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AN INEQUALITY FOR THE TAKAGI FUNCTION

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(*communicated by Z. Daróczy*)

Abstract. The well-known Takagi function $T(x) = \sum_{k=0}^{\infty} 2^{-k}\mathrm{dist}\,(x,\mathbb{Z})$ plays a crucial role in the theory of approximately convex functions. In order to establish the sharpness of some Bernstein–Doetsch type results for approximate convexity, we prove that the Takagi function fulfils the inequality

$$T\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\left(T(x)+T(y)+|x-y|\right)$$

for all real numbers x and y.

1. Introduction

Let, for every $x \in \mathbb{R}$, $d(x) = \mathrm{dist}\,(x,\mathbb{Z})$, and

$$T(x) = \sum_{k=0}^{\infty} \frac{d(2^k x)}{2^k} \tag{1}$$

(where, throughout this paper, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of real numbers, integers, and positive integers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathrm{dist}\,(x,\mathbb{Z}) = \inf\{\,|x-s| \;:\; s \in \mathbb{Z}\,\}$). Functions of this type have been investigated by several authors (e.g. [3], [7], and [10]) as convenient examples for continuous nowhere differentiable functions. In particular, function T is usually cited as "van der Waerden's function" (e.g. [1], [2]). However, as it was also mentioned by Knopp [7], function T had been constructed earlier by Takagi [9] on the interval $[0,1]$ in a somewhat different way. Namely, Takagi determined $T(x)$ with the aid of the dyadic expansion of x. It seems therefore historically correct to call T the Takagi function. More historical and mathematical details can be found, for instance, in Kairies' paper [6].

Recently, Házy and Páles have discovered that the Takagi function plays a specific role in the theory of approximately convex functions. Namely, in order to extend the celebrated theorem of Bernstein and Doetsch for approximately midconvex functions, they have proved the following result [4, Theorem 4].

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THEOREM A. *Let X be a normed linear space and let $D \subset X$ be an open convex set. Suppose that ε and δ are nonnegative real numbers, $f : D \to \mathbb{R}$ fulfils the inequality*

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + \delta + \varepsilon|x-y| \tag{2}$$

for every $x, y \in D$, and f is locally bounded from above at a point of D. Then

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + 2\delta + 2\varepsilon T(\lambda)|x-y| \tag{3}$$

holds for all $x, y \in D$ and $\lambda \in [0,1]$.

The authors also investigate but do not completely justify the optimality of the coefficient $T(\lambda)$ in the inequality (3). Zsolt Páles observed that this problem can be reduced to the verification of the inequality

$$T\left(\frac{x+y}{2}\right) \leqslant \frac{T(x)+T(y)}{2} + \frac{1}{2}|x-y| \tag{4}$$

for every $x, y \in \mathbb{R}$. Namely, if (4) holds, then Theorem A can be applied with $f = T$, $\delta = 0$, and $\varepsilon = \frac{1}{2}$. Substituting these values with $x = 1$ and $y = 0$ into (3), we obtain $T(\lambda)$ on both sides of the inequality. Therefore, if the Takagi function T fulfils the inequality (4) identically, then the coefficient $T(\lambda)$ in (3) cannot be replaced with a smaller one for any $\lambda \in [0,1]$.

The aim of this note is to prove inequality (4), which was presented in 2003 by Páles as a conjecture [8].

We note that analogous questions arise for a two-parameter family of Takagi type functions when we consider a generalization of Theorem A (see [5]).

2. Basic properties

For every $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, let

$$\phi_n(x) = \frac{d(2^n x)}{2^n} \quad \text{and} \quad T_n(x) = \sum_{k=0}^{n} \phi_k(x). \tag{5}$$

Obviously,

$$T(x) = \lim_{n\to\infty} T_n(x) \tag{6}$$

for every $x \in \mathbb{R}$. Moreover, each T_n and also T is continuous. Our main idea is that first we prove a discrete version of inequality (4) for the functions T_n by induction on n. We support our arguments by encountering some basic properties of these functions.

REMARK. Clearly, $T_0 = d$, and

$$T_n(x) = T_{n-1}(x) + \phi_n(x) \tag{7}$$

holds for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

This recursion is also shown by Figure 1.

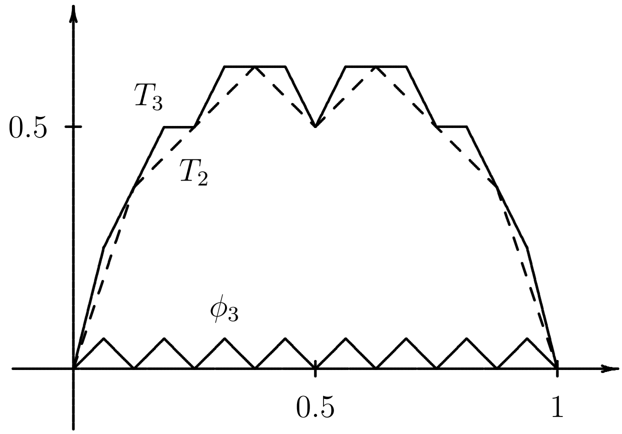

FIGURE 1. *The restrictions of the functions* T_2 , ϕ_3 , *and* $T_3 = T_2 + \phi_3$ *to the interval* $[0,1]$.

LEMMA 2.1. *For every* $n \in \mathbb{N}_0$ *and* $x, y \in \mathbb{R}$ *we have*

$$|T_n(x) - T_n(y)| \leqslant (n+1)|x-y|. \tag{8}$$

Proof. Clearly, the definition of the function d yields $|d(x) - d(y)| \leqslant |x-y|$ for all $x, y \in \mathbb{R}$. Hence, by (5) we have

$$|\phi_n(x) - \phi_n(y)| = \frac{1}{2^n}\,|d(2^n x) - d(2^n y)| \leqslant \frac{1}{2^n}\,|2^n x - 2^n y| = |x-y|$$

and

$$|T_n(x) - T_n(y)| \leqslant \sum_{k=0}^{n} |\phi_k(x) - \phi_k(y)| \leqslant (n+1)|x-y|$$

for every $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$. □

In order to describe the local shape of several functions in consideration, we call a function $f\ :\ \mathbb{R} \to \mathbb{R}$ *affine on the interval* I if there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for every $x \in I$.

PROPOSITION 2.2. *If* $n \in \mathbb{N}_0$ *and* $k \in \mathbb{Z}$ *, then* T_n *is affine on the interval* $\left[\frac{k}{2^{n+1}}\,,\,\frac{k+1}{2^{n+1}}\right]$.

Proof. The statement can be easily verified by induction on n. Obviously, $d(x) = x$ for $x \in [0, \frac{1}{2}]$, $d(x) = 1 - x$ for $x \in [\frac{1}{2}, 1]$, and $d(x+1) = d(x)$ for every $x \in \mathbb{R}$. Thus $T_0 = d$ is affine on the interval $\left[\frac{k}{2}\,,\,\frac{k+1}{2}\right]$ for all $k \in \mathbb{Z}$. Now let us assume that $n \in \mathbb{N}$ such that T_{n-1} is affine on the interval $\left[\frac{l}{2^n}\,,\,\frac{l+1}{2^n}\right]$ for all $l \in \mathbb{Z}$. It follows from (5) and our above observation that, for every $k \in \mathbb{Z}$, the function ϕ_n is affine on the interval $I_{n,k} = \left[\frac{k}{2^{n+1}}\,,\,\frac{k+1}{2^{n+1}}\right]$. Hence, $T_n = T_{n-1} + \phi_n$ is also affine on $I_{n,k}$. □

LEMMA 2.3. *If* $n \in \mathbb{N}$ *and* $s \in \mathbb{Z}$ *, then*

$$T_n\left(\frac{s}{2^n}\right) = T_{n-1}\left(\frac{s}{2^n}\right) \tag{9}$$

and

$$2\,T_n\left(\frac{s}{2^n} + \frac{1}{2^{n+1}}\right) - T_n\left(\frac{s}{2^n}\right) - T_n\left(\frac{s+1}{2^n}\right) = \frac{1}{2^n}\,. \tag{10}$$

Proof. Both identities can be obtained from the recursion (7) if we observe, additionally, that

$$\phi_n\left(\frac{s}{2^n}\right) = \frac{1}{2^n}d(s) = 0\,,$$

similarly, $\phi_n\left(\frac{s+1}{2^n}\right) = 0$,

$$\phi_n\left(\frac{s}{2^n} + \frac{1}{2^{n+1}}\right) = \frac{1}{2^n}d\left(s + \frac{1}{2}\right) = \frac{1}{2^{n+1}}\,,$$

and

$$2\,T_{n-1}\left(\frac{s}{2^n} + \frac{1}{2^{n+1}}\right) - T_{n-1}\left(\frac{s}{2^n}\right) - T_{n-1}\left(\frac{s+1}{2^n}\right) = 0\,,$$

since T_{n-1} is affine on the interval $\left[\frac{s}{2^n}\,,\ \frac{s+1}{2^n}\right]$. □

3. Inequalities

THEOREM 3.1. *If* $k\,,\, n \in \mathbb{N}_0$ *,* $p \in \mathbb{Z}$ *, and* $q \in \{\,p\,,\, p+1\,\}$*, then*

$$T_n\left(\frac{p}{2^{n+1}}\right) + T_n\left(\frac{q}{2^{n+1}}\right) - T_n\left(\frac{p-k}{2^{n+1}}\right) - T_n\left(\frac{q+k}{2^{n+1}}\right) \leqslant \frac{k}{2^n}\,. \tag{11}$$

Proof. Clearly, if $k = 0$, then both sides of the inequality (11) equal zero. Therefore, we have to show the validity of (11) in the case when k is a positive integer. We proceed with induction on n. In order to verify (11) for $n = 0$, we observe that all values of the function $T_0 = d$ belong to the interval $[0, \frac{1}{2}]$, hence

$$\begin{aligned} &T_0\left(\tfrac{p}{2}\right) + T_0\left(\tfrac{q}{2}\right) - T_0\left(\tfrac{p-k}{2}\right) - T_0\left(\tfrac{q+k}{2}\right) \\ &\quad = \left(d\left(\tfrac{p}{2}\right) - d\left(\tfrac{p-k}{2}\right)\right) + \left(d\left(\tfrac{q}{2}\right) - d\left(\tfrac{q+k}{2}\right)\right) \leqslant \tfrac{1}{2} + \tfrac{1}{2} = 1 \leqslant k\,. \end{aligned}$$

In the next step we assume that $n \geqslant 1$ and the inequality

$$T_{n-1}\left(\frac{p_0}{2^n}\right) + T_{n-1}\left(\frac{q_0}{2^n}\right) - T_{n-1}\left(\frac{p_0-k_0}{2^n}\right) - T_{n-1}\left(\frac{q_0+k_0}{2^n}\right) \leqslant \frac{k_0}{2^{n-1}}\,. \tag{12}$$

holds for all $k_0 \in \mathbb{N}_0$, $p_0 \in \mathbb{Z}$, and $q_0 \in \{\,p_0\,,\, p_0+1\,\}$. Using this assumption, we have to prove that (11) is fulfilled for every $k \in \mathbb{N}$, $p \in \mathbb{Z}$, and $q \in \{\,p\,,\, p+1\,\}$. We distinguish eight cases according to the evenness of these parameters.

Case I: $p=2r$, $q=p$, and $k=2j$ for some $r\in\mathbb{Z}$ and $j\in\mathbb{N}$. Applying (9) and (12) we obtain

$$\begin{aligned}&2\,T_n\left(\tfrac{2r}{2^{n+1}}\right)-T_n\left(\tfrac{2r-2j}{2^{n+1}}\right)-T_n\left(\tfrac{2r+2j}{2^{n+1}}\right)\\&\qquad=2\,T_{n-1}\left(\tfrac{r}{2^n}\right)-T_{n-1}\left(\tfrac{r-j}{2^n}\right)-T_{n-1}\left(\tfrac{r+j}{2^n}\right)\leqslant\tfrac{j}{2^{n-1}}=\tfrac{2j}{2^n}=\tfrac{k}{2^n}\,.\end{aligned}$$

In the following seven cases we replace each term of the form $T_n\left(\frac{2s+1}{2^{n+1}}\right)$, where s denotes some integer, with the arithmetic mean of $T_n\left(\frac{s}{2^n}\right)$ and $T_n\left(\frac{s+1}{2^n}\right)$, adding also the error given by (10). Then (9) allows us to apply the inequality (12).

Case II: $p=2r$, $q=p+1$, and $k=2j$ for some $r\in\mathbb{Z}$ and $j\in\mathbb{N}$. Then we have

$$\begin{aligned}&T_n\left(\frac{2r}{2^{n+1}}\right)+T_n\left(\frac{2r+1}{2^{n+1}}\right)-T_n\left(\frac{2r-2j}{2^{n+1}}\right)-T_n\left(\frac{2r+1+2j}{2^{n+1}}\right)\\&\quad=\frac12\left(2\,T_{n-1}\left(\frac{r}{2^n}\right)-T_{n-1}\left(\frac{r-j}{2^n}\right)-T_{n-1}\left(\frac{r+j}{2^n}\right)\right)\\&\qquad+\frac12\left(T_{n-1}\left(\frac{r}{2^n}\right)+T_{n-1}\left(\frac{r+1}{2^n}\right)-T_{n-1}\left(\frac{r-j}{2^n}\right)-T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right)\\&\qquad+\frac12\left(2\,T_n\left(\frac{r}{2^n}+\frac{1}{2^{n+1}}\right)-T_n\left(\frac{r}{2^n}\right)-T_n\left(\frac{r+1}{2^n}\right)\right)\\&\qquad+\frac12\left(T_n\left(\frac{r+j}{2^n}\right)+T_n\left(\frac{r+j+1}{2^n}\right)-2\,T_n\left(\frac{r+j}{2^n}+\frac{1}{2^{n+1}}\right)\right)\\&\quad\leqslant\frac12\cdot\frac{j}{2^{n-1}}+\frac12\cdot\frac{j}{2^{n-1}}+\frac12\cdot\frac{1}{2^n}+\frac12\left(-\frac{1}{2^n}\right)=\frac{2j}{2^n}=\frac{k}{2^n}\,.\end{aligned}$$

Case III: $p=2r+1$, $q=p$, and $k=2j$ for some $r\in\mathbb{Z}$ and $j\in\mathbb{N}$. Then we have

$$\begin{aligned}&2\,T_n\left(\frac{2r+1}{2^{n+1}}\right)-T_n\left(\frac{2r+1-2j}{2^{n+1}}\right)-T_n\left(\frac{2r+1+2j}{2^{n+1}}\right)\\&\quad=\frac12\left(2\,T_{n-1}\left(\frac{r}{2^n}\right)-T_{n-1}\left(\frac{r-j}{2^n}\right)-T_{n-1}\left(\frac{r+j}{2^n}\right)\right)\\&\qquad+\frac12\left(2\,T_{n-1}\left(\frac{r+1}{2^n}\right)-T_{n-1}\left(\frac{r+1-j}{2^n}\right)-T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right)\\&\qquad+\left(2\,T_n\left(\frac{r}{2^n}+\frac{1}{2^{n+1}}\right)-T_n\left(\frac{r}{2^n}\right)-T_n\left(\frac{r+1}{2^n}\right)\right)\\&\qquad+\frac12\left(T_n\left(\frac{r-j}{2^n}\right)+T_n\left(\frac{r-j+1}{2^n}\right)-2\,T_n\left(\frac{r-j}{2^n}+\frac{1}{2^{n+1}}\right)\right)\\&\qquad+\frac12\left(T_n\left(\frac{r+j}{2^n}\right)+T_n\left(\frac{r+j+1}{2^n}\right)-2\,T_n\left(\frac{r+j}{2^n}+\frac{1}{2^{n+1}}\right)\right)\\&\quad\leqslant\frac12\cdot\frac{j}{2^{n-1}}+\frac12\cdot\frac{j}{2^{n-1}}+\frac{1}{2^n}+\frac12\left(-\frac{1}{2^n}\right)+\frac12\left(-\frac{1}{2^n}\right)=\frac{2j}{2^n}=\frac{k}{2^n}\,.\end{aligned}$$

Case IV: $p = 2r + 1$, $q = p + 1$, and $k = 2j$ for some $r \in \mathbb{Z}$ and $j \in \mathbb{N}$. Then we have

$$\begin{aligned}
&T_n\left(\frac{2r+1}{2^{n+1}}\right) + T_n\left(\frac{2r+2}{2^{n+1}}\right) - T_n\left(\frac{2r+1-2j}{2^{n+1}}\right) - T_n\left(\frac{2r+2+2j}{2^{n+1}}\right) \\
&\quad = \frac{1}{2}\left(T_{n-1}\left(\frac{r}{2^n}\right) + T_{n-1}\left(\frac{r+1}{2^n}\right) - T_{n-1}\left(\frac{r-j}{2^n}\right) - T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right) \\
&\qquad + \frac{1}{2}\left(2\,T_{n-1}\left(\frac{r+1}{2^n}\right) - T_{n-1}\left(\frac{r+1-j}{2^n}\right) - T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right) \\
&\qquad + \frac{1}{2}\left(2\,T_n\left(\frac{r}{2^n} + \frac{1}{2^{n+1}}\right) - T_n\left(\frac{r}{2^n}\right) - T_n\left(\frac{r+1}{2^n}\right)\right) \\
&\qquad + \frac{1}{2}\left(T_n\left(\frac{r-j}{2^n}\right) + T_n\left(\frac{r-j+1}{2^n}\right) - 2\,T_n\left(\frac{r-j}{2^n} + \frac{1}{2^{n+1}}\right)\right) \\
&\quad \leqslant \frac{1}{2}\cdot\frac{j}{2^{n-1}} + \frac{1}{2}\cdot\frac{j}{2^{n-1}} + \frac{1}{2}\cdot\frac{1}{2^n} + \frac{1}{2}\left(-\frac{1}{2^n}\right) = \frac{2j}{2^n} = \frac{k}{2^n}\,.
\end{aligned}$$

Case V: $p = 2r$, $q = p$, and $k = 2j + 1$ for some $r \in \mathbb{Z}$ and $j \in \mathbb{N}_0$. Then we have

$$\begin{aligned}
&2\,T_n\left(\frac{2r}{2^{n+1}}\right) - T_n\left(\frac{2r-2j-1}{2^{n+1}}\right) - T_n\left(\frac{2r+2j+1}{2^{n+1}}\right) \\
&\quad = \frac{1}{2}\left(2\,T_{n-1}\left(\frac{r}{2^n}\right) - T_{n-1}\left(\frac{r-j}{2^n}\right) - T_{n-1}\left(\frac{r+j}{2^n}\right)\right) \\
&\qquad + \frac{1}{2}\left(2\,T_{n-1}\left(\frac{r}{2^n}\right) - T_{n-1}\left(\frac{r-j-1}{2^n}\right) - T_{n-1}\left(\frac{r+j+1}{2^n}\right)\right) \\
&\qquad + \frac{1}{2}\left(T_n\left(\frac{r-j-1}{2^n}\right) + T_n\left(\frac{r-j}{2^n}\right) - 2\,T_n\left(\frac{r-j-1}{2^n} + \frac{1}{2^{n+1}}\right)\right) \\
&\qquad + \frac{1}{2}\left(T_n\left(\frac{r+j}{2^n}\right) + T_n\left(\frac{r+j+1}{2^n}\right) - 2\,T_n\left(\frac{r+j}{2^n} + \frac{1}{2^{n+1}}\right)\right) \\
&\quad \leqslant \frac{1}{2}\cdot\frac{j}{2^{n-1}} + \frac{1}{2}\cdot\frac{j+1}{2^{n-1}} + \frac{1}{2}\left(-\frac{1}{2^n}\right) + \frac{1}{2}\left(-\frac{1}{2^n}\right) = \frac{2j+1}{2^n} - \frac{1}{2^n} < \frac{2j+1}{2^n} = \frac{k}{2^n}\,.
\end{aligned}$$

We note that strict inequality occurs only in this case.

Case VI: $p = 2r$, $q = p+1$, and $k = 2j+1$ for some $r \in \mathbb{Z}$ and $j \in \mathbb{N}_0$. Then we have

$$\begin{aligned}
&T_n\left(\frac{2r}{2^{n+1}}\right) + T_n\left(\frac{2r+1}{2^{n+1}}\right) - T_n\left(\frac{2r-2j-1}{2^{n+1}}\right) - T_n\left(\frac{2r+1+2j+1}{2^{n+1}}\right) \\
&\quad = \frac{1}{2}\left(2\,T_{n-1}\left(\frac{r}{2^n}\right) - T_{n-1}\left(\frac{r-j-1}{2^n}\right) - T_{n-1}\left(\frac{r+j+1}{2^n}\right)\right) \\
&\qquad + \frac{1}{2}\left(T_{n-1}\left(\frac{r}{2^n}\right) + T_{n-1}\left(\frac{r+1}{2^n}\right) - T_{n-1}\left(\frac{r-j}{2^n}\right) - T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right)
\end{aligned}$$

$$+\frac{1}{2}\left(2\,T_n\left(\frac{r}{2^n}+\frac{1}{2^{n+1}}\right)-T_n\left(\frac{r}{2^n}\right)-T_n\left(\frac{r+1}{2^n}\right)\right)$$
$$+\frac{1}{2}\left(T_n\left(\frac{r-j-1}{2^n}\right)+T_n\left(\frac{r-j}{2^n}\right)-2\,T_n\left(\frac{r-j-1}{2^n}+\frac{1}{2^{n+1}}\right)\right)$$
$$\leqslant \frac{1}{2}\cdot\frac{j+1}{2^{n-1}}+\frac{1}{2}\cdot\frac{j}{2^{n-1}}+\frac{1}{2}\cdot\frac{1}{2^n}+\frac{1}{2}\left(-\frac{1}{2^n}\right)=\frac{2j+1}{2^n}=\frac{k}{2^n}\,.$$

Case VII: $p=2r+1$, $q=p$, and $k=2j+1$ for some $r\in\mathbb{Z}$ and $j\in\mathbb{N}_0$. Then we have $p-k=2r-2j$, $q+k=2r+2j+2$, and

$$2\,T_n\left(\frac{2r+1}{2^{n+1}}\right)-T_n\left(\frac{2r-2j}{2^{n+1}}\right)-T_n\left(\frac{2r+2j+2}{2^{n+1}}\right)$$
$$=\left(T_{n-1}\left(\frac{r}{2^n}\right)+T_{n-1}\left(\frac{r+1}{2^n}\right)-T_{n-1}\left(\frac{r-j}{2^n}\right)-T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right)$$
$$+\left(2\,T_n\left(\frac{r}{2^n}+\frac{1}{2^{n+1}}\right)-T_n\left(\frac{r}{2^n}\right)-T_n\left(\frac{r+1}{2^n}\right)\right)$$
$$\leqslant \frac{j}{2^{n-1}}+\frac{1}{2^n}=\frac{2j+1}{2^n}=\frac{k}{2^n}\,.$$

Case VIII: $p=2r+1$, $q=p+1$, and $k=2j+1$ for some $r\in\mathbb{Z}$ and $j\in\mathbb{N}_0$. Then we have $q=2r+2$, $p-k=2r-2j$, $q+k=2r+2j+3$, and

$$T_n\left(\frac{2r+1}{2^{n+1}}\right)+T_n\left(\frac{2r+2}{2^{n+1}}\right)-T_n\left(\frac{2r-2j}{2^{n+1}}\right)-T_n\left(\frac{2r+2j+3}{2^{n+1}}\right)$$
$$=\frac{1}{2}\left(T_{n-1}\left(\frac{r}{2^n}\right)+T_{n-1}\left(\frac{r+1}{2^n}\right)-T_{n-1}\left(\frac{r-j}{2^n}\right)-T_{n-1}\left(\frac{r+1+j}{2^n}\right)\right)$$
$$+\frac{1}{2}\left(2\,T_{n-1}\left(\frac{r+1}{2^n}\right)-T_{n-1}\left(\frac{r-j}{2^n}\right)-T_{n-1}\left(\frac{r+j+2}{2^n}\right)\right)$$
$$+\frac{1}{2}\left(2\,T_n\left(\frac{r}{2^n}+\frac{1}{2^{n+1}}\right)-T_n\left(\frac{r}{2^n}\right)-T_n\left(\frac{r+1}{2^n}\right)\right)$$
$$+\frac{1}{2}\left(T_n\left(\frac{r+j+1}{2^n}\right)+T_n\left(\frac{r+j+2}{2^n}\right)-2\,T_n\left(\frac{r+j+1}{2^n}+\frac{1}{2^{n+1}}\right)\right)$$
$$\leqslant \frac{1}{2}\cdot\frac{j}{2^{n-1}}+\frac{1}{2}\cdot\frac{j+1}{2^{n-1}}+\frac{1}{2}\cdot\frac{1}{2^n}+\frac{1}{2}\left(-\frac{1}{2^n}\right)=\frac{2j+1}{2^n}=\frac{k}{2^n}\,.$$

This completes the proof of Theorem 3.1. □

It is worth noting that in the sequel we apply only the particular case $q=p$ of Theorem 3.1. However, the calculations in Case VII show that we cannot restrict our inductive argument to the verification of the statement only for $q=p$.

COROLLARY 3.2. *If $n \in \mathbb{N}_0$ and u, $h \in \mathbb{R}$ such that $h \geqslant 0$, then*

$$2\,T_n(u) - T_n(u-h) - T_n(u+h) \leqslant \frac{2^{n+1}h + 3(n+1)}{2^n}\,. \tag{13}$$

Proof. Let $p_n = [2^{n+1}u]$ and $k_n = [2^{n+1}h]$, where $[x]$ denotes the lower integer part of the real number x. Clearly, $p_n \in \mathbb{Z}$ and $k_n \in \mathbb{N}_0$. Applying the inequalities (8) and (11) we obtain

$$\begin{aligned}
&2\,T_n(u) - T_n(u-h) - T_n(u+h) \\
&\quad = \left(2\,T_n\left(\frac{p_n}{2^{n+1}}\right) - T_n\left(\frac{p_n-k_n}{2^{n+1}}\right) - T_n\left(\frac{p_n+k_n}{2^{n+1}}\right)\right) + 2\left(T_n(u) - T_n\left(\frac{p_n}{2^{n+1}}\right)\right) \\
&\qquad + \left(T_n\left(\frac{p_n-k_n}{2^{n+1}}\right) - T_n(u-h)\right) + \left(T_n\left(\frac{p_n+k_n}{2^{n+1}}\right) - T_n(u+h)\right) \\
&\quad \leqslant \frac{k_n}{2^n} + (n+1)\left(2\left|u - \frac{p_n}{2^{n+1}}\right| + \left|\frac{p_n-k_n}{2^{n+1}} - (u-h)\right| + \left|\frac{p_n+k_n}{2^{n+1}} - (u+h)\right|\right) \\
&\quad \leqslant \frac{k_n}{2^n} + (n+1)\left(4\left|u - \frac{p_n}{2^{n+1}}\right| + 2\left|h - \frac{k_n}{2^{n+1}}\right|\right) \leqslant \frac{k_n}{2^n} + \frac{6(n+1)}{2^{n+1}} \\
&\quad = \frac{[2^{n+1}h] + 3(n+1)}{2^n} \leqslant \frac{2^{n+1}h + 3(n+1)}{2^n}\,.
\end{aligned}$$

□

COROLLARY 3.3. *The inequality* (4) *holds for all x, $y \in \mathbb{R}$.*

Proof. If we let n tend to infinity, the inequality (13) yields

$$2T(u) - T(u-h) - T(u+h) \leqslant 2h \tag{14}$$

for all u, $h \in \mathbb{R}$ such that $h \geqslant 0$. Substituting $u = \frac{x+y}{2}$ and $h = \frac{1}{2}|x-y|$ into (14) and rearranging the terms appropriately we obtain the inequality (4). □

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