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Mathematical modeling on the propagation of guided elastic waves

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ABSTRACT

The main cause of train derailment is related to transverse defects that arise in the railhead. These consist typically of opened or internal flaws that develop generally in a plane that is orthogonal to the rail direction. Most of the actual inspection techniques of rails rely on eddy currents, electromagnetic induction, and ultrasounds. Ultrasounds based testing is performed according to the excitation-echo procedure [1]. It is conducted conventionally by using a contact excitation probe that rolls on the railhead or by a contact-less system using a laser as excitation and air-coupled acoustic sensors for wave reception. The ratio of false predictions either positive or negative is yet too high due to the low accuracy of the actual devices. The inspection rate is also late; new numerical method has been developed in this context: The semi-analytical finite element method SAFE. This method has been applied in the case of anisotropic media [2], composite plates [3] and media in contact with fluids [4]. This method has been used successfully for several structures and especially in the case of beams of any cross-section such as rails that are the subject of this work and we were interested in wave propagation in waveguides of any arbitrary cross-section in the case of beams or rails.

KEYWORDS

train derailment, transverse defects, railhead, SAFE method

1. INTRODUCTION

The propagation of elastic waves is widely used in the field of non-destructive testing [5], in particular for quality control and detection defect in mechanical components of machines and industrial installations. Inspection methods of rails that are based on ultrasonic wave propagation were widely used [6].

The SAFE approach to determine the dispersive curves is to discretize the domain cross-section by the finite element method, in a two-dimensional problem (2D). In the propagation direction of the wave, which is orthogonal to the cross-section, the displacements are modeled using harmonic analytic functions. Hence the name semi-analytical method of finite elements. The great merit of this approach is the reduction of computation time by comparison with a purely three-dimensional computation, and in particular for high frequencies or what amounts to the same at the small wavelengths [7].

2. PROPAGATION OF GUIDED ELASTIC WAVES IN A RAIL

Let us consider an elastic waveguide with waves propagating along the direction x_3 with the wave number k and frequency ω . The cross-section of the waveguide is in the plane (O, x_1, x_2) . The equation of motion is written [8]:

$$C_{iplm} \frac{\partial^2 u_l}{\partial x_p \partial x_m} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad i, p, l, m = 1, 2, 3 \quad (1)$$

It is envisaged to search for the general solution of the waveguide in the form

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$$u_i(x_1, x_2, x_3, t) = U_i(x_1, x_2) \exp[j(kx_3 - \omega t)] \quad i = 1, 2, 3 \quad (2)$$

where $j = \sqrt{-1}$ is the imaginary unit.

The expression (2) expresses the separation between the movement in the plane of the cross-section and the off-plane motion which is considered purely harmonic. This requires that the material properties remain constant in the middle section of (O, x_1, x_2)

By substituting Eq. (2) in Eq. (1), it comes

$$C_{iplm} \frac{\partial^2 u_l}{\partial x_p \partial x_m} + \rho \omega^2 u_i = 0 \quad i, p, l, m \in \{1, 2, 3\} \quad (3)$$

where C_{iplm} are the terms of the elasticity tensor, ρ the density of the material, u_i the components of the displacement vector.

By explaining the calculation of derivation with respect to x_3 we obtain:

$$C_{iplm} \frac{\partial^2 U_l}{\partial x_p \partial x_m} + jk(C_{i3lp} + C_{ipl3}) \frac{\partial U_l}{\partial x_p} - k^2 C_{i3l3} U_l + \rho \omega^2 \delta_{il} U_l = 0 \quad i, l \in \{1, 2, 3\} \quad p, m \in \{1, 2\} \quad (4)$$

To Eq. (4), we must add the boundary conditions on the free boundary of the right section which the stress on the boundaries are written:

$$T_i = \sigma_{ip} n_p = C_{iplm} \frac{\partial U_l}{\partial x_m} n_p + jk C_{ipl3} U_l n_p = 0 \quad i, l \in \{1, 2, 3\} \quad p, m \in \{1, 2\} \quad (5)$$

where n_p is the k th component of the unit normally coming out of the domain boundary defining the cross-section.

The problem defined by Eq. (4) has the form of a quadratic problem with eigenvalues.

To formulate it discretely, the evaluation of spatial derivatives is necessary. The numerical approximation by the finite element method can be used to evaluate these derivatives.

The finite element method can be implemented directly from a variational principle such as the principle of virtual works. For a system subjected to the action of distribution of surface forces noted \vec{T} , the principle of virtual works applied to a continuous medium occupying the boundary domain ∂V makes it possible to write

$$\int_{\partial V} \delta[\vec{u}]^T [\vec{T}] dS = \int_V \delta[\varepsilon]^T [\sigma] dV + \int_V \delta[\vec{u}]^T \rho [\ddot{\vec{u}}] dV \quad (6)$$

For a free elastic system, the left part of the Eq. (6) vanishes:

$$\int_V \delta[\varepsilon]^T [C][\varepsilon] dV + \int_V \delta[\vec{u}]^T \rho [\ddot{\vec{u}}] dV = 0 \quad (7)$$

In practice, it is perfectly possible to account for dissipation in the formulation of the variational problem via the introduction of elastic constitutive constants that are complex [9]. We then introduce the following star constants

$$C_{iplm}^* = C_{iplm} + j\eta C_{iplm} \quad (8)$$

where η is a constant that reports the depreciation level.

The Eq. (7) becomes:

$$\int_V \delta[\varepsilon]^T [C][\varepsilon] dV - \int_V \rho \delta[\dot{\vec{u}}]^T [\dot{\vec{u}}] dV = 0 \quad (9)$$

In the case of a slender system following the direction x_3 , the SAFE method introduces the discretization of the problem by considering on an element e defined in the intersection of the domain V with a plane parallel to (x_1, x_2) the following interpolation

$$[\vec{u}]^e = [N(x_1, x_2)]^e [\vec{q}]^e \exp[j(kx_3 - \omega t)] \quad (10)$$

where

$$[N(x_1, x_2)]^e = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \ddots & N_{ne} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \ddots & 0 & N_{ne} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \ddots & 0 & 0 & N_{ne} \end{bmatrix} \quad (11)$$

and

$$[\vec{q}]^e = [\bar{U}_{11} \bar{U}_{21} \bar{U}_{31} \bar{U}_{12} \bar{U}_{22} \bar{U}_{32} \cdots \bar{U}_{1ne} \bar{U}_{2ne} \bar{U}_{3ne}]^T \quad (12)$$

where ne is the number of nodes of the finite element chosen to perform the discretization of the domain.

We will note in the following the basic section which serves as a domain to the formulation SAFE by $\Omega = V \cap (x_1, x_2)$ and its boundary which is a curve by $\partial\Omega$.

The tensor of the deformations can be represented by a six-dimensional matrix. Using Eq. (10) and the definition of the small deformation tensor, we obtain the deformation field on the element e in the form [9]

$$[\varepsilon]^e = [B][\vec{q}]^e \exp[j(kx_3 - \omega t)] \quad (13)$$

where

$$[B] = [B_1] + jk[B_2] \\ [B_1] = L_1 \frac{\partial [N]}{\partial x_1} + L_2 \frac{\partial [N]}{\partial x_2} \quad (14) \\ [B_2] = L_3 [N]$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ L_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (15)$$



By substituting Eq. (13) for the elementary integral that appears in Eq. (7), we get:

$$\int_{\Omega^e} \delta[\varepsilon]^T [C] [\varepsilon] d\Omega + \int_{\Omega^e} \rho \delta[\vec{u}]^T [\ddot{\vec{u}}] d\Omega = \delta[q]^e T \{ [K^e] - \omega^2 [M^e] \} [q]^e \quad (16)$$

where

$$[K^e] = \underbrace{\int_{\Omega^e} [B_1]^T [C] [B_1] d\Omega}_{[K_1^e]} + jk \underbrace{\int_{\Omega^e} \{ [B_1]^T [C] [B_2] - [B_2]^T [C] [B_1] \} d\Omega}_{[K_2^e]} + k^2 \underbrace{\int_{\Omega^e} [B_2]^T [C] [B_2] d\Omega}_{[K_3^e]}$$

and

$$[M^e] = \int_{\Omega^e} [N]^T \rho [N] d\Omega$$

By assembly, the discretized problem is written

$$\bigcup_{e=1}^{Ne} \delta[\vec{q}]^e T \{ [K^e] - \omega^2 [M^e] \} [\vec{q}]^e = 0 \quad (17)$$

where Ne is the total number of elements that discretize the cross-section and $\bigcup_{e=1}^{Ne}$ refers to the assembly operation.

Here the following global assembled system of dimension is equal to the total number of nodes nn

$$\{ [K] - \omega^2 [M] \} [\vec{q}] = [0] \quad (18)$$

where

$$[K] = [K_1] + jk[K_2] + k^2[K_3] \quad (19)$$

It is possible to make a base change in order to eliminate the pure imaginary complex that appears in Eq. (19). The following transformation matrix is thus introduced [7]

$$T_n = \begin{bmatrix} j & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & \ddots & \\ & & & & & & j & 0 & 0 \\ & & & & & & 0 & 1 & 0 \\ 0 & \dots & & & & & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

The new matrix K is written

$$[\widehat{K}] = [K_1] + k[\widehat{K}_2] + k^2[K_3] \quad (21)$$

where

$$[\widehat{K}_2] = iT^T [K_2] T_n$$

where $j^T = -j$ is used.

The new quadratic problem with the proper values to be solved in k for ω fixed is written

$$\{ [K_1] + k[\widehat{K}_2] + k^2[K_3] - \omega^2[M] \} [\widehat{q}] = [0] \quad (22)$$

where \widehat{q} is the new eigenvector to get the old eigenvector in the form

$$[q] = T_n^T [\widehat{q}]$$

The eigenvalue problem (22) can be solved by fixing the frequency ω and the dispersion branches are determined in the form $k = k(\omega)$. The real part of k describes the velocity of propagation of the wave and the imaginary part describes the attenuation that the wave undergoes. If the real part of k is zero the wave is evanescent and does not propagate.

The quadratic problem (22) of the eigenvalue to be solved by writing in the following form

$$[K' + \lambda K''] [\vec{Q}] = 0 \quad (23)$$

where

$$K' = \begin{bmatrix} \widehat{K}_2 & K_1 - \omega^2 M \\ K_1 - \omega^2 M & 0 \end{bmatrix}; \quad K'' = \begin{bmatrix} -K_3 & 0 \\ 0 & K_1 - \omega^2 M \end{bmatrix} \text{ and } Q = \begin{bmatrix} k\widehat{q} \\ \widehat{q} \end{bmatrix}$$

The matrices K' and K'' are dimension $2nn$ where nn is the total number of nodes.

From a numerical point of view, the eigenvalue problem defined by Eq. (18) or (21) is more convenient than the problem (23). The latter is slower than the first and is justified only in the case of depreciation in the structure [10].

The eigenvalue problem (23) is solved by specific numerical methods. The Matlab command *polyeig* solves it with the following command: *polyeig*(A_1, A_2, A_3) with $A_1 = K_1 - \omega^2 M$; $A_2 = \widehat{K}_2$ et $A_3 = -K_3$. The wavenumber is then obtained in the form $k = -\lambda$. The associated eigenvectors represent the propagation modes in the forward direction $x_3 \geq 0$ or retrograde $x_3 \leq 0$.

The eigenvalue problem (23) allows parametric definition for each value of ω a set of $2nn$ a number of waves $\{k_i\}_{i=1,\dots,2nn}$ and two sets of eigenvectors associated respectively on the left (L) and on the right (R): $\{Q_i^L\}_{i=1,\dots,2nn}$ et $\{Q_i^R\}_{i=1,\dots,2nn}$.

3. NUMERICAL CALCULATION OF THE FORCED REGIME SOLUTION

For a forced system in the context of the SAFE formulation, the stress vector is interpolated in the harmonic form

$$[\vec{T}]^e = [N(x_1, x_2)]^e [T(x_3)]^e \exp(-j\omega t) \quad (24)$$

where $[T(x_3)]^e$ is the vector of elemental nodal forces. This vector can be decomposed by inverse Fourier transform in the form



$$[T(x_3)]^e = \int_{-\infty}^{+\infty} [\bar{T}(k)]^e \exp(jkx_3) dk \quad (25)$$

where $[\bar{T}(k)]^e$ is the transformed Fourier vector of $[T(x_3)]^e$ which is defined by

$$[\bar{T}(k)]^e = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [T(x_3)]^e \exp(-jkx_3) dx_3 \quad (26)$$

Considering the spectral component k , the stress vector is interpolated in the form

$$[\bar{T}]^e = [N(x_1, x_2)]^e [\bar{T}]^e \exp[j(kx_3 - \omega t)] \quad (27)$$

By substituting Eq. (27) in Eq. (6) and using Eqs. (16) and (17), it comes by assembling

$$\bigcup_{e=1}^{Ne} \delta[q]^e T [(K^e) - \omega^2 (M^e)] [\bar{q}]^e - [\bar{F}^e] = 0 \quad (28)$$

where $[\bar{F}^e] = \int_{\Omega^e} [N]^T [N] [\bar{T}]^e d\Omega$

After assembly and multiplication on the left by the matrix T_n defined by Eq. (20), it comes

$$\{[K_1] + k[\hat{K}_2] + k^2[K_3] - \omega^2[M]\} [\hat{q}] = [\hat{F}] \quad (29)$$

where $[\hat{F}] = T_n[\bar{F}]$

Using Eq. (23), we obtain

$$[K' - kK''] [\bar{Q}] = [\bar{P}] \quad (30)$$

where $[\bar{P}] = \begin{bmatrix} [\hat{F}] \\ 0 \end{bmatrix}$

The solution of Eq. (30) is written

$$[\bar{Q}] = \sum_{i=1}^{2nn} \lambda_i Q_i^R \quad (31)$$

where

$$\lambda_i = -\frac{Q_i^L p}{(k - k_i) Q_i^L K'' Q_i^R} \quad (32)$$

As the displacement represents the lower part of the vector $[\bar{Q}]$, it is written

$$[\hat{q}] = \sum_{i=1}^{2nn} \lambda_i Q_i^{Rinf} \quad (33)$$

Considering a load punctual in $x_3 = x_3^0$, Eq. (32) is rewritten

$$\lambda_i = -\frac{Q_i^L \bar{p}}{(k - k_i) Q_i^L K'' Q_i^R} \delta(x_3 - x_3^0) \quad (34)$$

where δ is the distribution of Dirac.

Nodal displacement in physical space is then given by

$$[q(x_3)] = -T_n \sum_{i=1}^{2nn} \frac{Q_i^L \bar{p}}{Q_i^L K'' Q_i^R} \exp[jk_i(x_3 - x_3^0)] \quad (35)$$

Then, by interpolation, we obtain the harmonic displacement field in the form

$$[\vec{u}(x_1, x_2, x_3, t)]^e = [N(x_1, x_2)]^e [q(x_3)]^e \exp[j(kx_3 - \omega t)] \quad (36)$$

Note that calculating the displacement field by Eqs. (35) and (36) is a hard job.

It is necessary to determine the vector $[\bar{p}]$ by temporal and spatial Fourier transformation on x_3 of the applied stress vector field, and solve for each frequency the eigenvalue problem (23) in order to determine k_i , Q_i^L et Q_i^R . Once the harmonic displacement field is calculated by Eq. (36), we return to physical space by executing the inverse Fourier transform in time.

4. NUMERICAL CALCULATION OF THE GROUP VELOCITY OF THE GUIDED ELASTIC WAVES IN A RAIL

It is possible to implement the finite element method to calculate the global matrices \hat{K} and M appearing in Eq. (29). Then the eigenvalue problem can be solved to determine the dispersion curves. We can then calculate the phase velocity defined for the propagating modes by

$$c_p = \frac{\omega}{k} \quad (37)$$

where we take real wave numbers, knowing that in general, they are complex. If the wave number is pure imaginary, the wave is said to be evanescent and it does not propagate. When $real(k) \neq 0$ the propagation is possible, the imaginary part $imag(k)$ introduces then an attenuation and the wave is damped.

The group velocity that characterizes the shape of the wave is defined by

$$c_g = \frac{\partial \omega}{\partial k} \quad (38)$$

In the conservative case, it is possible to explain the velocity group for each particular solution (ω, k) in the following form [10].

$$c_g = \frac{(\hat{q}^L)^T (\hat{K}_2 + 2kK_3) \hat{q}^R}{2\omega (\hat{q}^L)^T M \hat{q}^R} \quad (39)$$

where \hat{q}^R are the eigenvectors on the right and \hat{q}^L the eigenvectors on the left obtained by solving the eigenvalue problem (29).

Cutoff frequencies are obtained by posing $k = 0$, that is, by solving the problem with the following eigenvalues

$$([K_1] - \omega^2[M])[q] = [0] \quad (40)$$

In the case of a damping system for which the coefficients C_{iplm} are complex, which has not been considered here, the wavenumbers are complex numbers and it is no longer possible to distinguish between propagating and evanescent waves. The notion of cutoff frequency no longer makes sense.



5. CONCLUSION

This generalist method applies to any beam type waveguide and allows to parametrically analyze the various possibilities of excitation of the structure able to highlight targeted defects present in the structure. The semi-analytical finite element method offers the possibility of calculating the displacement field and the reflection and transmission coefficients when a certain defect is considered on the cross-section of the rail.

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