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# FK-DLR properties of a quantum multi-type Bose-gas with a repulsive interaction

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The paper extends earlier results from Suhov and Kelbert ["FK-DLR states of a quantum Bose-gas with a hardcore interaction," arXiv:1304.0782] and Suhov *et al.* ["Shift-invariance for FK-DLR states of a 2D quantum Bose-gas," arXiv:1304.4177] about infinite-volume quantum bosonic states (FK-DLR states) to the case of multi-type particles with non-negative interactions. (An example is a quantum Widom–Rowlinson model.) Following the strategy from Suhov and Kelbert and Suhov *et al.*, we establish that, for the values of fugacity $z \in (0, 1)$ and inverse temperature $\beta > 0$, finite-volume Gibbs states form a compact family in the thermodynamic limit. Next, in dimension two we show that any limit-point state (an FK-DLR state in the terminology adopted in Suhov and Kelbert and Suhov *et al.*) is translation-invariant. *© 2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4886478]

## I. LIMIT-POINT GIBBS STATES AND REDUCED DENSITY MATRICES

The present paper is a continuation of earlier works Refs. 19 and 20. As in Refs. 19 and 20, we attempt at establishing a working definition of an infinite-volume quantum bosonic Gibbs state and justify it by checking natural properties such as shift-invariance in dimension two. In addition, the paper lays a foundation for future research into phase transitions in quantum Widom–Rowlinson (WR) models with several types of particles (following a recent progress in classical WR models; see Refs. 11 and 12), cf. the earlier work.[4] The class of states under consideration is formed by the so-called FK-DLR states (a more general concept is an FK-DLR functional): these states satisfy a quantum analog of the DLR equation (after Dobrushin–Lanford–Ruelle).

In fact, introducing and studying FK-DLR states is an attempt to expose quantum problems to established methods of classical statistical mechanics. In particular, we aim at a working definition of an infinite-volume equilibrium state for (important) quantum systems that do not fit formal requirements of the Kubo–Martin–Schwinger (KMS) theory. The role of a particle is played here by a trajectory of a varying time-length; these trajectories are subject to a self-interaction and an interaction involving several (in this paper – two) trajectories. The initial breakthrough in this direction was achieved in Refs. 5–7; it had (and has to this day) a strong impact on the whole of quantum statistical mechanics. However, the lack of a universal concept of an infinite-volume state leaves a gap in the picture; viz., in Ref. 4, a non-uniqueness of an infinite-volume state was, in essence, correctly shown, but could not be stated in a suitable form. The FK-DLR states allow to patch this gap.

Throughout the paper, we refer to Refs. 19 and 20 by adding the Roman numerals **I** and **II**, respectively: Theorem 1.2.**I**, formula (4.1.**II**), etc. The difference between the present work and Refs. 19 and 20 is in assumptions upon the interaction potential, which implies different conditions on the thermodynamic variables $z$ (the fugacity) and $\beta$ (the inverse temperature). Besides, in this

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paper we consider systems with several particle types $i \in \{1, \dots, q\}$. We suppose that the (two-body) interaction potentials $V_{ij}$ between types $i, j$ are non-negative (i.e., generate a repulsion of particles); they may also include hard cores. The non-negativity assumption allows us to work in the open domain $z_1, \dots, z_q \in (0, 1)$, $\beta > 0$: in the border of this domain (where $z_j = 1$ for some $j$), one may expect a Bose–Einstein condensation (which occurs for $V_{ij} \equiv 0$). However, even for $z_j$ less than (but close to) 1, one cannot exclude (at least at a rigorous level) a non-uniqueness of an infinite-volume Gibbs state as it has been defined in Refs. 19 and 20 and in the current paper. In the quantum bosonic WR model with $z_1 = \dots = z_q = z$, we expect a first-order phase transition for $z \in (1 - \eta, 1)$ (cf. Ref. 4).

*Remark.* Historically, the assumption $V \geq 0$ was used, elegantly and to a great effect, by Ginibre in Ref. 5 and became popular in quantum Statistical Mechanics (cf. the Refs. 2 and 18, to name a few). Admittedly, this assumption was termed in Ref. 5 "a severe physical limitation," and it was declared that the "next task is to get rid of it." To a certain extent, it was achieved in Ref. 6. Indeed, it can be noted that in Refs. 6 and 7 (where a number of different conditions upon the potential were introduced and intermittently used), the assumption of non-negativity was not present. However (and perhaps, consequently), the conditions upon $z$ and $\beta$ guaranteeing the key result of Refs. 6 and 7 (convergence to a unique infinite-volume limit and cluster expansion of the quantum Gibbs state) became notably less transparent than in Ref. 5.

The present paper follows the approach adopted in Refs. 19 and 20; this allows us to use pre-requisites and technical tools from the above references. However, we attempted at making this paper, to a degree, self-sufficient, as far as the statements of the main theorems are concerned. In Secs. I A–I C, we introduce the models, state the main results, and discuss the principal tool of the work: the Feynman–Kac (FK) representation. In Secs. II A–II C, we prove the existence of an infinite-volume FK-DLR state. Finally, in Sec. III, we focus on the 2D case and check that any FK-DLR functional is shift-invariant.

### A. The local Hamiltonian

A model of a quantum Bose-gas in $\mathbb{R}^d$ with $q$ types of particles is determined by a family of local Hamiltonians. Given a vector $\underline{n} = (n(1), \dots, n(q))$ with non-negative integer entries $n(j)$. Consider a system with $n(j)$ particles of type $j \in \{1, \dots, q\}$ in a finite "box" (a $d$-dimensional cube)

$$\Lambda (= \Lambda_L) = [-L, +L]^{\times d}.$$

The Hamiltonian, $H_{\underline{n},\Lambda}$, is a linear operator acting on functions $\phi_{\underline{n}} \in \mathcal{H}_{\underline{n}}(\Lambda) := \bigotimes_{1\le j\le q} \mathrm{L}_2^{\mathrm{sym}}(\Lambda^{n(j)})$:

$$\begin{aligned} \left(H_{\underline{n},\Lambda}\phi_{\underline{n}}\right)\left(\underline{x}^{\underline{n}}\right) &= -\frac{1}{2}\sum_{1\le j\le q}\;\sum_{1\le l\le n(j)} \left(\Delta_{j,l}\phi_{\underline{n}}\right)\left(\underline{x}^{\underline{n}}\right) \\ &+ \sum_{1\le j\le j'\le q}\;\sum_{1\le l\le n(j)}\;\sum_{1\le l'\le n(j')} V_{j,j'}\left(\left|x_{j,l} - x_{j',l'}\right|\right)\phi_{\underline{n}}\left(\underline{x}^{\underline{n}}\right), \\ \underline{x}^{\underline{n}} &= (\underline{x}(1), \dots, \underline{x}(q)),\;\; \underline{x}(j) = (x_{j,1}, \dots, x_{j,n(j)}) \in \Lambda^{n(j)}. \end{aligned} \tag{1.1}$$

Here function $\phi_{\underline{n}}$ is symmetric under permutations of variables $x_{j,l}$ within each group $\underline{x}(j)$. The symbol $\left|x_{j,l} - x_{j',l'}\right|$ stands for the Euclidean distance between points $x_{j,l}, x_{j',l'} \in \mathbb{R}^d$. (Sometimes we use an alternative notation $x(j, l), x(j', l')$.)

Operator $\Delta_{j,l}$ in (1.1) acts as a Laplacian in the variable $x_{i,l}$. Further, $V_{j,j'} : r \in [0, +\infty) \mapsto V_{j,j'}(r) \in [0, +\infty]$ is assumed to be a $C^2$-function at each point where $V_{j,j'} < +\infty$ and with a compact support. Function $V_{j,j'}$ describes a two-body interaction potential between type $j$ and type $j'$ particles, depending upon the distance between the particles involved. The value

$$\mathtt{R} = \inf\left[r > 0:\;\; V_{j,j'}(\widetilde{r}) \equiv 0 \;\text{ for }\; \widetilde{r} \geq r,\;\; 1 \leq j \leq j' \leq q\right] \tag{1.2}$$

is called the interaction radius (or the interaction range). We also assume that $V_{j,j'}(r)$ may take the value $+\infty$ on a closed set (e.g., when $0 \le r \le D(j, j')$, with $D(j, j') = D(j', j) \in [0, +\infty)$ representing the diameter of a hard-core repulsion between particles of types $j$ and $j'$). In the case $V_{j,j'}$ takes the value $+\infty$, the operator $H_{\underline{n},\Lambda}$ acts on functions $\phi_{\underline{n}}$ vanishing at every $\underline{x}^{\underline{n}}$ where $V_{j,j'}(|x_{j,l} - x_{j',l'}|) = +\infty$ for some $j, j'$ and $l, l'$. The set of such points $\underline{x}^{\underline{n}} \in \Lambda^{\underline{n}}$ is denoted by $\Lambda^{\underline{n}}_{\infty}$. Next, we suppose, for definiteness, that

$$\overline{V}^{(0)} = \max\Big[V_{j,j'}(r):\ 0 \le r \le \mathtt{R},\quad V_{j,j'}(r) < +\infty,\ 1 \le j \le j' \le q\Big] < +\infty, \tag{1.3}$$

and

$$\overline{V}^{(1)} = \max\Big[\,\big|V'_{j,j'}(r)\big| : 0 \le r \le \mathtt{R},\quad V_{j,j'}(r) < +\infty,\ 1 \le j \le j' \le q\Big] < +\infty, \tag{1.4}$$

$$\overline{V}^{(2)} = \max\Big[\,\big|V''_{j,j'}(r)\big| : 0 \le r \le \mathtt{R},\quad V_{j,j'}(r) < +\infty,\ 1 \le j \le j' \le q\Big] < +\infty.$$

*Remark.* As was said above, the assumption that $V_{j,j'} \ge 0$ means that the interaction potential generates repulsion between particles. Such a condition was repeatedly used in the works on quantum systems of Statistical Mechanics, see, e.g., Refs. 2, 5, and 18. It covers the case of a free gas (where $V_{j,j'} \equiv 0$). Removing the non-negativity assumption (without introducing a hard-core of a positive diameter) represents some challenges and remains an open question. On the other hand, the finite range assumption is used in this paper for simplifying some technicalities and can be relaxed to a controlled decay of $V_{j,j'}(r)$ for large $r$; this will be the subject of a forthcoming research.

Operator $H_{\underline{n},\Lambda}$ is determined by a boundary condition on $\partial\Lambda^{\underline{n}}$. Here $\Lambda^{\underline{n}} = \mathop{\times}\limits_{1\le j\le q} \Lambda^{n(j)}$ and

$$\partial\Lambda^{\underline{n}} = \Big\{\underline{x}^{\underline{n}} = \big(\underline{x}(1), \dots, \underline{x}(q)\big) \in \Lambda^{\underline{n}} :\quad \max\Big[\big|x_{jl}\big|_{\mathrm{m}} :\ 1 \le j \le q,\ 1 \le l \le n(j)\Big] = L\Big\}. \tag{1.5}$$

Here $|\ |_{\mathrm{m}}$ stands for the maximum norm in $\mathbb{R}^d$. More precisely, we initially consider $H_{\underline{n},\Lambda}$ as a symmetric operator given by the RHS of Eq. (1.1) on the set of C$^2$-functions $\phi = \phi_{\underline{n}}$ vanishing in a neighborhood of $\partial\Lambda^{\underline{n}}$, i.e., have the support within the interior of $\Lambda^{\underline{n}}$ (but outside $\Lambda^{\underline{n}}_{\infty}$). According to the Krein theory, this operator has a (monotone) family of self-adjoint extension identified via boundary conditions. We consider the self-adjoint extension (denoted by the same symbol $H_{\underline{n},\Lambda}$) which is determined by Dirichlet's boundary condition:

$$\phi_{\underline{n}}(\underline{x}^{\underline{n}}) = 0, \quad \underline{x}^{\underline{n}} \in \partial\Lambda^{\underline{n}}. \tag{1.6}$$

The domain of this extension is formed by symmetric functions $\phi_{\underline{n}}$ which are (i) C$^2$ at every point $\underline{x}^{\underline{n}} \in \Lambda$ which lies in the interior of $\Lambda^{\underline{n}} \setminus \Lambda^{\underline{n}}_{\infty}$, (ii) vanish on $\partial\Lambda^{\underline{n}} \cup \Lambda^{\underline{n}}_{\infty}$. For more detailed comments on the issue of the self-adjoint extensions, cf. Refs [2, 3, and 5–7].

However, the methods of this paper allow us to consider a broad class of conditions, viz., elastic boundary conditions (where a linear combination of the value of $\phi$ at the boundary and the value of its normal derivative vanishes); periodic boundary conditions can also included. Considering various boundary conditions endeavors towards including possible phase transitions; this question can be left for forthcoming works.

Under the above assumptions, $H_{\underline{n},\Lambda}$ is a self-adjoint operator, bounded from below and with a pure point spectrum. Moreover, $\forall\ \beta \in (0, +\infty)$, the operator $G_{\beta,\underline{n},\Lambda} = \exp\big[-\beta H_{\underline{n},\Lambda}\big]$ (the Gibbs operator in $\mathcal{H}_{\underline{n}}(\Lambda)$ at the inverse temperature $\beta$) is a positive-definite operator in $\mathcal{H}_{\underline{n}}(\Lambda)$, of the trace class. The trace

$$\Xi(\beta,\underline{n},\Lambda) := \mathrm{tr}_{\mathcal{H}_{\underline{n}}(\Lambda)}\, G_{\beta,\underline{n},\Lambda} \in (0,+\infty) \tag{1.7}$$

represents the $\underline{n}$-particle partition function in $\Lambda$.

As in Refs. 19 and 20, we work with the grand canonical Gibbs ensemble. Namely, we consider, $\forall$ vector $\underline{z} = (z_1,\dots,z_q)\in(0,1)^q$, the direct sum

$$G_{\underline{z},\beta,\Lambda} = \bigoplus_{\underline{n}\geq 0} \underline{z}^{\underline{n}}\, G_{\beta,\underline{n},\Lambda}. \tag{1.8}$$

Here and below, we set $\underline{z}^{\underline{n}} = \prod\limits_{1\leq j\leq q} z_j^{n(j)}$. This determines a positive-definite trace-class operator $G_{\underline{z},\beta,\Lambda}$ in the bosonic Fock space

$$\mathcal{H}(\Lambda) = \bigoplus_{\underline{n}\geq 0} \mathcal{H}_{\underline{n}}(\Lambda). \tag{1.9}$$

The quantity

$$\Xi_{\underline{z},\beta}(\Lambda) := \sum_{\underline{n}\geq 0} \underline{z}^{\underline{n}}\, \Xi(\beta,\underline{n},\Lambda) = \mathrm{tr}_{\mathcal{H}(\Lambda)} G_{\underline{z},\beta,\Lambda} \in (0,+\infty) \tag{1.10}$$

yields the grand canonical partition function in $\Lambda$ at fugacity $\underline{z}$ and the inverse temperature $\beta$. Further, the operator

$$R_{\underline{z},\beta,\Lambda} = \frac{1}{\Xi_{\underline{z},\beta}(\Lambda)} G_{\underline{z},\beta,\Lambda} \tag{1.11}$$

is called the (grand-canonical) density matrix (DM) in $\Lambda$; this is a positive-definite operator in $\mathcal{H}(\Lambda)$ of trace 1. Operator $R_{\underline{z},\beta,\Lambda}$ determines the Gibbs state (GS), i.e., a linear positive normalized functional $\varphi_{\underline{z},\beta,\Lambda}$ on the C*-algebra $\mathfrak{B}(\Lambda)$ of bounded operators in $\mathcal{H}(\Lambda)$:

$$\varphi_{\underline{z},\beta,\Lambda}(A) = \mathrm{tr}_{\mathcal{H}(\Lambda)}\big(A R_{\underline{z},\beta,\Lambda}\big), \quad A\in\mathfrak{B}(\Lambda). \tag{1.12}$$

*Remark.* The assumption that $0 < z_j < 1$ means that we avoid a (possible) "critical" regime. Namely, it is the values $z_j \nearrow 1$ that generates a Bose-condensation in the free gas ($V_{j,j'}\equiv 0$) in dimension $d\geq 3$. Compare, Ref. 3 and the references therein.

As in Refs. 19 and 20, the object of interest in this paper is the reduced DM (briefly, RDM), in cube $\Lambda_0\subset\Lambda$ centered at a point $c_0 = (\mathrm{c}_0^1,\dots,\mathrm{c}_0^d)$:

$$\Lambda_0 = [-L_0 + \mathrm{c}_0^1, \mathrm{c}_0^1 + L_0]\times\cdots\times[-L_0 + \mathrm{c}_0^d, \mathrm{c}_0^d + L_0]. \tag{1.13}$$

As in the aforementioned papers, the term RDM is used here for the operator $R^{\Lambda_0}_{\underline{z},\beta,\Lambda}$ defined via partial trace

$$R^{\Lambda_0}_{\underline{z},\beta,\Lambda} = \mathrm{tr}_{\mathcal{H}(\Lambda\setminus\Lambda_0)} R_{\underline{z},\beta,\Lambda}, \tag{1.14}$$

it is based on the tensor-product representation $\mathcal{H}(\Lambda) = \mathcal{H}(\Lambda_0)\otimes\mathcal{H}(\Lambda\setminus\Lambda_0)$. (This definition the RDM is different from (although close to) that used in Refs. 5 and 6.) Operator $R^{\Lambda_0}_{\underline{z},\beta,\Lambda}$ acts in $\mathcal{H}(\Lambda_0)$ is positive-definite and has trace 1. Furthermore, the partial trace operation generates an important compatibility property for RDMs. Suppose cubes $\Lambda_1\subset\Lambda_0\subset\Lambda$, then

$$R^{\Lambda_1}_{\underline{z},\beta,\Lambda} = \mathrm{tr}_{\mathcal{H}(\Lambda_0\setminus\Lambda_1)} R^{\Lambda_0}_{\underline{z},\beta,\Lambda}. \tag{1.15}$$

Throughout this paper, we use the upper indices $\Lambda_0$ and $\Lambda_1$ to indicate the corresponding "volumes" have been not affected by the partial trace.

To shorten the notation, the indices/arguments $\underline{z}$ and $\beta$ will be omitted whenever it does not produce a confusion. A straightforward modification of the above concepts emerges by including an external potential field induced by an external classical multi-type configuration (CC) $\mathbf{x}(\Lambda^{(\mathrm{R})})$.

Such a configuration is represented by a collection $(\mathbf{x}(\Lambda^{(\mathtt{R})}, 1), \ldots, \mathbf{x}(\Lambda^{(\mathtt{R})}, q))$ of finite subset in an "external" annulus encircling cube $\Lambda$:

$$\Lambda^{(\mathtt{R})}(=\Lambda_L^{(\mathtt{R})}) = \{x \in \mathbb{R}^d \setminus \Lambda : \ \mathrm{dist}(x, \Lambda) \leq \mathtt{R}\}. \tag{1.16}$$

Namely, the Hamiltonian $H_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}$ is given by

$$\begin{aligned}\left(H_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}\phi_{\underline{n}}\right)\left(\underline{x}^{\underline{n}}\right) &= \left(H_{\underline{n},\Lambda}\phi_{\underline{n}}\right)\left(\underline{x}^{\underline{n}}\right)\\ &+ \sum_{1\leq j\leq q}\ \sum_{1\leq l\leq n(j)}\ \sum_{1\leq j'\leq q}\ \sum_{\overline{x}\in\mathbf{x}(\Lambda^{(\mathtt{R})},j')} V_{j,j'}\left(\left|x_{j,l}-\overline{x}\right|\right)\phi_{\underline{n}}\left(\underline{x}^{\underline{n}}\right)\end{aligned} \tag{1.17}$$

and has all properties that have been listed above for $H_{\underline{n},\Lambda}$ (including self-adjointness). This enables us to introduce the Gibbs operators $G_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}$ and $G_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}$, the partition functions $\Xi_{\underline{n}}(\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})}))$ and $\Xi(\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})}))$, the DM $R_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}$, the GS $\varphi_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}$, and the RDMs $R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}$, where $\Lambda_0\subset\Lambda$, viz.,

$$\begin{aligned}G_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})} = \exp\left[-\beta H_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}\right],\ G_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})} &= \bigoplus_{\underline{n}\geq 0} \underline{z}^{\underline{n}}\, G_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})},\\ \Xi(\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})) &:= \mathrm{tr}_{\mathcal{H}_{\underline{n}}(\Lambda)} G_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})}},\\ \Xi(\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})) := \sum_{\underline{n}\geq 0}\underline{z}^{\underline{n}}\ \Xi(\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})) &= \mathrm{tr}_{\mathcal{H}(\Lambda)} G_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})},\\ R_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})} = \frac{G_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}}{\Xi(\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})}))},\ R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})} &= \mathrm{tr}_{\mathcal{H}(\Lambda\setminus\Lambda_0)} R_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})},\\ \varphi_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}(A) = \mathrm{tr}_{\mathcal{H}(\Lambda)}&\big(AR_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}\big),\quad A\in\mathfrak{B}(\Lambda).\end{aligned} \tag{1.18}$$

The previous definitions (1.1)–(1.15) correspond to the case of an empty exterior CC $\mathbf{x}(\Lambda^{(\mathtt{R})}) = \emptyset$.

*Remark.* External CCs could be considered as a part of a general notion of a boundary condition for a system in $\Lambda$. They will allow us to consider a variety of limiting infinite-volume objects (FK-DLR states and functionals) naturally associated with a given quantum model. However, it is not the most general concept expected here, and the quest to find further generalizations should be pursued in future studies. The question is to find a structure compatible with the local Hamiltonians (1.1) and (1.17).

As in Refs. 19 and 20, the Fock spaces $\mathcal{H}(\Lambda)$ and $\mathcal{H}(\Lambda_0)$ (see (1.9)) will be represented as $\mathrm{L}_2(\mathcal{C}(\Lambda))$ and $\mathrm{L}_2(\mathcal{C}(\Lambda_0))$, respectively. Here and below, $\mathcal{C}(\Lambda)$ denotes the space formed by collections $\overline{\mathbf{x}} = \{\mathbf{x}_1, \ldots, \mathbf{x}_q\}$ of finite (unordered) subsets $\mathbf{x}(j) \subset \Lambda$ (including the empty set) with the Lebesgue–Poisson measure

$$\mathrm{d}\overline{\mathbf{x}} = \prod_{1\leq j\leq q}\frac{1}{(\sharp\,\mathbf{x}(j))!}\prod_{x\in\mathbf{x}(j)}\mathrm{d}x,\quad \Big(\text{with} \int_{\mathcal{C}(\Lambda)}\mathrm{d}\overline{\mathbf{x}} = \exp\left[q\ell(\Lambda)\right],$$
$$\text{where } \ell \text{ is the Lebesgue measure on } \mathbb{R}^d\Big). \tag{1.19}$$

Here and later on, the symbol $\sharp$ is used for the cardinality of a given set. In accordance with this notation, we write that $\mathbf{x}(\Lambda^{(\mathtt{R})}) \in \mathcal{C}(\Lambda^{(\mathtt{R})})$. Points $\overline{\mathbf{x}}$, $\overline{\mathbf{x}}'$, $\mathbf{x}(\Lambda^{(\mathtt{R})})$ (and $\overline{\mathbf{y}}$ later on) are called, as before, classical multi-type configurations (CCs). We also introduce the subset $\mathcal{C}(\Lambda, \underline{n})$ formed by CCs $\overline{\mathbf{x}} \in \mathcal{C}(\Lambda)$ with $\sharp\,\mathbf{x}(j) = n(j)$. Next, the external CCs $\mathbf{x}(\Lambda^{(\mathtt{R})})$ have to be controlled, up to a degree, as $\Lambda \to \mathbb{R}^d$; see below. The methods developed in this paper allow us to introduce several methods of such control. Throughout the paper, we will refer to the following condition upon a family $\{\mathbf{x}(\Lambda^{(\mathtt{R})})\}$: for given $(z_1, \ldots, z_q) \in (0, 1)^q$ and $\beta > 0$, $\forall$ constant $\mathrm{c} \in (0, +\infty)$, the quantity

$$B(\mathrm{c}) := \sup\left[\sum_{1\leq i\leq q}\sharp\,\mathbf{x}(\Lambda_L^{(\mathtt{R})}, i)\sum_{k\geq 1} z_i^k k\ \exp\left(-\frac{L^2-\mathrm{c}L}{2\beta k}\right):\ L\geq 1\right] < \infty. \tag{1.20}$$

This assumption will be used without stressing it every time again. However, we do not consider it as a final one; in our opinion, it can be weakened.

### B. The thermodynamic limit and the shift-invariance property in two dimensions

The thermodynamic limit is the key concept of rigorous Statistical Mechanics; in the context of this work it is $\lim\limits_{\Lambda\nearrow\mathbb{R}^d}$, the family of standard cubes ordered by inclusion. In the literature, the quantities and objects identified as limiting points in the course of this limit are often referred to as infinite-volume ones (e.g., an infinite-volume RDM or GS). Traditionally, the existence and uniqueness of such a limiting object is treated as absence of a phase transition. On the other hand, a multitude of such objects (viz., depending on the boundary conditions for the Hamiltonian or the choice of external CCs) is considered as a manifestation of a phase transition.

However, since late 1960s, there is known an elegant alternative where infinite-volume objects are identified in terms that do not explicitly invoke the thermodynamic limit. For classical systems, this is the DLR equations and for so-called quantum spin systems—the KMS boundary condition. The latter is not applicable to the class of quantum systems under consideration, since the Hamiltonians $H_{\underline{n},\Lambda}$ and $H_{\underline{n},\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$ are not bounded.

In this paper, we propose a construction generalizing the classical DLR equation (see Sec. II C). A justification of this construction is given in Sec. III where we establish the shift-invariance property for the emerging objects (the RDMs and GSs) in dimension two (i.e., for $d=2$).

The first result claimed in this work is

**Theorem 1.1.** *Given $\beta\in(0,+\infty)$ and $\underline{z}\in(0,1)^q$, $\forall$ cube $\Lambda_0$ (see Eq. (1.13)), the family of RDMs $\left\{R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})},\ \Lambda\nearrow\mathbb{R}^d\right\}$ is compact in the trace-norm operator topology in $\mathcal{H}(\Lambda_0)$, for any choices of CCs $\mathbf{x}(\Lambda^{(\mathrm{R})})\in\mathcal{C}(\Lambda^{(\mathrm{R})})$ satisfying (1.20). Any limit-point operator $R^{\Lambda_0}$ for $\left\{R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}\right\}$ is a positive-definite operator in $\mathcal{H}(\Lambda_0)$ of trace 1. Further, let $\Lambda_1$, $\Lambda_0$ be a pair of cubes, $\Lambda_1\subset\Lambda_0$, and $R^{\Lambda_1}$, $R^{\Lambda_0}$ be a pair of limit-point RDMs such that*

$$R^{\Lambda_1}=\lim_{k\to+\infty}R^{\Lambda_1}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{(\mathrm{R})})}\ \textit{and}\ R^{\Lambda_0}=\lim_{k\to+\infty}R^{\Lambda_0}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{(\mathrm{R})})}. \tag{1.21}$$

*Here $\Lambda(k)=[-L(l),L(k)]^{\times d}$, $l=1,2,\dots$, are increasing cubes, with $L(k)\nearrow\infty$ and $\mathbf{x}(\Lambda(k)^{(\mathrm{R})})\in\mathcal{C}(\Lambda(k)^{(\mathrm{R})})$ are external CCs. Then $R^{\Lambda_1}$ and $R^{\Lambda_0}$ satisfy the compatibility property*

$$R^{\Lambda_1}=\mathrm{tr}_{\mathcal{H}(\Lambda_0\setminus\Lambda_1)}R^{\Lambda_0}. \tag{1.22}$$

□

A direct consequence of Theorem 1.1 yields the construction of a limit-point infinite-volume Gibbs state $\varphi$. For this purpose, it is enough to consider a countable family of cubes $\Lambda_0(l_0)=[-L_0l_0,L_0l_0]^{\times d}$ centered at the origin, of side-length $2L_0l_0$, where $L_0\in(0,\infty)$ is fixed and $l_0=1,2,\dots$ . By virtue of a diagonal process, we can ensure that, given a family of external CCs $\mathbf{x}(\Lambda^{(\mathrm{R})})$, one can excerpt a sequence $\Lambda(l)\nearrow\mathbb{R}^d$ such that (a) $\forall$ natural $l_0$ $\exists$ the trace-norm limit

$$R^{\Lambda_0(l_0)}=\lim_{l\to+\infty}R^{\Lambda_0(l_0)}_{\Lambda(l)|\mathbf{x}(\Lambda(l)^{(\mathrm{R})})} \tag{1.23}$$

and (b) for the limiting operators $R^{\Lambda_0(l_0)}$ relation (1.21) is satisfied, with $\Lambda_1=\Lambda_0(l_1)$ and $\Lambda_0=\Lambda_0(l_0)$ whenever $l_1<l_0$. This allows us to define an infinite-volume Gibbs state $\varphi$ by setting

$$\varphi(A)=\lim_{l\to\infty}\varphi_{\Lambda(l)}(A)=\mathrm{tr}_{\mathcal{H}(\Lambda_0(l_0))}\big(AR^{\Lambda_0(l_0)}\big),\quad A\in\mathfrak{B}(\Lambda_0(l_0)). \tag{1.24}$$

More exactly, $\varphi$ is a state of the quasilocal C*-algebra $\mathfrak{B}(\mathbb{R}^d)$ defined as the norm-closure of the inductive limit $\mathfrak{B}^0(\mathbb{R}^d)$:

$$\mathfrak{B}=\big(\mathfrak{B}^0(\mathbb{R}^d)\big)^-,\quad \mathfrak{B}^0(\mathbb{R}^d)=\mathop{\mathrm{ind\,lim}}_{\Lambda\nearrow\mathbb{R}^d}\mathfrak{B}(\Lambda). \tag{1.25}$$

What is more, $\phi$ is defined by a family of finite-volume RDMs $R^{\Lambda_0}$ acting in $\mathcal{H}(\Lambda_0)$, with $\Lambda_0 \subset \mathbb{R}^d$ being an arbitrary cube of the form (1.13), and obeying the compatibility property (1.22).

As we mentioned earlier, in two dimensions we prove the property of shift-invariance of the limit-point Gibbs states $\varphi$. Note that $\forall$ cube $\Lambda_0$ as in (1.13) and vector $s = \left(\mathtt{s}^1, \ldots, \mathtt{s}^d\right) \in \mathbb{R}^d$, the Fock spaces $\mathcal{H}(\Lambda_0)$ and $\mathcal{H}(\mathtt{S}(s)\Lambda_0)$ can be related via mutually inverse shift isomorphisms:

$$\mathtt{U}^{\Lambda_0}(s):\ \mathcal{H}(\Lambda_0) \to \mathcal{H}(\mathtt{S}(s)\Lambda_0) \quad \text{and} \quad \mathtt{U}^{\mathtt{S}(s)\Lambda_0}(-s):\ \mathcal{H}(\mathtt{S}(s)\Lambda_0) \to \mathcal{H}(\Lambda_0).$$

Here $\mathtt{S}(\mathtt{s})$ denotes the shift isometry $\mathbb{R}^d \to \mathbb{R}^d$:

$$\mathtt{S}(s):\ y \mapsto y + s, \quad y \in \mathbb{R}^d, \tag{1.26}$$

while $\mathtt{S}(s)\Lambda_0$ stands for the image of $\Lambda_0$:

$$\begin{aligned}\mathtt{S}(s)\Lambda_0 = &\left[-L_0 + \mathtt{c}_0^1 + \mathtt{s}^1, \mathtt{s}^1 + \mathtt{c}_0^1 + L^0\right]\\ &\times \cdots \times \left[-L_0 + \mathtt{c}_0^d + \mathtt{s}^d, \mathtt{s}^d + \mathtt{c}_0^d + L^0\right].\end{aligned} \tag{1.27}$$

The isomorphisms $\mathtt{U}^{\Lambda_0}(s)$ and $\mathtt{U}^{\mathtt{S}(s)\Lambda_0}(-s)$ are defined as follows:

$$\begin{aligned}&\left(\mathtt{U}^{\Lambda_0}(s)\phi_{\underline{n}}\right)(x^{\underline{n}}) = \phi_{\underline{n}}\left(\mathtt{S}(-s)x^{\underline{n}}\right), \quad x^{\underline{n}} \in \mathtt{S}\Lambda_0^{\underline{n}},\ \phi_{\underline{n}} \in \mathcal{H}_{\underline{n}}(\Lambda_0),\\ &\left(\mathtt{U}^{\mathtt{S}(s)\Lambda_0}(-s)\phi_{\underline{n}}\right)(x^{\underline{n}}) = \phi_{\underline{n}}(\mathtt{S}(s)x^{\underline{n}}), \quad x^{\underline{n}} \in \Lambda_0^{\underline{n}},\ \phi_{\underline{n}} \in \mathcal{H}_{\underline{n}}(\Lambda_0),\end{aligned} \tag{1.28}$$

with $\underline{n} = (n(1), \ldots n(q)),\ n(j) = 0, 1, \ldots$.

**Theorem 1.2.** *Let $d = 2$ and $\beta \in (0, +\infty)$, $\underline{z} \in (0, 1)^q$. Then any limit-point infinite-volume Gibbs state $\varphi$ is shift-invariant: $\forall\ s = (\mathtt{s}^1, \mathtt{s}^2) \in \mathbb{R}^2$*

$$\varphi(A) = \varphi(\mathtt{S}(s)A), \quad A \in \mathfrak{B}(\mathbb{R}^2). \tag{1.29}$$

*Here $\mathtt{S}(s)A$ stands for the shift of the argument $A$: if $A \in \mathfrak{B}(\Lambda_0)$ where $\Lambda_0$ is a square $[-L_0 + \mathtt{c}_1, \mathtt{c}_1 + L_0] \times [-L_0 + \mathtt{c}_2, \mathtt{c}_2 + L_0]$ then*

$$\mathtt{S}(s)A = \mathtt{U}^{\mathtt{S}(s)\Lambda_0}(-s)A\,\mathtt{U}^{\Lambda_0}(s) \in \mathfrak{B}(\mathtt{S}(s)\Lambda_0).$$

*In terms of the RDMs $R^{\Lambda_0}$:*

$$R^{\mathtt{S}(s)\Lambda_0} = \mathtt{U}^{\Lambda_0}(s)R^{\Lambda_0}\,\mathtt{U}^{\mathtt{S}(s)\Lambda_0}(-s). \tag{1.30}$$

□

### C. The FK-representation for the RDMs

Let us return to a general value of dimension $d$. We will assume that $\beta \in (0, +\infty)$ and $\underline{z} \in (0, 1)^q$. According to the featured realization of the Fock space $\mathcal{H}(\Lambda)$ as $\mathrm{L}_2(\mathcal{C}(\Lambda))$ (see (1.19)), its elements are identified as functions $\phi_\Lambda :\ \mathbf{x}(\Lambda) \in \mathcal{C}(\Lambda) \mapsto \phi_\Lambda(\mathbf{x}(\Lambda)) \in \mathbb{C}$, with

$$\int_{\mathcal{C}(\Lambda)} |\phi_\Lambda(\mathbf{x}(\Lambda))|^2\, \mathrm{d}\mathbf{x}(\Lambda) < \infty. \tag{1.31}$$

The space $\mathcal{H}(\Lambda_0)$ is represented in a similar manner: here we will use a short-hand notation $\overline{\mathbf{x}}_0$ and $\overline{\mathbf{y}}_0$ instead of $\mathbf{x}(\Lambda_0), \mathbf{y}(\Lambda_0) \in \mathcal{C}(\Lambda_0)$. (When it is convenient, $\overline{\mathbf{x}}_0$ and $\overline{\mathbf{y}}_0$ are understood as ordered arrays and identified with $x^{\underline{n}}$ and $y^{\underline{n}}$, points from $\Lambda_0^{\underline{n}}$.)

The first step in the proof of Theorems 1.1 is to reduce its assertions to statements about the integral kernels $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$ and $F^{\Lambda_0}$ which define the RDMs $R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$ and their infinite-volume counterpart $R^{\Lambda_0}$; we call these kernels RDMKs for short. Indeed, $R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$ and $R^{\Lambda_0}$ are integral operators:

$$\begin{aligned}\left(R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}\phi_\Lambda\right)(\overline{\mathbf{x}}_0) &= \int_{\mathcal{C}(\Lambda)} F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)\phi_\Lambda(\overline{\mathbf{y}}_0)\mathrm{d}\overline{\mathbf{y}}_0\\ \left(R^{\Lambda_0}\phi_\Lambda\right)(\overline{\mathbf{x}}_0) &= \int_{\mathcal{C}(\Lambda)} F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)\phi_\Lambda(\overline{\mathbf{y}}_0)\mathrm{d}\overline{\mathbf{y}}_0.\end{aligned} \tag{1.32}$$

The RDMKs $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$ and $F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$ are investigated through an FK representation. Properties of these kernels are listed in Theorem 1.3 where we adopt a setting from Theorem 1.1. We refer in Theorem 1.3 to the Hilbert–Schmidt (HS) metric generated by the norm $\|A\|_{\mathrm{HS}} = [\mathrm{tr}(AA^*)]^{1/2}$ expressed as

$$\|A\|^2_{\mathrm{HS}} = \int_{\mathcal{C}(\Lambda_0)\times\mathcal{C}(\Lambda_0)} \left|\mathrm{A}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)\right|^2 \mathrm{d}\overline{\mathbf{x}}_0 \mathrm{d}\overline{\mathbf{y}}_0. \tag{1.33}$$

Here $(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \mapsto \mathrm{A}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$ is an integral kernel (in general, complex) representing an HS operator $A$ in $\mathcal{H}(\Lambda_0)$. (Equivalently, $\mathrm{A} \in \mathcal{H}(\Lambda_0) \otimes \mathcal{H}(\Lambda_0)$.)

**Theorem 1.3.** *Any pair of cubes $\Lambda_0 \subset \Lambda$ and a family of CCs $\mathbf{x}(\Lambda^{(\mathrm{R})}) \in \mathcal{C}(\Lambda^{(\mathrm{R})})$ obeying (1.20), the family of RDMKs $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$ is compact in the HS metric. Any limit-point function*

$$(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \in \mathcal{C}(\Lambda_0) \times \mathcal{C}(\Lambda_0) \mapsto F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \tag{1.34}$$

*determines a positive-definite operator $R^{\Lambda_0}$ in $\mathcal{H}(\Lambda_0)$ of trace 1 (a limit-point RDM). Furthermore, given a cube $\Lambda_1 \subset \Lambda_0$, suppose $F^{\Lambda_1}$, $F^{\Lambda_0}$ are limit-point RDMKs such that the limits*

$$F^{\Lambda_0} = \lim_{k\to+\infty} F^{\Lambda_0}_{\Lambda(l)|\mathbf{x}(\Lambda(k)^{\mathrm{c}})}, \quad F^{\Lambda_1} = \lim_{k\to+\infty} F^{\Lambda_1}_{\Lambda(l)|\mathbf{x}(\Lambda(k)^{\mathrm{c}})} \tag{1.35}$$

*hold in $C^0\left(\mathcal{C}(\Lambda_0) \times \mathcal{C}(\Lambda_0)\right)$ for a sequence of cubes $\Lambda(k) \nearrow \mathbb{R}^d$ and external CCs $\overline{\mathbf{x}}\left(\Lambda(k)^{(\mathrm{R})}\right)$. Then the corresponding limit-point RDMs $R^{\Lambda_1}$ and $R^{\Lambda_0}$ obey (1.22).* □

Theorem 1.1 is deduced from Theorem 1.3 with the help of Theorem 1.4 below. The latter is a slight generalization of Lemma 1.1 from Ref. 8 (going back to Lemma 1 in Ref. 17).

**Theorem 1.4.** *Let $\mathtt{M}$ be a Polish space with a finite measure $\mu$ ($\mu(\mathtt{M}) < +\infty$) and $\rho_m(\mathrm{x}, \mathrm{y})$, $\mathrm{x}, \mathrm{y} \in \mathtt{M}$, be a sequence of kernels defining positive-definite operators $R_m$ of trace class and with trace 1 in a Hilbert space $L_2(\mathtt{M}, \nu)$. Suppose that as $m \to \infty$, $\rho_m(\mathrm{x}, \mathrm{y})$ converge to a limit kernel $\rho(\mathrm{x}, \mathrm{y})$ in the Hilbert–Schmidt (HS) norm:*

$$\|\rho_m - \rho\|^2_{\mathrm{HS}} = \int_{\mathtt{M}\times\mathtt{M}} \left[\rho_m(\mathrm{x}, \mathrm{y}) - \rho(\mathrm{x}, \mathrm{y})\right]^2 \nu(\mathrm{dx})\nu(\mathrm{dy}) \to 0, \tag{1.36}$$

*and $\rho(\mathrm{x}, \mathrm{y})$ defines a positive-definite trace-class operator $R$ of trace 1. Then*

$$\lim_{m\to\infty} \|R_m - R\|_{\mathrm{tr}} = 0 \tag{1.37}$$

*where $\|A\|_{\mathrm{tr}} = \mathrm{tr}[(AA^*)^{1/2}]$.* □

The proof Theorem 1.4 repeats that of the aforementioned lemmas, and for shortness we do not reproduce it here.

Therefore we focus from now on upon the proof of Theorems 1.2 and 1.3. In fact, we will establish similar facts for more general objects—FK-DLR functionals. As in Ref. 19, we use the terms a (multi-type) path configuration (PC) and a (multi-type) loop configuration (LC). The concept of FK-DLR functionals is based in our context on a series of definitions from Secs. II **I** and IV **I** related to PCs and LCs. We will not repeat here Definitions 2.1.1.**I**–2.1.4.**I** but give the list of the relevant notation used below. As to Definitions 2.1.1.**I**, 2.1.2.**I**, the corresponding objects are grouped into pairs: items with the symbol $\mathcal{W}$ represent path spaces whereas items with the symbol $\mathbb{P}$ represent path measures:

(i) $\overline{\mathcal{W}}^{k\beta}(x, y)$ (the space of paths $\omega : [0, k\beta] \to \mathbb{R}^d$ with endpoints $x, y$) $\leftrightarrow \mathrm{d}\overline{\mathbb{P}}^{k\beta}_{x,y}(\overline{\omega})$ (the unnormalized Wiener measure on $\overline{\mathcal{W}}^{k\beta}(x, y)$);

(ii) $\overline{\mathcal{W}}^*(x, y) = \bigcup_{k\geq 1} \overline{\mathcal{W}}^{k\beta}(x, y) \leftrightarrow \mathrm{d}\overline{\mathbb{P}}^*_{x,y}(\overline{\omega}^*)$ (the sum-measure $\sum_{k\geq 1} \mathrm{d}\overline{\mathbb{P}}^{k\beta}_{x,y}$ on $\overline{\mathcal{W}}^*(x, y)$);

(iii) $\mathcal{W}^*(x) = \overline{\mathcal{W}}^*(x, x)$ (the space of loops with an endpoint $x$) $\leftrightarrow \mathrm{d}\mathbb{P}^*_x(\omega^*) = \mathrm{d}\overline{\mathbb{P}}^*_{x,x}(\omega^*)$;

(iv) $\overline{\mathcal{W}}^*(\underline{x}(j),\underline{y}(j)) = \underset{1\le i\le n(j)}{\times}\overline{\mathcal{W}}^*(x_{j,i},y_{j,i})$ (the space of path configurations $\overline{\Omega}^*(j) = (\overline{\omega}^*_{j,1},\dots,\overline{\omega}^*_{j,n(j)})$ starting/finishing at CCs $\underline{x}(j)$, $\underline{y}(j)$) $\leftrightarrow \overline{\mathbb{P}}^*_{\underline{x}(j),\underline{y}(j)}(\overline{\Omega}^*(j)) = \underset{1\le i\le n(j)}{\times}\overline{\mathbb{P}}^*_{x_{j,i},y_{j,i}}(\overline{\omega}^*_{j,i})$;

(v) $\underline{\mathcal{W}}^*(\underline{x}(j),\underline{y}(j)) = \bigcup_{\pi_{n(j)}\in\mathfrak{S}_{n(j)}}\overline{\mathcal{W}}^*(\underline{x}(j),\pi_{n(j)}\underline{y}(j))$(the space of PCs with permuted finishing CCs) $\leftrightarrow \mathrm{d}\underline{\mathbb{P}}^*_{\underline{x}(j),\underline{y}(j)}(\overline{\Upsilon}^*(j)) = \sum_{\pi_{n(j)}\in\mathfrak{S}_{n(j)}}\mathrm{d}\overline{\mathbb{P}}^*_{\underline{x}(j),\,\pi_{n(j)}\underline{y}(j)}(\overline{\Omega}^*(j))$. Here $\mathfrak{S}_{n(j)}$ is the permutation group on $n(j)$ elements, and $\pi_{n(j)}\underline{y}(j) = (y_{j,\pi_{n(j)}(1)},\dots,y_{j,\pi_{n(j)}(n(j))})$. Symbol $\overline{\Upsilon}^*(j)$ covers all type $j$ PCs $\overline{\Omega}^*(j)\in\overline{\mathcal{W}}^*(\underline{x}(j),\pi_{n(j)}\underline{y}(j))$ where $\underline{x}(j)$, $\underline{y}(j)$ are fixed and $\pi_{n(j)}\in\mathfrak{S}_{n(j)}$ varies;

(vi) $\underline{\mathcal{W}}^*(\underline{x}^{\underline{n}},\underline{y}^{\underline{n}}) = \underset{1\le j\le q}{\times}\underline{\mathcal{W}}^*(\underline{x}(j),\underline{y}(j))$ (the space of multi-type PCs) $\leftrightarrow \mathrm{d}\underline{\mathbb{P}}^*_{\underline{x}^{\underline{n}},\underline{y}^{\underline{n}}}(\overline{\mathbf{\Upsilon}}^*) = \underset{1\le j\le q}{\times}\mathrm{d}\underline{\mathbb{P}}^*_{\underline{x}(j),\underline{y}(j)}(\overline{\Upsilon}^*(j))$;

(vii) $\mathcal{W}^*(\mathbf{x}(j)) = \underset{x\in\mathbf{x}(j)}{\times}\mathcal{W}^*(x)$ (the space of loop configurations $\mathbf{\Omega}^*(j) = \{\omega^*_x,\ x\in\mathbf{x}(j)\}$ starting/finishing at a CC $\mathbf{x}(j)$) $\leftrightarrow \mathrm{d}\mathbb{P}^*_{\mathbf{x}(j)}(\mathbf{\Omega}^*(j)) = \underset{x\in\mathbf{x}(j)}{\times}\mathrm{d}\mathbb{P}^*_x(\omega^*_x)$;

(viii) $\mathcal{W}^*(\overline{\mathbf{x}}) = \underset{1\le j\le q}{\times}\mathcal{W}^*(\mathbf{x}(j))$ (the space of multi-type LCs $\mathbf{\Omega}^* = (\mathbf{\Omega}^*(1),\dots,\mathbf{\Omega}^*(q))$ starting/finishing at $\overline{\mathbf{x}}$) $\leftrightarrow \mathrm{d}\mathbb{P}^*_{\overline{\mathbf{x}}}(\mathbf{\Omega}^*) = \underset{1\le j\le q}{\times}\mathrm{d}\mathbb{P}^*_{\mathbf{x}(j)}(\mathbf{\Omega}^*(j))$;

(ix) $\mathcal{W}^*(\Lambda) = \bigcup_{\overline{\mathbf{x}}\in\mathcal{C}(\Lambda)}\mathcal{W}^*(\overline{\mathbf{x}}) \leftrightarrow \mathrm{d}\overline{\mathbf{x}}\times\mathbb{P}^*_{\overline{\mathbf{x}}}(\mathrm{d}\mathbf{\Omega}^*_\Lambda) =: \mathrm{d}\mathbf{\Omega}^*_\Lambda$. Here $\mathcal{W}^*(\Lambda)$ is the space of (finite) multi-type LCs $\mathbf{\Omega}^*_\Lambda$ with varying initial/end points in $\Lambda$ (however, the loops constituting $\mathbf{\Omega}^*_\Lambda$ do not need to stay in $\Lambda$);

(x) $\mathcal{W}^*(\mathbb{R}^d)$: the set of countable multi-type LCs $\mathbf{\Omega}^* = \mathbf{\Omega}^*_{\mathbb{R}^d}$ such that their initial/end point CCs $\overline{\mathbf{x}} = (\mathbf{x}(1),\dots,\mathbf{x}(q))$ have no accumulation points in $\mathbb{R}^d$. A similar meaning is assigned to the notation $\mathcal{W}^*(\Lambda^{\complement})$ and $\mathbf{\Omega}^*_{\Lambda^{\complement}}$.

To recapitulate, we list once again most of frequently used symbols below:

$\overline{\Omega}^*(j) = (\overline{\omega}^*(j,1),\dots,\overline{\omega}^*(j,n(j)))$ a type $j$ PC (ordered), with fixed initial/end points,
$\overline{\Upsilon}^*(j) = (\overline{\omega}^*(j,1),\dots,\overline{\omega}^*(j,n(j)))$ a type $j$ PC (ordered), with permuted end points,
$\overline{\mathbf{\Upsilon}}^* = (\overline{\Upsilon}^*(1),\dots,\overline{\Upsilon}^*(q))$ a multi-type PC (ordered), with permuted end points,
$\Omega^*(j)$ a type $j$ loop collection (unordered), with a fixed initial/end CC,
$\mathbf{\Omega}^* = (\Omega^*(1),\dots,\Omega^*(q))$ a multi-type LC, with a fixed initial/end CC,
$\Omega^*_\Lambda(j)$ a finite type $j$ LC with a varying initial/end CC in $\Lambda$,
$\mathbf{\Omega}^*_\Lambda = (\Omega^*_\Lambda(1),\dots,\Omega^*_\Lambda(q))$ a finite multi-type LC with a varying initial /end CC in $\Lambda$,
$\mathbf{\Omega}^*_{\Lambda^{\complement}} = (\Omega^*_{\Lambda^{\complement}}(1),\dots,\Omega^*_{\Lambda^{\complement}}(q))$ a countable multi-type LC with a varying initial/end point CC in $\Lambda^{\complement}$.

As in Ref. 19, we also use the term a $\mathtt{t}$-section (of a path/loop, and of a PC/LC) and employ the notation

$$\{\overline{\Omega}^*\}(j,\mathtt{t}),\ \{\overline{\Upsilon}^*\}(j,\mathtt{t}),\ \{\overline{\mathbf{\Upsilon}}^*\}(\mathtt{t}),\ \{\Omega^*\}(j,\mathtt{t}),\ \{\mathbf{\Omega}^*\}(\mathtt{t}),\ \{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t}),\ \{\mathbf{\Omega}^*{}_{\Lambda^{\complement}}\}(\mathtt{t}),$$

similarly to Sec. II A of Ref. 19. Next, for a concatenation of two or more configurations (CC, PC, and/or LC) we use the symbol $\vee$.

Furthermore, we need a host of (integral) energy functionals $h(\,\cdot\,)$ and $h(\,\cdot\,|\,\cdot\,)$, viz.,

$$h(\mathbf{\Omega}^*),\ h(\mathbf{\Omega}^*_\Lambda),\ h(\mathbf{\Omega}^*|\mathbf{\Omega}^*_{\Lambda^{\complement}}),\ h(\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\complement}}),\ h(\overline{\mathbf{\Upsilon}}^*),\ \text{etc.},\tag{1.38}$$

and their versions $h(\,\cdot\,||\mathbf{x}(\Lambda^{(\mathrm{R})})$.

Next, "counting" functionals are needed, e.g.,

$$K(\overline{\Upsilon}^*_0(j)),\, K(\Omega^*_0(j)),\, K(\Omega^*_\Lambda(j)),\, L(\Omega^*_0(j)),\, L(\Omega^*_\Lambda(j)),\quad 1\le j\le q. \tag{1.39}$$

Also, indicator functionals $\alpha_\Lambda$ and $\chi^{\Lambda_0}$ will be used, e.g.,

$$\alpha_\Lambda(\overline{\mathbf{\Omega}}^*),\, \alpha_\Lambda(\mathbf{\Omega}^*),\, \alpha_\Lambda(\mathbf{\Omega}^*_\Lambda). \tag{1.40}$$

Recall, functionals $h$, $K$, and $L$ use the (aggregated) time-length multiplicities. Next, $\alpha_\Lambda$ requires that the PC/LC in the argument does not leave $\Lambda$, whereas $\chi^{\Lambda_0}$ prevents it from entering $\Lambda_0$ at ceratin time points.

The above functionals are also used with concatenated arguments, viz.,

$$h\left(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}}\right),\quad h\left(\mathbf{\Omega}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}}\right),$$

$$\chi^{\Lambda_0}\left(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}}\right),\quad \chi^{\Lambda_0}\left(\mathbf{\Omega}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}}\right) \tag{1.41}$$

where $\overline{\mathbf{\Upsilon}}^*_0\in\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ and $\mathbf{\Omega}^*_0=\mathbf{\Omega}^*_{\Lambda_0}\in\mathcal{W}^*(\Lambda_0)$; see (1.42)–(1.48). Compare also Eqs. (2.1.1.**I**)–(2.1.14.**I**).

For instance, let us be given:

(a) Cubes $\Lambda_0$, $\Lambda$, with $\Lambda_0\subset\Lambda$.

(b) CCs $\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0\in\mathcal{C}(\Lambda_0)$ with $\sharp\,\overline{\mathbf{x}}_0=\sharp\,\overline{\mathbf{y}}_0$ and a multi-type PC $\overline{\mathbf{\Upsilon}}^*_0=(\overline{\Upsilon}^*_0(1),\dots,\overline{\Upsilon}^*_0(q))\in\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ where $\overline{\Upsilon}^*_0(j)=(\overline{\omega}^*_0(j,1),\dots,\overline{\omega}^*_0(j,n(j)))$ is a configuration of type $j$ paths $\overline{\omega}^*_0(j,l):[0,k(j,l)\beta]\to\mathbb{R}^d$ for some given $k(j,l)=1,2,\dots$.

(c) A (finite) multi-type LC $\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}=(\Omega^*_{\Lambda\setminus\Lambda_0}(1),\dots,\Omega^*_{\Lambda\setminus\Lambda_0}(q))\in\mathcal{W}^*(\Lambda\setminus\Lambda_0)$ where each $\Omega^*_{\Lambda\setminus\Lambda_0}(j)$ is a configuration of type $j$ loops $\omega^*:[0,k(\omega^*)\beta]\to\mathbb{R}^d$ for given values $k(\omega^*)=1,2,\dots$.

(d) A (countable) LC $\mathbf{\Omega}^*_{\Lambda^{\complement}}=(\Omega^*_{\Lambda^{\complement}}(1),\dots,\Omega^*_{\Lambda^{\complement}}(q))=\in\mathcal{W}^*(\mathbb{R}^d)$ where each $\Omega^*_{\Lambda^{\complement}}(j)$ is a configuration of type $j$ loops $\widetilde{\omega}^*:[0,k(\widetilde{\omega}^*)\beta]\to\mathbb{R}^d$ for given values $k(\widetilde{\omega}^*)=1,2,\dots$...

The energy functional $h(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}})$ figuring in (1.47) is determined by

$$h(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}})=h(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}})+h(\overline{\mathbf{\Upsilon}}^*_0\big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}}). \tag{1.42}$$

Here $h(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}})$ represents the conditional energy of LC $\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}$ in the potential field generated by LC $\mathbf{\Omega}^*_{\Lambda^{\complement}}$ and $h(\overline{\mathbf{\Upsilon}}^*_0\big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}})$ the conditional energy of PC $\overline{\mathbf{\Upsilon}}^*_0$ in the potential field generated by the concatenation of LCs $\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}}$:

$$\begin{aligned}
&h(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}})\\
&=\sum_{1\le j\le q}\ \sum_{\omega^*\in\Omega^*_{\Lambda\setminus\Lambda_0}(j)}U_{j,j}(\omega^*,\omega^*)+\frac{1}{2}\sum_{1\le j\le q}\ \sum_{\substack{\omega^*,\omega^{*\prime}\in\Omega^*_{\Lambda\setminus\Lambda_0}(j)\\ \omega^*\ne\omega^{*\prime}}}U_{j,j}(\omega^*,\omega^{*\prime})\\
&\quad+\sum_{1\le j<j'\le q}\ \sum_{\substack{\omega^*\in\Omega^*_{\Lambda\setminus\Lambda_0}(j)\\ \omega^{*\prime}\in\Omega^*_{\Lambda\setminus\Lambda_0}(j')}}U_{j,j'}(\omega^*,\omega^{*\prime})+\sum_{1\le j,\widetilde{j}\le q}\ \sum_{\substack{\omega^*\in\Omega^*_{\Lambda\setminus\Lambda_0}(j)\\ \widetilde{\omega}^*\in\Omega^*_{\Lambda^{\complement}}(\widetilde{j})}}U_{j,\widetilde{j}}(\omega^*,\widetilde{\omega}^*),
\end{aligned} \tag{1.43}$$

$$\begin{aligned}
&h(\overline{\mathbf{\Upsilon}}^*_0\big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}})\\
&=\sum_{1\le j\le q}\ \sum_{\overline{\omega}^*_{j,l}\in\overline{\Upsilon}^*_0(j)}U_{j,j}(\overline{\omega}^*_{j,l},\overline{\omega}^*_{j,l})+\sum_{1\le j\le q}\ \sum_{\substack{\overline{\omega}^*_{j,l},\overline{\omega}^*_{j,l'}\in\overline{\Upsilon}^*_0(j)\\ 1\le l<l'\le n(j)}}U_{j,j}(\overline{\omega}^*_{j,l},\overline{\omega}^*_{j,l'})\\
&\quad+\sum_{1\le j<j'\le q}\ \sum_{\substack{\overline{\omega}^*_{j,l}\in\overline{\Upsilon}_0(j)\\ \overline{\omega}^*_{j',l'}\in\overline{\Upsilon}_0(j')}}U_{j,j'}(\overline{\omega}^*_{j,l},\overline{\omega}^*_{j',l'})+\sum_{1\le j,\widetilde{j}\le q}\ \sum_{\substack{\overline{\omega}^*_{j,l}\in\overline{\Upsilon}_0(j)\\ \widetilde{\omega}^*\in\Omega^*_{\Lambda\setminus\Lambda_0}\vee\Omega^*_{\Lambda^{\complement}}(\widetilde{j})}}U_{j,\widetilde{j}}(\overline{\omega}^*_{j,l},\widetilde{\omega}^*).
\end{aligned} \tag{1.44}$$

Here $U_{j,j'}(\cdot\,,\,\cdot)$ is the (integral) contribution of a pair of paths/loops; viz., in (1.43),

$$U_{j,\widetilde{j}}(\omega^*,\widetilde{\omega}^*)$$
$$=\int_0^\beta \sum_{\substack{1\leq m\leq k(\omega^*)\\ 1\leq\widetilde{m}\leq k(\widetilde{\omega}^*)}} V_{j,j}\Big(\big|\omega^*((m-1)\beta+\mathtt{t})-\widetilde{\omega}^*((m'-1)\beta+\mathtt{t})\big|\Big)\mathtt{dt} \tag{1.45}$$

and in (1.44),

$$U_{j,j}(\overline{\omega}^*_{j,l},\overline{\omega}^*_{j,l})$$
$$=\int_0^\beta \sum_{1\leq m<m'\leq k(j,l)} V_{j,j}\Big(\big|\overline{\omega}^*_{j,l}((m-1)\beta+\mathtt{t})-\overline{\omega}^*_{j,l}((m'-1)\beta+\mathtt{t})\big|\Big)\mathtt{dt}. \tag{1.46}$$

We also use the (standard) representation of the partition function $\Xi\big[\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})\big]$ (see Eq. (2.2.1.**I**)) and RDMK $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ (see Eqs. (2.2.2.**I**)–(2.2.5.**I**)). Next, Lemma 2.2.1.**I**, Definition 2.4.**I**, Lemma 2.2.2.**I**, and Definitions 2.5.**I**–2.7.**I** introduce the concepts of the infinite-volume FK-DLR functionals, states, and measures.

We employ the same notation $\mathfrak{F}$ (for infinite-volume FK-DLR functionals), $\mathfrak{F}_+$ (for infinite-volume FK-DLR states) and $\mathfrak{K}$ (for infinite-volume FK-DLR probability measures) as in Ref. 19. Recall, $\mu\in\mathfrak{K}$ is a probability measure (PM) on $(\mathcal{W}(\mathbb{R}^d),\mathfrak{M}(\mathbb{R}^d))$. Here $\mathfrak{M}(\mathbb{R}^d)$ is the sigma-algebra generated by cylinder events. In a probabilistic terminology, $\mu$ is a random marked point process with marks from $\mathcal{W}^*(0)$, the space of loops starting and ending up at the origin. Formally, $\mathfrak{M}(\mathbb{R}^d)$ is the smallest sigma-algebra contained the "local" sigma-algebras $\mathfrak{M}(\Lambda)\forall$ cube $\Lambda$. Compare Sec. III **I**.

For any PM $\mu\in\mathfrak{K}$, the Ruelle bound (see Eqs. (2.3.18.**I**)–(2.3.20.**I**)) holds true, with $\overline{\rho}=z$. Finally, the statements of Theorems 2.1.**I** and 2.2.**I** are carried through.

To summarize the FK-DLR representation: $\forall$ functional $\varphi\in\mathfrak{F}$, the RDMK of an RDM $R^{\Lambda_0}$ in $\Lambda_0$ has the form: $\forall\ \Lambda\supseteq\Lambda_0$ and $\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0\in\mathcal{C}(\Lambda)$ with $\sharp\,\mathbf{x}_0(j)=\sharp\,\mathbf{y}_0(j)$,

$$\begin{aligned}F^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)&=\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\\
&\times\int_{\mathcal{W}^*(\mathbb{R}^d)}\mathrm{d}\mu(\mathbf{\Omega}^*_{\Lambda^{\complement}})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\complement}}\in\mathcal{W}^*(\Lambda^{\complement})\Big)\\
&\qquad\times\int_{\mathcal{W}^*(\Lambda\setminus\Lambda_0)}\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\chi^{\Lambda_0}(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}})\\
&\times\prod_{1\leq j\leq q}z_j^{K(\overline{\Upsilon}^*_0(j))}\frac{z_j^{K(\Omega^*_{\Lambda\setminus\Lambda_0}(j))}}{L(\Omega^*_{\Lambda\setminus\Lambda_0}(j))}\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}}\Big)\Big].\end{aligned} \tag{1.47}$$

Here $\mu$ is an FK-DLR measure (i.e., $\mu\in\mathfrak{K}$). This means that the restriction $\mu\upharpoonright_{\mathfrak{M}(\Lambda_0)}$ is determined by the Radon–Nikodym derivative admitting the following representation: $\forall\ \Lambda\supseteq\Lambda_0$ and $\mathbf{\Omega}^*_0\in\mathcal{W}^*(\Lambda_0)$,

$$\begin{aligned}\frac{\mathrm{d}\mu\upharpoonright_{\mathfrak{M}(\Lambda_0)}(\mathbf{\Omega}^*_0)}{\mathrm{d}\mathbf{\Omega}^*_0}&=\int_{\mathcal{W}^*(\mathbb{R}^d)}\mathrm{d}\mu(\mathbf{\Omega}^*_{\Lambda^{\complement}})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\complement}}\in\mathcal{W}^*(\Lambda^{\complement})\Big)\\
&\times\int_{\mathcal{W}^*(\Lambda\setminus\Lambda_0)}\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\chi^{\Lambda_0}(\mathbf{\Omega}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^{\complement}})\\
&\times\prod_{1\leq j\leq q}\frac{z_j^{K(\Omega^*_0(j))}}{L(\Omega^*_0(j))}\frac{z_j^{K(\Omega^*_{\Lambda\setminus\Lambda_0}(j))}}{L(\Omega^*_{\Lambda\setminus\Lambda_0}(j))}\exp\ \Big[-h\left(\mathbf{\Omega}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda^{\complement}}\right)\Big].\end{aligned} \tag{1.48}$$

(Recall, we use the notation $\overline{\mathbf{\Upsilon}}^*_0=(\overline{\Upsilon}^*_0(1),\dots,\overline{\Upsilon}^*_0(q))\in\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ and $\mathbf{\Omega}^*_0=(\Omega^*_0(1),\dots,\Omega^*_0(q))\in\mathcal{W}^*(\Lambda_0)$.) Similar formulas hold true for RDMKs $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ and the PMs $\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$; see below.

The rest of the paper is organized as follows. In Sec. II, we analyze compactness properties and prove Theorem 1.3. Section III gives a brief sketch of the proof of Theorem 1.2. Throughout the argument, a number of properties of Wiener trajectories are employed; cf. the guidebook Ref. 1.

## II. THE COMPACTNESS ARGUMENT: PROOF OF THEOREM 1.3

### A. Uniform boundedness and HS convergence

Let us fix a cube $\Lambda_0$ of side length $2L_0$ centered at $c_0=(\mathrm{c}^1_0,\dots,\mathrm{c}^d_0)$: cf. Eq. (1.13). The first step in the proof is to verify that, as $\Lambda_0\subset\Lambda$ and cube $\Lambda\nearrow\mathbb{R}^d$, the RDMK $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ (see (1.32)) form a compact family in $C^0(\mathcal{C}(\Lambda_0,\underline{n})\times\mathcal{C}(\Lambda_0,\underline{n}))$ $\forall$ given $\underline{n}$. (We want to stress that we work with pairs $(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ with $\sharp\,\mathbf{x}_0(j)=\sharp\,\mathbf{y}_0(j)$; otherwise, $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)=0$.) Clearly, the Cartesian product $\mathcal{C}(\Lambda_0,\underline{n})\times\mathcal{C}(\Lambda_0,\underline{n})$ (the range of variable $(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ with given $\sharp\,\mathbf{x}_0(j)=\sharp\,\mathbf{y}_0(j)=n(j)$) is compact. As in Refs. 8–10, it is convenient to employ the Ascoli–Arzela theorem, i.e., verify that, for a given $\underline{n}$, the functions $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\restriction_{\mathcal{C}(\Lambda_0,\underline{n})\times\mathcal{C}(\Lambda_0,\underline{n})}$ are uniformly bounded and equi-continuous.

Checking uniform boundedness for a fixed $\underline{n}$ proceeds as follows: $\forall$ $(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\in\mathcal{C}(\Lambda_0,\underline{n})\times\mathcal{C}(\Lambda_0,\underline{n})$, the RDMK $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ satisfies, $\forall$ $\Lambda\subseteq\Lambda'\supseteq\Lambda_0$,

$$\begin{aligned}F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)&=\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\\&\times\int_{\mathcal{W}^*(\Lambda)}\mathrm{d}\mu(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\in\mathcal{W}^*(\Lambda\setminus\Lambda')\Big)\\&\times\int_{\mathcal{W}^*(\Lambda'\setminus\Lambda_0)}\mathrm{d}\mathbf{\Omega}^*_{\Lambda'\setminus\Lambda_0}\chi^{\Lambda_0}\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda'\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\Big)\\&\times\alpha_\Lambda\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda'\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\Big)\prod_{1\le j\le q}z_j^{K(\overline{\Upsilon}^*_0(j))}\frac{z_j^{K(\Omega^*_{\Lambda'\setminus\Lambda_0}(j))}}{L(\mathbf{\Omega}^*_{\Lambda'\setminus\Lambda_0}(j))}\\&\times\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda'\setminus\Lambda_0}\big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\mathbf{x}(\Lambda^{(\mathrm{R})})\Big)\Big].\end{aligned}\tag{2.1}$$

When $\Lambda'=\Lambda_0$, this simplifies to

$$\begin{aligned}F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)&=\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le j\le q}z_j^{K(\overline{\Upsilon}^*_0(j))}\\&\times\int_{\mathcal{W}^*(\Lambda)}\mathrm{d}\mu(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\in\mathcal{W}^*(\Lambda\setminus\Lambda_0)\Big)\chi^{\Lambda_0}\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\Big)\\&\times\alpha_\Lambda(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{(\mathrm{R})})\Big)\Big]\end{aligned}\tag{2.2}$$

and leads to the bound

$$F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\le Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0).\tag{2.3}$$

where function $Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ is specified below.

Namely, for $\overline{\mathbf{x}}_0=(\underline{x}_0(1),\dots,\underline{x}_0(q)),\overline{\mathbf{y}}_0=(\underline{y}_0(1),\dots,\underline{y}_0(q))\in\mathop{\times}\limits_{1\le j\le q}\Lambda_0^{n(j)}$, with $\underline{x}_0(j)=(x(j,1),\dots,x(j,n(j)))$ and $\underline{y}_0(j)=(y(j,1),\dots,y(j,n(j)))$, the RHS in (2.3) is given by

$$Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)=\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le j\le q}z_j^{K(\overline{\Upsilon}^*_0(j))}\chi^{\Lambda_0}\big(\overline{\Upsilon}^*_0(j)\big)$$
$$=\prod_{1\le j\le q}\sum_{\pi\in\mathfrak{S}_{n(j)}}\prod_{1\le l\le n(j)}\sum_{k\ge 1}{z_j}^k$$
$$\times\int_{\mathcal{W}^{\beta k}(x(j,l),y(j,\pi l))}\mathbb{P}^{\beta k}_{x(j,l),y(j,\pi l)}(\mathrm{d}\omega^*_{j,l})\chi^{\Lambda_0}(\overline{\omega}^*_{j,l}). \tag{2.4}$$

Whenever $\sharp\,\underline{x}(j)\neq\sharp\,\underline{y}(j)$, the quantity $Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$ is set to be 0. Recall, in (2.1), (2.2) and (2.4), we work with path configurations

$$\overline{\mathbf{\Upsilon}}^*_0=(\overline{\Upsilon}^*_0(1),\dots,\overline{\Upsilon}^*_0(q)),\quad \overline{\Upsilon}^*_0(j)=(\overline{\omega}^*_{j,1},\dots,\overline{\omega}^*_{j,n(j)})$$

with permuted endpoints. Accordingly, $\mathfrak{S}_{n(j)}$ denotes the symmetric group on $n(j)$ elements; $\pi=\pi_{n(j)}$ is a permutation of order $n(j)$ acting on "digits" 1, …, $n(j)$. Compare part (v) in the series of definitions (i)–(x) in Sec. I C. The integral $\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}$ in (2.4) (more precisely, the presence of the indicator $\chi^{\Lambda_0}(\overline{\Upsilon}^*_0(j)))$ yields the $\mathbb{P}^{\beta k}_{x(j,l),y(j,\pi_{n(j)}l)}$-probabilities that the paths $\overline{\omega}^*_{j,l}$, of a varying time-length $\beta k$ $(=\beta k(j,l))$, issued from point $x(j,l)$ and ending up at point $y(j,\pi l)$ do not enter cube $\Lambda_0$ at times $\beta,2\beta,\dots,\beta(k(j,l)-1)$. Formally,

$$\prod_{1\le l\le n(j)}\int_{\overline{\mathcal{W}}^{\beta k}(x(j,l),y(j,\pi l))}\mathbb{P}^{\beta k}_{x(j,l),y(j,\pi l)}(\mathrm{d}\overline{\omega}^*_{j,l})\chi^{\Lambda_0}(\overline{\omega}^*_{j,l})$$
$$=\prod_{1\le l\le n(j)}\mathbb{P}^{\beta k}_{x(j,l),y(j,\pi l)}\Big(\overline{\omega}^*_{j,l}(m\beta)\not\in\Lambda_0\ \forall\ m=1,\dots,k-1\Big).$$

For the future proof of the HS compactness we need to check that

$$\sum_{\underline{n}\,\ge\,\underline{0}}\frac{1}{(\underline{n}!)^2}\int_{(\Lambda_0)^{\times\underline{n}}\times(\Lambda_0)^{\times\underline{n}}}\prod_{1\le j\le q}\mathrm{d}\mathbf{x}_0(j)\mathrm{d}\mathbf{y}_0(j)\Big[Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\Big]^2$$
$$=\int_{\mathcal{C}(\Lambda_0)\times\mathcal{C}(\Lambda_0)}\mathrm{d}\overline{\mathbf{x}}_0\mathrm{d}\overline{\mathbf{y}}_0\Big[Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\Big]^2<\infty, \tag{2.5}$$

where $\underline{n}!=\prod_{1\le j\le q}n(j)!$, and

$$\mathrm{d}\underline{x}_0(j)=\prod_{1\le l\le n(j)}\mathrm{d}x(j,l),\quad \mathrm{d}\underline{y}_0(j)=\prod_{1\le j\le(j)}\mathrm{d}y,$$
$$\mathrm{d}\overline{\mathbf{x}}_0=\prod_{1\le j\le q}\frac{1}{\sharp\,\underline{x}_0(i)!}\mathrm{d}\underline{x}_0(j),\quad \mathrm{d}\overline{\mathbf{y}}_0=\prod_{1\le j\le q}\frac{1}{\sharp\,\underline{x}_0(i)!}\mathrm{d}\underline{y}_0(j).$$

First, we estimate the integral in $\mathrm{d}\mathbf{y}_0$, $\int_{\mathcal{C}(\Lambda_0,\underline{n})}\mathrm{d}\overline{\mathbf{y}}_0\Big[Q^{\Lambda_0}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\Big]^2$, for $\sharp\,\mathbf{x}_0(j)=\sharp\,\mathbf{y}_0(j)=n(j)$. This integral does not exceed (again with $k=k(j,l)$) the expression

$$\prod_{1\le j\le q}\sum_{\pi\in\mathfrak{S}_{n(j)}}\prod_{1\le l\le n(j)}\sum_{k\ge 1}\int_{\mathcal{W}^{\beta k}(x(j,l),x(j,\pi l))}\sum_{1\le m<k}{z_j}^k$$
$$\times\mathbb{P}^{\beta k}_{x(j,l),x(j,\pi l)}\Big(\overline{\omega}^*_{j,l}(m\beta)\in\Lambda_0,\ \overline{\omega}^*_{j,l}(m'\beta)\not\in\Lambda_0,\ 1\le m'<k,\ m'\neq m\Big)$$
$$=\prod_{1\le j\le q}\sum_{\pi\in\mathfrak{S}_{n(j)}}\prod_{1\le l\le n(j)}\sum_{k\ge 1}z_j^k\ \mathbb{P}^{\beta k}_{x(j,l),x(j,\pi l)}\Big(\overline{\omega}^*_{j,l}(m\beta)\in\Lambda_0$$
$$\text{just for one value } m\in\{1,\dots,k-1\}\Big). \tag{2.6}$$

The next step in the proof of (2.5) is to decompose the permutation $\pi_{n(j)}$ into the product of cycles: $\pi_{n(j)} = \gamma_1\cdots\gamma_s$, a cycle $\gamma_i$ having length $n_i$ where $n_1 + \ldots + n_s = n(j)$ and starting at digit $t_i$ (say). Next, we take into account such a decomposition, and for each cycle $\gamma_i$ merge the paths $x_0(t_i) \to x_0(\gamma_i t_i)$, $x_0(\gamma_i t_i) \to x_0(\gamma_i^2 t_i), \ldots, x_0(\gamma_i^{n_i-1} t_i) \to x_0(t_i)$ into a loop with the identical initial and endpoint $x_0(t_i)$ lying within $\Lambda_0$. In addition, each among the above paths contains precisely one intermediate time point of the form $\beta m$, where $m$ is a positive integer such that the path at this point lies in $\Lambda_0$. It is not hard to see that for the emerging loop $\overline{\omega}^*$, of the time-length $\beta M$ (say), the total number of time-points $\beta m$ such that $m$ is a positive integer, $1 \le m < M$ and $\overline{\omega}^*(\beta m) \in \Lambda_0$ is always odd. So,

$$\begin{aligned}\int_{\mathcal{C}(\Lambda_0)\times\mathcal{C}(\Lambda_0)} \mathrm{d}\overline{\mathbf{x}}_0 \mathrm{d}\overline{\mathbf{y}}_0 \Big[ Q^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \Big]^2 &\le \prod_{1\le j\le q} \sum_{s\ge 0} \frac{1}{s!} \\ \times \Bigg[ \sum_{M\ge 2} {z_j}^M \int_{\Lambda_0} \mathrm{d}x \int_{\mathcal{W}^{\beta M}(x,x)} & \mathbb{P}^{\beta M}_{x,x}(\mathrm{d}\overline{\omega}^*) \mathbf{1}\Big( \overline{\omega}^*(m\beta) \in \Lambda_0 \\ &\text{for an odd number of values } m \in \{1, \ldots, M\} \Big) \Bigg]^s \\ \le \prod_{1\le j\le q} \sum_{s\ge 0} \frac{1}{s!} \Bigg[ \upsilon(\Lambda_0) \sum_{M\ge 1} {z_j}^M \Bigg]^s < \infty, \end{aligned} \tag{2.7}$$

where $\upsilon(\Lambda_0)$ stands for the Euclidean volume of cube $\Lambda_0$.

A similar argument remains valid for the limiting RDMK $F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$, beginning with the representation

$$\begin{aligned} F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) = & \int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0) \prod_{1\le j\le q} z_j^{K(\overline{\Upsilon}^*_0(j))} \\ & \times \int_{\mathcal{W}^*(\mathbb{R}^d)} \mathrm{d}\mu(\mathbf{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0}) \mathbf{1}\Big( \mathbf{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0} \in \mathcal{W}^*(\mathbb{R}^d \setminus \Lambda_0) \Big) \\ & \times \chi^{\Lambda_0}\Big( \overline{\mathbf{\Upsilon}}^*_0 \vee \mathbf{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0} \Big) \exp\Big[ -h\Big( \overline{\mathbf{\Upsilon}}^*_0 \mid \mathbf{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0} \Big) \Big], \end{aligned} \tag{2.8}$$

this again leads to the bound

$$F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \le Q^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \tag{2.9}$$

similar to (2.3).

Accordingly, we can write:

$$\Big[ F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) - F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \Big]^2 \le 4 \Big[ Q^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \Big]^2. \tag{2.10}$$

Let us outline the argument of compactness in the HS norm. After checking that the family of the RDMKs $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$ satisfies, for given $\underline{n}$ and $\Lambda_0$, the assumptions of the Ascoli–Arzela theorem, we guaranty compactness in $C^0(\mathcal{C}(\Lambda_0, \underline{n}) \times \mathcal{C}(\Lambda_0, \underline{n}))$. Hence, $\forall\ \Lambda_0$ and $\underline{n}$, we can extract a sequence $\{\Lambda(s), \mathbf{x}((\Lambda(s))^{(\mathrm{R})}\}$ along which we have a convergence

$$F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \to F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0)$$

as $s \to \infty$ uniform in $(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \in \mathcal{C}(\Lambda_0, \underline{n}) \times \mathcal{C}(\Lambda_0, \underline{n})$. By invoking the diagonal process, we obtain a sequence $\{\Lambda(s), \mathbf{x}((\Lambda(s))^{(\mathrm{R})}\}$ along which we have convergence for a given $\Lambda_0$ but $\forall\ \underline{n}$. Next, by using the Lebesgue dominated convergence theorem, we get from (2.5) and (2.10) that along our sequence,

$$\int_{\mathcal{C}(\Lambda_0)\times\mathcal{C}(\Lambda_0)} \Big[ F^{\Lambda_0}_{\Lambda(s)|\mathbf{x}((\Lambda(s))^{(\mathrm{R})}}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) - F^{\Lambda_0}(\overline{\mathbf{x}}_0, \overline{\mathbf{y}}_0) \Big]^2 \to 0. \tag{2.11}$$

Then Theorem 1.4 implies that the RDMs $R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}$ converge to $R^{\Lambda_0}$ in the trace norm. Finally, by inspecting a countable family of cubes $\Lambda_0$, we get convergence for all given $\Lambda_0$, i.e., the compactness of states.

### B. Equicontinuity

To verify the equi-continuity property of RDMKs $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$, we have to check uniform bounds upon the gradients

$$\nabla_x F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\ \text{ and }\ \nabla_y F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0).$$

Here $x=x(j,l)$ is one of the points in $\underline{x}_0(j)$ and $y=y(j,\pi l)$ one of the points in $\underline{y}_0(j)$, $1\le l\le n(j)$, $1\le j\le q$. Both cases are treated in a similar fashion; for definiteness, we consider gradients $\nabla_y F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)$.

It can be seen from representation (2.2), (2.8) that there are two contributions into the gradient. The first contribution comes from varying the functional $\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\Big|\mathbf{x}(\Lambda^{(\mathbb{R})})\Big)\Big]$. The second one emerges from varying the measure $\overline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}$. (We are interested only in variations related to a chosen point $y$.) Symbolically,

$$\begin{aligned}
&\nabla_y F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)\\
&\quad=\int_{\mathcal{W}^*(\Lambda)}\mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\chi^{\Lambda_0}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\in\mathcal{W}^*(\Lambda\setminus\Lambda_0)\Big)\\
&\quad\times\Bigg\{\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\nabla_y\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\Big|\ \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{R}})\Big)\Big]\\
&\quad+\Big(\nabla_y\int_{\underline{\mathcal{W}}^*(\mathbf{x}_0,\mathbf{y}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\overline{\mathbf{\Upsilon}}^*_0)\Big)\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\Big|\ \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{R}})\Big)\Big]\Bigg\}\\
&\quad\times\prod_{1\le j\le q}z_j^{K(\overline{\Upsilon}^*_0(j))}\frac{z_j^{K(\Omega_{\Lambda\setminus\Lambda_0}(j))}}{L(\Omega_{\Lambda\setminus\Lambda_0}(j))}\chi^{\Lambda_0}(\overline{\mathbf{\Upsilon}}^*_0)\alpha_\Lambda\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\Big).
\end{aligned}\tag{2.12}$$

Let us analyze the parts involving the gradient. As before, write $\overline{\mathbf{x}}_0=(\underline{x}_0(1),\dots,\underline{x}_0(q))$, $\overline{\mathbf{y}}_0=(\underline{y}_0(1),\dots,\underline{y}_0(q))$ and $\overline{\mathbf{\Upsilon}}^*_0=(\overline{\Upsilon}^*_0(1),\dots,\overline{\Upsilon}^*_0(q))$. Here

$$\underline{x}_0(j)=(x(j,1),\dots,x(j,n(j))),\ \ \underline{y}_0(j)=(y(j,1),\dots,y(j,n(j)))$$

and $\overline{\Upsilon}_0(j)$ is a type $j$ PC formed by paths $\overline{\omega}^*_{j,l}$, of varying time-lengths $k=k(j,l)$ and with permuted end-points: $\overline{\Upsilon}_0(j)=(\overline{\omega}^*_{j,1},\dots,\overline{\omega}^*_{j,n(j)})$.

We have to focus on the following expression:

$$\begin{aligned}
&\nabla_y\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\overline{\mathbf{\Upsilon}}^*_0)\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\Big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{(\mathbb{R})})\Big)\Big]\\
&\quad=\prod_{1\le j\le q}\sum_{\pi\in\mathfrak{S}_{n(j)}}\nabla_y\prod_{1\le l\le n(j)}\sum_{k\ge1}z_j^k\int_{\mathcal{W}^{\beta k}(x(j,l),x(j,l))}\mathbb{P}^{\beta k}_{x(j,l),x(j,l)}(\mathrm{d}\omega^*_{j,l})\\
&\qquad\times\exp\Big[-h\Big(\mathbf{\Omega}^*_0+\overline{\mathbf{Z}}^*_0\Big|\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{(\mathbb{R})})\Big)\Big]\\
&\qquad\qquad\times\exp\left\{-\frac{|x(j,l)-y(j,\pi l)|^2_{\mathrm{Eu}}}{2k\beta}\right\}.
\end{aligned}\tag{2.13}$$

(The indicators $\chi^{\Lambda_0}$ and $\alpha_\Lambda$ do not contribute and are omitted.) Here $\overline{\mathbf{Z}}_0^*$ is a collection of straight paths: $\overline{\mathbf{Z}}_0^* = (\overline{Z}^*(1), \ldots, \overline{Z}^*(q))$, with $\overline{Z}^*(j) = (\zeta^*_{j,1}, \ldots, \zeta^*_{j,n(j)})$ where each $\zeta^*_{j,l}$ is a linear function

$$\zeta^*_{j,l} : \mathtt{t} \in \big[0, k\beta\big] \mapsto \frac{\mathtt{t}}{k\beta}(y(j,\pi l) - x(j,l)), \quad 1 \le j \le q, \quad 1 \le l \le n(j). \tag{2.14}$$

Observe that the argument $\mathbf{\Omega}_0^*$ in (2.13) represents a collection of loops $\omega^*(j,l) = \omega^*_{j.l}$ beginning and ending at coinciding points $x(j, l)$.

Of course, the gradient will only affect the expression

$$\exp\Bigg[-\frac{|x(j,l)-y(j,\pi l)|^2_{\mathrm{Eu}}}{2k\beta} \\ -h\Big(\overline{\omega}^*_{j,l} + \zeta^*_{j,l} \;\big|\; \big[\{\mathbf{\Omega}_0^* + \overline{\mathbf{Z}}_0^*\} \setminus \{\overline{\omega}^*_{j,l} + \zeta^*_{j,l}\}\big] \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \vee \mathbf{x}(\Lambda^{(\mathrm{R})})\Big)\Bigg]$$

where $\pi \in \mathfrak{S}_{n(j)}$ and $y(j, \pi l) = y$. The subscript Eu stresses that we work with the Euclidean norm/distance.

The first aforementioned contribution to the gradient emerges when we differentiate the term

$$\exp\Big[-h\Big(\overline{\omega}^*_{j,l} + \zeta^*_{j,l} \;\big|\; \big[\{\mathbf{\Omega}_0^* + \overline{\mathbf{Z}}_0^*\} \setminus \{\overline{\omega}^*_{j,l} + \zeta^*_{j,l}\}\big] \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \vee \mathbf{x}(\Lambda^{(\mathrm{R})})\Big)\Big];$$

this contribution is more difficult to estimate. The second comes from differentiating the term

$$\exp\Big[-|x(j,l)-y(j,\pi l)|^2_{\mathrm{Eu}}\big/(2k\beta)\Big].$$

It is easier to assess, and we refer the reader to Ref. 19 for a detailed argument about it.

Thus, we concentrate on the first contribution and write the corresponding expression down: for $y = y(j, \pi l)$, and with $k = k(j, l)$,

$$\begin{aligned}
&\nabla_y \exp\Big[-h\Big(\overline{\omega}^*_{j,l} + \zeta^*_{j,l}\big|\big[\{\mathbf{\Omega}_0^* + \overline{\mathbf{Z}}_0^*\} \setminus \{\overline{\omega}^*_{j,l} + \zeta^*_{j,l}\}\big] \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \vee \mathbf{x}(\Lambda^{(\mathrm{R})})\Big)\Big]\\
&= -\exp\Big[-h\Big(\overline{\omega}^*_{j,l} + \zeta^*_{j,l}\big|\big[\{\mathbf{\Omega}_0^* + \overline{\mathbf{Z}}_0^*\} \setminus \{\overline{\omega}^*_{j,l} + \zeta^*_{j,l}\}\big] \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \vee \mathbf{x}(\Lambda^{(\mathrm{R})})\Big)\Big]\\
&\quad\times \int_0^\beta \mathrm{d}\mathtt{t} \sum_{1\le m<k(j,l)} \Bigg\{ \sum_{\substack{1\le m'<k(j,l)\\ m'\ne m}} \nabla_y V_{j,j}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)\\
&\qquad\quad + \zeta^*_{j,l}(\mathtt{t}+m\beta) - \overline{\omega}^*_{j,l}(\mathtt{t}+m'\beta) - \zeta^*_{j,l}(\mathtt{t}+m'\beta)|\Big)\\
&\quad + \sum_{\substack{1\le j'\le q\\ j'\ne j}} \sum_{1\le l'\le n_{j'}} \sum_{1\le m'<k(j',l')} \nabla_y V_{j,j'}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)\\
&\qquad\quad + \zeta^*_{j,l}(\mathtt{t}+m\beta) - \overline{\omega}^*_{j',l'}(\mathtt{t}+m'\beta) - \zeta^*_{j',l'}(\mathtt{t}+m'\beta)|\Big)\\
&\quad + \sum_{1\le j'\le q} \Bigg[\sum_{\omega^*\in\Omega^*_{\Lambda\setminus\Lambda_0}(j')} \sum_{1\le m'<k(\omega^*)} \nabla_y V_{j,j'}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)\\
&\qquad\qquad\qquad + \zeta^*_{j,l}(\mathtt{t}+m\beta) - \omega^*(\mathtt{t}+m'\beta)|\Big)\\
&\quad + \sum_{\overline{x}\in\mathbf{x}(\Lambda^{(\mathrm{R})},j')} \nabla_y V_{j,j'}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta) + \zeta^*_{j,l}(\mathtt{t}+m\beta) - \overline{x}|\Big)\Bigg]\Bigg\}.
\end{aligned} \tag{2.15}$$

The initial observation is that the two first sums, in the RHS of Eq. (2.15), $\sum\limits_{1\le m<k(j,l)}\ \sum\limits_{\substack{1\le m'<k(j,l)\\ m'\ne m}}$ and $\sum\limits_{1\le m<k(j,l)}\ \sum\limits_{\substack{1\le j'\le q\\ j'\ne j}}\ \sum\limits_{1\le l'<n(j')}\ \sum\limits_{1\le m'<k(j',l')}$, can be controlled uniformly in $\Lambda$ in a straightforward manner. Their input to (2.13) is bounded, respectively, by

$$\begin{aligned}2\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} & \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le i\le q} z_i^{K(\overline{\Upsilon}^*_0(i))}\chi^{\Lambda_0}\overline{\mathbf{\Upsilon}}^*_0\\ &\times\int_0^\beta \mathrm{dt}\sum_{1\le m<m'<k(j,l)}\mathbf{1}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)+\zeta^*_{j,l}(\mathtt{t}+m\beta)\\ &\qquad -\overline{\omega}^*_{j,l}(\mathtt{t}+m'\beta)-\zeta^*_{j,l}(\mathtt{t}+m'\beta)|<\mathtt{R}\Big)\end{aligned}$$

and – for $n(j)>1$ –

$$\begin{aligned}\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} & \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le i\le q} z_i^{K(\overline{\Upsilon}^*_0(i))}\chi^{\Lambda_0}\overline{\mathbf{\Upsilon}}^*_0\\ &\times 1\int_0^\beta \mathrm{dt}\sum_{1\le m<k(j,l)}\ \sum_{\substack{1\le j'\le q\\ j'\ne j}}\ \sum_{1\le l'<n(j')}\ \sum_{1\le m'<k(j',l')}\mathbf{1}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)\\ &\qquad +\zeta^*_{j,l}(\mathtt{t}+m\beta)-\overline{\omega}^*_{j',l'}(\mathtt{t}+m'\beta)-\zeta^*_{j',l'}(\mathtt{t}+m'\beta)|<\mathtt{R}\Big).\end{aligned}$$

(The fact that potentials $V_{j,j'}$ may take the value $+\infty$ does not play a role in this bound.)

We only need to assess these expressions for given $\underline{n}$ and $\Lambda_0$. Indeed, we upper-bound them by a "brute force":

$$\text{by}\quad \frac{\beta}{2}\overline{V}^{(1)}\Theta_2(z_j)\prod_{1\le i\le q} n(i)!\,(1+\Theta_0(z_i))^{n(i)} := A_1(\underline{n})$$

and

$$\begin{aligned}\text{by}\quad \beta(q-1)\overline{V}^{(1)} & \left(\sum_{1\le i'\le q}\Theta_1(z_{i'})\right)\Theta_1(z_j)\\ &\times\prod_{1\le i\le q} n(i)!\,n(i)\,(1+\Theta_0(z_i))^{n(i)} := A_2(\underline{n}).\end{aligned}$$

(At this stage we did not use the fact that potentials $V_{j,j'}$ have a finite radius.) Here and below we use a host of quantities $\Theta_a(z)=\Theta_a(z,\beta)$:

$$\Theta_a(z)=\sum_{k\ge 1}\frac{z^k k^a}{(2\pi\beta k)^{d/2}},\quad a=-1,0,1,2. \tag{2.16}$$

The third sum, $\sum\limits_{1\le m<k(j,l)}\ \sum\limits_{1\le j'\le q}\ \sum\limits_{\omega^*\in\Omega^*_{\Lambda\setminus\Lambda_0}(j')}\ \sum\limits_{1\le m'<k(\omega^*)}$, in the RHS of (2.15) involves loops $\omega^*$ from the LC $\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}$. (Here $k(\omega^*)$ stands for the time-multiplicity of $\omega^*$.) The contribution of this

sum into (2.12) is bounded from above in norm by

$$\begin{aligned}
&\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Omega}}^*_0)\\
&\quad\times\int_{\mathcal{W}^*(\Lambda)} \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathtt{R})})}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\in\mathcal{W}^*(\Lambda\setminus\Lambda_0)\Big)\\
&\qquad\times\chi^{\Lambda_0}\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\Big)\alpha_\Lambda\Big(\overline{\mathbf{\Upsilon}}^*_0\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\Big)\\
&\quad\times\prod_{1\le i\le q} z_i^{K(\overline{\Upsilon}^*_0(i))}\frac{z_i^{K(\Omega^*_0(i))}}{L(\Omega^*_0(i))}\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathtt{R}})\Big)\Big]\\
&\times\sum_{\omega^*\in\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}}\int_0^\beta \mathrm{d}\mathtt{t}\sum_{\substack{1\le m<k(j,l)\\1\le m'<k(\omega^*)}}\mathbf{1}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)-\omega^*(\mathtt{t}+m'\beta)|<\mathtt{R}\Big).
\end{aligned}\tag{2.17}$$

By using the Campbell theorem, the Ruelle bound and the fact that

$$h\left(\overline{\Upsilon}^*_0+\overline{\mathbf{Z}}^*_0\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathtt{R}})\right)\ge 0$$

and omitting unused indicators, the quantity (2.17) does not exceed

$$\begin{aligned}
&\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le i\le q} z_i^{K(\overline{\Upsilon}^*_0(i))}\\
&\times\Bigg[\sum_{1\le i'\le q}\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathcal{W}^*(x,x)}\mathrm{d}\mathbb{P}^*_{x,x}(\omega^*)\,\frac{z_{i'}^{k(\omega^*)}}{k(\omega^*)}\\
&\times\int_0^\beta\mathrm{d}\mathtt{t}\sum_{\substack{1\le m<k(j,l)\\1\le m'<k(\omega^*)}}\mathbf{1}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)-\omega^*(\mathtt{t}+m'\beta)|<\mathtt{R}\Big)\Bigg].
\end{aligned}\tag{2.18}$$

Observe, that the expression (2.18) does not depend upon $\Lambda\supset\Lambda_0$.

In turn, (2.18) is less than or equal to

$$\begin{aligned}
&\beta\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le i\le q} z_i^{K(\overline{\Upsilon}^*_0(i))}\Bigg[\sum_{1\le i'\le q}\sum_{k\ge1}\frac{z_{i'}^k}{k}\\
&\times\sum_{1\le m<k(j,l)}\int_{\mathbb{R}^d}\mathrm{d}x\sum_{1\le m'<k}\mathbb{P}^{\beta k}_{x,x}\Big(\omega^*\in\mathcal{W}^{\beta k}(x,x):\ \omega^*(\mathtt{t}+m'\beta)\\
&\text{lies within distance}\le\mathtt{R}\ \text{ from }\ \overline{\omega}^*_{j,l}(\mathtt{t}+m\beta),\text{ for some }\mathtt{t}\in[0,\beta]\Big)\Bigg].
\end{aligned}\tag{2.19}$$

Next, by moving the starting/end points of both paths, $\omega^*$ and $\overline{\omega}^*_{j,l}$, we obtain that (2.19) does not exceed

$$\begin{aligned}
&\beta\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)} \mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le i\le q} z_i^{K(\overline{\Upsilon}^*_0(i))}k(j,l)\sum_{1\le i'\le q}\sum_{k\ge1}\frac{z_{i'}^k}{k}\\
&\times\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathcal{W}^{\beta k}(x,x)}\mathrm{d}\mathbb{P}^{\beta k}_{x,x}(\omega^*)\mathbf{1}\Big(\omega^*(\mathtt{t})\text{ lies within distance}\\
&\le\mathtt{R}\ \text{ from }\ \overline{\omega}^*_{j,l}(\mathtt{t}),\text{ for some }\mathtt{t}\in[0,\beta]\Big).
\end{aligned}\tag{2.20}$$

To assess (2.20), we use the requirement that the path $\overline{\omega}^*_{j,l}$ and the loop $\omega^*$ must come close to each other on the time interval $[0,\beta]$. A necessary condition for this is that—when $\mathrm{dist}_{\mathrm{Eu}}(x,\Lambda_0)>\mathtt{R}$

– at least one of them must travel at least a half of the distance $\mathrm{dist}_{\mathrm{Eu}}(x, \Lambda_0) - \mathtt{R}$ over the time interval $[0, \beta]$. For a point $x \in \mathbb{R}^d$ with a large value of $\mathrm{dist}_{\mathrm{Eu}}(x, \Lambda_0)$ it generates a sum of two small probabilities: one coming from $\mathbb{P}^{\beta k}_{x,x}$, the other from $\overline{\mathbb{P}}^{\beta k(j,l)}_{x(j,l),y(j,\pi l)}$. (Recall that $y(j, \pi l) = y$, the varying point from $\overline{\mathbf{y}}_0$.)

Formally, we use Lemma 2.1:

*Lemma 2.1. The following bounds hold true.*(i)$\forall$ $x \in \mathbb{R}^d$ *and* $a > 0$*,*

$$\underline{\mathbb{P}}^{\beta k}_{x,x}\Big( \sup \Big[ \mathrm{dist}_{\mathrm{Eu}}(\omega^*(\mathtt{t}), \Lambda_0) \Big] :\ 0 \leq \mathtt{t} \leq \beta > a \Big) \leq c_0 e^{-c_1 a^2},$$

*where* $c_0 = c_0(\beta)$ *and* $c_1 = c_1(\beta)$ *are finite positive constants.*

(ii)$\forall$ $x, y \in \Lambda_0$ *and* $a > 0$,

$$\underline{\mathbb{P}}^{\beta k}_{x,y}\Big( \sup \Big[ \mathrm{dist}_{\mathrm{Eu}}(\overline{\omega}^*(\mathtt{t}), \Lambda_0) \Big] :\ 0 \leq \mathtt{t} \leq \beta > a \Big) \leq c_0 e^{-c_1 a^2},$$

*where* $c_0 = c_0(\beta, \Lambda_0)$ *and* $c_1 = c_1(\beta, \Lambda_0)$ *are finite positive constants.*

*Proof of Lemma* 2.1. The starting point is the Skorohod formula for the Brownian bridge on the time interval $[0, \beta]$ in one dimension: given $a > 0$ and $|\mathrm{x} - \mathrm{y}| < a$,

$$\begin{aligned} P^{\beta}_{\mathrm{x},\mathrm{y}}\Big( \omega : \sup \Big[ |\omega(\mathtt{t}) - \mathrm{x}| :\ 0 \leq \mathtt{t} \leq \beta \Big] > a \Big) \\ = \frac{1}{\sqrt{2\pi\beta}} \sum_{l \in \mathbb{Z}:\, l \neq 0} (-1)^{l-1} \exp\Big[ -\frac{1}{2\beta}(\mathrm{y} - \mathrm{x} - 2la)^2 \Big]. \end{aligned} \tag{2.21}$$

Compare Ref. 16, Chap. 6, Sec. 27, the formulas in and below Eq. (27.1). We convert it to the following equality:

$$P^{\beta k}_{\mathrm{x},\mathrm{y}}\Big( \sup \Big[ |\omega(\mathtt{t}) - \mathrm{x}| : 0 \leq \mathtt{t} \leq \beta \Big] > a \Big) = \frac{1}{2\pi\beta} \frac{1}{\sqrt{k-1}} \tag{2.22}$$

$$\begin{aligned} \times \int \mathrm{d}\mathrm{u} \Big\{ \mathbf{1}\big( |\mathrm{x} - \mathrm{u}| > a \big) \exp\Big[ -\frac{(\mathrm{u}-\mathrm{x})^2}{2\beta} - \frac{(\mathrm{u}-\mathrm{y})^2}{2\beta(k-1)} \Big] \\ -\mathbf{1}\big( |\mathrm{x} - \mathrm{u}| < a \big) \sum_{l \in \mathbb{Z}:\, l \neq 1} \exp\Big[ -\frac{(\mathrm{u} - \mathrm{x} - 2la)^2}{2\beta} - \frac{(\mathrm{u}-\mathrm{y})^2}{2\beta(k-1)} \Big] \Big\}. \end{aligned} \tag{2.23}$$

(We agree that for $k = 1$, (2.22) morphs back to (2.21).)

(i) Take $\mathrm{x} = \mathrm{y}$. By the Cauchy–Schwarz inequality, the contribution of the integral $\int \mathrm{d}\mathrm{u}\ \mathbf{1}\big( |\mathrm{x} - \mathrm{u}| > a \big)$ is

$$\begin{aligned} &\leq \frac{1}{(4\pi^2\beta^2(k-1))^{1/4}} \left( \frac{1}{\sqrt{\pi\beta}} \int \mathrm{d}\mathrm{u}\ \mathbf{1}\big( |\mathrm{x} - \mathrm{u}| > a \big) \exp\Big[ -\frac{(\mathrm{u}-\mathrm{x})^2}{\beta} \Big] \right)^{1/2} \\ &= \frac{2}{(4\pi^2\beta^2(k-1))^{1/4}} \Phi\left( \frac{a}{2\sqrt{\beta}} \right)^{1/2} \text{ where } \Phi(b) = \frac{1}{\sqrt{2\pi}} \int_b^{+\infty} e^{-\mathrm{v}^2/2} \mathrm{d}\mathrm{v}. \end{aligned} \tag{2.24}$$

Next, consider the contribution of the integral $\int \mathrm{du}\, \mathbf{1}\big(|\mathrm{x}-\mathrm{u}|<a\big)$. When $a > \mathrm{u} - \mathrm{x} > 0$, we can write

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}:\, l\neq 1} (-1)^{l-1}\exp\left[-\frac{(\mathrm{u}-\mathrm{x}-2la)^2}{2\beta}-\frac{(\mathrm{u}-\mathrm{x})^2}{2\beta(k-1)}\right]\\
&\quad\leq \exp\left[-\frac{(\mathrm{u}-\mathrm{x})^2}{2\beta(k-1)}\right]\Bigg\{\exp\left[-\frac{(\mathrm{u}-\mathrm{x}+2a)^2}{2\beta}\right]\\
&\qquad -\exp\left[-\frac{(\mathrm{u}-\mathrm{x}+4a)^2}{2\beta}\right]+\exp\left[-\frac{(\mathrm{u}-\mathrm{x}+6a)^2}{2\beta}\right]\\
&\quad+\exp\left[-\frac{(\mathrm{u}-\mathrm{x}-2a)^2}{2\beta}\right]-\exp\left[-\frac{(\mathrm{u}-\mathrm{x}-4a)^2}{2\beta}\right]\\
&\quad+\exp\left[-\frac{(\mathrm{u}-\mathrm{x}-6a)^2}{2\beta}\right]\Bigg\}\leq 6\exp\left[-\frac{(\mathrm{u}-\mathrm{x})^2}{2\beta(k-1)}\right]\exp\left(-\frac{a^2}{2\beta}\right). \qquad (2.25)
\end{aligned}$$

A similar bound holds when $\mathrm{u} < \mathrm{x}$. Integrating in $\mathrm{du}$ yields a finite value, with the factor $e^{-a^2/(2\beta)}$ in front.

Going back to (2.23), we can write

$$P^{\beta k}_{\mathrm{x},\mathrm{y}}\left(\sup\Big[|\omega(\mathtt{t})-\mathrm{x}|:0\leq \mathtt{t}\leq\beta\Big]>a\right)\leq c_0\exp(c_1a^2)$$

where $c_0, c_1 \in (0, +\infty)$ are constants depending upon $\beta$. The rest of the argument completing the proof assertion (i) is standard and omitted.

The proof of statement (ii) is similar. ■

By virtue of Lemma 2.1, we can upper-bound (2.20) by

$$\begin{aligned}
&2\beta\overline{V}^{(1)}\left(\sum_{1\leq i'\leq q}\Theta_0(z_{i'})\right)\Theta_1(z_j)\prod_{1\leq i\leq q} n(i)!\,n(i)\,(1+\Theta_0(z_i))^{n(i)}\\
&\qquad\times\left\{c+c_0\int_{\mathbb{R}^d}\mathrm{d}x\exp\left[-c_1\mathrm{dist}_{\mathrm{Eu}}(x,\Lambda_0)^2\right]\right\}:=A_3(\underline{n},\Lambda_0). \qquad (2.26)
\end{aligned}$$

Here $c\in(0,\infty)$, $c_0\in(0,\infty)$ and $c_1\in(0,\infty)$ are constants.

Let us now focus on the forth sum, $\sum\limits_{1\leq m<k(j,l)}\ \sum\limits_{\substack{1\leq j'\leq q\\ j'\neq j}}\ \sum\limits_{\overline{x}\in\mathbf{x}(\Lambda^{(\mathrm{R})},j')}$, in the RHS of (2.15). This sum contributes into (2.12) a quantity whose norm is

$$\begin{aligned}
&\leq\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\leq i\leq q} z_i^{K(\overline{\Upsilon}^*_0(i))}\chi^{\Lambda_0}\overline{\mathbf{\Upsilon}}^*_0\alpha_\Lambda\big(\overline{\mathbf{\Upsilon}}^*_0\big)\\
&\times\int_{\mathcal{W}^*(\Lambda)}\mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\chi^{\Lambda_0}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\in\mathcal{W}^*(\Lambda\setminus\Lambda_0)\Big)\\
&\times\prod_{1\leq i'\leq q}\frac{z_{i'}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})}\alpha_\Lambda\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\Big)\exp\Big[-h\left(\overline{\mathbf{\Upsilon}}^*_0\,\big|\,\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{R}})\right)\Big]\\
&\qquad\times\int_0^\beta\mathrm{d}\mathtt{t}\sum_{\substack{1\leq m<k(l,j)\\ \overline{x}\in\mathbf{x}(\Lambda^{(\mathrm{R})})}}\mathbf{1}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)-\overline{x}|<\mathtt{R}\Big). \qquad (2.27)
\end{aligned}$$

The middle integral in (2.27) is

$$\int_{\mathcal{W}^*(\Lambda)} \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\chi^{\Lambda_0}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\in\mathcal{W}^*(\Lambda\setminus\Lambda_0)\Big)$$
$$\times\prod_{1\le i'\le q}\frac{z_{i'}^{K(\Omega^*_{\Lambda\setminus\Lambda_0}(i'))}}{L(\Omega^*_{\Lambda\setminus\Lambda_0}(i'))}\alpha_\Lambda(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\exp\Big[-h\Big(\overline{\mathbf{\Upsilon}}^*_0\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{R}})\Big)\Big]\le 1.$$

Indeed, $\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$ is a probability distribution, the values $z_{i'}\in(0,1)$, functionals $K(\Omega^*_{\Lambda\setminus\Lambda_0}(i'))$, $L(\Omega^*_{\Lambda\setminus\Lambda_0}(i'))\ge 1$ and $h\Big(\overline{\mathbf{\Upsilon}}^*_0\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{R}})\Big)\ge 0$, and the rest are indicators. Therefore, (2.2.7) does not exceed

$$\overline{V}^{(1)}\int_{\underline{\mathcal{W}}^*(\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0)}\mathrm{d}\underline{\mathbb{P}}^*_{\overline{\mathbf{x}}_0,\overline{\mathbf{y}}_0}(\overline{\mathbf{\Upsilon}}^*_0)\prod_{1\le i\le q}z_i^{K(\overline{\Upsilon}^*_0(i))}\chi^{\Lambda_0}(\overline{\mathbf{\Upsilon}}^*_0)$$
$$\times\int_0^\beta\mathrm{dt}\sum_{\substack{1\le m<k(j,l)\\ \overline{x}\in\mathbf{x}(\Lambda^{(\mathrm{R})})}}\mathbf{1}\Big(|\overline{\omega}^*_{j,l}(\mathtt{t}+m\beta)-\overline{x}|<\mathtt{R}\Big).\tag{2.28}$$

To bound (2.28) from above, we use the following argument. The sum $\sum\limits_{\substack{1\le m<k(j,l)\\ \overline{x}\in\mathbf{x}(\Lambda^{(\mathrm{R})})}}$ is not zero only if the path $\overline{\omega}^*_{j,l}$ reaches the "internal" annulus

$$\Lambda_{(\mathrm{R})}=\{x\in\Lambda:\ \mathrm{dist}_{\mathrm{Eu}}(x,\partial\Lambda)\le\mathtt{R}\};$$

in this case, the sum does not exceed $k(j,l)\sharp\,\mathbf{x}(\Lambda^{(\mathrm{R})})$. The probability that $\overline{\omega}^*_{j,l}$ reaches $\Lambda_{(\mathrm{R})}$ is

$$\le\frac{1}{(2\pi\beta k(j,l))^{d/2}}\exp\left[-\frac{\mathrm{dist}_{\mathrm{Eu}}(\Lambda_0,\Lambda_{(\mathrm{R})})^2}{2\beta k(j,l)}\right].$$

In turn, for $\Lambda\supset\Lambda_0$ we have that

$$\mathrm{dist}_{\mathrm{Eu}}(\Lambda_0,\Lambda_{(\mathrm{R})})\ge L-\mathtt{R}-L_0-\mathrm{dist}_{\mathrm{Eu}}(0,\Lambda_0),$$

where $\mathrm{dist}_{\mathrm{Eu}}(0,\Lambda_0)$ is the distance between $\Lambda_0$ and the origin. Going back to the external annulus $\Lambda^{(\mathrm{R})}=\Lambda^{(\mathrm{R})}_L$ (see (1.16)), the quantity (2.16) is

$$\le\beta\overline{V}^{(1)}\sharp\,\mathbf{x}(\Lambda^{(\mathrm{R})})\prod_{1\le i\le q}n(i)!\left(1\vee\sum_{k\ge1}\frac{z_i^k}{(\sqrt{2\pi\beta k})^d}\right)^{n(i)}$$
$$\times\sum_{k\ge1}\frac{z_j^k k}{(2\pi\beta k)^{d/2}}\exp\left[-\frac{\big(L-\mathtt{R}-L_0-\mathrm{dist}_{\mathrm{Eu}}(0,\Lambda_0)\big)^2}{2\beta k}\right],$$

which in turn does not exceed

$$\beta\overline{V}^{(1)}\prod_{1\le i\le q}n(i)!\,(1+\Theta_0(z_i))^{n(i)}\,B(\mathrm{c}):=A_4(\underline{n},\Lambda_0).\tag{2.29}$$

Here the quantity $B(\mathrm{c})$ has been introduced in Eq. (1.20), and the argument c is specified as

$$\mathrm{c}=(\mathtt{R}+L_0+\mathrm{dist}_{\mathrm{Eu}}(0,\Lambda_0))^2.\tag{2.30}$$

We see that the norm of the gradient vector represented by (2.12) is upper-bounded by

$$\overline{V}^{(1)}\Big[A_1(\underline{n},\Lambda_0)+A_2(\underline{n},\Lambda_0)+A_3(\underline{n},\Lambda_0)+A_4(\underline{n},\Lambda_0)\Big],$$

which yields the equi-continuity property required. Hence, the family of RDMKs $\Big\{F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}\Big\}$ is compact in space $C^0(\mathcal{C}(\Lambda_0,\underline{n})\times\mathcal{C}(\Lambda_0,\underline{n}))$. This closes the argument that the set of Gibbs states $\varphi_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$ is compact.

### C. Weak compactness of FK-DLR measures

A version of the above argument is applicable for proving that, for any given cube $\Lambda_0$, the probability measures (PMs) $\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}$ on $\mathcal{W}^*(\Lambda_0)$ form a compact family as $\Lambda \nearrow \mathbb{R}^d$. According to the Prokhorov theorem, it is enough to verify that the family $\left\{\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}\right\}$ is tight.

The proof of tightness proceeds along steps (a)–(d); see below.

(a) Let $\epsilon > 0$ be given. Then we can find $k^0 = k^0(\epsilon, \Lambda_0)$ such that the value

$$\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}\Big(\mathbf{\Omega}^*_0 = (\Omega^*_0(1), \dots, \Omega^*_0(q)) \in \mathcal{C}(\Lambda_0) : \\ \max\left[K(\Omega^*_0(j)) :\ 1 \le j \le q\right] \ge k^0\Big) \tag{2.31}$$

can be made as small as desired. In fact,

$$\begin{aligned}
&\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}\left(\mathbf{\Omega}^*_0 :\ \max\left[K(\Omega^*_0(j)) :\ 1 \le j \le q\right] \ge k^0\right)\\
&= \int_{\mathcal{W}^*(\Lambda)} \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathbb{R})})}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}) \mathbf{1}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \in \mathcal{W}^*(\Lambda\setminus\Lambda_0))\\
&\times \int_{\mathcal{W}^*(\Lambda_0)} \mathrm{d}\mathbf{\Omega}^*_0 \prod_{1\le j\le q} \frac{z_j^{K(\Omega^*_0(j))}}{L(\Omega^*_0(j))} \mathbf{1}(\max\left[K(\Omega^*_0(j)) :\ 1 \le j \le q\right] \ge k^0)\\
&\times \chi^{\Lambda_0}(\mathbf{\Omega}^*_0 \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\, \alpha_\Lambda(\mathbf{\Omega}^*_0 \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}) \exp\Big[-h(\mathbf{\Omega}^*_0 \,\big|\, \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \vee \mathbf{x}(\Lambda^{(\mathbb{R})}))\Big]\\
&\le \int_{\mathcal{W}^*(\Lambda_0)} \mathrm{d}\mathbf{\Omega}^*_0 \prod_{1\le j\le q} \frac{z_j^{K(\Omega^*_0(j))}}{L(\Omega^*_0(j))} \mathbf{1}(\max\left[K(\Omega^*_0(j)) :\ 1 \le j \le q\right] \ge k^0)\\
&= \prod_{1\le j\le q} \exp\Big[\upsilon(\Lambda_0)\left(1 + \Theta_0(z_j)\right)\Big]\\
&\times \sum_{1\le i\le q}\sum_{n\ge 0} \frac{\upsilon(\Lambda_0)^n}{n!} \prod_{1\le l\le n}\sum_{k(l)\ge 1} \frac{z_i^{k(l)}}{k(l)(2\pi\beta k(l))^{d/2}} \mathbf{1}\left(\sum_{1\le l\le n} k(l) \ge k^0\right).
\end{aligned} \tag{2.32}$$

Like before, $\upsilon(\Lambda_0)$ stands here for the Euclidean volume of $\Lambda_0$. For the definition of $\Theta_0$ (and $\Theta_{-1}$ below), see (2.16).

The sum $\sum\limits_{n\ge 0}$ in the RHS of (2.32) is divided into two: $\sum_1 := \sum\limits_{n>\sqrt{k^0}}$ and $\sum_2 := \sum\limits_{n\le\sqrt{k^0}}$. The contribution of the former to the last line in (2.32) is

$$\le \sum_{1\le i\le q}\sum_{n>\sqrt{k^0}} \frac{\upsilon(\Lambda_0)^n}{n!}\Theta_{-1}(z_i)^n, \tag{2.33}$$

which can be made arbitrarily small for large $k^0$. Next, in the latter at least one $k(l)$ must satisfy $k(l) \ge k^0/n \ge \sqrt{k^0}$. So, the contribution from $\sum_2$ to the last line in (2.32) does not exceed

$$\sum_{1\le i\le q}\left(\sum_{k\ge\sqrt{k^0}} \frac{z_i^k}{k(2\pi\beta k)^{d/2}}\right)\sum_{n\le\sqrt{k^0}} n\frac{\upsilon(\Lambda_0)^n}{n!}\Theta_{-1}(z_i)^{n-1}, \tag{2.34}$$

which again is small for large $k^0$.

(b) The second step is the remark that the Radon-Nikodym derivative is bounded uniformly in $\Lambda$ and $\mathbf{x}(\Lambda^{(\mathrm{R})})$ (since $z \in (0, 1)$):

$$\begin{aligned}\frac{\mathrm{d}\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\mathbf{\Omega}^*_0)}{\mathrm{d}\mathbf{\Omega}^*_0} &= \int_{\mathcal{W}^*(\Lambda)} \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0})\mathbf{1}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0} \in \mathcal{W}^*(\Lambda \setminus \Lambda_0)) \\ &\quad \times \chi^{\Lambda_0}(\mathbf{\Omega}^*_0 \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}) \prod_{1\leq j\leq q} \frac{z_j^{K(\Omega^*_0(j))}}{L(\Omega^*_0(j))} \alpha_\Lambda(\mathbf{\Omega}^*_0 \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}) \leq 1. \end{aligned} \tag{2.35}$$

(c) By virtue of property (b), it suffices to prove that, for given $\delta > 0$ and positive integer $k^0$, there exists a compact set $\mathcal{J} \subset \mathcal{C}(\Lambda_0)$ such that

$$\mathcal{J} \subset \mathcal{K}(k^0) := \left\{\mathbf{\Omega}^*_0 : \ \max\left[K(\Omega^*_0(j))\right] \leq k^0\right\} \ \text{and} \ \int_{\mathcal{C}(\Lambda_0)\setminus\mathcal{J}} \mathrm{d}\mathbf{\Omega}^*_0 < \delta. \tag{2.36}$$

As before, this is achieved with the help of the Ascoli–Arzela theorem, connecting compactness with uniform boundedness and equi-continuity. First, we guarantee the uniform boundedness by claiming that $\forall\delta$ and $k_0$ there exists an $\ell^0 \in (0, \infty)$ such that

$$\int_{\mathcal{K}(k^0)} \mathrm{d}\mathbf{\Omega}^*_0 \, \mathbf{1}\left(\max_{\omega^*\in\mathbf{\Omega}^*_0} \sup\left[|\omega^*(\mathtt{t}) - \omega^*(0)| : 0 \leq \mathtt{t} \leq \beta k(\omega^*)\right] \geq \ell^0\right) \leq \frac{\delta}{2}. \tag{2.37}$$

This claim holds because on the set $\mathcal{K}(k^0)$ the number of loops $\omega^*$ constituting the LC $\mathbf{\Omega}^*_0$ and their time-multiplicities $k(\omega^*)$ do not exceed $k^0$.

(d) Finally, we need to verify the equi-continuity property. But this fact holds true since the reference measure $\mathrm{d}\mathbf{\Omega}^*_0$ on the set $\mathcal{K}(k^0)$ is supported by LCs $\mathbf{\Omega}^*_0$ such that all loops $\omega^* \in \mathbf{\Omega}^*$ have a (global) continuity modulus not exceeding $\sqrt{2k^0\beta\epsilon \ln(1/\epsilon)}$.

This completes the proof of compactness for PMs $\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{(\mathrm{R})})}$.

As a result, the family of limit-point PMs $\{\mu^{\Lambda_0} : \ \Lambda_0 \subset \mathbb{R}^d\}$ has the compatibility property and therefore satisfies the assumptions of the Kolmogorov theorem. This implies that there exists a unique PM $\mu$ on $(\mathcal{W}^*(\mathbb{R}^d), \mathfrak{W}(\mathbb{R}^d))$ such that the restriction of $\mu$ on the sigma-algebra $\mathfrak{W}(\Lambda_0)$ coincides with $\mu^{\Lambda_0}$.

The fact that $\mu$ is an FK-DLR PM follows from the above construction. Hence, each limit-point state $\varphi$ falls in class $\mathfrak{F}_+(z, \beta)$. This completes the proof of Theorem 1.2.

*Remark.* In the course of the proof of compactness of measures $\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda R)}$ we did not use the condition (1.20).

## III. THE SHIFT-INVARIANCE OF AN FK-DLR PM IN A PLANE

In this section, we establish the following theorem (cf. Theorem 1.2.**II**).

**Theorem 3.1.** *In dimension two ($d = 2$), any FK-DLR PM $\mu \in \mathfrak{K}$ is translation invariant: $\forall\ s = (\mathtt{s}^1, \mathtt{s}^2) \in \mathbb{R}^2$, square $\Lambda_0 = [-L_0, L_0]^{\times 2}$ and event $\mathcal{D} \in \mathcal{W}^*(\mathbb{R}^2)$ localized in $\Lambda_0$ (i.e., belonging to a sigma-algebra $\mathfrak{W}(\Lambda_0)$; cf. Definition 2.4.***I***), we have that*

$$\mu(\mathtt{S}(s)\mathcal{D}) = \mu(\mathcal{D}).$$

*Here $\mathtt{S}(s)\mathcal{D}$ stands for the shifted event localized in the shifted square*
$\mathtt{S}(s)\Lambda_0 = [-L_0 + \mathtt{s}^1, \mathtt{s}^1 + L_0] \times [-L_0 + \mathtt{s}^2, \mathtt{s}^2 + L_0].$ □

Our Theorem 1.2 is a direct corollary of Theorem 3.1. As in Ref. 20, the principal step in the proof of Theorem 3.1 is

**Theorem 3.2.** *Let $\mu$ be an FK-DLR PM, $\Lambda_0$ be a square $[-L_0, L_0]^{\times 2}$ and an event $\mathcal{D}\subset \mathcal{W}^*(\mathbb{R}^2)$ be given, localized in $\Lambda_0$: $\mathcal{D}\in\mathfrak{W}(\Lambda_0)$. Then*

$$\mu(\mathtt{S}(s)\mathcal{D})+\mu(\mathtt{S}(-s)\mathcal{D})-2\mu(\mathcal{D})\geq 0. \tag{3.1}$$

□

Compare Theorem 2.1.**II**. The proof of Theorem 3.2 is basically a repetition of that of Theorem 2.1.**II** (its main ideas go back to Refs. 13–15, particularly Ref. 14). Consequently, we will omit various technical details referring the reader to the above publications. Let $L > L_0$ be given, and set $\Lambda = [-L, L]\times[-L, L]$. The main ingredient of the proof is a family of maps $\mathtt{T}^{\pm}_L=\mathtt{T}^{\pm}_{L,L_0}(s): \mathcal{W}^*(\mathbb{R}^2)\to\mathcal{W}^*(\mathbb{R}^2)$, $s=(\mathtt{s}^1,\mathtt{s}^2)$, featuring properties (i)–(vi) listed in Sec. II **II**. The formal definition of maps $\mathtt{T}^{\pm}_L$ follows Sec. III **II** and is given in terms of $\mathtt{t}$-sections of LCs $(\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})$. As in Ref. 20, Theorem 3.2 can be deduced from Theorem 3.3:

**Theorem 3.3.** *For any $\delta > 0$ there exists $L^*_0 = L^*_0(\delta) > 0$ such that for $L\geq L^*_0$ there exists a subset $\mathcal{G}_L\subset\mathcal{W}^*(\mathbb{R}^2)$ such that $\mathcal{G}_L\in\mathfrak{M}$ and the following properties are satisfied:*

$$\text{(A)}\qquad \mu(\mathcal{G}_L)=\int_{\mathcal{W}^*(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\,\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}}\in\mathcal{W}^*(\Lambda^{\mathtt{C}})\Big)$$
$$\times\int_{\mathcal{W}^*(\Lambda)}\mathrm{d}\mathbf{\Omega}^*_\Lambda\;\mathbf{1}\Big(\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}}\in\mathcal{G}_L\Big)$$
$$\times\;\frac{z^{K(\mathbf{\Omega}^*_\Lambda)}}{L(\mathbf{\Omega}^*_\Lambda)}\;\exp\Big[-h(\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\Big]\geq 1-\delta. \tag{3.2}$$

(B) *The probabilities $\mu(\mathtt{S}(\pm s)(\mathcal{D}\cap\mathcal{G}_L))$ are represented in the form*

$$\mu(\mathtt{S}(\pm s)(\mathcal{D}\cap\mathcal{G}_L))=\int_{\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}}\in\mathcal{W}_{\mathtt{r}}\left(\Lambda^{\mathtt{C}}\right)\Big)$$
$$\times\int_{\mathcal{W}^*_{\mathtt{r}}(\Lambda)}\mathrm{d}\mathbf{\Omega}^*_\Lambda\;\mathbf{1}\Big(\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}}\in\mathcal{G}_L\cap\mathcal{D}\Big)\,\frac{z^{K(\mathbf{\Omega}^*_\Lambda)}}{L(\mathbf{\Omega}^*_\Lambda)}$$
$$\times\,J^{\pm}_L(\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\exp\Big[-h(\mathtt{T}^{\pm}_L(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\Big], \tag{3.3}$$

*where functions $J^{\pm}_L=J^{\pm}_{L,s}$ give the Jacobians of maps $\mathtt{T}^{\pm}_L(s)$.*

(C)*Furthermore, the following properties hold true: $\forall\ \mathbf{\Omega}^*_\Lambda\in\mathcal{W}^*(\Lambda)$, $\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}}\in\mathcal{W}^*(\Lambda^{\mathtt{C}})$ with $\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}}\in\mathcal{G}_L$,*

$$\Big[J^+_L(\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})J^-_L(\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\Big]^{1/2}\geq 1-\delta, \tag{3.4a}$$

*and*

$$h(\mathtt{T}^+_L(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})+h(\mathtt{T}^-_L(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})-2h(\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})\leq\delta. \tag{3.4b}$$

□

*Remark.* As in Ref. 20, the dimension 2 is crucial for properties (3.4a) and (3.4b). Theorem 3.3 is the only place where condition $\overline{V}^{(2)}<+\infty$ is used. See (1.4).

Theorem 3.2 is deduced from Theorem 3.3 in a standard fashion (see Eqs. (2.10.**II**)–(2.12.**II**)).

The proof of Theorem 3.2 goes in parallel with that of Theorem 2.2.**II**; a particular role is played by a specific form of the Jacobians $J^{\pm}_L(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\mathtt{C}}})$; cf. Eq. (3.23.**II**). Here we mark the places where the proof of Theorem 2.2.**II** (see Secs. III **II**–V **II**) has to be modified, because of the assumption of non-negativity for the potentials $V_{j,j'}$ and the condition that fugacities $z_j\in(0, 1)$, $1\leq j\leq q$. (a) Every time we use the Ruelle bound (cf. Eqs. (3.27.**II**), (4.12.**II**), (4.21.**II**)), we should employ $z_j$ instead of $\overline{\rho}$ (defined in Eqs. (1.1.19.**I**) and (1.4.**II**). (b) The quantity $\mathtt{r}$ appearing in Eqs. (3.13.**II**), (4.4.**II**), (4.5.**II**), (4.8.**II**), (4.9.1.**II**), (4.9.2.**II**), (4.10.**II**) (4.13.**II**), (4.14.**II**), (4.17.**II**), (4.19.1.**II**), (4.19.2.**II**). (4.20.1.**II**) (4.20.2.**II**), (4.21.**II**), (5.8.**II**), and (5.9.**II**) should be set to be 0.

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