Integral bases and monogenity of pure fields

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April 26, 2016

Abstract

Let m be a square-free integer ($m \neq 0, 1$). We show that the structure of the integral bases of the fields $K = \mathbb{Q}(\sqrt[n]{m})$ are periodic in m. For $3 \leq n \leq 9$ we show that the period length is n^2. We explicitly describe the integral bases, and for $n = 3, 4, 5, 6, 8$ we explicitly calculate the index forms of K. This enables us in many cases to characterize the monogenity of these fields. Using the explicit form of the index forms yields a new technics that enables us to derive new results on monogenity and to get several former results as easy consequences. For $n = 4, 6, 8$ we give an almost complete characterization of the monogenity of pure fields.

1 Introduction

Let m be a square-free integer ($m \neq 0, 1$) and $n \geq 2$ a positive integer. There is an extensive literature of pure fields of type $K = \mathbb{Q}(\sqrt[n]{m})$. (Describing the

[*]Research supported in part by K115479 from the Hungarian National Foundation for Scientific Research

2010 *Mathematics Subject Classification*: Primary 11R04; Secondary 11Y50

Key words and phrases: pure fields, integral basis, power integral basis, monogenity

following results on pure fields we use some basic concepts on monogenity and power integral bases that are detailed in Section 2.)

B.K.Spearman and K.S.Williams [14] gave an explicit formula for the integral basis of pure cubic fields. B.K.Spearman, Y.Qiduan and J.Yoo [13] showed that if i is a cubefree positive integer then there exist infinitely many pure cubic fields with minimal index equal to i. I.Gaál and T.Szabó [11] studied the behaviour of the minimal indices of pure cubic fields in terms of the discriminant. L. El Fadil [12] gave conditions for the existence of power integral bases of pure cubic fields in terms of the index form equation.

T.Funakura [7] studied the integral basis in pure quartic fields. I.Gaál and L.Remete [9] calculated elements of index 1 (with coefficients $< 10^{1000}$) in pure quartic fields $K = \mathbb{Q}(\sqrt[4]{m})$ for $1 < m < 10^7$, $m \equiv 2, 3 \pmod 4$.

S.Ahmad, T.Nakahara and S.M.Husnine [3] showed that if $m \equiv 1 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$ then $\mathbb{Q}(\sqrt[6]{m})$ is not monogenic. On the other hand [4], if $m \equiv 2, 3 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$ then $\mathbb{Q}(\sqrt[6]{m})$ is monogenic.

A.Hameed and T.Nakahara [1] constructed integral bases of pure octic fields $\mathbb{Q}(\sqrt[8]{m})$. They proved [2] that if $m \equiv 1 \pmod 4$ then $\mathbb{Q}(\sqrt[8]{m})$ is not monogenic. On the other hand A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [5] proved that if $m \equiv 2, 3 \pmod 4$ then $\mathbb{Q}(\sqrt[8]{m})$ is monogenic.

A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [5] showed that if $m \equiv 2, 3 \pmod 4$ then $\mathbb{Q}(\sqrt[2^n]{m})$ is monogenic, this involves the pure quartic and pure octic fields, as well. Moreover, they showed [5] that if all the prime factors of n divide m then $\mathbb{Q}(\sqrt[n]{m})$ is monogenic.

Our purpose is for $3 \leq n \leq 9$ to give a general characterization of the integral basis of $K = \mathbb{Q}(\sqrt[n]{m})$. We prove that the integral bases of $K = \mathbb{Q}(\sqrt[n]{m})$ is periodic in m. For $3 \leq n \leq 9$ the period length is n^2.

The knowledge of the integral bases makes possible also to compete the sporadic results on the monogenity of these fields. Our method applying the explicit form of the index forms yields a new technics that enables us to obtain new results on the monogenity of these fields and to obtain several former results as easy consequences.

In our Theorems 4, 7, 8 we give an almost complete characterization of the monogenity pure quartic, sextic and octic fields, respectively. The cubic case is well-known and easy, much less is known about the quintic, septic and nonic cases.

2 Basic concepts about the monogenity of number fields

We recall those concepts [8] that we use throughout. Let α be a primitive integral element of the number field K (that is $K = \mathbb{Q}(\alpha)$) of degree n with ring of integers $\mathbb{Z}_K$. The **index** of α is

$$I(\alpha) = (\mathbb{Z}_K^+ : \mathbb{Z}[\alpha]^+) = \sqrt{\left|\frac{D(\alpha)}{D_K}\right|} = \frac{1}{\sqrt{|D_K|}} \prod_{1\le i<j\le n} \left|\alpha^{(i)} - \alpha^{(j)}\right| \quad ,$$

where D_K is the discriminant of K and $\alpha^{(i)}$ denote the conjugates of α. The minimal index of K is

$$i_K = \min I(\alpha)$$

where α runs through the primitive integral elements of K.

If $B = (b_1 = 1, b_2, \ldots, b_n)$ is an integral basis of K, then the **index form** corresponding to this integral basis is

$$I(x_2, \ldots, x_n) = \frac{1}{\sqrt{|D_K|}} \prod_{1\le i<j\le n} \left((b_2^{(i)} - b_2^{(j)})x_2 + \ldots + (b_n^{(i)} - b_n^{(j)})x_n\right)$$

(where $b_j^{(i)}$ denote the conjugates of b_j) which is a homogeneous polynomial with integral coefficients. For the integral element

$$\alpha = x_1 + b_2 x_2 + \ldots + b_n x_n$$

we have

$$I(\alpha) = |I(x_2, \ldots, x_n)|$$

independently of x_1. α generates a **power integral basis** $(1, \alpha, \ldots, \alpha^{n-1})$ if and only if $I(\alpha) = 1$ that is $(x_2, \ldots, x_n) \in \mathbb{Z}^{n-1}$ is a solution of the **index form equation**

$$I(x_2, \ldots, x_n) = \pm 1 \quad \text{in} (x_2, \ldots, x_n) \in \mathbb{Z}^{n-1}. \tag{1}$$

In this case

$$\mathbb{Z}_K = \mathbb{Z}[\alpha]$$

and K is called **monogenic**.

3 Basic results

Throughout we assume that m is a square-free integer with $m \neq 0, 1$ and $n > 2$ an integer. Let $K = \mathbb{Q}(\sqrt[n]{m})$ and $\vartheta = \sqrt[n]{m}$.

Our first theorem is on the prime divisors of the denominators of the integral basis elements:

Theorem 1. *If $(1, \vartheta, \ldots, \vartheta^{n-1})$ is not an integral basis in K, then for any element*

$$\alpha = \frac{a_0 + a_1\vartheta + \ldots + a_{n-1}\vartheta^{n-1}}{q} \tag{2}$$

of the integral basis (with $a_0, \ldots, a_{n-1}, q \in \mathbb{Z}$, $q \neq 0$) the denominator q can only be divisible by primes dividing n, the prime factors of q do not divide m.

Proof

The discriminant of $\vartheta = \sqrt[n]{m}$ is $\pm n^n m^{n-1}$. If $(1, \vartheta, \ldots, \vartheta^{n-1})$ is not an integral basis in $\mathbb{Q}(\vartheta)$, then there must be a number q dividing $n^n m^{n-1}$ and an element α of type (2) such that α is an algebraic integer and an element of $(1, \vartheta, \ldots, \vartheta^{n-1})$ can be replaced by α to get a basis with smaller discriminant.

Let p be a prime divisor of q. Then obviously

$$\alpha' = \frac{q}{p}\alpha = \frac{e_0 + e_1\vartheta + \ldots + e_{n-1}\vartheta^{n-1}}{p} \tag{3}$$

is also an algebraic integer. We can also assume that $0 \leq e_i < p$ $(0 \leq i \leq n-1)$ by taking each e_i modulo p.

We show that p is a divisor of n.

Assume on the contrary that $p|m$. The element

$$\alpha'\vartheta = \frac{e_0\vartheta + e_1\vartheta^2 + \ldots + e_{n-2}\vartheta^{n-1} + e_{n-1}m}{p}$$

is obviously an algebraic integer. By $p|m$ the element

$$\frac{e_0\vartheta + e_1\vartheta^2 + \ldots + e_{n-2}\vartheta^{n-1}}{p}$$

is also an algebraic integer. We proceed by multiplying this element by ϑ and omitting the analogous integral part. Finally we obtain that

$$\varrho = \frac{e_0 \vartheta^{n-1}}{p}$$

is an algebraic integer. The element ϱ is the root of the polynomial

$$f_\varrho(x) = px^n - e_0^n \left(\frac{m}{p} \right)^{n-1}.$$

This polynomial is irreducible over $\mathbb{Q}$ if and only if its reciprocal polynomial

$$f_{1/\varrho}(x) = e_0^n \left(\frac{m}{p} \right)^{n-1} x^n - p.$$

is irreducible. Here m/p is an integer, not divisible by p because m is square-free. e_0 is also not divisible by p (otherwise we did not have e_0 in (3) and we had the same result with the first non-zero e_i). Hence $f_{1/\varrho}(x)$ is an Eisentein polynomial, therefore $f_\varrho(x)$ is irreducible. Then $f_\varrho(x)$ is the defining polynomial of ϱ. This contradicts to ϱ being an algebraic integer. □

Remark. Theorem 1 implies Theorem 3.1. of A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [5]: if all the prime factors of n divide m then $\mathbb{Q}(\sqrt[n]{m})$ is monogenic.

Next we show that the integral bases of $K = \mathbb{Q}(\sqrt[n]{m})$ are periodic. First we prove this statement with a period length much larger than n^2 but this result is valid for any n.

Theorem 2. *Let $n = p_1^{h_1} \dots p_k^{h_k}$ and $n_0 = p_1^{[nh_1/2]} \dots p_k^{[nh_k/2]}$ where $[x]$ denotes the lower integer part of x. Let $\vartheta = \sqrt[n]{m}$ and $\gamma = \sqrt[n]{m + n_0^n}$. Then the structure of the integral bases of the fields $\mathbb{Q}(\vartheta)$ and $\mathbb{Q}(\gamma)$ is the same in terms of ϑ and γ, respectively.*

Remark. Under the "same structure" we mean that if the integral basis of $\mathbb{Q}(\vartheta)$ has an element

$$\frac{a_0 + a_1\vartheta + \ldots + a_{n-1}\vartheta^{n-1}}{q},$$

then the integral basis of $\mathbb{Q}(\gamma)$ has an element

$$\frac{a_0 + a_1\gamma + \ldots + a_{n-1}\gamma^{n-1}}{q}$$

and vice versa.

Proof

Assume that $(1, \vartheta, \ldots, \vartheta^{n-1})$ is not an integral basis in $\mathbb{Q}(\vartheta)$. Then there must be integer elements of type

$$\alpha = \frac{a_0 + a_1\vartheta + \ldots + a_{n-1}\vartheta^{n-1}}{q} \tag{4}$$

which can replace elements of $(1, \vartheta, \ldots, \vartheta^{n-1})$ to obtain an integral basis. We show that the existence of analogous algebraic integers of type (4) is equivalent in the fields generated by $\vartheta = \sqrt[n]{m}$ and by $\gamma = \sqrt[n]{m + n_0^n}$.

If we replace an element of the basis $(1, \vartheta, \ldots, \vartheta^{n-1})$ by α of (4) then the discriminant of the basis decreases by a factor q^2. Hence q^2 divides $\pm n^n m^{n-1}$ (the discriminant of $\vartheta = \sqrt[n]{m}$ is $\pm n^n m^{n-1}$). By Theorem 1 the prime divisors p of q do not divide m. Hence $q^2 | n^n$, which implies that q divides n_0.

Denote the conjugates of α by $\alpha^{(j)}, j = 1, \ldots, n$. The defining polynomial of α is

$$\prod_{j=1}^{n}(x - \alpha^{(j)}) = \frac{1}{q^n}\prod_{j=1}^{n}(qx - a_0 - a_1\vartheta^{(j)} - \ldots - a_{n-1}(\vartheta^{(j)})^{n-1}).$$

The product is a symmetrical polynomial of $\vartheta^{(1)}, \ldots, \vartheta^{(n)}$, hence its coefficients can be expressed as polynomials (with integer coefficients) of the defining polynomial of ϑ, that is $x^n - m$. Hence there exist polynomials $P_0, \ldots, P_{n-1} \in \mathbb{Z}[x]$ such that

$$\prod_{j=1}^{n}(x - \alpha^{(j)}) = \frac{1}{q^n}((qx)^n + P_{n-1}(m)(qx)^{n-1} + \ldots + P_1(m)(qx) + P_0(m)).$$

Therefore the element α is an algebraic integer if and only if $q^n | q^j P_j(m)$ that is

$$q^{n-j} | P_j(m) \quad (j = 0, 1, \ldots, n-1). \tag{5}$$

Replace now m by $m' = m + n_0^n$ and consider integral bases in the field $\mathbb{Q}(\gamma) = \mathbb{Q}(\sqrt[n]{m + n_0^n})$. In this field an element of type (4), that is

$$\delta = \frac{a_0 + a_1\gamma + \ldots + a_{n-1}\gamma^{n-1}}{q} \tag{6}$$

is an algebraic integer if and only if

$$q^{n-j} | P_j(m + n_0^n) \ \ (j = 0, 1 \ldots, n-1) \tag{7}$$

with the same polynomials P_j. By $q|n_0$ the conditions (5) are equivalent to (7). Therefore $\mathbb{Q}(\vartheta)$ and $\mathbb{Q}(\gamma)$ contains the same type of integer elements. Elements of that type are linearly independent in the first case if and only if they are linearly independent in the second case. Therefore $\mathbb{Q}(\vartheta)$ and $\mathbb{Q}(\gamma)$ admits the same type of integral bases. □

4 The structure of the integral bases is periodic in m modulo n^2 for $3 \le n \le 9$

Theorem 2 implies that the integral bases of $K = \mathbb{Q}(\sqrt[n]{m})$ are periodic modulo n_0^n. This number is of magnitude $n^{n^2/2}$. For small values of n we have a much sharper assertion.

Theorem 3. *For $3 \le n \le 9$ the integral bases of $\mathbb{Q}(\sqrt[n]{m})$ are periodic in m modulo n^2.*

Proof
For $n = 3, 4, 5$ the n_0^n is 27, 655536, 9765625, respectively. Calculating the integral bases of $\mathbb{Q}(\sqrt[n]{m})$ for square-free m up to n_0^n it is easily seen that the structure of the integral bases of $\mathbb{Q}(\sqrt[n]{m})$ are periodic modulo n^2. One can easily detect a few types of integral bases that are repeated for square-free values of m, $m + n^2$, $m + 2n^2$ etc.

Let now $n > 5$. Then n_0^n is far too large for the calculations described above. However for $n = 6, 7, 8, 9$ we managed to prove the same assertion.

Let $1 < r < n^2$. If r is square-free, then set $r' = r$. If r has a common square factor with n, then none of $r + kn^2$ is square-free, we omit r. If r has no common square factor with n but contains another square factor, then we set $r' = r + n^2$ or $r' = r + 2n^2$ etc. which is already square-free.

Let $\vartheta = \sqrt[n]{r'}$, calculate the integral bases of $\mathbb{Q}(\sqrt[n]{r'})$ and denote the basis elements by $(b_1 = 1, b_2, \ldots, b_n)$, where b_j is of the form

$$b_j = \frac{a_{j0} + a_{j1}\vartheta + \ldots + a_{j,n-1}\vartheta^{n-1}}{q}$$

(with $a_{j0}, a_{j1}, \ldots, a_{j,n-1} \in \mathbb{Z}$ and with a non-zero denominator q the prime factors of which divide n).

Let $m = r + kn^2$ be a square-free integer, $\gamma = \sqrt[n]{m}$ and

$$b'_j = \frac{a_{j0} + a_{j1}\gamma + \ldots + a_{j,n-1}\gamma^{n-1}}{q}.$$

We wonder

I. if the analogues of the elements b_j, that is the elements b'_j remain algebraic integer for any square-free $m = r + kn^2$, further

II. if for some square-free $m = r + kn^2$ some of the basis elements $(b'_1 = 1, b'_2, \ldots, b'_n)$ can be replaced by an integral element of type

$$d = \frac{e_1 b'_1 + \ldots + e_n b'_n}{p} \tag{8}$$

(where $0 \leq e_1, \ldots, e_n \leq p - 1$ and p is a prime divisor of n) to obtain a basis with smaller discriminant.

I. The defining polynomial of $e_1 b'_1 + \ldots + e_n b'_n$ is

$$G(x) = \prod_{j=1}^{n} (x - e_1 b_1'^{(j)} - \ldots - e_n b_n'^{(j)}).$$

This polynomial is symmetrical in the conjugates of $\gamma = \sqrt[n]{m}$, hence its coefficients will be polynomials in m:

$$G(x) = x^n + G_{n-1}(m)x^{n-1} + \ldots + G_1(m)x + G_0(m).$$

Since there are denominators in the b_i', the polynomials G_j (depending also on $e_1,\dots,e_n$) are not necessarily of integer coefficients. Let us substitute $m=r+kn^2$. We obtain

$$G(x)=x^n+H_{n-1}(k)x^{n-1}+\ldots+H_1(k)x+H_0(k).$$

For all possible residues r we have explicitly calculated these polynomials $H_j(k)$ which also depend on $e_1,\dots,e_n$. In all cases we found that these are polynomials in $k,e_1,\dots,e_n$ with integer coefficients. Substituting $e_i=1$ and $e_j=0, j=1,\dots,n,\ j\neq i$ this implies that b_i' is integer for any square-free $m=r+kn^2$ $(1\leq i\leq n)$.

II.Consider now the defining polynomial of d (8), that is

$$P(x)=\frac{1}{p^n}G(px)=\frac{1}{p^n}\left((px)^n+H_{n-1}(k)(px)^{n-1}+\ldots+H_1(k)(px)+H_0(k)\right).$$

This polynomial has integer coefficients if and only if p^n divides $p^jH_j(k)$ that is

$$p^{n-j}|H_j(k)=H_j(k,e_1,\dots,e_n)\ \ (0\leq j\leq n-1). \tag{9}$$

Let now $(e_1,\dots,e_n)\in\mathbb{Z}^n$ be an arbitrary given fixed tuple. Obviously by (9) the validity of the statement if d is integral or not, depends only on the behaviour of k modulo p^n and not on the value of k. This allows us to test the fields $\mathbb{Q}(\sqrt[n]{m})$ for square-free $m=r+kn^2$ where k runs though all residue classes modulo p^n. These fields were tested directly, calculating their integral bases. We found that in all cases the fields had the same structure of integral basis in terms of $\gamma=\sqrt[n]{m}$ like the field $\mathbb{Q}(\sqrt[n]{r'})$ in terms of $\vartheta=\sqrt[n]{r'}$. This proves our assertion. $\Box$

Remark The test described at the end of the above proof required to calculate the integral bases of
$24(2^6+3^6)=18239$ sextic fields,
$48\cdot 7^7=39530064$ septic fields,
$48\cdot 2^8=12288$ octic fields and
$72\cdot 3^9=1417176$ nonic fields.

5 Integral bases and monogenity of pure fields

In this section we give a list of our results on the fields $K = \mathbb{Q}(\sqrt[n]{m})$ for $3 \leq n \leq 9$. According to Theorem 3 we set $m = r + kn^2$ where $1 < r < n^2$ and m is square free. If r has a common square factor with n, then none of $r + kn^2$ is square-free, we omit r. If r has no common square factor with n but contains another square factor, then in our computations it is replaced by $r + n^2$ or $r + 2n^2$ etc. which is already square-free, but the case will still be represented by r and will cover fields $K = \mathbb{Q}(\sqrt[n]{m})$ with square-free $m = r + kn^2$.

For all m we give the integral basis B and discriminant D of K. As far as it is possible we display the index form $I(x_2, \ldots, x_n)$ corresponding to the integral basis and discuss the monogenity of K. Mentioning here the index form equation we always mean the equation (1).

Stating that the index form equation is not solvable modulo q in certain cases $m = r + kn^2$ (with a fixed r) we mean that if we let $x_2, \ldots, x_n$ and k run through all residue classes modulo q we never have $I(x_2, \ldots, x_n) \equiv \pm 1 \pmod{q}$. This implies that the corresponding fields admit no power integral bases, are not monogenic.

5.1 Pure cubic fields, $K = \mathbb{Q}(\sqrt[3]{m})$

In these cases the index form equation is a cubic Thue equation.

Case 3.1. $r = 2, 3, 4, 5, 6, 7$, $m = r + 9k$ square-free

$$B = \{1, x, x^2\}, \quad D = -27m^2$$

$$I(x_2, x_3) = x_2^3 - mx_3^3$$

These fields are obviously monogenic, $(1, 0)$ is a solution of the index form equation.

Case 3.2. $r = 1$, $m = 1 + 9k$ square-free

$$B = \left\{1, x, \frac{1 + x + x^2}{3}\right\}, \quad D = -3m^2$$

$$I(x_2, x_3) = 3x_2^3 + 3x_2^2 x_3 + x_2 x_3^2 - kx_3^3$$

In this case the index form equation is solvable e.g. for $k = 27, 37$ but not solvable e.g. for $k = 10, 11, 12$.

Case 3.3. $r = 8$, $m = 8 + 9k$ square-free

$$B = \left\{1, x, \frac{1 + 2x + x^2}{3}\right\}, \quad D = -3m^2$$

$$I(x_2, x_3) = 3x_2^3 + 6x_2^2 x_3 + 4x_2 x_3^2 - kx_3^3$$

In this case the index form equation is solvable e.g. for $k = 1, 4, 12$ but not solvable e.g. for $k = 2, 3, 5, 6, 7$.

5.2 Pure quartic fields, $K = \mathbb{Q}(\sqrt[4]{m})$

In these cases the index form is the product of a quadratic and a quartic form.

Case 4.1. $r = 2, 3, 6, 7, 10, 11, 14, 15$, $m = r + 16k$ square-free

$$B = \{1, x, x^2, x^3\}, \quad D = -256m^3$$

$$I(x_2, x_3, x_4) = (x_2^2 - mx_4^2)(x_2^4 + 2mx_2^2 x_4^2 + m^2 x_4^4 + 4mx_3^4 - 8mx_2 x_4 x_3^2)$$

These fields are monogenic, $(1, 0, 0)$ is a solution of the index form equation. This follows also from A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [5].

Case 4.2. $r = 1, 9$, $m = 1 + 8k$ square-free

$$B = \left\{1, x, \frac{1+x^2}{2}, \frac{1+x+x^2+x^3}{4}\right\}, \quad D = -4m^3$$

$$I(x_2, x_3, x_4) = (-x_2x_4 - 2x_2^2 + x_4^2k)\cdot$$
$$(x_4^4k^2 - 2x_4^3x_2k - 16x_4^2x_3x_2k + 4x_4^2x_2^2k + 8x_4^2x_3^2k - 16x_4x_3^2x_2k + 16x_4x_3^3k + 8x_3^4k$$
$$+2x_4^2x_2^2 - 2x_4^2x_3x_2 + x_4^2x_3^2 - 2x_4x_3^2x_2 + 4x_2^3x_4 + 2x_4x_3^3 + x_3^4 + 4x_2^4)$$

If $m = 1 + 16\ell$, that is $k = 2\ell$ then the index form equation is not solvable modulo 2.
If $m = 9 + 16\ell$, that is $k = 2\ell + 1$ then the index form equation has a solution for $\ell = 4, 5$ that is for $m = 73, 89$ (the solution is (2,1,1)). For other parameters we conjecture that the minimal index of K is 8.

Case 4.3. $r = 5, 13$, $m = 5 + 8k$ square-free

$$B = \left\{1, x, \frac{1+x^2}{2}, \frac{x+x^3}{2}\right\}, \quad D = -16m^3$$

$$I(x_2, x_3, x_4) = (-x_2x_4 - x_2^2 + 2x_4^2k + x_4^2)\cdot$$
$$(16x_4^4k^2 + 24x_4^4k + 16x_4^3x_2k - 16x_4^2x_3^2k + 16x_4^2x_2^2k - 32x_4x_3^2x_2k + 8x_3^4k$$
$$+9x_4^4 + 12x_4^3x_2 - 10x_4^2x_3^2 + 16x_4^2x_2^2 + 8x_2^3x_4 - 20x_4x_3^2x_2 + 4x_2^4 + 5x_3^4)$$

Denote by f_1 and f_2 the first and second factor of the index form, respectively. Then we have

$$f_2 - 4f_1^2 = (8k+5)(2x_2x_4 - x_3^2 + x_4^2)^2.$$

If K is monogenic, then for some x_2, x_3, x_4 we have $f_1, f_2 = \pm 1$, hence $f_2 - 4f_1^2 = -3$ or -5. This number can only be divisible by $8k + 5$ for $k = -1$. The field $K = \mathbb{Q}(\sqrt[4]{-3})$ is monogenic, e.g. $(1, 1, 0)$ is a solution of the index form equation. All other fields of this type are not monogenic.

Summarizing the above statements we have

Theorem 4. *For the following values of r let $m = r + 16k$ ($k \in \mathbb{Z}$) be a square-free integer. The field $K = \mathbb{Q}(\sqrt[4]{m})$ is monogenic for $r = 2, 3, 6, 7, 10, 11, 14, 15$ and is not monogenic for $r = 1, 5, 13$ with the exception of $\mathbb{Q}(\sqrt[4]{-3})$ which is monogenic.*

Remark 5. *We conjecture that for $m = 9 + 16k$ the only monogenic fields are $\mathbb{Q}(\sqrt[4]{73})$, $\mathbb{Q}(\sqrt[4]{89})$.*

5.3 Pure quintic fields, $K = \mathbb{Q}(\sqrt[5]{m})$

Case 5.1. $r = 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23$, $m = r + 25k$ square-free

$$B = \{1, x, x^2, x^3, x^4\}, \quad D = 3125m^4.$$

These fields are obviously monogenic. The index form is very complicated:

$$\begin{gathered}
I(x_2, x_3, x_4, x_5) = -75m^4x_2x_3^2x_4^3x_5^4+45m^4x_2^2x_3x_4^2x_5^5+40m^4x_2x_3^3x_4x_5^5-40m^4x_2x_3x_4^5x_5^3 \\
-75m^2x_2^4x_3^3x_4^2x_5-40m^2x_2^3x_3^5x_4x_5+45m^2x_2^5x_3^2x_4x_5^2+40m^2x_2^5x_3x_4^3x_5+75m^3x_2^3x_3x_4^4x_5^2 \\
+75m^3x_2^2x_3^4x_4x_5^3 + 50m^3x_2^4x_3x_4x_5^4 - 200m^3x_2^3x_3^2x_4^2x_5^3 + 200m^3x_2^2x_3^3x_4^3x_5^2 \\
-45m^3x_2^2x_3^2x_4^5x_5-45m^3x_2x_3^5x_4^2x_5^2-50m^3x_2x_3^4x_4^4x_5-20m^5x_3^2x_4x_5^7+5m^5x_2x_3x_5^8 \\
+35m^5x_3x_4^3x_5^6 - 15m^5x_2x_4^2x_5^7 - 5m^4x_2^3x_4x_5^6 + 20m^4x_2x_4^7x_5^2 + 25m^4x_2^2x_4^4x_5^4 \\
-25m^4x_3^4x_4^2x_5^4 + 25m^4x_3^3x_4^4x_5^3 - 5m^4x_3x_4^8x_5 - 10m^4x_2^2x_3^2x_5^6 + 10m^4x_3^2x_4^6x_5^2 \\
-15mx_2^7x_3^2x_5-20mx_2^7x_3x_4^2+5mx_2^8x_4x_5+35mx_2^6x_3^3x_4+20m^2x_2^2x_3^7x_5-5m^2x_2^6x_3x_5^3 \\
+25m^2x_2^4x_3^4x_5^2 - 25m^2x_2^4x_3^2x_4^4 + 25m^2x_2^3x_3^4x_4^3 - 5m^2x_2x_3^8x_4 - 10m^2x_2^6x_4^2x_5^2 \\
+10m^2x_2^2x_3^6x_4^2-35m^3x_2^3x_4^6x_5+15m^3x_2^2x_3x_4^7-35m^3x_2x_3^6x_5^3+5m^3x_2x_3^3x_4^6+15m^3x_3^7x_4x_5^2 \\
+5m^3x_3^6x_4^3x_5 - 25m^3x_2^4x_4^3x_5^3 - 25m^3x_2^3x_3^3x_5^4 + x_2^{10} - m^4x_4^{10} - m^2x_3^{10} + x_5^10m^6 \\
-11m^5x_4^5x_5^5 + 11m^4x_3^5x_5^5 - 11mx_2^5x_3^5 + 11m^2x_2^5x_4^5 - 2m^3x_2^5x_5^5 + 2m^3x_3^5x_4^5
\end{gathered}$$

Case 5.2. $r = 1$, $m = 1 + 25k$ square-free

$$B = \left\{1, x, x^2, x^3, \frac{1 + x + x^2 + x^3 + x^4}{5}\right\}, \quad D = 125m^4$$

Case 5.3. $r = 7$, $m = 7 + 25k$ square-free

$$B = \left\{1, x, x^2, x^3, \frac{1 + 3x + 4x^2 + 2x^3 + x^4}{5}\right\}, \quad D = 125m^4$$

Set $m = 7 + 25k$. For $k = 0$ the tuple $(0, -2, -1, 2)$ is a solution of the index form equation, therefore $K = \mathbb{Q}(\sqrt[5]{7})$ is monogenic. (For $k = 1$ the $m = 32$ is not square free.)

Case 5.4. $r = 18$, $m = 18 + 25k$ square-free

$$B = \left\{1, x, x^2, x^3, \frac{1 + 2x + 4x^2 + 3x^3 + x^4}{5}\right\}, \quad D = 125m^4$$

Case 5.5. $r = 24$, $m = 24 + 25k$ square-free

$$B = \left\{1, x, x^2, x^3, \frac{1 + 4x + x^2 + 4x^3 + x^4}{5}\right\}, \quad D = 125m^4$$

Remark 6. *For the following values of r let $m = r + 25k$ ($k \in \mathbb{Z}$) be a square-free integer. $K = \mathbb{Q}(\sqrt[5]{m})$ is monogenic for $r = 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23$. We conjecture that for $r = 1, 7, 18, 24$ the fields $\mathbb{Q}(\sqrt[5]{m})$ are not monogenic having minimal index 5 with the exception of $\mathbb{Q}(\sqrt[5]{7})$ which is monogenic.*

5.4 Pure sextic fields, $K = \mathbb{Q}(\sqrt[6]{m})$

In all these cases the index form is the product of three factors of degrees 3,6,6, respectively. We shall denote these factors by f_1, f_2, f_3. These depend on the parameter m and on the variables $x_2, \ldots, x_6$.

Case 6.1. $r = 2, 3, 6, 7, 11, 14, 15, 22, 23, 30, 31, 34$, $m = r + 36k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5\}, \quad D = 6^6 m^5$$

In this case K is monogenic. This follows also from the result of S.Ahmad, T.Nakahara and S.M.Husnine [4] since $m \equiv 2, 3 \pmod{4}$ and $m \not\equiv \pm 1 \pmod{9}$. We have

$$I(x_2, x_3, x_4, x_5) = (-3mx_2x_4x_6 + x_2^3 + mx_4^3 + m^2x_6^3)$$

$$\begin{gathered}
(18m^2x_2^2x_3x_5x_6^2 - 18m^2x_2x_3^2x_5^2x_6 - 3m^3x_3^2x_6^4 - 2m^2x_2^3x_6^3 + 3m^2x_2^2x_5^4 + 3m^2x_3^4x_6^2 \\
+2m^2x_3^3x_5^3 - 3mx_2^4x_5^2 - 6m^3x_2x_5^2x_6^3 + 6m^3x_3x_5^3x_6^2 - 6mx_2^3x_3^2x_6 + 6mx_2^2x_3^3x_5 + m^4x_6^6 \\
-m^3x_5^6 - mx_3^6 + x_2^6)
\end{gathered}$$

$$\begin{gathered}
(x_2^6 + 64m^2x_4^6 + m^4x_6^6 + 27m^3x_5^6 + 27mx_3^6 - 216m^2x_3^3x_4x_5x_6 - 72mx_2^3x_3x_4x_5 + 12m^3x_2x_4x_6^4 \\
+108m^3x_4^2x_5^2x_6^2 - 108m^3x_4x_5^4x_6 + 36m^2x_2^2x_4^2x_6^2 - 96m^2x_2x_4^4x_6 + 144m^2x_2x_4^3x_5^2 \\
+144m^2x_3^2x_4^3x_6 + 324m^2x_3^2x_4^2x_5^2 - 288m^2x_3x_4^4x_5 + 12mx_2^4x_4x_6 + 108mx_2^2x_3^2x_4^2 \\
-108mx_2x_3^4x_4 - 18m^3x_2x_5^2x_6^3 + 54m^3x_3x_5^3x_6^2 - 18mx_2^3x_3^2x_6 + 54mx_2^2x_3^3x_5 \\
-72m^3x_3x_4x_5x_6^3 - 216m^2x_2x_3x_4x_5^3 - 54m^2x_2^2x_3x_5x_6^2 + 162m^2x_2x_3^2x_5^2x_6 \\
-16m^3x_4^3x_6^3 - 16mx_2^3x_4^3 + 9m^3x_3^2x_6^4 + 2m^2x_2^3x_6^3 + 27m^2x_2^2x_5^4 + 27m^2x_3^4x_6^2 \\
-54m^2x_3^3x_5^3 + 9mx_2^4x_5^2)
\end{gathered}$$

Case 6.2. $r = 5, 13, 21, 25, 29, 33$, $m = r + 36k$ square-free

$$\left\{1, x, x^2, \frac{1+x^3}{2}, \frac{x+x^4}{2}, \frac{x^2+x^5}{2}\right\}, \quad D = 3^6 m^5$$

Calculating the index form it is easily seen that the index form equation is not solvable modulo 2, hence these fields are not monogenic. This also follows (in a much more complicated way) from the theorem of S. Ahmad, T. Nakahara and S. M. Husnine [3] since all these m are of the form $m \equiv 1 \pmod{4}$ and $m \not\equiv \pm 1 \pmod{9}$.

Case 6.3. $r = 10, 19$, $m = r + 36k$ square-free

$$\left\{1, x, x^2, x^3, \frac{1+x^2+x^4}{3}, \frac{x+x^3+x^5}{3}\right\}, \quad D = 2^6 3^2 m^5$$

Calculating the index form it is easily seen that the index form equation is not solvable modulo 3, hence these fields are not monogenic. This case is not covered by S. Ahmad, T. Nakahara and S. M. Husnine [3], [4] since $m \equiv 1 \pmod 9$.

Case 6.4. $r = 26, 35$, $m = r + 36k$ square-free

$$\left\{1, x, x^2, x^3, \frac{1+2x^2+x^4}{3}, \frac{x+2x^3+x^5}{3}\right\}, \quad D = 2^6 3^2 m^5$$

This case is not covered by S. Ahmad, T. Nakahara and S. M. Husnine [3], [4] since $m \equiv -1 \pmod 9$.

If $m = 26 + 36k$ then $4m | (f_2 - 9f_3)$. If K is monogenic then for a solution of the index form equation these factors are equal to ± 1. The possible values of $f_2 - 9f_3$ are $\pm 8, \pm 10$, hence the above divisibility can not hold.

If $m = 35 + 36k$ then $4(35 + 36k) | (f_2 - 9f_3)$. If K is monogenic then for a solution of the index form equation these factors are equal to ± 1. The possible values of $f_2 - 9f_3$ are $\pm 8, \pm 10$, hence the above divisibility can only hold for $k = -1$, that is $m = -1$. It is easily seen that the relative index [10] of elements of $K = \mathbb{Q}(\sqrt[6]{-1})$ is divisible by 9, therefore this field is not monogenic, either.

Case 6.5. $r = 17$, $m = r + 36k$ square-free

$$\left\{1, x, x^2, \frac{1+x^3}{2}, \frac{4+3x+2x^2+x^4}{6}, \frac{4x+3x^2+2x^3+x^5}{6}\right\}, \quad D = 3^2 m^5$$

This case is not covered by S. Ahmad, T. Nakahara and S. M. Husnine [3], [4] since $m \equiv -1 \pmod 9$. Calculating the index form we can easily see that the index form equation is not solvable modulo 6.

Case 6.6. $m = 1$, $m = 1 + 36k$ square-free

$$\left\{1, x, x^2, \frac{1+x^3}{2}, \frac{4+3x+4x^2+x^4}{6}, \frac{3+4x+3x^2+x^3+x^5}{6}\right\}, \quad D = 3^2 m^5$$

This case is not covered by S. Ahmad, T. Nakahara and S. M. Husnine [3], [4] since $m \equiv 1 \pmod 9$. Calculating the index form we can easily see that the index form equation is not solvable modulo 3.

Summarizing the above statements we have

Theorem 7. *For the following values of* r *let* $m = r + 36k$ $(k \in \mathbb{Z})$ *be a square-free integer. The field* $K = \mathbb{Q}(\sqrt[6]{m})$ *is monogenic for*
$r = 2, 3, 6, 7, 11, 14, 15, 22, 23, 30, 31, 34$ *and is not monogenic for*
$r = 1, 5, 10, 13, 17, 19, 21, 25, 26, 29, 33, 35.$

6 Pure septic fields, $K = \mathbb{Q}(\sqrt[7]{m})$

Case 7.1. $r = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47,$
$m = r + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, x^6\}, \quad D = -m^6 7^7$$

These fields are obviously monogenic.

Case 7.2. $r = 18$, $m = 18 + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, \frac{1 + 2x + 4x^2 + x^3 + 2x^4 + 4x^5 + x^6}{7}\}, \quad D = -m^6 7^5$$

Case 7.3. $r = 19$, $m = 19 + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, \frac{1 + 3x + 2x^2 + 6x^3 + 4x^4 + 5x^5 + x^6}{7}\}, \quad D = -m^6 7^5$$

Case 7.4. $r = 30$, $m = 30 + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, \frac{1 + 4x + 2x^2 + x^3 + 4x^4 + 2x^5 + x^6}{7}\}, \quad D = -m^6 7^5$$

Case 7.5. $r = 31$, $m = 31 + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, \frac{1 + 5x + 4x^2 + 6x^3 + 2x^4 + 3x^5 + x^6}{7}\}, \quad D = -m^6 7^5$$

Case 7.6. $r = 48$, $m = 48 + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, \frac{1 + 6x + x^2 + 6x^3 + x^4 + 6x^5 + x^6}{7}\}, \quad D = -m^6 7^5$$

Case 7.7. $r = 1$, $m = 1 + 49k$ square-free

$$B = \{1, x, x^2, x^3, x^4, x^5, \frac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{7}\}, \quad D = -m^6 7^5$$

7 Pure octic fields, $K = \mathbb{Q}(\sqrt[8]{m})$

In all these cases the index form is the product of three factors of degrees 4,8,16, respectively. We shall denote these factors by f_1, f_2, f_3. These depend on the parameter m and on the variables $x_2, \ldots, x_8$.

Case 8.1. $r = 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31, 34, 35, 38, 39, 42, 43, 46, 47, 50, 51, 54, 55, 58, 59, 62, 63$, $m = r + 64k$ square-free

$$\left\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\right\}, \quad D = -8^8 m^7$$

These fields are obviously monogenic. This also follows from A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [5].

Case 8.2. $r = 1, 17, 33, 49$, $m = r + 64k$ square-free

$$\left\{1, x, x^2, x^3, \frac{1+x^4}{2}, \frac{x+x^5}{2}, \frac{1+x^2+x^4+x^6}{4},\right.$$
$$\left.\frac{1+x+x^2+x^3+x^4+x^5+x^6+x^7}{8}\right\}, \quad D = -2^{10}m^7$$

These fields are not monogenic by the theorem of A.Hameed and T.Nakahara [2]. We conjecture that in these fields the minimal index is 128.

Case 8.3. $r = 5, 13, 21, 29, 37, 45, 53, 61$, these cases can be included by $m = 5 + 8k$, square-free

$$\left\{1, x, x^2, x^3, \frac{1+x^4}{2}, \frac{x+x^5}{2}, \frac{x^2+x^6}{2}, \frac{x^3+x^7}{2}\right\}, \quad D = -2^{16}m^7$$

Calculating and factorizing $f_3 - 16f_2^2$ we find that it is divisible by m. If there existed a power integral basis then for a solution of the index form equation we would have $f_1, f_2, f_3 = \pm 1$, hence $f_3 - 16f_2^2 = \pm 1 - 16$ is either -15 or -17. The possible divisors are $\pm 3, \pm 5, \pm 15, \pm 17$ but only $m = -3, 5$ is of type $m = 5 + 8k$.

For $m = -3$ then element $(-1, -1, 0, 1, 1, 0, -1)$ has index one, hence $K = \mathbb{Q}(\sqrt[8]{-3})$ is monogenic.

For $m = 5$ the least index we found in $K = \mathbb{Q}(\sqrt[8]{5})$ was 16.

Note that A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [2] assert that these fields are not monogenic, they certainly did not involve $K = \mathbb{Q}(\sqrt[8]{-3})$.

Case 8.4. $r = 9, 25, 41, 57$, these cases can be included by $m = 9 + 16k$, square-free

$$\left\{1, x, x^2, x^3, \frac{1+x^4}{2}, \frac{x+x^5}{2}, \frac{1+x^2+x^4+x^6}{4}, \frac{x+x^3+x^5+x^7}{4}\right\}, \quad D = -2^{12}m^7$$

Calculating and factorizing $f_2 - 4f_1^2$ we find that it is divisible by m. If there existed a power integral basis then for a solution of the index form equation we would have $f_1, f_2, f_3 = \pm 1$, hence $f_2 - 4f_1^2 = \pm 1 - 4$ is either -3 or -5. The possible divisors are $\pm 3, \pm 5$ but none of them is of type $m = 9 + 16k$. Therefore these fields are not monogenic. This also follows from the theorem of A.Hameed, T.Nakahara, S.M.Husnine and S.Ahmad [2].

Summarizing the above statements we have

Theorem 8. *For the following values of* r *let* $m = r + 64k$ $(k \in \mathbb{Z})$ *be a square-free integer. The field* $K = \mathbb{Q}(\sqrt[8]{m})$ *is monogenic for*
$r = 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31, 34, 35, 38, 39, 42, 43, 46, 47, 50, 51, 54, 55, 58, 59, 62, 63$ *and is not monogenic for*
$r = 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61$*,* $m \neq 5$*, with the exception of* $K = \mathbb{Q}(\sqrt[8]{-3})$ *which is monogenic.*

Remark 9. *We conjecture that the minimal index of* $K = \mathbb{Q}(\sqrt[8]{5})$ *is 16. Octic fields of this type will be considered in a forecoming paper.*

8 Pure nonic fields, $K = \mathbb{Q}(\sqrt[9]{m})$

Case 9.1. $r = 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, 29, 30, 31, 32, 33, 34, 38, 39, 40, 41, 42, 43, 47, 48, 49, 50, 51, 52, 56, 57, 58, 59, 60, 61, 65, 66, 67, 68, 69, 70, 74, 75, 76, 77, 78, 79$, $m = r + 81k$, square-free

$$\left\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\right\}, \quad D = 3^{18} m^8$$

These fields are obviously monogenic.

Case 9.2. $r = 1, 28, 55$, $m = r + 81k$, square-free

$$\left\{1, x, x^2, x^3, x^4, x^5, \frac{1 + x^3 + x^6}{3}, \frac{x + x^4 + x^7}{3}, \right.$$

$$\left.\frac{1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8}{9}\right\}, \quad D = 3^{10}m^8$$

Case 9.3. $r = 8, 17, 35, 44, 62, 71$, $m = r + 81k$, square-free

$$\left\{1, x, x^2, x^3, x^4, x^5, \frac{1+2x^3+x^6}{3}, \frac{x+2x^4+x^7}{3}, \frac{x^2+2x^5+x^8}{3}\right\}, \quad D = 3^{12}m^8$$

Case 9.4. $r = 10, 19, 37, 46, 64, 73$, $m = r + 81k$, square-free

$$\left\{1, x, x^2, x^3, x^4, x^5, \frac{1+x^3+x^6}{3}, \frac{x+x^4+x^7}{3}, \frac{x^2+x^5+x^8}{3}\right\}, \quad D = 3^{12}m^8$$

Case 9.5. $r = 26, 53, 80$, $m = r + 81k$, square-free

$$\left\{1, x, x^2, x^3, x^4, x^5, \frac{1+2x^3+x^6}{3}, \frac{x+2x^4+x^7}{3},\right.$$

$$\left.\frac{1+2x+x^2+8x^3+7x^4+8x^5+x^6+2x^7+x^8}{9}\right\}, D = 3^{10}m^8$$

9 Computational remarks

In all our calculations we used Maple [6] and most of our programs executed a couple of seconds or a few minutes on an average laptop. For $n = 4, 6, 8$ we needed a very careful calculation of the factors of the index forms, which may take extremely long otherwise.

The tests corresponding to Theorem 3 took also a few minutes for $n = 3, 4, 5, 6, 8$. For $n = 9$ it executed 5 hours. For $n = 7$ we executed our Malpe program on a supercomputer with nodes having 24 CPU-s. The running time on one node was 10 hours per remainder. We had 48 remainders and the program was running on 10 nodes parallelly.

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