonstrated in [8].

In multi-criteria group decision-making problems, the hypersoft set, an extension of the soft set when many sets of qualities occur, is highly beneficial. One function with multiple arguments is the hypersoft set. The fuzzy hypersoft set (FHSS) is the hybridization of the fuzzy set and hypersoft set. The authors of [9], had described the use of a suitable example to demonstrate the dependability and authenticity of the basic set-theoretic operations on FHSS. The article [10], aimed to introduce a novel mathematical model for pandemic diagnosis and treatment. The model was based on a special flexible fuzzy-like structure known as the complex fuzzy hypersoft (CFHS) set, which was a glued structure of hypersoft sets (an extension of soft sets) and complex fuzzy (CF) sets.

Topological numbers are a graph-theoretic approach used to describe the topology of a graph. The articles [11,12], are very significant in the field of topological numbers. The primary goal of the article [13], is to define fuzzy graphs' topological numbers with examples. Certain numbers had proven bounds. Researchers in [14], proved the best possible lower bound for the Randić number for a triangle-free graph, for which the minimum degree of the graph was given. Two fuzzy topological numbers, fuzzy Randić, and fuzzy harmonic, were considered by mathematicians in [15]. They determined a number of upper bounds on these fuzzy topological numbers. In a fuzzy soft framework, some well-known graph families and the degrees of their nodes were defined, and topological numbers for those graph families were also computed in [16]. Some new concepts related to fuzzy soft graphs were initiated, and their properties were discussed in [17]. Connectivity number, average connectivity number, and Randic number in vague graph were initiated in [18].

In many areas of mathematics and chemistry, as well as in network theory, spectral graph theory, and molecular chemistry, the Zagreb numbers are crucial graph parameters. The first and second Zagreb numbers for a range of fuzzy graphs, such as path, cycle, star, fuzzy subgraph, etc., were studied by the authors of [19,20], and provided a large number of findings.

One popular multicriteria decision-making (MCDM) technique for raising the standard of decision-making is the VIKOR method. The paper [21], offered a new version of the VIKOR approach that emphasizes compromise solutions while covering all kinds of criteria. The authors of the research [22], used the VIKOR approach to choose a renewable energy project that aligned with the Spanish government's Renewable Energy Plan. A three-dimensional computational fluid dynamics (CFD) simulation model of a urea injection system including a four-channel cyclone mixer was developed [23].

### 1.1. Theoretical and methodological advancements

The goal of MCDM is to organize and handle preference and planning concerns, including multi-criteria. Certain kinds of problems cannot be solved using fuzzy soft set environments if the characteristics are further divided and have multiple values. Because of this, there was a critical need to find a novel solution to these issues, which led to the establishment of a new setting known as FHSS. In multi-criteria group decision-making problems, the hypersoft set, an extension of the soft set when many sets of qualities occur, is highly beneficial. A hypersoft set is a multi-argument function. The idea of an FSS was first presented in earlier research studies and was effectively used in a number of fields. Compared to FSS, the FHSS theory offers greater flexibility in addressing parameterized uncertainty concerns. FHSS is desperately needed to address the problem where FSS failed to explain uncertainty and incompleteness [24]. It performs best when the parametric data, that is, the data involving nebulous concepts, is more complex. A hypersoft set is the generalization of the soft set, as in it a single set of distinct attributes is replaced by their corresponding disjoint attribute-valued sets. In this hypersoft set becomes a more effective tool to handle vague problems. Fuzzy hypersoft graphs are formulated to handle fuzzy hypersoft information by combining fuzzy hypersoft sets with graph theory. Fuzzy hypersoft graphs are more adaptable in managing complex, multi-attribute scenarios than fuzzy soft graphs because they allow for multi-parameter approximate functions, whereas fuzzy soft graphs describe relationships between objects using a single parameter set. So, integrating FHS graphs improves decision-making, and the use of MCDM in ranking optimal solutions would generate more precise and generic results. This reveals information about our approach's resilience and flexibility. Topological numbers are based on fundamental formulation; topological numbers were defined for crisp graphs only. We have extended these topological numbers to the FHS framework with some amendments, i.e. make use of multi-attributed parameters. So, integrating the FHS graph with topological numbers, the framework provides a more structured way to evaluate and rank alternatives in MCDM. Topological numbers help in understanding node importance, connectivity, and influence in complex networks. FHS graphs handle higher-dimensional uncertainties better than traditional fuzzy graphs, making them useful for real-world applications in business, healthcare, and energy optimization. This integration extends existing fuzzy graph models by incorporating new mathematical properties, allowing for a more flexible and robust theoretical foundation. This integration is useful in supply chain management, communication networks, and logistics, where relationships between entities are uncertain and multi-valued.

### 1.2. Importance of Fuzzy hypersoft topological numbers in artificial intelligence

The FHS topological number is a good choice for AI applications requiring ambiguity since it may be utilized to model imprecise and uncertain data. Fuzzy logic and soft sets are combined in soft computing to address challenging issues. FHS topological numbers-based algorithms improve picture segmentation, object detection, and feature extraction in image processing. It can be applied to decision-making as well. It can improve the precision of data classification and clustering. FHS topological numbers address uncertainty and non-linearity in robotics and control systems.

### 1.3. The VIKOR method

Depending on the goals of the problem, material choices are evaluated in engineering design using varying criteria. Different units are used to quantify performance ratings for different criteria, but all of the decision matrix's parts must be dimensionless in order for comparisons to be legitimate. Though several normalization techniques have been established for cost and benefit criteria, engineering design scenarios where achieving the target values is desirable have not received enough attention, and the existing techniques have drawbacks. The VIKOR technique was created to optimize complicated systems using multiple criteria. For preference stability of the compromise solution derived from the original (provided) weights, it determines the compromise ranking list, the compromise solution, and the weight stability intervals. This approach concentrates on prioritizing and choosing among options when there are competing standards. It presents the multicriteria ranking index, which is based on a specific metric of "closeness" to the "optimal" answer.

Solving the potential supplier evaluation and ranking issue has emerged as a crucial strategic consideration for commercial organizations. In the information age, as automated and intelligent information systems proliferate, there is an increasing demand for more effective decision-making techniques. To address MCDM techniques for raising the standard of decision-making, the VIKOR method was developed by Wang and Tzeng. In addition to normalizing outcomes, it decreases personal regret, enhances social usefulness, and offers a middle ground. In order to handle MCDM issues with competing and non-commensurable (different units) criteria, the VIKOR approach was developed. It operates under the suppositions that compromise is a viable means of resolving conflicts, that the decision-maker favors a solution that comes close to the ideal, and that all accepted standards are taken into account while weighing the options. This method concentrates on evaluating and selecting from a variety of possibilities and offering a compromise solution where there are conflicting requirements.

### 1.4. Structure of study

A summary of several articles will be provided in Section 2, followed by contribution and novelty in Section 2.1, diagrammatical explanation of the complete manuscript in Section 2.2, in Sections 2.3 and 2.4, Nomenclature and parameters and their impact on performance are provided with basic definitions of each subsection's fundamental term in Section 2.5. In the same section, three new FHS topological numbers will also be stated. The three topological numbers specified in Section 2 will be computed for FHS graph families when certain graph families are defined in an FHS environment for the first time, and each node's degree is determined. The VIKOR method's algorithm will be established in 4, where each step will be explained in great detail. A collaborative venture about the suggested technique will be presented in 5, wherein the outcomes are initially examined using topological numbers and subsequently validated through the utilization of an MCDM technique, VIKOR. Section 5.5, will provide a comparison between the suggested technique and previously used techniques. Finally, a conclusion will be drawn on the entire project, along with recommendations for future study directions and limitations of the proposed work.

## 2. Literature review

A prominent article in graph theory is [25]. To depict the global point of interest (POI) properties and the geographic connections among all POIs, two graphs are created from a global viewpoint [26]. A generic recommendation paradigm that specifically took into account correlations between user/item subgraphs was the Graph Cross-correlated Network for Recommendation (GCR) [27]. The concept of fuzzy node graphs was introduced in the study [28], with reference to several predecessing models. The article [29], offered two alternative methods to the graph coloring problem of a fuzzy graph: the successive coloring technique and the extension to the concept of coloring function. A field of applied engineering called agricultural robotics has been compared to computer science and machine tool technology [30]. The cycle connectivity of a fuzzy graph depends on the strengths of all strong cycles in the graph [31]. A type of fuzzy graph that is widely used in medical and psychological sciences is the vague graph (VG) [18]. The article [32], examined the regularity of nodes using the first perfectly regular vague graph (PRVG). The notions of partial cubic connection enhancing node and partial cubic connectivity reducing node were defined in the study [33], and some associated findings were demonstrated. Using certain examples, the authors covered a variety of ideas and characteristics pertaining to dominance in ambiguous graphs, including an edge dominating set, an edge independent set, a regular dominating set, a regular independent set, and a global dominating set in [34].

The main idea of [35], was to use hybrid models of fuzzy soft sets and fuzzy soft graphs to explore granular structures in order to investigate the undetectable division of the set of universes. The notion of fuzzy soft graphs, different techniques for creating them, and an examination of some of their associated characteristics were all covered in the article [36]. A brand-new soft product for soft sets is called the "soft star-product", as well as all of its algebraic characteristics with regard to different kinds of soft equalities and subsets. Some varieties of irregular fuzzy soft graphs were covered by the authors. Additionally, they discussed the use of fuzzy soft graphs in road networks and social networks. In [37], uniform node and uniform edge fuzzy soft graphs, as well as the degree and total degree of a node, were computed. A decision-making method for the picture fuzzy soft graph has been presented using Domination Integrity to suggest an algorithm for determining the ideal location for establishing a city diagnosis center [38].

The connectivity number and average connectivity number of fuzzy graphs were presented in [39]. Numerous outcomes concerning these parameters were discovered. In [40], the first and second Zagreb numbers, the Randić number, and the harmonic number of the fuzzy chemical graph phenylene were estimated by the researchers. In [41], authors discussed some newly developed degree-based topological numbers. The article [42], presented a comparison between the Randić number and the logarithm of its multiplicative version. Some fuzzy soft topological numbers for popular graphs were calculated, and an application in a fuzzy soft environment originated in [43]. In order to maximize the total system throughput, a game-theoretic approach-based plan for simultaneous mode selection and power adaption was put out [44]. The articles [45–48], are very prominent in the context of energy system modeling.

The fuzzy hypergraph theory is a significant framework for effectively handling the uncertain information prevailing in the multi-dimensional relationships of real-world systems [49]. The work [50], explored the innovative concept of the complex multi-fuzzy hypersoft set (CMFHSS), which could simultaneously account for the amplitude and phase elements of the complex numbers to address uncertainties, vagueness, and unclearness of data contained in the information. In order to more effectively allocate investments based on product sales, the article [51], tried to assess the risk associated with investing in various product types in supermarkets. Innovative picture fuzzy hypersoft graphs were used to accomplish this. The most extended form of a fuzzy set, called the Bipolar Linear Diophantine, the FHSS is implemented with some basic operations [52].

The study [53], employed language expressions to collect decision makers' viewpoints while analyzing supplier selection as a group MCDM problem. These linguistic terms were then translated into fuzzy trapezoidal numbers. The challenges in Enterprise Resource Planning (ERP) are analyzed by using the DEMATEL approach [54]. Twenty-four companies are ranked based on fifteen criteria using six MCDM techniques [55]. Each vaccine's significance was gauged by its Zagreb number, and the scores were normalized using a weighted aggregate technique. A calculated scoring function was used to determine the final ranking in [56]. This research contributes to the field of healthcare supply chain management by presenting a robust MADM methodology called lattice-ordered q-rung ortho-pair multi-fuzzy soft set for supplier evaluation and ranking amidst the challenges posed by the COVID-19 pandemic [57]. Some fuzzy soft topological numbers were calculated before and after vertex deletion [58]. The fuzzy TOPSIS and the fuzzy AHP were employed to rank the factors affecting a construction company's performance [59]. The mining of cryptocurrencies has recently sparked environmental worries about increased carbon emissions, and as cryptocurrency mining grows, so does the demand for energy production. Using the mathematical model, one can assess and lessen the environmental issues brought on by cryptocurrency mining operations [60]. The most well-known technique for resolving Linear Assignment Problems (LAP) is the Hungarian algorithm. One of the most widely used strategies for resolving MCDM problems is the Linear Assignment Method (LAM), which is an application of LAP [61]. In order to choose the most desirable alternative or alternatives, the authors created a vague neutrosophic multiple criterion decision making approach. The assessment values of the alternatives on the qualities were expressed as ambiguous neutrosophic numbers in [62]. The authors of article [63], introduced a novel data normalization approach, which made the VIKOR method more accurate and acceptable for such data in the medical profession. The article [64], presented a novel VIKOR approach for material ranking that had taken into account the simultaneous availability of interval data and several criteria. The VIKOR technique was expanded to include a trade-off analysis and a stability analysis that established the weight stability intervals. Three multicriteria decision-making techniques were contrasted with the expanded VIKOR method [65]. With an emphasis on socioeconomic considerations, two MCDM techniques: the fuzzy analytic hierarchy process (AHP) and fuzzy VIKOR are used to determine the best site for a nuclear power facility [66].

### 2.1. Contribution and novelty of this work

This study introduces a novel approach by integrating FHS graph theory with the VIKOR decision-making algorithm, which has not been explored in previous literature. The key contributions and novel aspects of this work are as follows: Traditional VIKOR methods primarily operate in crisp or fuzzy domains. Our work extends VIKOR into the FHS framework, allowing it to handle multi-attribute, and multi-subattribute decision problems with higher flexibility and granularity. This extension improves decision-making in complex, uncertain, and imprecise environments, which are common in business networks and industrial applications. By extending the ideas of fuzzy graphs and hypersoft sets, a "fuzzy hypersoft graph" makes it possible to depict intricate, multi-dimensional interactions with several parameters and intrinsic uncertainties. By handling multi-argument approximate functions and incorporating numerous parameters, FHS graphs are more versatile and dependable for modeling complicated, uncertain scenarios than fuzzy graphs and fuzzy soft graphs. FHS graphs can model complex relationships that involve multiple parameters or attributes, unlike traditional graphs that often focus on single-attribute relationships. Unlike existing works that focus on standard multi-criteria decision-making (MCDM) applications, our study demonstrates how this approach can be effectively applied in the business sector. We further highlight its potential for energy modeling and optimization, paving the way for future interdisciplinary applications. These graphs have a wide range of applications in multi-criteria decision-making, risk analysis, and knowledge representation. Since topological numbers are important mathematical tools for describing the topology of any graph in a single numerical value and can be used to make decisions, this manuscript takes a novel approach to defining and computing topological numbers in an FHS environment for the first time. No literature is present in which any MCDM is defined in a fuzzy hypersoft environment. This manuscript is also the first to use the decision-making algorithm VIKOR in the FHS framework. We provide a detailed sensitivity analysis to show the stability of decision rankings under different input variations. This is crucial for validating the robustness of the proposed method, an aspect often overlooked in previous studies. This framework enables better representation of uncertain and imprecise data, leading to more realistic and reliable decision-making. This study bridges the gap between fuzzy hypersoft graphs and VIKOR-based MCDM, provides higher adaptability in uncertain decision environments, and demonstrates real-world applicability with an example in the business industry while suggesting extensions to energy systems.

### 2.2. Step-by-sep algorithm using flow chart

The flow chart in Fig. 1, will help in ensuring a clear step-by-step understanding of our approach.

### 2.3. Nomenclature

See Tables 1 and 2.

### 2.4. Algorithm parameters and their impact on performance

In this study, we have utilize the fuzzy hypersoft topological numbers and VIKOR algorithm to analyze decision-making scenarios. A comparison of VIKOR algorithm and analytical Hierarchy process (AHP) will also be presented in application section. Since the dataset consists of assumed values, we carefully selected the parameters based on theoretical justifications and logical reasoning. Below, we will describe the key parameters, their selection criteria, and their impact on overall performance.

**Impact of Parameter Variation on Performance**

To evaluate how changes in parameters influence decision-making, we conducted a sensitivity analysis. The following observations were made:

- **Variation in Membership Values** $\curlyvee'(\mathfrak{r})$: A higher membership value shifts decisions toward dominant options, while lower values introduce more neutrality.
- **Changes in Weighting Criteria** $(\mathfrak{W}_i)$: When weights are adjusted, rankings in the VIKOR algorithm shift, leading to different optimal choices.
- **Value of** $\mathfrak{Y}$: A higher $\mathfrak{Y}$ emphasizes the best solution, whereas a lower $\mathfrak{Y}$ gives more importance to compromise solutions.
- **Number of FHS sub-graphs, depending on cardinality of** $\mathfrak{R}$: By varying the cardinality of $\mathfrak{R}$ and increasing the number of vertices or number of parameters in attributed sets $\mathfrak{R}_\kappa$ can generate a very complex network, which will increase the computational complexity of the proposed work.

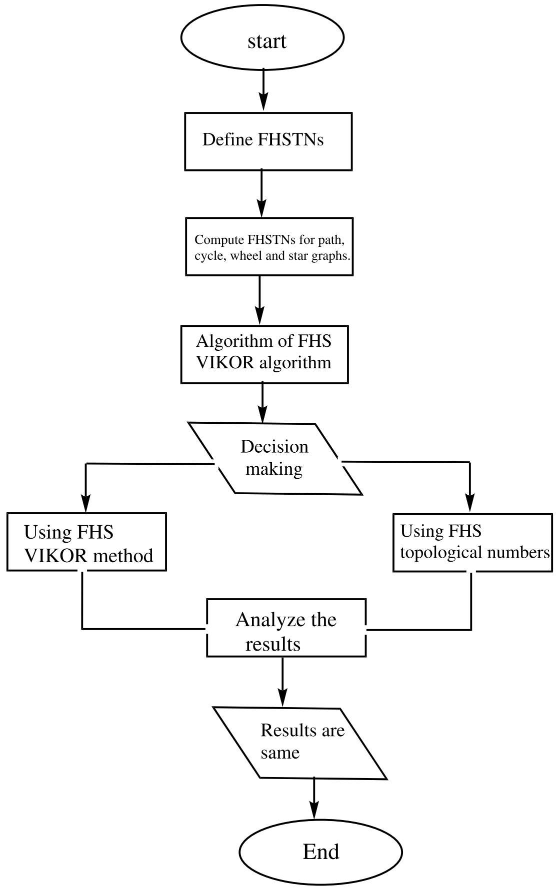


**Fig. 1.** Step-by-step algorithm by flow chart.

**Table 1**
List of symbols.

| Symbols | Description | Symbols | Description |
|---|---|---|---|
| $\tilde{\Gamma}^*$ | Crisp graph | $\mathfrak{X}$ | Universe of discourse |
| $\mathfrak{x}$ | representation of node | $\mathscr{G}$ | Fuzzy soft graph |
| $\rhd$ | Node set | $\models$ | Edge set |
| $J$ | Fuzzy graph | $\Psi$ | Membership function for vertices |
| $\daleth$ | Membership value of edges in fuzzy graph | $\tilde{\Gamma}$ | Fuzzy soft graph |
| $\sqcup'$ | Set of parameters | $\wedge$ | minimum |
| $\tilde{Q}_b$ | soft graph | $\varsigma_\Lambda$ | parameters for FS graph |
| $\curlyvee''$ | Membership function for nodes of FS graph | $\mathfrak{H}'$ | Membership function for edges of FS graph |
| $\mathcal{H}$ | FHS graph | $\curlyvee'$ | Membership function for nodes of FHS graph |
| $\curlyvee'$ | Membership function for nodes of FHS graph | $\mathfrak{H}$ | Membership function for edges of FHS graph |
| $\mathfrak{R}_\kappa$ | Disjoint attributed valued sets | $\star$ | Cardinality of set $\mathfrak{R}$ |
| $\curlywedge_\star$ | Parameters for FHS graph | $S_i$ | Utility measure in VIKOR |
| $R_i$ | Regret measure in VIKOR | $Q_i$ | Final VIKOR index |
| $\mathfrak{Y}$ | Strategy weight of majority criteria | $\mathfrak{W}_i$ | weights assigned to different criteria |

### *2.5. Preliminaries*

An object is referred to as a node in a graph, and the edge (link) that connects two entities is the source of the relationship. A crisp graph $\tilde{\Gamma}^*(\rhd, \models)$ is represented in a traditional manner as follows, the finite set of links is denoted by $\models$, and the number of links is **size**, whereas its finite set of nodes is represented by $\rhd$, and the number of nodes is **Order**.

**Definition 2.1** (*[2]*)**.** A fuzzy set is a pair $(\mho, \Psi)$, where $\mho$ is a set on set of vertices, $\rhd$ and $\Psi : \mho \to [0,1]$, a membership function. The set $\mho$ is called the universe of discourse, and for each $\mathfrak{x}, \psi(\mathfrak{x})$ is called the membership value of $\mathfrak{x}$ in $\rhd$.

**Definition 2.2** (*[67]*)**.** A fuzzy graph $J = (\Psi, \daleth)$, is a pair of functions $\Psi : \rhd \to [0,1]$ and $\daleth : \models \to [0,1]$, where, $\models \subseteq \rhd \times \rhd$, such that $\daleth\left(\mathfrak{x}_i\mathfrak{x}_j\right) \leq \left\{\Psi(\mathfrak{x}_i), \Psi(\mathfrak{x}_j)\right\}_\wedge$, $\forall\ \mathfrak{x}_i, \mathfrak{x}_j \in \rhd$. The order and size of fuzzy graph $J$ are $\sum_{\mathfrak{x}\in\rhd(J)} \Psi(\mathfrak{x})$, and $\sum_{\mathfrak{x}_i\mathfrak{x}_j\in\models(J)} \daleth\left(\mathfrak{x}_i\mathfrak{x}_j\right)$ respectively. Degree of node $\mathfrak{x}_i$ in $J$ is defined as, $D(\mathfrak{x}_i) \leq \sum_{(\mathfrak{x}_i,\mathfrak{x}_j)\in\models(J)} \left\{\Psi(\mathfrak{x}_i), \Psi(\mathfrak{x}_j)\right\}_\wedge$.

**Table 2**
Selected algorithm parameters and their justifications.

| Parameter | Symbol | Selected value(s) | Justification |
|---|---|---|---|
| Membership values of nodes in FHS graph | $\curlyvee'(\mathfrak{x})$ | [0,1] | Ensures values remain within the valid fuzzy range. |
| Weighting criteria in VIKOR | $\mathfrak{W}_i$ | Assumed values | Represents the importance of each criterion in decision-making. |
| $\mathfrak{y}$ in VIKOR | $\mathfrak{y}$ | 0.5 (Default) | Balances weight between best and compromise solution. |
| Eigen value in AHP | $\lambda$ | Depending on weights | Important to find consistency criteria. |
| Number of FHS sub-graphs, depending on cardinality of $\mathfrak{R}$ | $\curlywedge_\star$ | Number, equal to the cardinality of Cartesian product of attributed sets $\mathfrak{R}_\kappa$ | Defines the strength of node degree in FHS graphs. |

**Definition 2.3** ([68]). An illustration $\left(\tilde{Q}^*, \mathfrak{Y}', \mathfrak{A}', \sqcup'\right) = \tilde{Q}_b$, is a soft graph with $\tilde{Q}^* = (\rhd, \vDash)$ is defined as a classic graph. $\sqcup'$ is a set of parameters that is not empty, $\left(\sqcup', \mathfrak{Y}'\right)$ is a soft set over $\rhd$ and $\left(\sqcup', \mathfrak{A}'\right)$ is over $\vDash$, and $\left(\mathfrak{A}'(b'), \mathfrak{Y}'(b')\right)$, is a sub-graph $\forall\, b' \in \sqcup'$.

**Definition 2.4** ([69]). Assume that the first universe discourse is $\mathfrak{X}$, the collection of all parameters is $\sqcup'$, and $\sqcup' \subseteq \mathfrak{O}^*$ and all fuzzy subsets of $\mathfrak{X}$ are gathered under $\mathfrak{I}(\mathfrak{X})$. The fuzzy approximation function is mapping $\Psi^* : \mathfrak{O}^* \to \mathfrak{I}^*(\mathfrak{X})$. Then, a FS set is defined as $(\Psi^*, \mathfrak{O}^*)$.

**Definition 2.5** ([70]). For a classic graph $\mathcal{G}^* = (\rhd, \vDash)$, and for a parameterized set $\sqcup'$, $\tilde{\Gamma}$ is a FS graph, such that $\tilde{\Gamma} = \left(\tilde{\Gamma}^*, \curlyvee'', \mathfrak{H}', \sqcup'\right)$. Then, the FS set over $\rhd$ is $\left(\curlyvee'', \sqcup^*\right)$, and $\left(\mathfrak{H}', \sqcup^*\right)$ is the fuzzy soft sets over set of edges $\vDash$. The FS sub-graphs for each $\varsigma_\Lambda \in \sqcup'$ are denoted as, $\left(\curlyvee''_{\varsigma_\Lambda}(\mathfrak{x}_i), \mathfrak{H}'_{\varsigma_\Lambda}(\mathfrak{x}_i\mathfrak{x}_j)\right)$, and are defined as:

$$\mathfrak{H}'_{\varsigma_\Lambda}(\mathfrak{x}_i\mathfrak{x}_j) \le \left\{\curlyvee''_{\varsigma_\Lambda}(\mathfrak{x}_i), \curlyvee''_{\varsigma_\Lambda}(\mathfrak{x}_j)\right\}_I, \forall\, \mathfrak{x}_i, \mathfrak{x}_j \in \rhd(\tilde{\Gamma}).$$

The order and size of a fuzzy soft graph are defined as, $\sum_{\varsigma_\Lambda \in \sqcup'}\left(\sum_{i=1}^{n} \curlyvee''_{\varsigma_\Lambda}(\mathfrak{x}_i)\right)$, $\forall \mathfrak{x}_i \in \rhd(\Gamma)$ and $\sum_{\varsigma_\Lambda \in \sqcup'}\left(\sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\tilde{\Gamma})} \mathfrak{H}'_{\varsigma_\Lambda}(\mathfrak{x}_i\mathfrak{x}_j)\right)$, respectively and the degree of any node $\mathfrak{x}_i$ is less than or equal to
$\sum_{\varsigma_\Lambda \in \sqcup'}\left(\sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\tilde{\Gamma})}\left\{\curlyvee''_{\varsigma_\Lambda}(\mathfrak{x}_i), \curlyvee''_{\varsigma_\Lambda}(\mathfrak{x}_j)\right\}_\wedge\right)$

**Definition 2.6.** Hypersoft set is the generalization of soft set, in which mapping is transformed into a multi-attribute. Suppose $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ for $\kappa \ge 1$ are $\kappa-$distinct traits, whose corresponding trait values are respectively the sets $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ with $\mathfrak{R}_i \cap \mathfrak{R}_j = \emptyset$, $\forall\, i \ne j$. Then the pair $\left(\mathfrak{y}^{\mathfrak{d}}, \mathfrak{R}\right)$, where $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, such that $\mathfrak{y}^{\mathfrak{d}} : \mathfrak{R} \to P^*(\mathfrak{U})$ is called hypersoft set over the universe of discourse $\mathfrak{U}$.

**Definition 2.7.** Let $\mathcal{H} = \left(\tilde{\Gamma}^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\right)$, be a fuzzy hypersoft graph with node set $\rhd = \left\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \ldots, \mathfrak{x}_n\right\}$. Suppose $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$. Then the graph $\mathcal{H} = \left(\tilde{\Gamma}^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\right)$, such that

- $\tilde{\Gamma}^* = (\rhd, \vDash)$, is a crisp graph,
- The Cartesian product of attributed valued sets corresponding to unique characteristics, is $\mathfrak{R}$,
- $\left(\curlyvee', \mathfrak{R}\right)$ and $(\mathfrak{H}, \mathfrak{R})$ are FHS sets over $\rhd$ and $\vDash$ accordingly,
- $\left(\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_i\mathfrak{x}_j)\right)$ are referred to as the fuzzy hypersoft sub-graphs of $\mathcal{H}$, $\forall\, \mathfrak{x}_i, \mathfrak{x}_j \in \rhd$, corresponding to every $\curlywedge_\star \in \mathfrak{R}$, such that,

$$\mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_i\mathfrak{x}_j) \le \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_j)\right\}_\wedge,$$

where, $\star$ is the cardinality of set $\mathfrak{R}$, $\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)$ is the membership value of node $\mathfrak{x}_i$, $\mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_i\mathfrak{x}_j)$ is the membership value of edge $\mathfrak{x}_i\mathfrak{x}_j$ and $\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_j)\right\}_\wedge$ represents the minimum of $\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)$ and $\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_j)$, for any $\curlywedge_\star \in \mathfrak{R}$.

**Definition 2.8.** A FHS graph $\mathcal{H}$ has the following order:

$$\hat{O}^{\curlywedge_\star}(\mathcal{H}) = \sum_{\curlywedge_\star \in \mathfrak{R}}\left(\sum_{i=1}^{n} \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)\right), \tag{2.1}$$

$\forall\, \mathfrak{x}_i \in \rhd(\Gamma^*)$.

**Definition 2.9.** A FHS graph $\mathcal{H}$ has size, denoted by $\mathcal{A}^{\curlywedge_\star}(\mathcal{H})$ and is defined as:

$$\mathcal{A}^{\curlywedge_\star}(\mathcal{H}) = \sum_{\curlywedge_\star \in \mathfrak{R}}\left(\sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\mathcal{H})} \mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_i\mathfrak{x}_j)\right), \tag{2.2}$$

$\forall\, \mathfrak{x}_i\mathfrak{x}_j \in \vDash(\Gamma^*)$ and $\forall\, \curlywedge_\star \in \mathfrak{R}$.

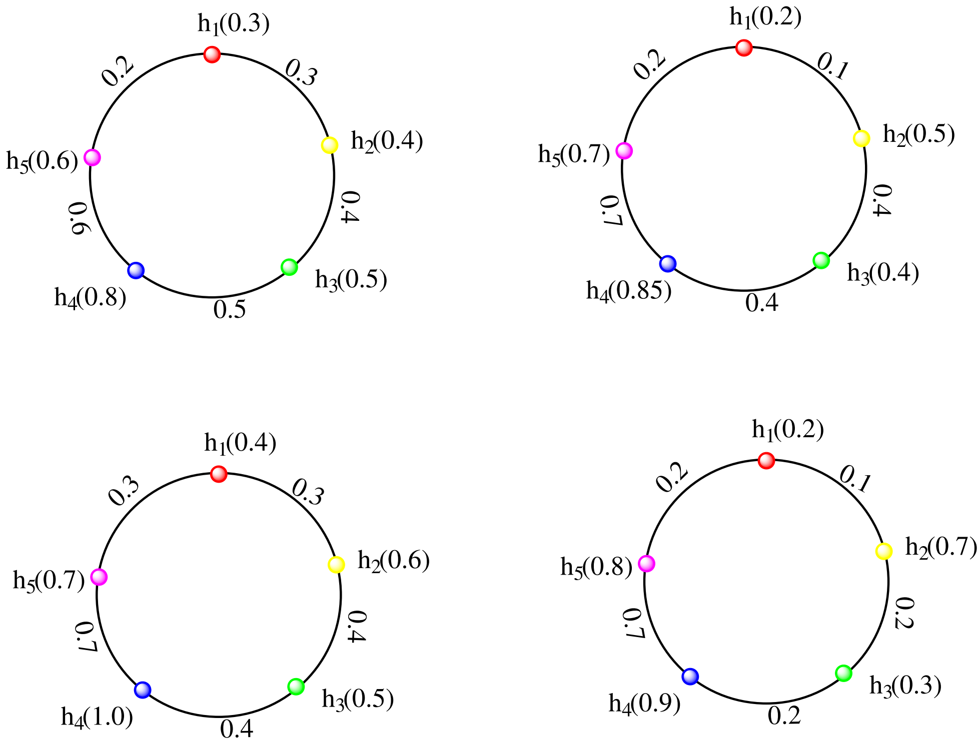


**Fig. 2.** Four fuzzy hypersoft cycle sub-graphs.

**Definition 2.10.** The degree of a node $\mathfrak{x}_i$ in an FHS network $\mathcal{H}$ is defined as:

$$D^{\curlywedge_\star}(\mathfrak{x}_i) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{x}_i \mathfrak{x}_j \in \vDash(\mathcal{H})} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_j) \right\}_{\wedge} \right), \tag{2.3}$$

$\forall \curlywedge_\star \in \mathfrak{R}$.

**Example** Suppose John wants to buy a house. He has shortlisted five houses.

$$\rhd = \{H_1, H_2, H_3, H_4, H_5\},$$

where, $\rhd$ represents five houses However, choosing the best house is not simple, it depends on various attributes, some related to location and environment, and others to cost and space. Since these attributes are not always crisp (some are subjective), John uses a fuzzy hypersoft graph model to make the best decision. Two attributes are selected, one for location and one for cost The corresponding attributed values are $\{\mathfrak{R}_1, \mathfrak{R}_2\}$.

- $\mathfrak{R}_1 = \{\mathfrak{f}_{11} = Near\ \ market, \mathfrak{f}_{12} = Peacefula\ \ area\}$,
- $\mathfrak{R}_2 = \{\mathfrak{f}_{21} = Affordable, \mathfrak{f}_{22} = Spacious\}$,

$$\begin{aligned} \mathfrak{R} =& \mathfrak{R}_1 \times \mathfrak{R}_2 \\ &= \{\mathfrak{f}_{11}, \mathfrak{f}_{12}\} \times \{\mathfrak{f}_{21}, \mathfrak{f}_{22}\}, \\ &= \{(\mathfrak{f}_{11}, \mathfrak{f}_{21}), (\mathfrak{f}_{11}, \mathfrak{f}_{22}), (\mathfrak{f}_{12}, \mathfrak{f}_{21}), (\mathfrak{f}_{12}, \mathfrak{f}_{22}), (\mathfrak{f}_{13}, \mathfrak{f}_{21}), (\mathfrak{f}_{13}, \mathfrak{f}_{22})\}, \end{aligned}$$

$$\mathfrak{R} = \left\{ \begin{array}{l} \curlywedge_1 = (\mathfrak{f}_{11}, \mathfrak{f}_{21}), \curlywedge_2 = (\mathfrak{f}_{11}, \mathfrak{f}_{22}), \\ \curlywedge_3 = (\mathfrak{f}_{12}, \mathfrak{f}_{21}), \curlywedge_4 = (\mathfrak{f}_{12}, \mathfrak{f}_{22}). \end{array} \right\}$$

There are four outcomes and the house will be purchased on the basis of these outcomes. The number of fuzzy hypersoft sub-graphs is equal to the cardinality of set $\mathfrak{R}$. Five houses are arranged in the form of a cycle graph networks (see Figs. 2–5).

Now, we will compute the order of the FHS graph by applying Eq. (2.1), i.e. order is obtained by adding membership value of all the nodes in all the subgraphs.

$$\begin{aligned} \hat{O}^{\curlywedge_\star}(\mathcal{H}) &= (0.3 + 0.4 + 0.5 + 0.8 + 0.6) + (0.2 + 0.5 + 0.4 + 0.85 + 0.7) \\ &+ (0.2 + 0.7 + 0.3 + 0.9 + 0.8) + (0.4 + 0.6 + 0.5 + 1.0 + 0.7) \\ &= 2.6 + 2.65 + 2.9 + 3.2 = 11.35. \end{aligned}$$

Size of the FHS graph is obtained by applying Eq. (2.2), i.e by adding membership values of all the edges in all of the subgraphs.

$$\mathcal{A}^{\curlywedge_\star}(\mathcal{H}) = (0.3 + 0.4 + 0.5 + 0.6 + 0.2) + (0.1 + 0.4 + 0.4 + 0.7 + 0.2)$$

$$+ (0.3 + 0.4 + 0.4 + 0.7 + 0.3) + (0.1 + 0.2 + 0.2 + 0.7 + 0.2)$$
$$= 2.0 + 1.4 + 2.1 + 1.4 = 6.9.$$

Now, the degree of each node will be computed by adding the membership values of the edges incident on the same node in all subgraphs, using Eq. (2.3), which will decide which house is the best choice.

$$D(h_1) = (0.3 + 0.2) + (0.2 + 0.1) + (0.3 + 0.3) + (0.2 + 0.1) = 1.7,$$
$$D(h_2) = (0.3 + 0.4) + (0.1 + 0.4) + (0.3 + 0.4) + (0.2 + 0.1) = 2.2,$$
$$D(h_3) = (0.4 + 0.5) + (0.4 + 0.4) + (0.2 + 0.2) + (0.4 + 0.4) = 2.9,$$
$$D(h_4) = (0.5 + 0.6) + (0.4 + 0.7) + (0.2 + 0.7) + (0.4 + 0.7) = 4.2,$$
$$D(h_5) = (0.6 + 0.2) + (0.7 + 0.2) + (0.7 + 0.2) + (0.7 + 0.3) = 3.6.$$

Hence, house $h_4$ i the best choice.

**Definition 2.11** (*[71]*). The first and second Zagreb numbers were first time initiated for crisp graph $\tilde{\Gamma}^*$ by Gutman and Trinajstic as:

$$\mathbb{M}_1(\tilde{\Gamma}^*) = \sum_{\mathfrak{x}_i \in \triangleright(\tilde{\Gamma}^*)} \left(D(\mathfrak{x}_i)\right)^2,$$

and

$$\mathbb{M}_2(\tilde{\Gamma}^*) = \sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\tilde{\Gamma}^*)} D(\mathfrak{x}_i) \times D(\mathfrak{x}_j).$$

**Definition 2.12** (*[72]*). The Randić number was first time introduced by Milan Randić, as:

$$\mathbb{R}(\tilde{\Gamma}^*) = \sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\tilde{\Gamma}^*)} \left[D(\mathfrak{x}_i) \times D(\mathfrak{x}_j)\right]^{-\frac{1}{2}}.$$

*2.6. Fuzzy hypersoft topological numbers*

Topological indices are based on formulation, and initiated from crisp graph. We have just extended them for FHS graph basing on fundamental degrees of nodes by allowing for multi parameter approximate function. The fuzzy hypersoft topological numbers (FHSTNs) measures, how much a fuzzy hypersoft topology is influenced by parameters. Fuzzy hypersoft topological numbers are derived by extending fuzzy soft graph theory and topological concepts, defining topological numbers (like Zagreb numbers) within a fuzzy hypersoft environment, and then calculating them for various graph families. This is the novelty of our work that we first time introduced topological numbers for FHS graph. In this article we have defined three topological numbers first Zagreb number, second Zagreb number and Randić number in FHS framework.

**Definition 2.13.** For a fuzzy hypersoft graph $\mathcal{H}$, the first Zagreb number is given by:

$$\mathbb{M}_1{}^{\wedge_\star}(\mathcal{H}) = \sum_{\mathfrak{x}_i \in \triangleright(\mathcal{H})} \left[\left(\sum_{\wedge_\star \in \mathfrak{R}} D^{\wedge_\star}(\mathfrak{x}_i)\right)^2\right], \tag{2.4}$$

where, $D^{\wedge_\star}(\mathfrak{x}_i)$ is the degree of node $\mathfrak{x}_i$, with respect to the sub-graphs, $\forall \wedge_\star \in \mathfrak{R}$.

**Definition 2.14.** For a fuzzy hypersoft graph $\mathcal{H}$, the second Zagreb number is given by:

$$\mathbb{M}_2{}^{\wedge_\star}(\mathcal{H}) = \sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\mathcal{H})} \left[\sum_{\wedge_\star \in \mathfrak{R}} D^{\wedge_\star}(\mathfrak{x}_i) \times D^{\wedge_\star}(\mathfrak{x}_j)\right]. \tag{2.5}$$

**Definition 2.15.** The FHS Randić number for a fuzzy hypersoft graph $\mathcal{H}$ is defined as:

$$\mathbb{R}^{\wedge_\star}(\mathcal{H}) = \sum_{\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\mathcal{H})} \left[\sum_{\wedge_\star \in \mathfrak{R}} D^{\wedge_\star}(\mathfrak{x}_i) \times D^{\wedge_\star}(\mathfrak{x}_j)\right]^{-\frac{1}{2}}. \tag{2.6}$$

In Definitions 2.13–2.15, $\mathfrak{R}$ is the Cartesian product of distinct attributed valued sets corresponding to attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$, and $\wedge_\star$ is the number of members of $\mathfrak{R}$.

*2.7. Step-by step explanation of the proposed work*

Fig. 1, explicitly explains all the steps involving in the proposed work, with the assumption that fuzzy graphs are bound to have membership values between ′0′ and ′1′

## 3. Main results

The novelty of this work is to define some graph families in FHS framework, determining the degrees of the nodes of these graph families and then initiating formulae for first Zagreb number, second Zagreb number and Randić number for same graph families in FHS framework.

*3.1. Path graph in Fuzzy hyper-soft framework*

A network $\mathcal{P}_\hbar : \{\mathcal{P}^*_\hbar, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$ with the sequence $\{\mathfrak{x}_n\}$ of nodes, that is $\mathfrak{x}_i \neq \mathfrak{x}_j, \forall\ i \neq j$ is FHS path network. In FHS path graph $\forall\ 1 \leq i \leq n, \mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_{i-1}, \mathfrak{x}_i) > 0$, here $\mathcal{P}^*_\hbar = (\rhd, \vDash)$ is the classic path graph. The FHS set throughout the node set, $\rhd$, is $(\curlyvee', \mathfrak{R})$. the FHS path graph's length is $n-1$, and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$.

**Lemma 3.1.** *The membership degree of $n$ nodes is shown as follows in a FHS path graph $\mathcal{P}_\hbar$, with $\mathfrak{x}_i \in \rhd(\mathcal{P}_\hbar)$,*

$$D^{\curlywedge_\star}(\mathfrak{x}_1) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2) \right\}_\wedge,$$

$$D^{\curlywedge_\star}(\mathfrak{x}_n) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n) \right\}_\wedge,$$

$$D^{\curlywedge_\star}(\mathfrak{x}_j) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left[ \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{j-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_j) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_j), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{j+1}) \right\}_\wedge \right],$$

*where, $2 \leq j \leq n-1$, and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ are distinct attributed valued sets related to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$.*

**Theorem 3.1.** *Let the FHS path graph be $\mathcal{P}_\hbar$, and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let the node set be $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_n\}$, and each node's degree is determined by Lemma 3.1. Next, the following is the first FHS Zagreb number for the path graph:*

$$\begin{aligned} \mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{P}_\hbar) \leq & \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2) \right\}_\wedge \right]^2 \\ & + \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n) \right\}_\wedge \right) \right]^2 \\ & + \sum_{i=2}^{n-1} \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1}) \right\}_\wedge \right) \right]^2. \end{aligned}$$

**Proof.** Let $\mathcal{P}_\hbar : \{\mathcal{P}^*_\hbar, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$ be a path graph in FHS environment, where $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, is the Cartesian product of attributed valued sets, such taht $\curlywedge_\star \in \mathfrak{R}$. Lemma 3.1, defines the degrees of each node. Then,

for node $\mathfrak{x}_1$ :

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_1) \leq \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2) \right\}_\wedge \right]^2$$

The first FHS Zagreb number for the node $\mathfrak{x}_2$ is demonstrated as:

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_2) \leq \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3) \right\}_\wedge \right) \right]^2,$$

for the node $\mathfrak{x}_3$ :

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_3) \leq \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4) \right\}_\wedge \right) \right]^2,$$

and so on, for the node $\mathfrak{x}_{n-1}$ :

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}) \leq \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n) \right\}_\wedge \right) \right]^2,$$

and for the node $\mathfrak{x}_n$, we have:

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_1) \leq \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n) \right\}_\wedge \right]^2$$

Therefore, after adding up all of the findings, the path graph's first FHS Zagreb number is:

$$\begin{aligned} \mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{P}_\hbar) \leq & \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2) \right\}_\wedge \right]^2 \\ & + \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n) \right\}_\wedge \right) \right]^2 \\ & + \sum_{i=2}^{n-1} \left[ \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i) \right\}_\wedge + \left\{ \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1}) \right\}_\wedge \right) \right]^2. \quad \square \end{aligned}$$

**Theorem 3.2.** *Let $\mathcal{P}_\hbar$ be a FHS path graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\wedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_2, \dots, \mathfrak{r}_n\}$ be the set of nodes, and let degree be determined by Lemma 3.1 for each node. Next, the following is the second FHS Zagreb number for the path graph:*

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathcal{P}_\hbar) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_1), \Upsilon'_{\wedge_\star}(\mathfrak{r}_2)\right\}_\wedge\right] \times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_1)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_3)\right\}_\wedge\right)\right] \\ &+ \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-2})\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_n), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1})\right\}_\wedge\right)\right] \times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_n), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1})\right\}_\wedge\right] \\ &+ \sum_{i=1}^{n} \le \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_i), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{i-1})\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{i+1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_i)\right\}_\wedge\right)\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{i+1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_i)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{i+2}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{i+1})\right\}_\wedge\right)\right].\end{aligned}$$

**Proof.** Let $\mathcal{P}_\hbar : \{\mathcal{P}^*_\hbar, \Upsilon', \mathfrak{H}, \mathfrak{R}\}$ is a path network in the context of FHS, where $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, is the Cartesian product of attributed valued sets s.t $\wedge_\star \in \mathfrak{R}$. Let the set of nodes be $\rhd = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_2, \dots, \mathfrak{r}_n\}$. Lemma 3.1, defines the degrees of each node.

Then, for the edge $\mathfrak{r}_1\mathfrak{r}_2$ :

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{r}_1\mathfrak{r}_2) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_1), \Upsilon'_{\wedge_\star}(\mathfrak{r}_2)\right\}_\wedge\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_1), \Upsilon'_{\wedge_\star}(\mathfrak{r}_2)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_3)\right\}_\wedge\right)\right],\end{aligned}$$

for the edge $\mathfrak{r}_2\mathfrak{r}_3$,

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{r}_2\mathfrak{r}_3) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_1), \Upsilon'_{\wedge_\star}(\mathfrak{r}_2)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_3)\right\}_\wedge\right)\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_3)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_3), \Upsilon'_{\wedge_\star}(\mathfrak{r}_4)\right\}_\wedge\right)\right],\end{aligned}$$

for the edge $\mathfrak{r}_3\mathfrak{r}_4$,

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{r}_3\mathfrak{r}_4) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_3)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_3), \Upsilon'_{\wedge_\star}(\mathfrak{r}_4)\right\}_\wedge\right)\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_3), \Upsilon'_{\wedge_\star}(\mathfrak{r}_4)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_4), \Upsilon'_{\wedge_\star}(\mathfrak{r}_5)\right\}_\wedge\right)\right],\end{aligned}$$

for the edge $\mathfrak{r}_4\mathfrak{r}_5$,

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{r}_4\mathfrak{r}_5) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_3), \Upsilon'_{\wedge_\star}(\mathfrak{r}_4)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_4), \Upsilon'_{\wedge_\star}(\mathfrak{r}_5)\right\}_\wedge\right)\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_4), \Upsilon'_{\wedge_\star}(\mathfrak{r}_5)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_5), \Upsilon'_{\wedge_\star}(\mathfrak{r}_6)\right\}_\wedge\right)\right],\end{aligned}$$

following suit, the second FHS Zagreb number for the edge $\mathfrak{r}_{n-2}\mathfrak{r}_{n-1}$,

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{r}_{n-2}\mathfrak{r}_{n-1}) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-3}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-2})\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-2}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1})\right\}_\wedge\right)\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-2}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1})\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right)\right],\end{aligned}$$

and for the edge $\mathfrak{r}_{n-1}\mathfrak{r}_n$,

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{r}_{n-1}\mathfrak{r}_n) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-2}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1})\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right)\right] \\ &\times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right].\end{aligned}$$

By adding all the results, we have:

$$\begin{aligned}\mathbb{M}_2{}^{\wedge_\star}(\mathcal{P}_\hbar) \le& \left[\sum_{\wedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_1), \Upsilon'_{\wedge_\star}(\mathfrak{r}_2)\right\}_\wedge\right] \times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_1), \Upsilon'_{\wedge_\star}(\mathfrak{r}_2)\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_2), \Upsilon'_{\wedge_\star}(\mathfrak{r}_3)\right\}_\wedge\right)\right] \\ &+ \left[\sum_{\wedge_\star \in \mathfrak{R}} \left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-2}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1})\right\}_\wedge + \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right)\right] \times \left[\sum_{\wedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\wedge_\star}(\mathfrak{r}_{n-1}), \Upsilon'_{\wedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]\end{aligned}$$

$$+\sum_{i=2}^{n-1}\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_{i-1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_i)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_i),\Upsilon'_{\wedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge\right)\right]$$
$$\times\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_i),\Upsilon'_{\wedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_{i+2})\right\}_\wedge\right)\right].\quad\square$$

**Theorem 3.3.** *Let $\mathcal{P}_\hbar$ be a FHS path graph and $\mathfrak{R}_1,\mathfrak{R}_2,\mathfrak{R}_3,\dots,\mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1,\epsilon_2,\epsilon_3,\dots,\epsilon_\kappa$ such that $\mathfrak{R}=\mathfrak{R}_1\times\mathfrak{R}_2\times\mathfrak{R}_3\times\cdots\times\mathfrak{R}_\kappa$, with $\wedge_\star\in\mathfrak{R}$. Let $\rhd=\{\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_2,\dots,\mathfrak{x}_n\}$be the set of nodes, and each node's degree is determined by Lemma 3.1. Next, the path graph's FHS Randić number is provided as follows:*

$$\mathbb{R}^{\wedge_\star}(\mathcal{P}_\hbar)\geq\left[\sum_{\wedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge\times\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_1)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]^{-\frac{1}{2}}$$
$$+\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\sum_{z=n,n-2}\Upsilon'_{\wedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\times\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right]^{-\frac{1}{2}}$$
$$+\sum_{i=1}^{n}\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\sum_{z=i,i+1}\Upsilon'_{\wedge_\star}(\mathfrak{x}_i),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\times\left(\left\{\sum_{z=i,i+2}\Upsilon'_{\wedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\right]^{-\frac{1}{2}}$$

**Proof.** First we will calculate the Randić index for the edge $\mathfrak{x}_1\mathfrak{x}_2$. As:

$$D^{\wedge_\star}(\mathfrak{x}_1)\leq\sum_{\wedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge,$$
$$\Longrightarrow\frac{1}{D^{\wedge_\star}(\mathfrak{x}_1)}\geq\frac{1}{\sum_{\wedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge},\tag{3.1}$$

and

$$D^{\wedge_\star}(\mathfrak{x}_2)\leq\sum_{\wedge_\star\in\mathfrak{R}}\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right],$$
$$\Longrightarrow\frac{1}{D^{\wedge_\star}(\mathfrak{x}_2)}\geq\frac{1}{\sum_{\wedge_\star\in\mathfrak{R}}\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]},\tag{3.2}$$

multiplying Eqs. (3.1) and (3.1), we have:

$$\frac{1}{D^{\wedge_\star}(\mathfrak{x}_1)\times D^{\wedge_\star}(\mathfrak{x}_2)}\geq\frac{1}{\left[\sum_{\wedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge\times\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_1)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]},$$
$$\Longrightarrow\frac{1}{\left[D^{\wedge_\star}(\mathfrak{x}_1)\times D^{\wedge_\star}(\mathfrak{x}_2)\right]^{\frac{1}{2}}}\geq\frac{1}{\left[\sum_{\wedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge\times\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_1)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]^{\frac{1}{2}}}.$$

Hence,

$$\mathbb{R}_2{}^{\wedge_\star}(\mathfrak{x}_1\mathfrak{x}_2)\geq\frac{1}{\left[\sum_{\wedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_1),\Upsilon'_{\wedge_\star}(\mathfrak{x}_2)\right\}_\wedge\times\left(\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_1)\right\}_\wedge+\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]^{\frac{1}{2}}},\tag{3.3}$$

similarly for the edge $\mathfrak{x}_2\mathfrak{x}_3$,

$$\mathbb{R}_2{}^{\wedge_\star}(\mathfrak{x}_2\mathfrak{x}_3)\geq\frac{1}{\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\sum_{z=1,3}\Upsilon'_{\wedge_\star}(\mathfrak{x}_2),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\times\left(\left\{\sum_{z=2,4}\Upsilon'_{\wedge_\star}(\mathfrak{x}_3),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\right]^{\frac{1}{2}}},\tag{3.4}$$

continuing in the same way, for the edge $\mathfrak{x}_{n-2}\mathfrak{x}_{n-1}$,

$$\mathbb{R}_2{}^{\wedge_\star}(\mathfrak{x}_{n-2}\mathfrak{x}_{n-1})\geq\frac{1}{\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\sum_{z=n-3,n-1}\Upsilon'_{\wedge_\star}(\mathfrak{x}_{n-2}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\times\left(\left\{\sum_{z=n,n-2}\Upsilon'_{\wedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\right]^{\frac{1}{2}}},\tag{3.5}$$

and, for the edge $\mathfrak{x}_{n-1}\mathfrak{x}_n$,

$$\mathbb{R}_2{}^{\wedge_\star}(\mathfrak{x}_{n-1}\mathfrak{x}_n)\geq\frac{1}{\left[\sum_{\wedge_\star\in\mathfrak{R}}\left(\left\{\sum_{z=n-2,n}\Upsilon'_{\wedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\times\left\{\Upsilon'_{\wedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\wedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right]^{\frac{1}{2}}}.\tag{3.6}$$

Hence, by adding Eqs. (3.1)–(3.6), we have:

$$\mathbb{R}^{\curlywedge_\star}(\mathcal{P}_\hbar) \geq \left[\sum_{\curlywedge_\star\in\mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge \times \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]^{-\frac{1}{2}}$$

$$+\left[\sum_{\curlywedge_\star\in\mathfrak{R}} \left(\left\{\sum_{z=n,n-2}\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right) \times \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right]^{-\frac{1}{2}}$$

$$+\sum_{i=1}^{n}\left[\sum_{\curlywedge_\star\in\mathfrak{R}} \left(\left\{\sum_{z=i,i+1}\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right) \times \left(\left\{\sum_{z=i,i+2}\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_z)\right\}_\wedge\right)\right]^{-\frac{1}{2}} \quad \square$$

### 3.2. Cycle graph in Fuzzy hyper-soft framework

Let $\mathcal{C}_\hbar : \{\mathcal{C}^*_\hbar, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$ is a FHS cycle graph, with a sequence $\{\mathfrak{x}_n\}$ of distinct nodes, with $\mathfrak{x}_1 = \mathfrak{x}_n$, s.t $\sum_{\curlywedge_\star\in\mathfrak{R}} \mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_{i-1}\mathfrak{x}_i) > 0, \forall\ 2 \leq i \leq n-1$, but $\mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_1\mathfrak{x}_n) = 0$, where $\mathcal{C}^*_\hbar = (\rhd, \models)$ is a simple cycle graph. $(\curlyvee', \mathfrak{R}')$ is the FHS set throughout the node set $\rhd$, the length of FHS cycle graph is $n$. Let $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ be sets of distinct attributed valued sets associated with different attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$, so as $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$.

**Lemma 3.2.** *The membership degree of $n$ nodes in a FHS cycle graph $\mathcal{C}_\hbar$, with $\mathfrak{x}_i \in \rhd(\mathcal{C}_\hbar)$, is:*

$$D^{\curlywedge_\star}(\mathfrak{x}_i) \leq \sum_{\curlywedge_\star\in\mathfrak{R}}\left[\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_l + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge\right],$$

*$\forall\ 1 \leq i \leq n$, here $\mathfrak{x}_0 = \mathfrak{x}_n$ and $\mathfrak{x}_1 = \mathfrak{x}_{n+1}$. $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ are disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$.*

**Theorem 3.4.** *Let $\mathcal{C}_\hbar$, be a FHS cycle graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\}$ with $\mathfrak{x}_1 = \mathfrak{x}_n$ be the node set. Then,*

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{C}_\hbar) \leq \sum_{i=1}^{n}\left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge\right)\right]^2,$$

*where, $\mathfrak{x}_0 = \mathfrak{x}_n$ and $\mathfrak{x}_1 = \mathfrak{x}_{n+1}$.*

**Proof.** Let $\mathcal{C}_\hbar : \{\mathcal{C}^*_\hbar, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$ is a cycle graph in FHS environment, where $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, is the cartesian product of attributed valued sets s.t $\curlywedge_\star \in \mathfrak{R}$. Let the node set be $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\}$, with $\mathfrak{x}_1 = \mathfrak{x}_n$. Lemma 3.2, defines the degrees of each node. Then,

for node $\mathfrak{x}_1$ :

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_1) \leq \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]^2,$$

the first FHS Zagreb number for the node $\mathfrak{x}_2$ is demonstrated as:

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_2) \leq \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]^2,$$

for the node $\mathfrak{x}_3$ :

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_3) \leq \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_\wedge\right)\right]^2,$$

and so on, for the node $\mathfrak{x}_{n-1}$ :

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}) \leq \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]^2,$$

and for the node $\mathfrak{x}_n$, we have:

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_n) \leq \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]^2.$$

As a result, after adding together all the findings, the cycle graph's first FHS Zagreb number is:

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{C}_\hbar) \leq \sum_{i=1}^{n}\left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i),\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge\right)\right]^2,$$

where, $\mathfrak{x}_0 = \mathfrak{x}_n$ and $\mathfrak{x}_1 = \mathfrak{x}_{n+1}$. $\square$

**Theorem 3.5.** *Let $\mathcal{C}_\hbar$, be a FHS cycle graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \ldots, \mathfrak{x}_n\}$ with $\mathfrak{x}_1 = \mathfrak{x}_n$ be the node set. Next, the cycle graph's second FHS Zagreb* *number is provided as follows:*

$$\mathbb{M}_2{}^{\curlywedge_\star}(C_\hbar) \leq \sum_{i=1}^{n} \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+2})\right\}_\wedge\right)\right],$$

*where,* $\mathfrak{x}_0 = \mathfrak{x}_n$ *and* $\mathfrak{x}_1 = \mathfrak{x}_{n+1}$.

**Proof.** Let $C_\hbar : \left\{C^*_\hbar, \curlyvee', \mathfrak{H}, \mathfrak{R}\right\}$ be a cycle graph in the context of FHS, where $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, is the cartesian product of attributed valued sets s.t $\curlywedge_\star \in \mathfrak{R}$. Assume that the node set be $\triangleright = \left\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\right\}$. Lemma 3.2 provides the degree of each node. Therefore,
for the edge $\mathfrak{x}_1\mathfrak{x}_2$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_1\mathfrak{x}_2) \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right],$$

for the edge $\mathfrak{x}_2\mathfrak{x}_3$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_2\mathfrak{x}_3) \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_\wedge\right)\right],$$

for the edge $\mathfrak{x}_3\mathfrak{x}_4$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_3\mathfrak{x}_4) \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_5)\right\}_\wedge\right)\right],$$

for the edge $\mathfrak{x}_4\mathfrak{x}_5$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_4\mathfrak{x}_5) \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_5)\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_4), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_5)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_5), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_6)\right\}_\wedge\right)\right],$$

Maintaining the same methodology, the second FHS Zagreb number for the edge $\mathfrak{x}_{n-2}\mathfrak{x}_{n-1}$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_{n-2}\mathfrak{x}_{n-1}) \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-3}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2})\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right],$$

and for the edge $\mathfrak{x}_{n-1}\mathfrak{x}_n$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}\mathfrak{x}_n) \leq \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-2}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]$$
$$\times \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_\wedge\right)\right].$$

By adding all the results, we have:

$$\mathbb{M}_2{}^{\curlywedge_\star}(C_\hbar) \leq \sum_{i=1}^{n} \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge\right)\right.$$
$$\left.\times \sum_{\curlywedge_\star \in \mathfrak{R}} \left(\left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge + \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{i+2})\right\}_\wedge\right)\right], \forall\, \mathfrak{x}_i\mathfrak{x}_{i+1} \in \models(C_\hbar),$$

where, $\mathfrak{x}_0 = \mathfrak{x}_n$ and $\mathfrak{x}_1 = \mathfrak{x}_{n+1}$. □

**Theorem 3.6.** *Let* $C_\hbar$ *be a FHS cycle graph and* $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ *be disjoint attributed valued sets corresponding to distinct attributes* $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ *such that* $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, *with* $\curlywedge_\star \in \mathfrak{R}$. *Let the node set be* $\triangleright = \left\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \ldots, \mathfrak{x}_n\right\}$ *with all distinct nodes except first and* $n$*th node which are* *coincident. Then, the cycle graph's FHS Randić number is provided as follows:*

$$\mathbb{R}^{\wedge_\star}(C_\hbar) \geq \sum_{i=1}^{n}\Bigg[\sum_{\wedge_\star \in \mathfrak{R}}\Big(\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_{i-1}), \curlyvee'_{\wedge_\star}(\mathfrak{x}_i)\Big\}_\wedge + \Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_i), \curlyvee'_{\wedge_\star}(\mathfrak{x}_{i+1})\Big\}_\wedge\Big)$$
$$\times \sum_{\wedge_\star \in \mathfrak{R}}\Big(\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_i), \curlyvee'_{\wedge_\star}(\mathfrak{x}_{i+1})\Big\}_\wedge + \Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_{i+1}), \curlyvee'_{\wedge_\star}(\mathfrak{x}_{i+2})\Big\}_\wedge\Big)\Bigg]^{-\frac{1}{2}}, \forall\, \mathfrak{x}_i\mathfrak{x}_{i+1} \in \vDash(C_\hbar),$$

*where,* $\mathfrak{x}_0 = \mathfrak{x}_n$ *and* $\mathfrak{x}_1 = \mathfrak{x}_{n+1}$.

### *3.3. Complete graph in Fuzzy hyper-soft framework*

Let $\mathcal{K}_\hbar : \{\mathcal{K}_\hbar^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$, is a graph in FHS environment. Let $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed collection corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, and associated crisp graph is $\mathcal{K}^* = \{\rhd, \vDash\}$ with $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$. The condition for FHS complete graph is $\curlyvee'_{\wedge_\star}(\mathfrak{x}_i\mathfrak{x}_j) = \{\mathfrak{H}_{\wedge_\star}(\mathfrak{x}_i), \mathfrak{H}_{\wedge_\star}(\mathfrak{x}_j)\}_\wedge$. $(\curlyvee', \mathfrak{R})$ and $(\mathfrak{H}, \mathfrak{R})$ is FHS set over set of nodes $\rhd$ and set of edges respectively.

**Lemma 3.3.** *If* $\mathcal{K}_\hbar : \{\mathcal{K}_\hbar^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$*, is a FHS complete network, with the set of attributes* $\mathfrak{R}$*, such that* $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$*, corresponding to distinct attributes* $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ *and* $\mathcal{K}_\hbar^* = \{\rhd, \vDash\}$ *is the complete crisp graph with* $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$*, then for any node* $\mathfrak{x}_i$*, degree is defined as:*

$$D^{\wedge_\star}(\mathfrak{x}_i) = \sum_{i,j=1}^{n-1}\left(\sum_{\wedge_\star \in \mathfrak{R}}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_i), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right), \forall\, i \neq j.$$

**Theorem 3.7.** *Let* $\mathcal{K}_\hbar$*, be a FHS complete graph and* $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ *be disjoint attributed valued sets corresponding to distinct attributes* $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ *such that* $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$*, with* $\wedge_\star \in \mathfrak{R}$*. Let* $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$ *constitute the collection of nodes, and Lemma 3.3, determines the degree of each node. Next, the first FHS Zagreb number for the FHS complete graph is as follows:*

$$\mathbb{M}_1{}^{\wedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{n}\left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=1}^{n}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_i), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2, \forall\, i \neq j.$$

**Proof.** Let the complete graph in FHS environment is $\mathcal{K}_\hbar : \{\mathcal{K}_\hbar^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$, with $\mathcal{K}_\hbar^*$ as a complete crisp graph, and the set of nodes is $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$, for $\mathcal{K}_\hbar^*$. The set representing the Cartesian product of attributed valued sets is $\mathfrak{R}$, with $\wedge_\star \in \mathfrak{R}$. Lemma 3.3 defines the degree of each node, after which the first FHS Zagreb number can be computed via induction.

**Case1. When n = 3**

Let $\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3\}$, the node set for triangle. Then, for the node $\mathfrak{x}_1$

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_1) = \left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=2,3}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_1), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2,$$

for the node $\mathfrak{x}_2$,

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_2) = \left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=1,3}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_2), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2,$$

and for node $\mathfrak{x}_3$,

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_2) = \left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=1,2}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_3), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2,$$

It is generally able to be expressed as:

$$\mathbb{M}_1{}^{\wedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{3}\left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=1}^{3}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_i), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2, \forall\, i \neq j. \tag{3.7}$$

Now we calculate for $n = 4$.

Let $\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4\}$, is the set of nodes for $\mathcal{K}_4$ graph. Then, for node $\mathfrak{x}_1$

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_1) = \left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=2,3,4}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_1), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2,$$

for the node $\mathfrak{x}_2$,

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_2) = \left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=1,3,4}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_2), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2,$$

and for node $\mathfrak{x}_3$,

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_3) = \left[\sum_{\wedge_\star \in \mathfrak{R}}\left(\sum_{j=1,2,4}\Big\{\curlyvee'_{\wedge_\star}(\mathfrak{x}_3), \curlyvee'_{\wedge_\star}(\mathfrak{x}_j)\Big\}_\wedge\right)\right]^2,$$

and for node $\mathfrak{x}_4$,

$$\mathbb{M}_1{}^{\wedge_\star}(\mathfrak{x}_4) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{j=1,2,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_4), \Upsilon'_{\wedge_\star}(\mathfrak{x}_j) \right\}_{\wedge} \right)\right]^2,$$

Which in general can be written as:

$$\mathbb{M}_1{}^{\wedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{4} \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{j=1}^{4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_i), \Upsilon'_{\wedge_\star}(\mathfrak{x}_j) \right\}_{\wedge} \right)\right]^2, \forall\, i \neq j. \tag{3.8}$$

Hence, from Eqs. (3.7) and (3.8), we conclude that

$$\mathbb{M}_1{}^{\wedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{n} \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{j=1}^{n} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_i), \Upsilon'_{\wedge_\star}(\mathfrak{x}_j) \right\}_{\wedge} \right)\right]^2, \forall\, i \neq j. \quad \square$$

**Theorem 3.8.** *Let $\mathcal{K}_\hbar$, be a FHS complete graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \ldots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\wedge_\star \in \mathfrak{R}$. Assume $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \ldots, \mathfrak{x}_n\}$ to be the node set, Lemma 3.3 defines the degree of each node. Next, the complete graph's second FHS Zagreb number is provided as follows:.*

$$\mathbb{M}_2{}^{\wedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{n} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_i), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{n} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_j), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right], \forall\, i \neq j \neq \mathfrak{z},$$
$$\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\mathcal{K}_\hbar).$$

**Proof.** Let the complete graph in FHS environment is $\mathcal{K}_\hbar : \{\mathcal{K}_\hbar{}^*, \Upsilon', \mathfrak{H}, \mathfrak{R}\}$, with $\mathcal{K}_\hbar{}^*$ as a complete crisp graph, and the set of nodes is $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \ldots, \mathfrak{x}_n\}$, for $\mathcal{K}_\hbar{}^*$. The set representing the Cartesian product of attributed valued sets is $\mathfrak{R}$, with $\wedge_\star \in \mathfrak{R}$. Degree of all the nodes are defined in Lemma 3.3, then the first FHS Zagreb number can be calculated by induction.

**Case1. When n = 3**

Assume the node set $\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3\}$, for $\mathcal{K}_\hbar$. Then for the edge $\mathfrak{x}_1\mathfrak{x}_2$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_1\mathfrak{x}_2) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=2,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_1), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_2), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_2\mathfrak{x}_3$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_2\mathfrak{x}_3) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_2), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,2} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_3), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_1\mathfrak{x}_3$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_1\mathfrak{x}_3) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=2,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_1), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,2} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_3), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

Which in general can be written as:

$$\mathbb{M}_2{}^{\wedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{3}\sum_{j=1}^{3} \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_i), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_j), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right], \forall\, i \neq j \neq \mathfrak{z}.$$

**Case2. When n = 4**

Consider the FHS complete graph having four nodes $\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4\}$. Then for edge $\mathfrak{x}_1\mathfrak{x}_2$, we have:

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_1\mathfrak{x}_2) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=2,3,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_1), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,3,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_2), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_2\mathfrak{x}_3$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_2\mathfrak{x}_3) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,3,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_2), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,2,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_3), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_3\mathfrak{x}_4$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_3\mathfrak{x}_4) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{k=1,2,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_3), \Upsilon'_{\wedge_\star}(\mathfrak{x}_k) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{k=1,2,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_4), \Upsilon'_{\wedge_\star}(\mathfrak{x}_k) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_1\mathfrak{x}_4$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_1\mathfrak{x}_4) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=2,3,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_1), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,2,3} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_4), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_1\mathfrak{x}_3$,

$$\mathbb{M}_2{}^{\wedge_\star}(\mathfrak{x}_1\mathfrak{x}_3) = \left[\sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=2,3,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_1), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right) \times \sum_{\wedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1,2,4} \left\{ \Upsilon'_{\wedge_\star}(\mathfrak{x}_3), \Upsilon'_{\wedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_{\wedge} \right)\right],$$

for the edge $\mathfrak{x}_2\mathfrak{x}_4$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_2\mathfrak{x}_4) = \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{k=1,3,4} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_k) \right\}_\wedge \right) \times \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{k=1,2,3} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_4), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3) \right\}_\wedge \right)\right],$$

Which in general can be written as:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{4}\sum_{j=1}^{4} \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{4} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_\wedge \right) \times \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{4} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_j), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_\wedge \right)\right], \forall\, i \neq j \neq \mathfrak{z}.$$

Hence, for a FHS complete graph of $n$ nodes, the second FHS Zagreb number is:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{n} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_\wedge \right) \times \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{n} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_j), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_\wedge \right)\right], \forall\, i \neq j \neq \mathfrak{z},$$
$$\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\mathcal{K}_\hbar). \quad \square$$

**Theorem 3.9.** *Let $\mathcal{K}_\hbar$, be a FHS complete graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let set of nodes is $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_n\}$, and each node's degree is already found in Lemma 3.3. Then the FHS Randić numeral for FHS complete network is:*

$$\mathbb{R}^{\curlywedge_\star}(\mathcal{K}_\hbar) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{n} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_\wedge \right) \times \sum_{\curlywedge_\star \in \mathfrak{R}} \left( \sum_{\mathfrak{z}=1}^{n} \left\{ \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_j), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_\mathfrak{z}) \right\}_\wedge \right)\right]^{-\frac{1}{2}}, \forall\, i \neq j \neq \mathfrak{z},$$
$$\mathfrak{x}_i\mathfrak{x}_j \in \vDash(\mathcal{K}_\hbar).$$

**Proof.** The procedures in Theorem 3.7, can be used to establish the proof of this theorem. $\square$

### 3.4. Wheel graph in Fuzzy hyper-soft framework

Let the FHS wheel map is $\mathcal{W}_\hbar : \{\mathcal{W}_\hbar{}^*, \Upsilon', \mathfrak{H}, \mathfrak{R}\}$, where $\mathcal{W}_\hbar{}^* = \{\rhd, \vDash\}$ is a crisp wheel graph with $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$, and $\mathfrak{R}$ is the Cartesian product of attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$. By placing a node in the center of the cycle graph and connecting it to every other node in the cycle graph, one can create a wheel graph $\mathcal{W}_\hbar$. The center of wheel graph is the name given to this new node. Let $\mathfrak{x}_n$ be the wheel graph's center. If a wheel graph meets certain requirements, it qualifies as a FHS graph:

- $\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i\mathfrak{x}_j) \leq \left\{\mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_i), \mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_j)\right\}_\wedge, \forall\, 1 \leq i, j \leq n-1,$
- $\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i\mathfrak{x}_n) \leq \left\{\mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_i), \mathfrak{H}_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge, \forall\, 1 \leq i \leq n-1.$

**Lemma 3.4.** *Let $\mathcal{W}_\hbar : \{\mathcal{W}^*_\hbar, \Upsilon', \mathfrak{H}, \mathfrak{R}\}$, be a wheel map in FHS structure, with $\mathcal{W}_\hbar{}^* = \{\rhd, \vDash\}$ as a classic map, having node set as $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \upsilon_n\}$, and $\mathfrak{R} = \mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, is the attributed valued set corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$, Then, $\mathfrak{x}_i$ nodes' degrees are provided by:*

$$D^{\curlywedge_\star}(\mathfrak{x}_i) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1}), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge,$$

*$\forall\, 1 \leq i \leq n-1$, also $\mathfrak{x}_0 = \mathfrak{x}_{n-1}$,*
*and degree of node $\mathfrak{x}_n$ is:*

$$D^{\curlywedge_\star}(\mathfrak{x}_n) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge$$
$$+ \dots \cdots + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge.$$

**Theorem 3.10.** *Let $\mathcal{W}_\hbar$, be a FHS wheel graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_n\}$, then,*

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{W}_\hbar) \leq \sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right]^2$$
$$+ \left[\sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]^2.$$

**Proof.** Let $\mathcal{W}_\hbar : \{\mathcal{W}_\hbar{}^*, \Upsilon', \mathfrak{H}, \mathfrak{R}\}$, is a FHS wheel graph, and $\mathcal{W}_\hbar{}^*$ is a classic wheel structure, with $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$. $\mathfrak{R}$ is the Cartesian product of attributed sets, so as, $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\curlywedge_\star \in \mathfrak{R}$. First we calculate the first FHS Zagreb number for the node $\mathfrak{x}_1$.

The following illustrates node $\mathfrak{x}_1$'s membership degree:

$$D^{\curlywedge_\star}(\mathfrak{x}_1) = \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1), \Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge.$$

So,

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_1) \le \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right]^2,$$

similarly, for node $\mathfrak{x}_2$

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_2) \le \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right]^2,$$

for node $\mathfrak{x}_3$,

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_3) \le \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right]^2,$$

Proceeding in the same manner, $\mathfrak{x}_{n-1}$ is the node's first FHS Zagreb number.

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}) \le \left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-2}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right]^2.$$

Now, the first FHS Zagreb number for the node $\mathfrak{x}_n$ is:

$$\begin{aligned}\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{x}_n) \le \Bigg[&\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge} + \\ &\cdots + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\Bigg]^2.\end{aligned}$$

Hence, first FHS Zagreb number for the wheel graph is:

$$\begin{aligned}\mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{W}_{\hbar}) \le &\sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right]^2 \\ &+ \left[\sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\right]^2. \quad \square\end{aligned}$$

**Theorem 3.11.** *Let $\mathcal{W}_{\hbar}$, be a FHS wheel graph and $\mathfrak{R}_1,\mathfrak{R}_2,\mathfrak{R}_3,\ldots,\mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1,\epsilon_2,\epsilon_3,\ldots,\epsilon_\kappa$ such that $\mathfrak{R}=\mathfrak{R}_1\times\mathfrak{R}_2\times\mathfrak{R}_3\times\cdots\times\mathfrak{R}_\kappa$, with $\curlywedge_\star\in\mathfrak{R}$. Let $\rhd=\left\{\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_2,\ldots,\mathfrak{x}_n\right\}$ be the node set. The second FHS Zagreb number for the wheel graph is given as:*

$$\begin{aligned}\mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{W}_{\hbar}) \le &\sum_{i=1}^{n-1}\Bigg[\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \\ &\times\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+2})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\Bigg] \\ &+\sum_{i=1}^{n-1}\Bigg[\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \\ &\times\sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\Bigg], \forall\ \mathfrak{x}_i\mathfrak{x}_{i+1}\in\vDash(\mathcal{W}_{\hbar}), \mathfrak{x}_i\mathfrak{x}_n\in\vDash(\mathcal{W}_{\hbar}),\end{aligned}$$

*where, $\mathfrak{x}_0=\mathfrak{x}_{n-1}$.*

**Proof.** Let $\mathcal{W}_{\hbar}:\left\{\mathcal{W}_{\hbar}{}^*,\Upsilon',\mathfrak{H},\mathfrak{R}\right\}$, is a FHS wheel graph, and $\mathcal{W}_{\hbar}{}^*$ is a classic wheel graph with $\rhd=\left\{\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{x}_3,\ldots,\mathfrak{x}_n\right\}$. The set $\mathfrak{R}$ of attributes is so as, $\mathfrak{R}=\mathfrak{R}_1\times\mathfrak{R}_2\times\mathfrak{R}_3\times\cdots\times\mathfrak{R}_\kappa$, corresponding to distinct attributes $\epsilon_1,\epsilon_2,\epsilon_3,\ldots,\epsilon_\kappa$ with $\curlywedge_\star\in\mathfrak{R}$, and each node's degree is stated in Lemma 3.4. So, second FHS Zagreb number for edge $\mathfrak{x}_1\mathfrak{x}_2$,

$$\begin{aligned}\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_1\mathfrak{x}_2) \le &\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \\ &\times\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right),\end{aligned} \tag{3.9}$$

for edge $\mathfrak{x}_2\mathfrak{x}_3$,

$$\begin{aligned}\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_2\mathfrak{x}_3) \le &\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \\ &\times\left(\sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\mathfrak{R}}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right),\end{aligned} \tag{3.10}$$

in the same manner, the second FHS Zagreb number for the edge $\mathfrak{x}_{n-1}\mathfrak{x}_1$, is demonstrated as:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}\mathfrak{x}_1) \leq \left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-2}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \times \left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right), \tag{3.11}$$

Now, we calculate the second FHS Zagreb number for the edges incident on central node $\mathfrak{x}_n$ for edge $\mathfrak{x}_n\mathfrak{x}_1$, is demonstrated as:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_n\mathfrak{x}_1) \leq \left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \times \sum_{i=1}^{n-1}\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right), \tag{3.12}$$

for edge $\mathfrak{x}_n\mathfrak{x}_2$, is demonstrated as:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_n\mathfrak{x}_2) \leq \left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \times \sum_{i=1}^{n-1}\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \tag{3.13}$$

for edge $\mathfrak{x}_n\mathfrak{x}_3$, is demonstrated as:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_n\mathfrak{x}_3) \leq \left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_2),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_4)\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_3),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \times \sum_{i=1}^{n-1}\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right), \tag{3.14}$$

similarly, for the edge $\mathfrak{x}_{n-1}\mathfrak{x}_n$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}\mathfrak{x}_n) \leq \left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_1),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-2}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{n-1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right) \times \sum_{i=1}^{n-1}\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right), \tag{3.15}$$

Combining the results of Eqs. (3.9)–(3.15), the second FHS Zagreb number for the wheel graph is:

$$\begin{aligned}\mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{W}_\hbar) \leq \sum_{i=1}^{n-1}\Bigg[&\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\\ &\times\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+2})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\Bigg]\\ +\sum_{i=1}^{n-1}\Bigg[&\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\\ &\times\sum_{i=1}^{n-1}\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\Bigg], \forall\ \mathfrak{x}_i\mathfrak{x}_{i+1}\in\vDash(\mathcal{W}_\hbar), \mathfrak{x}_i\mathfrak{x}_n\in\vDash(\mathcal{W}_\hbar),\end{aligned}$$

where, $\mathfrak{x}_0 = \mathfrak{x}_{n-1}$. □

**Theorem 3.12.** *Let $\mathcal{W}_\hbar$, be a FHS wheel graph and $\Re_1, \Re_2, \Re_3, \dots, \Re_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\Re = \Re_1 \times \Re_2 \times \Re_3 \times \cdots \times \Re_\kappa$, with $\curlywedge_\star \in \Re$. Let $\rhd = \left\{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_n\right\}$ be the node set. Then the FHS Randić number for wheel graph is given as:*

$$\begin{aligned}\mathbb{R}^{\curlywedge_\star}(\mathcal{W}_\hbar) \geq \sum_{i=1}^{n-1}\Bigg[&\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\\ &\times\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+2})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1}),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\Bigg]^{-\frac{1}{2}}\\ +\sum_{i=1}^{n}\Bigg[&\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i-1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_{i+1})\right\}_{\wedge} + \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\\ &\times\sum_{i=1}^{n-1}\left( \sum_{\curlywedge_\star\in\Re}\left\{\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_i),\Upsilon'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_{\wedge}\right)\Bigg]^{-\frac{1}{2}}, \forall\ \mathfrak{x}_i\mathfrak{x}_{i+1}\in\vDash(\mathcal{W}_\hbar), \mathfrak{x}_i\mathfrak{x}_n\in\vDash(\mathcal{W}_\hbar),\end{aligned}$$

*where,* $\mathfrak{x}_0 = \mathfrak{x}_{n-1}$.

### 3.5. Star graph in Fuzzy hyper-soft framework

Let $\mathcal{S}_\hbar : \{\mathcal{S}_\hbar{}^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$, be a FHS graph, where $\mathfrak{R}$ is the Cartesian product of attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$. Here $\mathcal{S}_\hbar{}^* = \{\rhd, \models\}$ is a classic graph with node set $\rhd$, is partitioned into two sets $\rhd_1 = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \dots \mathfrak{r}_{n-1}\}$ and $\rhd_2 = \{\mathfrak{r}_n\}$, such that $\rhd_1 \cup \rhd_2 = \rhd$ and $\rhd_1 \cap \rhd_2 = \emptyset$. This FHS network is a FHS star graph if:

- $\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i\mathfrak{r}_j) = 0 \ \forall \ \mathfrak{r}_i, \mathfrak{r}_j \in \rhd_1$, and $\curlywedge_\star \in \mathfrak{R}$,
- $\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i\mathfrak{r}_n) \le \left\{\mathfrak{H}_{\curlywedge_\star}(\mathfrak{r}_i), \mathfrak{H}_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge \ \forall \ \mathfrak{r}_i \in \rhd_1, \mathfrak{r}_n \in \rhd_2$ and $\curlywedge_\star \in \mathfrak{R}$.

**Lemma 3.5.** *Let $\mathcal{S}_\hbar : \{\mathcal{S}_\hbar^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$, be a FHS star graph, where $\mathcal{S}_\hbar{}^* = \{\rhd, \models\}$ is a simple graph with $\mathfrak{R} = \mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, is the attributed valued set corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$. The degrees of $\mathfrak{r}_i$ nodes such that $1 \le i \le n-1$ are given by:*

$$D^{\curlywedge_\star}(\mathfrak{r}_i) \le \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge, \forall \ \mathfrak{r}_i \in \rhd_1, \mathfrak{r}_n \in \rhd_2,$$

*and degree of node $\mathfrak{r}_n$ is:*

$$D^{\curlywedge_\star}(\mathfrak{r}_n) \le \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right).$$

**Theorem 3.13.** *Let $\mathcal{S}_\hbar$, be a FHS star graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_2, \dots, \mathfrak{r}_n\}$ be the node set, partitioned into two sets $\rhd_1 = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_2, \dots, \mathfrak{r}_{n-1}\}$ and $\rhd_2 = \{\mathfrak{r}_n\}$. Then the first FHS Zagreb number for star graph is given as:*

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{S}_\hbar) \sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2 + \left[\sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right)\right]^2.$$

**Proof.** Let $\mathcal{S}_\hbar : \{\mathcal{S}_\hbar{}^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$, is a FHS star graph, and $\mathcal{S}_\hbar{}^*$ is its crisp graph with $\rhd = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \dots, \mathfrak{r}_n\}$, is partitioned into two sets $\rhd_1 = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_2, \dots, \mathfrak{r}_{n-1}\}$ and $\rhd_2 = \{\mathfrak{r}_n\}$. $\mathfrak{R}$ is the set of attributes, such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\curlywedge_\star \in \mathfrak{R}$.

First we calculate the first FHS Zagreb number for the nodes $\mathfrak{r}_i, \forall \ 1 \le i \le n-1$.

Each $\mathfrak{r}_i$ has one connections, only with $\mathfrak{r}_n$. The degrees of $\mathfrak{r}_i$ nodes such that $1 \le i \le n-1$ are defined in Lemma 3.5.

So, first FHS Zagreb number for the node $\mathfrak{r}_1$ is:

$$\implies \mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{r}_1) \le \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2,$$

similarly, for node $\mathfrak{r}_2$

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{r}_2) \le \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2,$$

for node $\mathfrak{r}_3$,

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{r}_3) \le \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2,$$

proceeding in the similar manner, for the node $\mathfrak{r}_{n-1}$

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{r}_{n-1}) \le \left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2.$$

Now, the first FHS Zagreb number for the node $\mathfrak{r}_n$ is:

$$\begin{aligned}\mathbb{M}_1{}^{\curlywedge_\star}(\mathfrak{r}_n) \le &\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_3), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge + \right.\\ &\left.\cdots + \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2,\\ = &\left[\sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right)\right]^2,\end{aligned}$$

Hence, first FHS Zagreb number for the star graph is:

$$\mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{S}_\hbar) \sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right]^2 + \left[\sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{r}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{r}_n)\right\}_\wedge\right)\right]^2. \quad \square$$

**Theorem 3.14.** *Let $\mathcal{S}_\hbar$, be a FHS star graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_n\}$ be the node set, partitioned into two sets $\rhd_1 = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_{n-1}\}$ and $\rhd_2 = \{\mathfrak{x}_n\}$. Then, for the star graph:*

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{S}_\hbar) \leq \sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge \times \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right].$$

**Proof.** Let $\mathcal{S}_\hbar : \{\mathcal{S}_\hbar{}^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\}$, is a FHS star graph, and $\mathcal{S}_\hbar{}^*$ is a crisp graph with $n$ nodes, $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \dots, \mathfrak{x}_n\}$, is partitioned into two sets $\rhd_1 = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_{n-1}\}$ and $\rhd_2 = \{\mathfrak{x}_n\}$. $\mathfrak{R}$ is the set of attributes, such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ with $\curlywedge_\star \in \mathfrak{R}$.

First we calculate the first FHS Zagreb number for the nodes $\mathfrak{x}_i, \forall\ 1 \leq i \leq n-1$.

Each $\mathfrak{x}_i$ has one connections, only with $\mathfrak{x}_n$. The degrees of $\mathfrak{x}_i$ nodes such that $1 \leq i \leq n-1$ are defined in Lemma 3.5.

$\mathfrak{x}_1\mathfrak{x}_n$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_1\mathfrak{x}_n) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_1), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge \times \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right),$$

for edge $\mathfrak{x}_2\mathfrak{x}_n$,

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_2\mathfrak{x}_n) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_2), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge \times \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right),$$

following the same methodology, for edge $\mathfrak{x}_{n-1}\mathfrak{x}_1$, we have:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathfrak{x}_{n-1}\mathfrak{x}_n) \leq \sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_{n-1}), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge \times \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right).$$

Hence, second FHS Zagreb number for FHS star graph is:

$$\mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{S}_\hbar) \leq \sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge \times \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right], \ \forall\ \mathfrak{x}_i\mathfrak{x}_n \in \models(\mathcal{S}_\hbar). \quad \square$$

**Theorem 3.15.** *Let $\mathcal{S}_\hbar$, be a FHS star graph and $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \dots, \mathfrak{R}_\kappa$ be disjoint attributed valued sets corresponding to distinct attributes $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_\kappa$ such that $\mathfrak{R} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \times \cdots \times \mathfrak{R}_\kappa$, with $\curlywedge_\star \in \mathfrak{R}$. Let $\rhd = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_n\}$ be the node set, partitioned into two sets $\rhd_1 = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_2, \dots, \mathfrak{x}_{n-1}\}$ and $\rhd_2 = \{\mathfrak{x}_n\}$. Then the FHS Randić number for the star graph is given as:*

$$\mathbb{R}^{\curlywedge_\star}(\mathcal{S}_\hbar) \geq \sum_{i=1}^{n-1}\left[\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge \times \sum_{i=1}^{n-1}\left(\sum_{\curlywedge_\star \in \mathfrak{R}} \left\{\curlyvee'_{\curlywedge_\star}(\mathfrak{x}_i), \curlyvee'_{\curlywedge_\star}(\mathfrak{x}_n)\right\}_\wedge\right)\right]^{-\frac{1}{2}}, \forall\ \mathfrak{x}_i\mathfrak{x}_n \in \models(\mathcal{S}_\hbar).$$

*where,* $1 \leq i \leq n-1$.

## 4. FHS VIKOR method

This section introduces the fuzzy VIKOR technique, a Multi-Attribute Decision-Analysis (MADA) approach that solves MCDM problems by combining FHS numbers with the VIKOR method. To find the FHS multicriteria problem's compromise solution, the fuzzy VIKOR approach was created. The process for this method begins as follows:

### 1: Constructing a decision matrix

Suppose there are $m$ alternatives, which are being assessed on the basis of $n$ criteria. In our problem alternatives would be graphical networks and degrees of nodes are the criteria on the basis of which best graphical network can be chosen.

### 2: Finding the best and worst value

The best value $\mathfrak{A}^+$ and the worst value $\mathfrak{A}^-$ for all the criteria, which are calculated as:

$$\mathfrak{A}_i^+ = \max_{\mathrm{i}} \mathfrak{A}_{\mathrm{ij}+},$$
$$\mathfrak{A}_{\mathrm{i}}^- = \min_{i} \mathfrak{A}_{\mathrm{ij}}^-.$$

**3: Calculating the normalized weights**

For the criterion, the divergence value $\{D(\mathfrak{x}_i)\}$ represents the inherent contrast intensity. The more significant a criterion is for the problem, the larger its value in $\{D(\mathfrak{x}_i)\}$. The VIKOR approach seeks a compromise solution that strikes a balance between individual regret and group value by determining criteria weights either objectively, utilizing techniques such as the target-based standard deviation (SD) method, or subjectively, by decision-makers.

The weights assigned to each criterion can be computed using the following formula:

$$\mathfrak{W}_i = \frac{\max\left\{D(\mathfrak{x}_i)\right\}}{\sum_{i=1}^{n}\left\{D(\mathfrak{x}_i)\right\}}.$$

The weights associated with the nodes/hyperedges are assigned such that their sum equals one, following the standard normalization approach. This ensures the uncertainty in weight assignment is controlled and the total weight distribution remains consistent.

**4: Finding the values of $S_i$, $R_i$, $Q_i$**

$$S_i = \sum_{j=1}^{n}\left\{\mathfrak{W}_j \frac{\mathfrak{A}_{\mathrm{j}}^{+} - \mathfrak{A}_{\mathrm{ij}}^{+}}{\mathfrak{A}_{\mathrm{j}}^{+} - \mathfrak{A}_{\mathrm{j}}^{-}}\right\},$$

$$R_i = \max_j\left\{\mathfrak{W}_j \frac{\mathfrak{A}_{\mathrm{j}}^{+} - \mathfrak{A}_{\mathrm{ij}}^{+}}{\mathfrak{A}_{\mathrm{j}}^{+} - \mathfrak{A}_{\mathrm{j}}^{-}}\right\}.$$

After that, $Q_i$ can be computed using the following formula:

$$Q_i = \mathfrak{y} \times \left(\frac{S_i - S^*}{S^- - S^*}\right) + (1-\mathfrak{y}) \times \left(\frac{R_i - R^*}{R^- - R^*}\right),$$

where, $S^* = \min_i\left\{S_i\right\}, S^- = \max_i\left\{S_i\right\}, R^* = \min_i\left\{R_i\right\}$ and $R^- = \max_i\left\{R_i\right\}$.

The strategy weight of the majority of criteria is denoted by $\mathfrak{y}$ in this case, such that $\mathfrak{y} \in [0,1]$, while the likelihood of each individual loss is $(1-\mathfrak{y})$.

**5: Ranking the alternatives**

The alternatives are ranked on the basis of the values of $Q_i$ in a descending order. For example alternative $a_1$ better than $a_2$ if $Q_2 \leq Q_1$.

**6: Examination of two conditions**

After ranking the alternatives, the last step is to generate compromise solution. In which it is checked that whether the selected alternative $a_1$(with minimum $Q$) fulfills the following conditions or not.

- **Condition 1: acceptable advantage**
  Examine the condition $Q(a_2) - Q(a_1) \geq \frac{1}{m-1}$, where $m$ is the number of alternatives, $a_2$ is the second highest alternative in the ranking list.
- **Condition 2: acceptable stability**
  This condition is satisfied when the alternative $A_1$ is also ranked best in $S_i$ and/or in $R_i$ ranking list.
  When one of the above conditions are not satisfied, then the following compromise solution set would be formed by decision makers.
- If condition 1 is satisfied only, then $a_1$ and $a_2$ formed the final compromise solution.
- If condition 2 is satisfied only, $a_1, a_2, a_3, \ldots\ldots, a_h$ are the compromise solutions, which are obtained by the relation $Q(a_h) - Q(a_1) < \frac{1}{m-1}$.

*4.1. Comparison of VIKOR with TOPSIS, AHP, and ELECTRE in multi-criteria decision-making*

By emphasizing compromise solutions, maximizing collective utility, and minimizing individual regret, VIKOR excels at multi-criteria decision-making. This makes it especially appropriate for complicated, conflicting, and incommensurable issues where other approaches, such as TOPSIS, AHP, and ELECTRE, would not be as effective.

From Table 3, it can be concluded that, VIKOR stands out for its capacity to identify a compromise solution that takes into account the preferences of numerous stakeholders and is especially useful in circumstances with conflicting criteria and incommensurable qualities, even if TOPSIS, AHP, and ELECTRE are all useful MCDM strategies.

*4.2. Analytical hierarchy process (AHP)*

Hierarchy process can be used to calculate weights of criteria, the main algorithm of this process is as follows.

- Define the problem.
- Formation of decision matrix with given number of criteria and selected numbers of alternatives.
- Hierarchy structure formation.
- Determine the relative importance of different attributes or criteria with respect to goal, which the construction of pair-wise comparison matrix. Pair-wise comparison matrix is created with the help of scale of relative importance, i.e. 1. for equal importance, 3. for moderate importance, 5. strong importance, 7. very strong, 9. for extremely important value, $2, 4, 6, 8$ for intermediate values, whereas, $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ for inverse comparison values.

**Table 3**
Comparison of VIKOR, TOPSIS, AHP, and ELECTRE.

| Criteria | VIKOR (Best) | TOPSIS | AHP | ELECTRE |
|---|---|---|---|---|
| Approach | Compromise ranking [73] | Distance-based ranking [74] | Pairwise comparisons [75] | Outranking |
| Decision focus | Balances best and worst performance | Closest to ideal solution | Expert judgment consistency | Eliminates weak alternatives |
| Handling of conflict | Considers trade-offs between criteria | Only considers distance from ideal | Depends on subjective comparisons | Focuses on dominance |
| Subjectivity | Low (uses direct values) | Moderate (requires normalization) | High (expert-based weights) | Moderate (requires thresholds) |
| Ranking stability | High (stable ranking with clear decision rules) | Moderate (changes with weights) | Low (depends on expert consistency) | Low (unstable due to threshold settings) |
| Mathematical complexity | Moderate | Low | High | High |
| Best for MCDM | ✓(Yes) | ×(No) | ×(No) | ×(No) |

- Formation of normalized pair-wise matrix. Normalized pair-wise matrix is obtained by dividing all element of column with sum of columns.
- Finding criteria weights, which are calculated by averaging all th elements in the row. Sum of all criteria weights in a matrix must be equal to one.
- Calculating consistency, there we check whether the calculated values are correct or not, for this consider that decision matrix, which is not normalized and multiply each value in the column by criteria value.
- Calculate weighted sum values, which are obtained by taking sum of each value row-wise in the pair-wise decision matrix.
- Calculate the ratio of weighted sum value to criteria weights for each row, which give $\lambda$. $\lambda_{max}$ is obtained by taking average of all the values obtained by taking ratio.
- Calculate consistency index ($CI$), s.t.

$$CI = \frac{\lambda_{max} - m}{m - 1}, \tag{4.1}$$

where, $m$ is number of criteria taken.
- Calculate the consistency ratio (CR), with the formula:

$$CR = \frac{CI}{RI}.$$

## 5. Application

In this section, we will present an application of the use of fuzzy hypersoft topological numbers in the business industry. First we will discuss the importance of a network to start a new business. If someone wants to launch their own company and wants to grow it quickly by attracting more customers, generating more revenue, and improving conversion rates overall. However, he has no idea how to accomplish that. There are multiple approaches to achieving that. Nevertheless, developing a robust business network is the most efficient approach for doing this. Having a strong business network is crucial for him to get the success he desires, regardless of the size of the company he wants to operate. When a business is first starting out, many entrepreneurs do not think networking is that vital. By doing this, they overlook the fact that building a solid network is crucial to laying the groundwork for a profitable company. A strong business network helps building professional relationship and facilitates new networks. It also helps in staying informed about industry trends and provides a competitive edge.

To build a strong business network, consistency and continuous effort is required with good strategies. These strategies are the **use of social media**, **attending networking functions** and **get involved in related organizations**.

The four Americans, James, William, Daniel, and Charles, from the cities of New York, Las Vegas, California, and Los Angeles, intend to launch a new company. They get introduced to each other on social media. All of them have different ideas and small investments. They choose to collaborate with each other by establishing the branches of the company in their respective native cities. After establishing four cities to maximize profit, they complete their homework by analyzing various graphical networks and using fuzzy hypersoft topological numbers to identify the greatest feasible network of trade and transit between the four enterprises. The result obtained by using fuzzy hypersoft topological numbers on graphical networks will be considered as the profit earned by the mutual business.

Let $\mathcal{H} = \left(\tilde{\Gamma}^*, \curlyvee', \mathfrak{H}, \mathfrak{R}\right)$ be a FHSG, with node set $\rhd = \{j^*, w^*, d^*, c^*\}$. Suppose $\left(\curlyvee', \mathfrak{R}\right)$ be the FHSS over $\rhd$. Set $\rhd$ represents the four Americans, who are going to instigate a new company in four different cities. Let $\epsilon = \left\{\epsilon_1, \epsilon_2, \epsilon_3\right\}$ be the set of attributes, such that each $\epsilon_\kappa$ stands for skills, business ideas and developing social media strategy respectively. The corresponding attributed values are $\left\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\right\}$. For the business person to invest efficiently to get maximum profit, following key points must be taken into consideration.

- $\mathfrak{R}_1 = \left\{\mathfrak{f}_{11} = passion, \mathfrak{f}_{12} = budget, \mathfrak{f}_{13} = finance\right\}$,
- $\mathfrak{R}_2 = \left\{\mathfrak{f}_{21} = Company\ \ A, \mathfrak{f}_{22} = Company\ \ B\right\}$,
- $\mathfrak{R}_3 = \left\{\mathfrak{f}_{31} = best\ \ website\ \ building\right\}$.

$$\begin{aligned}
\mathfrak{R} =& \mathfrak{R}_1 \times \mathfrak{R}_2 \times \mathfrak{R}_3 \\
&= \left\{\mathfrak{f}_{11}, \mathfrak{f}_{12}, \mathfrak{f}_{13}\right\} \times \left\{\mathfrak{f}_{21}, \mathfrak{f}_{22}\right\} \times \left\{\mathfrak{f}_{31}\right\}, \\
&= \left\{\left(\mathfrak{f}_{11}, \mathfrak{f}_{21}\right), \left(\mathfrak{f}_{11}, \mathfrak{f}_{22}\right), \left(\mathfrak{f}_{12}, \mathfrak{f}_{21}\right), \left(\mathfrak{f}_{12}, \mathfrak{f}_{22}\right), \left(\mathfrak{f}_{13}, \mathfrak{f}_{21}\right), \left(\mathfrak{f}_{13}, \mathfrak{f}_{22}\right)\right\} \times \left\{\mathfrak{f}_{31}\right\}, \\
&= \left\{\left(\mathfrak{f}_{11}, \mathfrak{f}_{21}, \mathfrak{f}_{31}\right), \left(\mathfrak{f}_{11}, \mathfrak{f}_{22}, \mathfrak{f}_{31}\right), \left(\mathfrak{f}_{12}, \mathfrak{f}_{21}, \mathfrak{f}_{31}\right), \left(\mathfrak{f}_{12}, \mathfrak{f}_{22}, \mathfrak{f}_{31}\right), \left(\mathfrak{f}_{13}, \mathfrak{f}_{21}, \mathfrak{f}_{31}\right), \left(\mathfrak{f}_{13}, \mathfrak{f}_{22}, \mathfrak{f}_{31}\right)\right\}
\end{aligned}$$

**Table 4**
Nodes of FHSG.

| ⊳ | $\curlywedge_1$ | $\curlywedge_2$ | $\curlywedge_3$ | $\curlywedge_4$ | $\curlywedge_5$ | $\curlywedge_6$ |
|---|---|---|---|---|---|---|
| $j^*$ | 0.3 | 0.9 | 0.8 | 0.7 | 0.7 | 0.2 |
| $w^*$ | 0.4 | 0.8 | 0.7 | 0.6 | 0.6 | 0.3 |
| $d^*$ | 0.5 | 0.8 | 0.8 | 0.5 | 0.8 | 0.4 |
| $c^*$ | 0.2 | 0.7 | 0.9 | 0.6 | 0.7 | 0.5 |

**Table 5**
Edges of FHSG.

| $\mathfrak{R}$ | $\curlywedge_1$ | $\curlywedge_2$ | $\curlywedge_3$ | $\curlywedge_4$ | $\curlywedge_5$ | $\curlywedge_6$ |
|---|---|---|---|---|---|---|
| $j^*w^*$ | 0.3 | 0.8 | 0.7 | 0.6 | 0.5 | 0.1 |
| $w^*d^*$ | 0.4 | 0.7 | 0.6 | 0.4 | 0.6 | 0.2 |
| $d^*c^*$ | 0.2 | 0.6 | 0.85 | 0.5 | 0.7 | 0.3 |
| $c^*j^*$ | 0.15 | 0.7 | 0.8 | 0.6 | 0.65 | 0.2 |
| $j^*d^*$ | 0.3 | 0.8 | 0.8 | 0.5 | 0.7 | 0.2 |
| $w^*c^*$ | 0.1 | 0.7 | 0.7 | 0.6 | 0.5 | 0.3 |

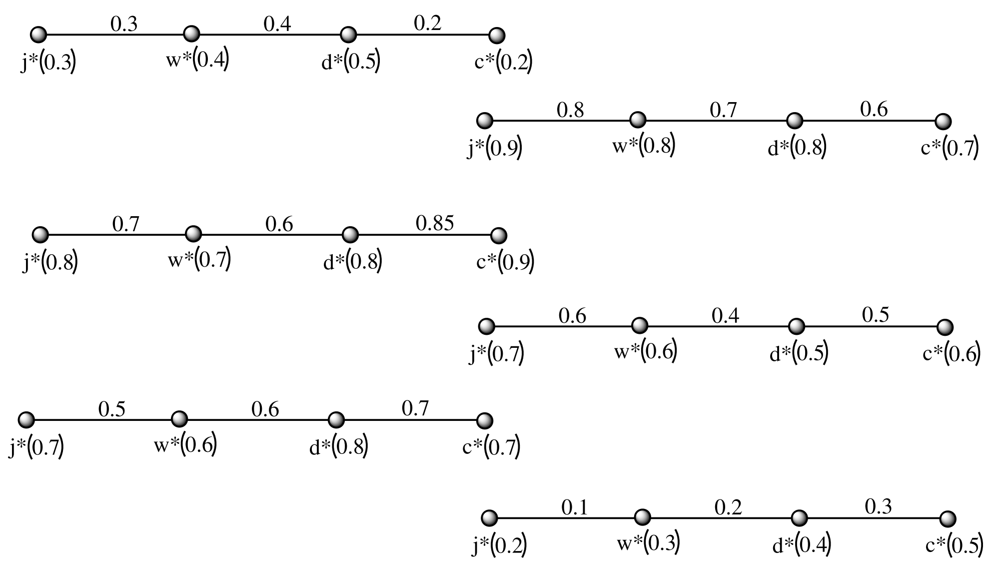


**Fig. 3.** Six fuzzy hypersoft path sub-graphs.

$$\mathfrak{R} = \left\{ \begin{matrix} \curlywedge_1 = (\mathfrak{f}_{11}, \mathfrak{f}_{21}, \mathfrak{f}_{31}), \curlywedge_2 = (\mathfrak{f}_{11}, \mathfrak{f}_{22}, \mathfrak{f}_{31}), \curlywedge_3 = (\mathfrak{f}_{12}, \mathfrak{f}_{21}, \mathfrak{f}_{31}), \\ \curlywedge_4 = (\mathfrak{f}_{12}, \mathfrak{f}_{22}, \mathfrak{f}_{31}), \curlywedge_5 = (\mathfrak{f}_{13}, \mathfrak{f}_{21}, \mathfrak{f}_{31}), \curlywedge_6 = (\mathfrak{f}_{13}, \mathfrak{f}_{22}, \mathfrak{f}_{31}). \end{matrix} \right\}$$

There are six outcomes and the business will be started on the basis of these outcomes. The number of fuzzy hypersoft sub-graphs is equal to the cardinality of set $\mathfrak{R}$. The node sets and edge sets of these six graphs are given in Table 4.

*Determining the best graphical network by the use of topological numbers:*

In this case, first, we will draw the FHS sub-graphs of different graphical networks related to each attributed value $\curlywedge_\star \in \mathfrak{R}$. Every node stands for a single partner from a single American city. The membership value of each node in each sub-graph indicates each person's interest in launching a business under the specified attributed value, and the membership value of each edge (connection) connecting two people indicates their desire to conduct business together.

After drawing the FHS sub-graphs of each network equal to the cardinality of the attributed valued set $\mathfrak{R}$, the degree of each node will be calculated by applying Eq. (2.1). The three topological numbers for each network may then be calculated by using Eqs. (2.4)–(2.6). This yields the profit that the four partners would make if they conducted business with each other in the form of that specific network.

As, the number of attributed sets belonging to $\mathfrak{R}$ are six, so six sub-graphs, will be drawn for each kind of network (see Tables 5 and 6). **For**

**Path Graph**

Degrees of nodes are:

$D(j^*) = 3.0, D(w^*) = 5.9, D(d^*) = 6.05, D(c^*) = 3.15.$

The profit earned is:

$$\begin{aligned} \mathbb{M}_1{}^{\curlywedge_\star}(\mathcal{P}_\hbar) &= 90.335, \\ \mathbb{M}_2{}^{\curlywedge_\star}(\mathcal{P}_\hbar) &= 72.45, \\ \mathbb{R}^{\curlywedge_\star}(\mathcal{P}_\hbar) &= 1.37. \end{aligned} \tag{5.1}$$

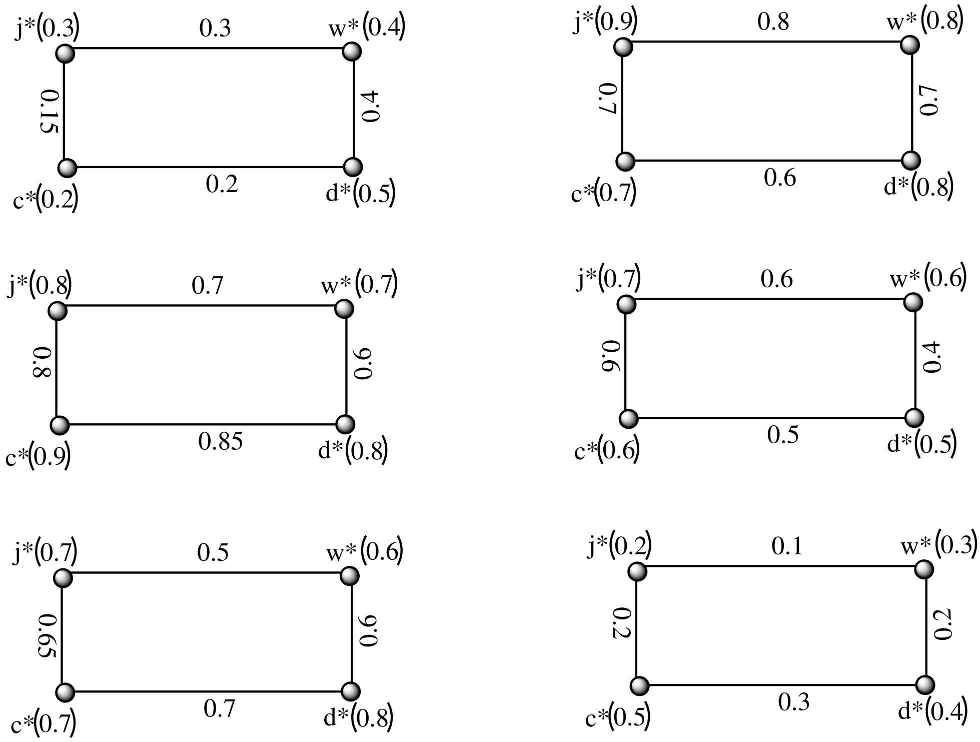


**Fig. 4.** Six fuzzy hypersoft cycle sub-graphs.

**For Cycle Graph**

Degrees of nodes are:
$D(j^*) = 6.1, D(w^*) = 5.9, D(d^*) = 6.05, D(c^*) = 6.25.$
The earned profit is:

$$\begin{aligned} \mathbb{M}_1{}^{\wedge\star}(\mathcal{C}_\hbar) &= 147.685, \\ \mathbb{M}_2{}^{\wedge\star}(\mathcal{C}_\hbar) &= 147.623, \\ \mathbb{R}^{\wedge\star}(\mathcal{C}_\hbar) &= 1.623. \end{aligned} \tag{5.2}$$

**For Complete Graph**

Degrees of nodes are:
$D(j^*) = 9.8, D(w^*) = 9.6, D(d^*) = 9.9, D(c^*) = 9.3.$
The network yields profit:

$$\begin{aligned} \mathbb{M}_1{}^{\wedge\star}(\mathcal{K}_\hbar) &= 372.7, \\ \mathbb{M}_2{}^{\wedge\star}(\mathcal{K}_\hbar) &= 558.63, \\ \mathbb{R}^{\wedge\star}(\mathcal{K}_\hbar) &= 1.93. \end{aligned} \tag{5.3}$$

**For Star Graph**

Degrees of nodes are:
$D(j^*) = 3.1, D(w^*) = 2.9, D(d^*) = 3.15, D(c^*) = 9.15.$
The profit is:

$$\begin{aligned} \mathbb{M}_1{}^{\wedge\star}(\mathcal{S}_\hbar) &= 111.66, \\ \mathbb{M}_2{}^{\wedge\star}(\mathcal{S}_\hbar) &= 83.72, \\ \mathbb{R}^{\wedge\star}(\mathcal{S}_\hbar) &= 1.30. \end{aligned} \tag{5.4}$$

**For Wheel Graph**

$D(j^*) = 8.6, D(w^*) = 8.4, D(d^*) = 8.55, D(c^*) = 9.15.$
The profit is:

$$\begin{aligned} \mathbb{M}_1{}^{\wedge\star}(\mathcal{W}_\hbar) &= 301.35, \\ \mathbb{M}_2{}^{\wedge\star}(\mathcal{W}_\hbar) &= 451.37, \\ \mathbb{R}^{\wedge\star}(\mathcal{W}_\hbar) &= 2.03. \end{aligned} \tag{5.5}$$

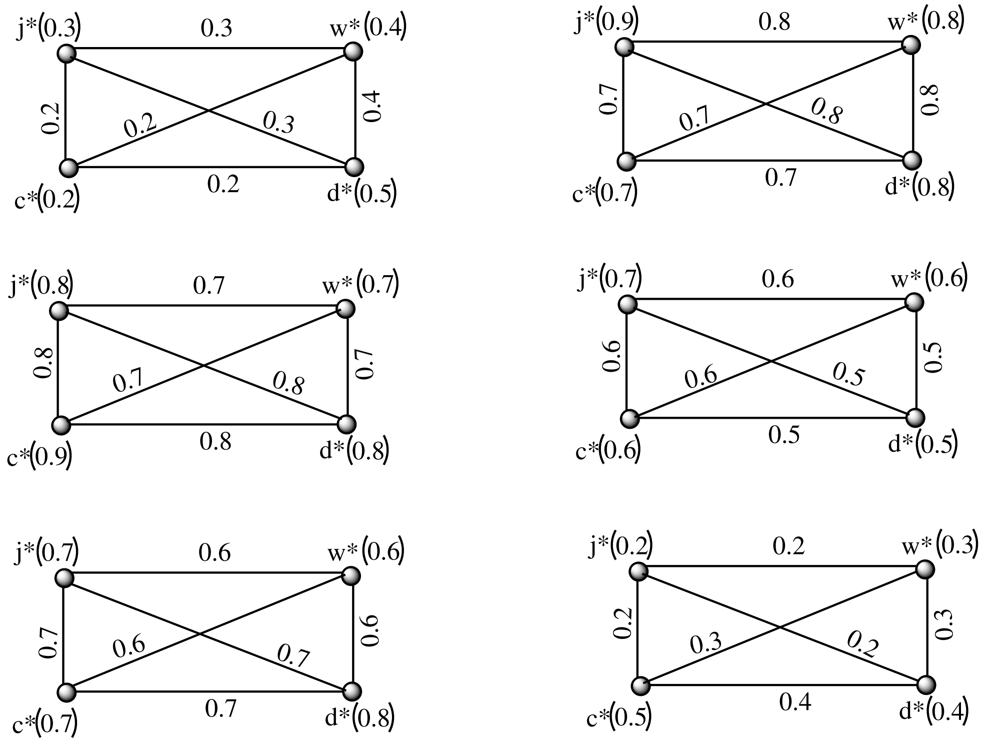


**Fig. 5.** Six fuzzy hypersoft complete sub-graphs.

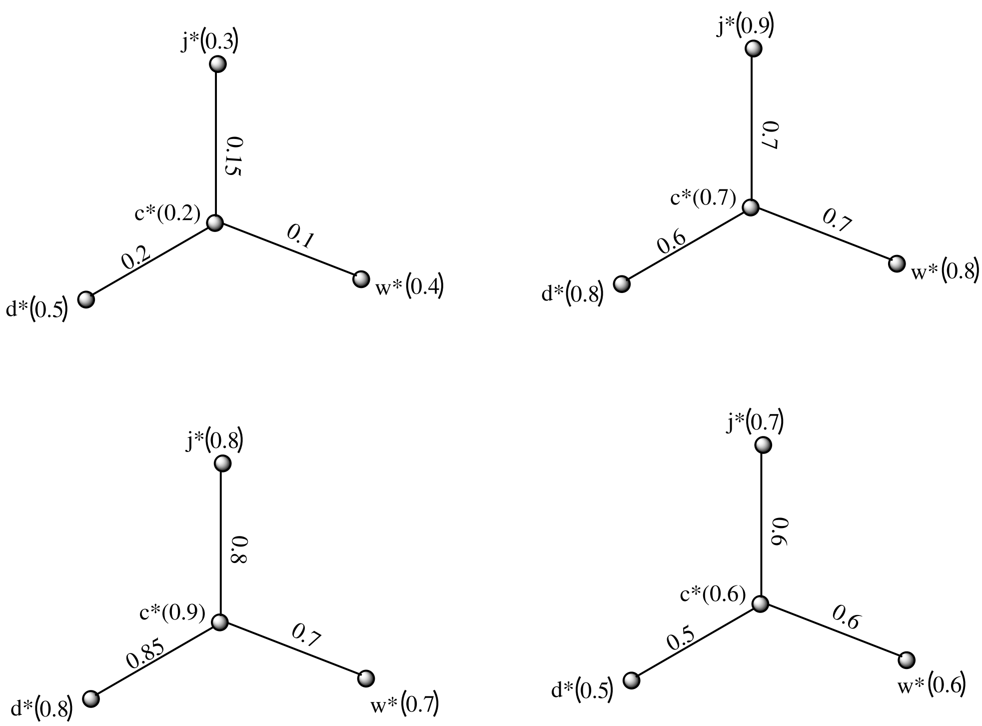


**Fig. 6.** 3D representation of fuzzy hypersoft complex subgraphs.

By Comparing the Results (5.1)–(5.5), we can analyze that complete graph network will produce the maximum profit. The ranking order with respect to the profit earned is $\mathcal{K}_\hbar > \mathcal{W}_\hbar > \mathcal{C}_\hbar > \mathcal{S}_\hbar > \mathcal{P}_\hbar$. Also second FHS Zagreb number is most efficient for the graphical networks yielding maximum profit, but in some cases first FHS Zagreb number is also efficient. Randić number yields very low profit in each case, so this number is not businesslike (see Figs. 6–8).

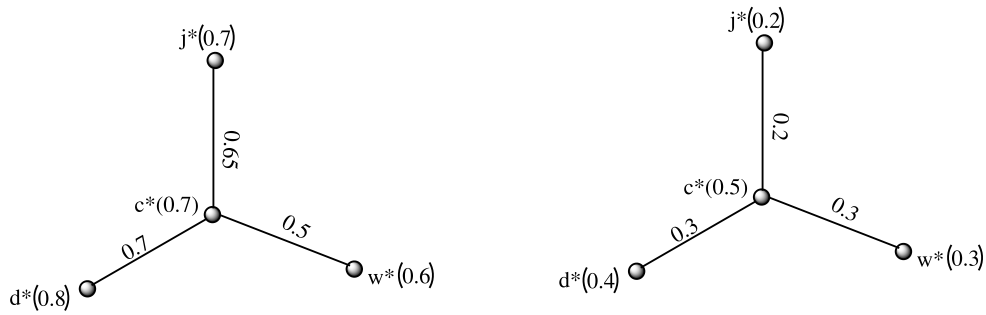


**Fig. 7.** Six fuzzy hypersoft star sub-graphs.

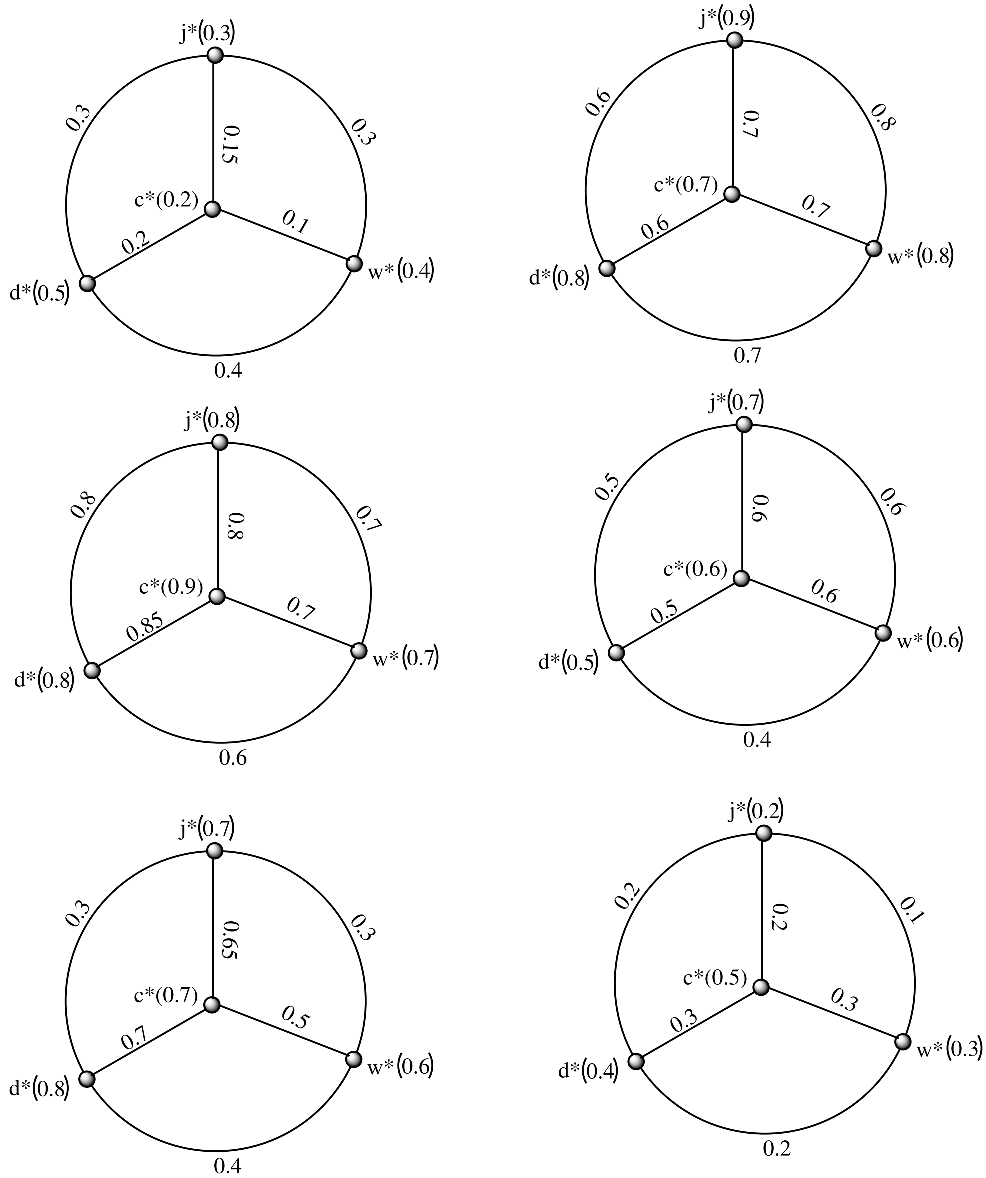


**Fig. 8.** Six fuzzy hypersoft wheel sub-graphs.

*The Vikor method:*

**Step1** : Table 10, is decision matrix. In above application graphical networks are alternatives and degrees of nodes are considered as criteria or decision makers.

**Table 6**
Decision matrix.

| Alternatives | D(j*) | D(w*) | D(d*) | D(c*) |
|---|---|---|---|---|
| Path | 3.0 | 5.9 | 6.05 | 3.15 |
| Cycle | 6.1 | 5.9 | 6.05 | 6.25 |
| Complete | 9.8 | 9.6 | 9.9 | 9.3 |
| Star | 3.1 | 2.9 | 3.15 | 9.15 |
| Wheel | 8.6 | 8.4 | 8.55 | 9.15 |

**Table 7**
Matrix of best and worst values.

| Alternatives | D(j*) | D(w*) | D(d*) | D(c*) |
|---|---|---|---|---|
| Best | 9.8 | 9.6 | 9.9 | 9.3 |
| Worst | 3.1 | 2.9 | 3.15 | 3.15 |

**Table 8**
Normalized matrix.

| Criteria | D(j*) | D(w*) | D(d*) | D(c*) |
|---|---|---|---|---|
| Path | 0.315224 | 0.173134 | 0.142593 | 0.25 |
| Cycle | 0.176716 | 0.160149 | 0.142593 | 0.123894 |
| Complete | 0 | 0 | 0 | 0 |
| Star | 0.32 | 0.29 | 0.25 | 0.0006098 |
| Wheel | 0.0537313 | 0.05194 | 0.05 | 0.0006098 |

**Table 9**
Ranking matrix.

| Criteria | Sj | Rj | Qj | Rank based on Qi |
|---|---|---|---|---|
| Path | 0.880951 | 0.315224 | 0.007463 | 5 |
| Cycle | 0.6034424 | 0.176716 | 0.381386 | 3 |
| Complete | 0 | 0 | 1 | 1 |
| Star | 0.866098 | 0.33 | 0.00843 | 4 |
| Wheel | 0.165351 | 0.057313 | 0.8166 | 2 |
| S*,R* | 0 | 0 | – | – |
| S-,R- | 0.880951 | 0.32 | – | – |

**Step2 :** In this step fuzzy best values and fuzzy worst values are determined from 10, as:

$$D_j^+ = \max_i D_{ij}^+,$$
$$D_j^- = \min_i D_{ij}^-.$$

These values are given in Table 7.

**Step3** In this step normalized fuzzy difference is calculated by applying formulae:

$$d_i = \mathfrak{W}_j \left\{ \frac{D_j^+ - D_{ij}}{D_j^+ - D_j^-} \right\},$$

where, $D_j^+$ and $D_j^-$ are fuzzy best and fuzzy worst values and $\mathfrak{W}_j$ are normalized weights. Applying this formula for all values of Table 10, with respect to best and worst values of Table 7, we get Table 8.

**Step4 :** In this step the values of $S_i$, $R_i$ and $Q_i$ would be calculated applying formulae:

$$S_i = \sum_{j=1}^{n} \left\{ \mathfrak{W}_j \frac{D_j^+ - D_{ij}}{D_j^+ - D_j^-} \right\},$$
$$R_i = \max_j \left\{ \mathfrak{W}_j \frac{D_j^+ - D_{ij}}{D_j^+ - D_j^-} \right\},$$

After that, $Q_i$ can be computed using the following formula:

$$Q_i = \eta \times \left( \frac{S_i - S^*}{S^- - S^*} \right) + (1-\eta) \times \left( \frac{R_i - R^*}{R^- - R^*} \right),$$

where, $S^* = \min_i \{S_i\}, S^- = \max_i \{S_i\}, R^* = \min_i \{R_i\}$ and $R^- = \max_i \{R_i\}$.

The strategy weight of the majority of criteria is denoted by $\eta$ in this case, such that $\eta \in [0,1]$, while the likelihood of each individual loss is $(1-\eta)$. In the end FHS alternatives are defuzzified by giving them ranks according to the values of $Q_i$. Higher the value of $Q_i$, greater will be the rank.

It is obvious from Table 9, that $\mathcal{K}_\hbar > \mathcal{W}_\hbar > \mathcal{C}_\hbar > \mathcal{S}_\hbar > \mathcal{P}_\hbar$.

In order to launch a cooperative venture, four Americans needed to interact with one another over graphical networks. Prior to launching their company, they aimed to do a mathematical analysis of the results to determine which network would make the most money. FHS topological numbers and MCDM VIKOR were taken into consideration as decision-makers in this case study. Both decision-makers come to the conclusion that the optimum option for maximizing profit would be to create a complete graph network between the four partners.

**Step5 :** Fig. 9, is the graphical representation of five networks with respect to the degree of their nodes.

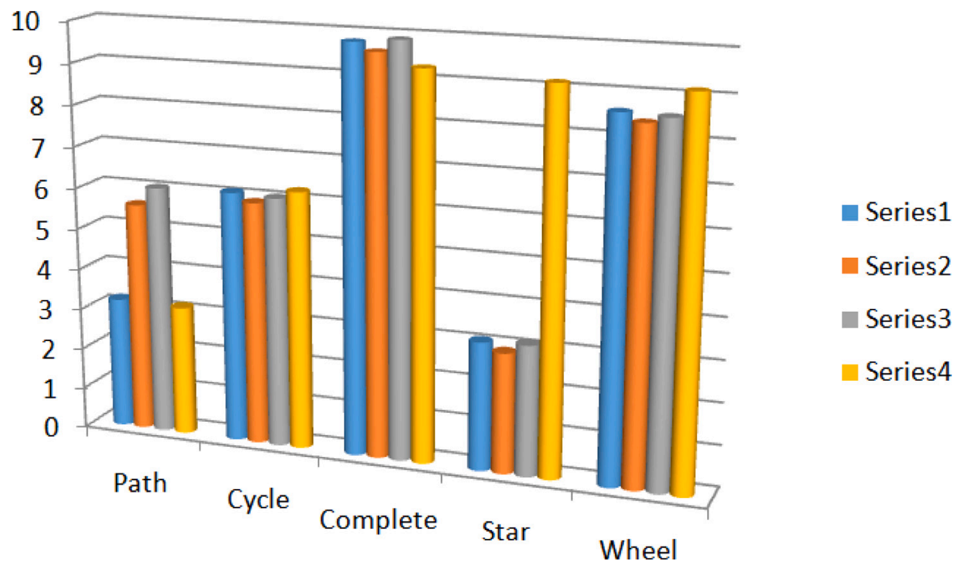


**Fig. 9.** Graphical behavior of networks.

**Table 10**
Decision matrix.

| Alternatives | D(j*) | D(w*) | D(d*) | D(c*) |
|---|---|---|---|---|
| Cycle | 1.00 | 0.96 | 0.99 | 1.02 |
| Complete | 1.02 | 1.00 | 1.03 | 0.96 |
| Wheel | 1.00 | 0.98 | 1.00 | 1.07 |
| Star | 0.34 | 0.32 | 0.34 | 1.00 |
| | 3.36 | 3.26 | 3.36 | 4.05 |

**Table 11**
Normalized matrix and criteria weights.

| Normalized matrix | | | | Criteria weights |
|---|---|---|---|---|
| 0.2976 | 0.2945 | 0.2945 | 0.2518 | 0.2846 |
| 0.3035 | 0.30674 | 0.30654 | 0.2370 | 0.2884 |
| 0.2976 | 0.3006 | 0.2976 | 0.2641 | 0.2900 |
| 0.1011 | 0.0982 | 0.1011 | 0.2469 | 0.1368 |

**Table 12**
Pair-wise comparison matrix weighted sum value and eigen value $\lambda$.

| Pair-wise comparison matrix | | | | W.sum values | Lambda |
|---|---|---|---|---|---|
| 0.2846 | 0.2769 | 0.2871 | 0.1396 | 0.9882 | 3.472 |
| 0.2903 | 0.2884 | 0.2987 | 0.1313 | 1.0089 | 3.4974 |
| 0.2846 | 0.2827 | 0.2900 | 0.1464 | 1.0038 | 3.4612 |
| 0.0967 | 0.0923 | 0.0986 | 0.1368 | 0.4245 | 3.1020 |

### *5.1. Comparison of VIKOR with AHP method:*

**Step1** : Table 10, is decision matrix. In above application graphical networks are alternatives and degrees of nodes are considered as criteria or decision makers.

**Step2: Construction of normalized pair-wise matrix:**

Each element of normalized matrix is obtained by dividing all the elements of every column by sum of column, i.e applying formula $\frac{D_{ij}}{\sum_1^n D_{ij}}$. Then calculate criteria weights applying formula $\mathfrak{W}_i = \frac{1}{n}\sum_1^n \frac{D_{ij}}{\sum_1^n D_{ij}}$ Table 11, provides normalized matrix and criteria weights are also computed in the same table.

**Step3: Construction of Pair-wise comparison matrix:**

Pair-wise comparison matrix is obtained after multiplying each element in decision matrix with criteria weights, i.e $D_{ij} \times \mathfrak{W}_i$. Then, weighted sum values are obtained by adding all entries in the respective rows. In the $\lambda$ is calculated by dividing criteria weights with weighted sum values.

From Table 12, $\lambda_{max} = 3.3831$.

**Step5: Calculation of consistency index:**

$CI = \frac{\lambda_{max}-m}{m-1}$,

$\Longrightarrow CI = -0.205$.

**Step6: Calculation of consistency ratio:**

AS, $CR = \frac{CI}{RI}$, and $RI = 0.9$ for $m = 4$.

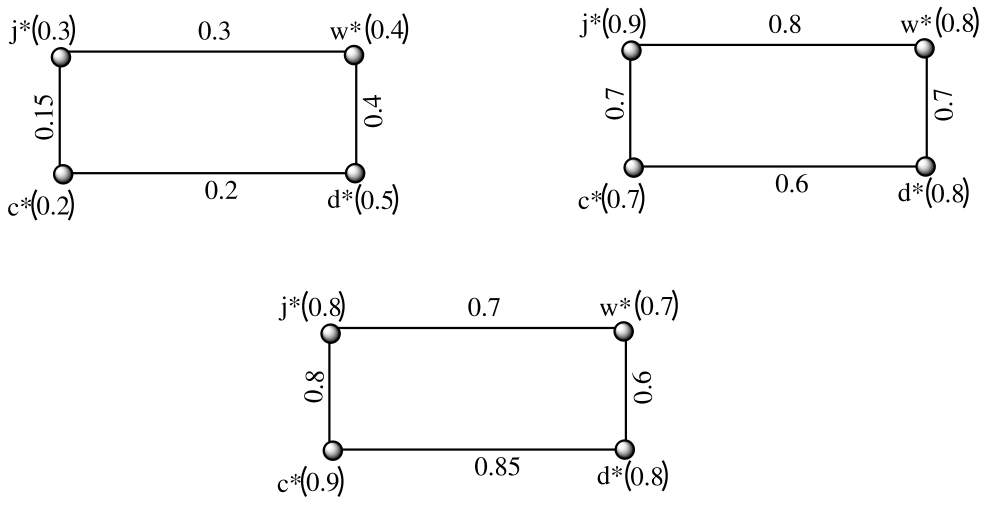


**Fig. 10.** Three fuzzy soft cycle sub-graphs.

$\implies CR = -0.22.$

As, $CR < 0.1$. So, AHP is consistent. The graphical networks can be ranked on the basis of values of $\lambda$. Higher the value of $\lambda$, maximum will be the profit yielded by that decision maker (graphical network).

$$\mathcal{K}_\hbar > \mathcal{W}_\hbar > \mathcal{C}_\hbar > \mathcal{S}_\hbar$$

**Note:** One drawback of AHP is that its matrix must always be square. Since each graph contains four vertices (criteria), we must take into account four graphical networks (alternatives) in order to meet this requirement, which requires us to omit the path graph network from the comparison. All remaining networks follow the same order of preference.

### 5.2. Comparison of Fuzzy hypersoft with Fuzzy soft graph

In order to calculate profit by considering any graphical network in fuzzy soft environment, only one set of parameters can be chosen. Suppose it be $\mathfrak{R}_1 = \{\mathfrak{f}_{11} = passion, \mathfrak{f}_{12} = budget, \mathfrak{f}_{13} = finance\}$. Then three sub-graphs corresponding to three parameters would be drawn. Which are given in Fig. 10.

Now we will calculate the degree of each node by applying the definition of node degree in fuzzy soft environment. Degrees of nodes are: $D(j^*) = 3.45, D(w^*) = 3.5, D(d^*) = 3.35, D(c^*) = 3.3.$

The earned profit is:

$$\begin{aligned} \mathbb{M}_1(C_\hbar) &= 46.26, \\ \mathbb{M}_2(C_\hbar) &= 46.24, \\ \mathbb{R}(C_\hbar) &= 2.16. \end{aligned} \tag{5.6}$$

By Comparing set of Eqs. (5.2) and (5.6), it is obvious that FHS graph is much more efficient as compare to the FS graph, and this is only due to the fact that FHS graph can handle multiple parameters and sub-attributes. Whereas, FS graph handles only single set of parameters.

### 5.3. Application in energy system and network optimization

We have demonstrated the above network optimization application using the VIKOR approach in addition to FHS topological numbers. The outcomes of both are the same. Since any MCDM technique is being used for the first time in a FHS environment, our findings will be crucial to the development of network optimization techniques. This is a significant step towards the application of numerous additional MCDM techniques in FHS and other fuzzy graph environments. Also, in energy systems, FHS approaches can be used to control imprecision and uncertainty in decision-making processes, especially in the selection and administration of renewable energy sources, enabling more flexible and reliable assessments.

- FHS graphs and VIKOR techniques, which optimize energy distribution, improve grid resilience, and manage the unpredictability of renewable energy sources, can be created using FHSS.
- In microgrids, FHS graphs and the VIKOR technique can be utilized to optimize demand-side management and control energy flow, enhancing grid stability and energy efficiency.
- A system can use FHS graphs and the VIKOR technique to manage the output of renewable energy sources, such as solar and wind, to ensure a stable power supply.

*Simulation environment*

To evaluate the effectiveness of the proposed approach, we conducted simulations using a well-defined computational setup.

- **Software and Programming Tools:** We have used only Excel software, but MATLAB and Python can also be used for the complex computations of FHS graphs.
- **Sampling Size:** Since our study is based on assumed values rather than real-world data, we designed a synthetic dataset with 5 criteria and 4 alternatives to evaluate the model's effectiveness.
- **Solver Type and Computational Setup:** The VIKOR calculations and fuzzy hypersoft computations were executed using a custom-built numerical solver in an Excel sheet.

### 5.4. Bridging of hypersoft topology with MCDM

Fuzzy graphs allow for incomplete or ambiguous connections by representing relationships between items with varying degrees of membership. By adding a parameterization tool, fuzzy soft graphs allow for the modeling of ambiguity and vagueness in practical situations. By enabling many parameters and multi-argument approximation mapping, this method improves fuzzy soft graphs and overcomes their shortcomings in managing complicated, multi-attribute data. Fuzzy and fuzzy soft graphs can struggle to model data with multiple attributes or complex relationships, where each attribute has its own level of fuzziness. Although fuzzy soft graphs introduce the idea of parameterization, they may still be inadequate for handling a large number of parameters. In contrast, fuzzy graphs do not provide a direct mechanism for parameterization. When an approximate mapping must take into account several variables or parameters at once, fuzzy and fuzzy soft graphs might not be the best choice. Through the use of numerous parameters and multi-argument approximation mapping, FHS graphs provide more flexibility and dependability for representing complicated, multi-attribute data. So, topological numbers defined for the FHS graph will offer a quantifiable measure of network properties, making decision-making more data-driven and interpretable. Similarly, integrating FHS graphs with MCDM techniques like VIKOR provides a systematic approach to ranking alternatives while handling higher-dimensional uncertainty. Also, algorithms of decision-making exist in fuzzy set theory but do not exist in graph theory. So, we cannot introduce them in graph theory; that is why, as a natural extension, we consider different environments of fuzzy graphs. One of our published articles on decision-making algorithms is [58].

### 5.5. Comparative analysis

A comparison analysis of proposed work with other environments of fuzzy graphs is provided in Table 13. When there are multi-attributed structures, previously published models in the literature are unable to handle some attributes using fuzzy parameters like membership and non-membership functions. Then, in order to address these, Hypersoft structures that are capable of resolving these kinds of problems are used, the hybrid structure that is suggested spans the entire range of features, enabling in-depth and comprehensive examination.

### 5.6. Sensitivity analysis

The purpose of fuzzy hypersoft graphs is to handle ambiguities, vagueness, and uncertainty. Sensitivity analysis evaluates how variations in input parameters influence the outcomes of a model. In this study, we have analyzed the impact of different parameters on topological numbers and the VIKOR algorithm in the FHS graph framework. A fuzzy hypersoft environment is dependent on a number of parameters, some of which are important and whose values might significantly affect the intended outcomes. FHS graphs and sensitivity analysis enable a more adaptable and sensitive review of data. It is possible to identify the input parameters that have the greatest influence on the output through sensitivity analysis. It is possible to observe changes in the results by adjusting the values of various parameters and setting their ranges. Better decision-making and resource allocation are made possible by sensitivity analysis, which can assist in identifying the most important aspects of implementing new technology.

As the results obtained by FHS graphs are consistent with the results yielded by the VIKOR algorithm, so changes in parameters will also influence the VIKOR algorithm's output **Computational Complexity and Sensitivity:**

- The proposed method has higher computational complexity due to fuzzy hypersoft properties.
- Increasing the number of alternatives and attributes exponentially increases processing time.

### 5.7. Conclusion

Starting a new business, especially in partnership and across different cities, carries considerable risk when procedures are not aligned. It is therefore essential to explore all potential opportunities and risks before entering the business world. Graph theory has consistently served as a valuable tool to support complex choices. Various fuzzy frameworks: such as fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic fuzzy graphs, fuzzy soft graphs, and fuzzy hypersoft graphs are useful for representing structured relationships. The fuzzy hypersoft graph is the most general among these, as it is based on the Cartesian product of separate sets of attribute values related to distinct features. A topological number is a numerical descriptor of a graph's structure and can be applied to problems that involve alternatives and criteria.

When comparing multiple options based on several criteria, methods like VIKOR are useful. VIKOR uses closeness to an ideal option to suggest a compromise that benefits the group overall while reducing individual disadvantage. In this work, we extend fuzzy soft graphs, where uncertainty in connections is handled using fuzzy membership to the more general fuzzy hypersoft graphs, which allow combinations of multiple parameters and reflect more complex attribute relationships.

Using this broader framework, we derived formulas for three topological numbers for various types of graphs. We then applied these formulas to a real-world situation involving four partners planning to open branches in four separate U.S. cities. Each possible business arrangement is represented by a different fuzzy hypersoft graph. The derived topological numbers help evaluate the expected outcome. Among all graph types,

**Table 13**
Comparison between different environments of fuzzy graph.

| Aspect | Fuzzy graph | Intuitionistic fuzzy graph | Neutrosophic fuzzy graph | Fuzzy soft graph | Fuzzy hypersoft graph |
|---|---|---|---|---|---|
| Definition | Nodes and edges have membership values between 0 and 1 [76]. | Considers both membership and non-membership degrees [77]. | Adds truth, indeterminacy, and falsity membership values ranging between 0 and 3 [78]. | Combines fuzzy graph with soft sets [79]. | Extends fuzzy soft graph by adding multi-attributes to parameters [9]. |
| Membership function | Single function for edges/nodes. | Membership and non-membership functions. | Truth, indeterminacy, and falsity functions. | Based on parameterized soft sets. | Multi-membership function with attributes of parameters. |
| Degree of uncertainty | Moderate, based on fuzzy membership. | Higher with both membership and non-membership. | Maximum, including indeterminacy. | Linked to fuzziness in parameters. | More uncertainty due to multiple attributes. |
| Computational complexity | Simple and take fewer minutes in calculations. | More complex with dual membership and some additional time is required in computation. | Very complex due to triple values. | More complex because of parameters, and take much time in computation, as number of sub-graphs are equal to parameters. | Most complex, with multi-attributes, in which number of sub-graphs are equal to the cardinality of multi attributed set, which is the cartesian product of the number of parameters in each attributed set. |
| Applications | Networks, decision-making, pattern recognition. | Improves decision-making with hesitation. | Suitable for indeterminate, vague scenarios. | Useful for soft set theory applications like medical diagnosis. | Complex decision-making, e.g., multi-criteria evaluation. |
| Handling of hesitation | Not handled. | Managed through non-membership function. | Explicitly handles hesitation and indeterminacy. | Managed indirectly by soft set parameters. | Managed through multi-attribute parameters. |
| Edge/node Representation | Single degree of fuzziness for edges and nodes. | Edges/nodes have membership and non-membership degrees. | Edges/nodes have truth, indeterminacy, and falsity. | Soft sets control relationships. | Multi-attribute soft sets govern relationships. |
| Key features | Simplifies uncertain relationships. | Handles opposition and hesitation. | Manages uncertainty, opposition, indeterminacy. | Adds soft set flexibility. | Introduces multi-attribute flexibility. |

the complete graph setup proved most beneficial, as it allows full connection among partners. In this context, the second fuzzy Zagreb number produced the highest profit estimate, while the Randić number showed the lowest.

We also used the VIKOR method to choose the best graph structure. Interestingly, both the VIKOR method and the topological numbers provided the same preference order. In addition, we applied the Analytic Hierarchy Process (AHP) within the fuzzy hypersoft setting. Due to AHP's limitations, one graph option had to be excluded. Still, both the VIKOR method and the topological numbers pointed to the same best choice among the remaining options. Lastly, we compared outcomes between fuzzy soft and fuzzy hypersoft approaches, showing the greater effectiveness of the hypersoft version.

### 5.8. Future directions

Researchers who are interested in working in an FHS setting can get an understanding from the literature [50,51]. These articles can provide directions to beginners, for working in diverse FHS contexts, such as intuitionistic FHS, neutrosophic FHS, and interval-valued FHS, etc. This study can serve as a landmark for the introduction of various topological numbers for various graph families. A deeper comparison with TOPSIS, ELECTRE, or fuzzy DEMATEL would provide further validation and highlight the strengths and limitations of our proposed model. While we presented a case in the business industry, future research could apply this approach to energy modeling, power distribution networks, and smart grid optimization, where uncertainty plays a crucial role. Integrating our fuzzy hypersoft VIKOR approach with machine learning or metaheuristic algorithms could enhance decision-making in large-scale and high-dimensional problems. FHS graphical networks can also be used in other fields, like risk analysis, energy production warehouse selection optimization, and graphical network modeling of uncertainties. As a result, its practical uses extend beyond optimizing business networks. Our study uses assumed values, but applying this approach to real-world datasets from industries like finance, supply chain, or healthcare would further demonstrate its effectiveness.

### 5.9. Limitations of proposed work

Despite their strength in managing uncertainty and intricate relationships, fuzzy hypersoft graphs can be limited by their computational complexity and requirement for specialized knowledge; as the size of the network increases, the computational burden grows significantly, which may limit its applicability to large-scale real-world problems. The results depend on parameter selection, which may impact its stability and reliability. Algorithms designed for fuzzy hypersoft graphs require powerful processing techniques and strong memory. It often requires domain expertise to accurately present vagueness and uncertainty in data. Subjective decisions about the parameters, membership functions, and aggregation operators may result in various interpretations and conclusions. To get accurate and significant findings, these parameters must be carefully chosen and adjusted. Additionally, a great deal of caution and a strong processing method is needed when using any decision algorithm in a fuzzy hypersoft environment.

## CRediT authorship contribution statement

**Shabana Anwar:** Writing – original draft, Validation, Software, Methodology, Conceptualization. **Muhammad Kamran Jamil:** Writing – original draft, Software, Resources, Methodology, Investigation, Conceptualization. **Hothefa Shaker Jassim:** Formal analysis, Conceptualization, Writing – review & editing, Visualization, Software, Methodology, Investigation. **Muhammad Azeem:** Writing – original draft, Visualization, Investigation, Data curation, Conceptualization. **Bandar Almohsen:** Writing – review & editing, Resources, Investigation, Formal analysis, Data curation. **Husam A. Neamah:** Writing – review & editing, Supervision, Software, Methodology, Funding acquisition, Formal analysis, Conceptualization.

## Consent for publication

All authors have reviewed and approved the final manuscript and consent to its submission to the journal.

## Ethics statement

Not applicable. This study does not involve human participants, animals, or data from social media.

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## Declaration of competing interest

The authors have no relevant financial or non-financial interests to disclose.
We confirm that this work is original and has not been published elsewhere, nor is it currently under consideration for publication elsewhere.

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## Data availability

All data generated or analyzed during this study are included in this published article. Additional materials may be available upon request from the corresponding author.

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