CLOSING THE DETECTION LOOPTHOLE IN MULTIPARTITE BELL TESTS USING GREENBERGER-HORNE-ZEILINGER STATES

Károly F. Pál, Tamás Vértesi, and Nicolas Brunner

1Institute of Nuclear Research of the Hungarian Academy of Sciences, H-4001 Debrecen, P.O. Box 51, Hungary
2Département de Physique Théorique, Université de Genève, 1211 Genève, Switzerland
3H.H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol, BS8 1TL, United Kingdom

(Received 7 August 2012; published 19 December 2012)

We investigate the problem of closing the detection loophole in multipartite Bell tests, and show that the required detection efficiencies can be significantly lowered compared to the bipartite case. In particular, we present Bell tests based on n-qubit Greenberger-Horne-Zeilinger states, which can tolerate efficiencies as low as 38% for a reasonable number of parties and measurements. Even in the presence of a significant amount of noise, efficiencies below 50% can be tolerated, which is encouraging given recent experimental progress.

Finally, we give strong evidence that, for a sufficiently large number of parties and measurements, arbitrarily small efficiencies can be tolerated, even in the presence of an arbitrary large amount of noise.

DOI: 10.1103/PhysRevA.86.062111 PACS number(s): 03.65.Ud

Quantum nonlocality is arguably one of the most counterintuitive aspects of quantum mechanics. According to quantum theory, separated parties sharing an entangled state and performing suitably chosen measurements are able to generate correlations which are unexplainable by any classical mechanism. These nonlocal correlations can be tested experimentally using Bell inequalities [1]. Numerous experiments have demonstrated Bell inequality violations giving strong evidence that nature is inherently nonlocal [2]. However, technical imperfections in these experiments open various loopholes, which make it still possible to explain the data with a local model. Given the fundamental importance of nonlocality, it is highly desirable to perform a loophole-free Bell test, which, despite recent theoretical proposals (see, e.g., [3]) and experimental progress [4,5], is still missing.

A loophole-free Bell test requires (i) a spacelike separation between the parties and (ii) a detection efficiency above a certain threshold (usually high). The first condition ensures that no communication between the parties is possible, hence closing the locality loophole. This was achieved in photonic experiments [2]. The second condition ensures that no classical model exploiting undetected events can reproduce the observed data, hence closing the detection loophole [6]. This was achieved in atomic experiments [7]. However, no experiments could yet close both loopholes simultaneously. On the one hand, atomic experiments are unsatisfactory from the locality point of view. On the other hand, typical photodetection efficiencies are still too low to close the detection loophole.

Addressing the detection loophole is also crucial for information-theoretic applications based on quantum nonlocality [8–10]. Failure in closing the detection loophole renders these protocols insecure as the observed Bell violation may have been produced by classical means, as nicely illustrated by recent experiments faker Bell violations [11].

In general, the required detection efficiency η depends on the Bell inequality and the quantum state which are considered. For the Clauser-Horne-Shimony-Holt (CHSH) inequality, an efficiency η > 82.8% is required for a maximally entangled qubit pair, while η > 66.7% for a partially entangled state [12]. More recently, improvements were reported using four-dimensional quantum systems, tolerating efficiencies ∼61% [13]. However, from a practical point of view, these results should be considered carefully, in particular when taking into account additional imperfections such as background noise. Importantly, even a small amount of noise increases significantly the threshold efficiencies; in the CHSH case for instance, adding 1% of noise to the state increases the threshold from 66.7% to 80% [12].

Another approach, which has received so far only little attention, is to consider multipartite Bell tests, that is, with n > 2 observers. Based on a combinatorial study, Buhrman et al. [14] showed that an arbitrarily small efficiency η can be tolerated as n becomes large. More recently, threshold efficiencies for the Mermin inequalities were shown to approach η = 50% for large n [15], but remain above 60% for any practical scenario. The same limit can be approached for the many-site generalization of the Clauser-Horne inequality [16]. Also, a multipartite Bell test based on single-photon entanglement was shown to approach η = 66.7% for large n [17]. However, until now, no practical Bell test featuring efficiencies lower than 60% for all observers was known [18].

Here we show that detection efficiencies as low as 38% can be tolerated in multipartite Bell tests featuring a reasonable number of parties and measurements, and lower than 50% even in the presence of noise. Specifically, we present a family of Bell tests, based on Bell inequalities, in which n observers perform m binary measurements on an n-qubit Greenberger-Horne-Zeilinger (GHZ) state [20]. Notably, efficiencies η < 50% can be tolerated already for a 6-qubit GHZ state and m = 7 or alternatively for a 5-qubit GHZ state and m = 11.

Furthermore, the measurements to be performed are equally distributed on an equator of the Bloch sphere, which is convenient from a practical point of view. Moreover our Bell tests appear to be robust to noise. For instance, for an 8-qubit GHZ state with 10% of noise, efficiencies η ∼ 50% can be tolerated for m = 7. From an experimental perspective, these results look encouraging, given recent experimental progress [21], in particular the observation of 8-qubit GHZ states [22]. Finally, we investigate the efficiency for our Bell tests in the asymptotic limit. We give strong evidence that η → 2/n when m → ∞ for a pure GHZ state. Moreover, we give evidence...
that arbitrarily low efficiencies can be tolerated, even if an arbitrary amount of noise is added to the GHZ state.

I. Setup

We consider a Bell scenario with \( n \) distant observers. Each observer may choose between a set of \( m \) measurements \( \{A_i\}, \{B_j\}, \{C_k\} \), and so on, with \( i, j, k, \ldots = 0, \ldots, m-1 \). All measurements have binary outcomes, +1 and −1. We use the shorthand notation \( P(A_i, B_j, C_k, \ldots) \equiv P(111 \cdots | A_i, B_j, C_k, \ldots) \) and similarly for any subset of parties. We start by defining a family of Bell inequalities:

\[
\sum_{i,j,k, \ldots = 0}^{m-1} P(A_i, B_j, C_k, \cdots)\left(y - x \delta_{i+j+k+\cdots \mod m}^0\right) - \sum_{j,k, \ldots = 0}^{m-1} P(B_j, C_k, \cdots) - \sum_{i,k, \ldots = 0}^{m-1} P(A_i, C_k, \cdots) - \sum_{i,j, \ldots = 0}^{m-1} P(A_i, B_j, \cdots) - \cdots \leq 0,
\]

(1)

where \( \delta_{x \mod m}^0 = 1 \) if \( x \) is divisible by \( m \), and is 0 otherwise. Note that the real parameters and \( v \) are chosen such that the local bound of the Bell inequality is 0. We shall see later how this condition can be enforced.

The observers share a noisy \( n \)-qubit GHZ state

\[
\hat{\rho} = v(\text{GHZ}) (\text{GHZ}) + (1-v) \frac{I}{2^n},
\]

(2)

with \( \text{GHZ} \equiv (|0\rangle^\otimes n + |1\rangle^\otimes n)\sqrt{2} \) and \( v \) is the visibility. This state is fully separable iff \( v \leq 1/(1 + 2^{-n-1}) \) [23] and violates a two-setting full-correlation inequality for \( v > 1/2^{(n-1)/2} \) [24].

Here we will focus on (projective) equatorial qubit measurements of the form

\[
\hat{A}_i = \cos \varphi_i^A \hat{\sigma}_x + \sin \varphi_i^A \hat{\sigma}_y, \quad \hat{B}_j = \cos \varphi_j^B \hat{\sigma}_x + \sin \varphi_j^B \hat{\sigma}_y, \quad \hat{C}_k = \cos \varphi_k^C \hat{\sigma}_x + \sin \varphi_k^C \hat{\sigma}_y,
\]

(3)

and so on for all parties; \( \hat{\sigma}_x, \hat{\sigma}_y \) denote the Pauli matrices.

With this choice of measurements and the state (2), it follows that (see, e.g., [25] for details)

\[
P(A_i, B_j, C_k, \cdots) = \frac{1 + v \cos (\varphi_i^A + \varphi_j^B + \varphi_k^C + \cdots)}{2^n}.
\]

(4)

Next, let us further simplify the structure of the measurement by choosing the \( m \) angles to be evenly distributed around the equator of the Bloch sphere, such that \( \varphi_i^A = \varphi_j^B = \varphi_k^C = 2\pi i/m + \pi/n \). With this choice we get

\[
P(A_i, B_j, C_k, \cdots) = \frac{1 - v \cos \left[ \frac{\pi}{m} (i + j + k + \cdots) \right]}{2^n},
\]

(5)

Finally, since the GHZ state has no \((n - 1)\) subcorrelations for equatorial measurements, it follows that all \((n - 1)\)-particle joint probabilities involved in our inequality take the value \(1/2^{n-1}\), independently of \( v \).

II. Threshold Efficiencies for Reasonable Number of Parties and Measurements

All observers detect their particles with the same limited efficiency \( \eta \). In case of nondetection, they agree to output −1. Hence the measurement outputs are still binary and the Bell inequality (1) can be used. However, the probabilities must be modified in the following way:

\[
P(A_i, B_j, C_k, \cdots) \rightarrow \eta^m P(A_i, B_j, C_k, \cdots)
\]

and similarly for any subset of parties. We shall see later how this condition can be enforced.

Note first that whenever one (or more) parties outputs \( 0 \), then it follows that (see, e.g., [25] for details)

\[
\sum_{i,j,k, \ldots = 0}^{m-1} P(A_i, B_j, C_k, \cdots) = \frac{1 + v \cos \left[ \frac{\pi}{m} (i + j + k + \cdots) \right]}{2^n}.
\]

(6)

Thus, in order to determine \( \eta^{*} \) for any given number of parties \( n \) and measurements \( m \), we must determine the parameters \( x \) and \( y \) of the Bell inequality (1) such that the local bound is 0. We shall see that, in general, the values of \( x \) and \( y \) leading to the lowest value of \( \eta^{*} \) may depend on the visibility \( v \) of the state.

We recall first that in order to find the maximal value of a linear Bell polynomial (such as (1)) it is sufficient to consider local deterministic strategies. For commodity we denote by \( a_i, b_j, c_k \), and so on the probabilities of getting outcome +1 for measurement \( A_i, B_j, C_k \), and so on. We now impose the following condition:

\[
\sum_{i,j,k, \ldots = 0}^{m-1} a_i b_j c_k \cdots \left(y - x \delta_{i+j+k+\cdots \mod m}^0\right) - \sum_{j,k, \ldots = 0}^{m-1} b_j c_k \cdots - \sum_{i,k, \ldots = 0}^{m-1} a_i c_k \cdots - \sum_{i,j, \ldots = 0}^{m-1} a_i b_j \cdots - \cdots \leq 0
\]

(8)

and so on for all parties; \( \delta_{x \mod m}^0 = 1 \) if \( x \) is divisible by \( m \), and is 0 otherwise. Hence, we can assume that \( \varphi_i^A = \varphi_j^B = \varphi_k^C = 2\pi i/m + \pi/n \). With this choice we get

\[
P(A_i, B_j, C_k, \cdots) = \frac{1 - v \cos \left[ \frac{\pi}{m} (i + j + k + \cdots) \right]}{2^n},
\]

(9)

where \( p = \alpha r^{-1} + \beta r^{-1} + y^{-1} + \cdots \) and \( q = S/(\alpha \beta \gamma \cdots) \) and

\[
S \equiv \sum_{i,j,k, \ldots = 0}^{m-1} a_i b_j c_k \cdots \delta_{i+j+k+\cdots \mod m}^0.
\]

(10)

For each choice of \( a_i, b_j, c_k, \ldots \), condition (9) is a linear constraint between \( x \) and \( y \), and hence defines a straight line (with positive or zero slope) in a plane with coordinates \( x \) and \( y \). For finite values of \( m \) and \( n \) we get a finite set of these lines. To ensure that the local bound of Bell inequality (1) is not greater than 0, \( x \) and \( y \) must be chosen such that the point
of the threshold detection efficiencies to be optimal. From our investigation for small values of a subset of all deterministic strategies which we conjecture investigate the behavior of CLOSING THE DETECTION LOOPHOLE IN ...

\[ \eta^* = \frac{2 + \frac{2}{m} - \frac{4}{mn}}{n} \]

Hence Bell inequality (1) can be violated using detectors with arbitrarily low efficiency \( \eta < 0 \), by choosing \( n \) and \( m \) large enough. Note that if either \( n \) or \( m \) is finite, \( \eta^* \) tends to a strictly positive value. Note also that for any given number of parties \( n \), we have that \( \eta \rightarrow 2/n \), for sufficiently large \( m \). This improves on the results of Ref. [14], which had \( \eta \rightarrow 8/n \). Finally, note that we have again considered only \( m \) prime. For \( m \) not prime it is possible to have \( \alpha + \beta + \gamma + \cdots > n + m - 2 \), such that \( \eta^* \rightarrow 0 \).

Next, we investigate the case in which the visibility of the state is limited, that is, \( 0 < v < 1 \), and give evidence that our Bell tests can tolerate arbitrarily low detection efficiencies even in the presence of an arbitrarily large amount of noise, when taking \( n \) and \( m \) large enough. We first determined for \( n = m \leq 59 \) the optimal Bell inequalities (i.e., parameters \( x, y \)), assuming that the optimal local deterministic strategy is a regular arrangement. The results, shown in Fig. 2, support qualitatively our above claim.

Then we consider the case \( m \ll n \). As we could not derive the complete set of conditions on \( x \) and \( y \), we focused our efforts, as in the noiseless case, on the horizontal line, that is, \( y = y_{\text{max}} = n + 1 - m/2 \) for \( m \ll n \). From Eq. (7) one can
Bell test based on photons. More generally, we believe that the multipartite setting offers possibilities for a loophole-free [5]. However, the main challenge is to achieve a heralded which seems within reach of current photonic experiments require a detection efficiency of qubit GHZ entanglement, with fidelities of all measurements, that is,$\alpha_m$ of the GHZ state, in the case experimentally. In particular, Ref. [22] recently reported 8-tolerated for a reasonable number of parties and measurements, the detection loophole. Notably, efficiencies below 50% can be the minimal detection efficiencies required in order to close $v$ visibilities of large $x, y$. Hence, even in the case of arbitrarily small visibility $v$, $\eta^*$ can become arbitrarily small by taking $x, y$, that is, $x_{\text{min}}$, such that all conditions (9) hold. We conjecture that $x_{\text{min}} \leq nm$ holds for $m \ll n$, leading to a threshold efficiency

$$\eta^* \simeq \frac{2}{mv} \quad (12)$$

Hence, even in the case of arbitrarily small visibility $v$, $\eta^*$ can become arbitrarily small by taking $m \ll n$ large enough. To support our conjecture that $x_{\text{min}} \leq nm$, we checked that, for $m \leq 199$ (prime) and $n \leq 199$, $x_{\text{min}}$ is always achieved by only two possible strategies: (i) All parties output +1 for all measurements, that is, $\alpha = \beta = \gamma = \cdots = m$, leading to $p = n/m$, $S = m^{n-1}$, and $q = 1/m$; this corresponds to a line reaching $y = y_{\text{max}}$ at $x_1 = m y_{\text{max}} - n$; or (ii) a regular arrangement with $\alpha + \beta + \gamma + \cdots = m + n$, leading to $p = n - m/2$, $S = 2$, and $q = 1/2^{m-1}$; this corresponds to a line reaching $y = y_{\text{max}}$ at $x_2 = (y_{\text{max}} - p)/q = 2^{m-1}$, independent of $n$. Indeed $x_1, x_2 \leq nm$ when $m \ll n$.

IV. CONCLUSION

We presented a family of multipartite Bell tests and derived the minimal detection efficiencies required in order to close the detection loophole. Notably, efficiencies below 50% can be tolerated for a reasonable number of parties and measurements, even in the presence of significant amount of noise. Our Bell tests are based on $n$-qubit GHZ states, which have been realized experimentally. In particular, Ref. [22] recently reported 8-qubit GHZ entanglement, with fidelities of $\sim 70\%$. This would require a detection efficiency of $\sim 60\%$ in our Bell tests, which seems within reach of current photonic experiments [5]. However, the main challenge is to achieve a heralded preparation of the GHZ state [27]. Nevertheless, this shows that the multipartite setting offers possibilities for a loophole-free Bell test based on photons. More generally, we believe that our findings open interesting experimental perspectives for multipartite nonlocality, and for its applications [8,28].

ACKNOWLEDGMENTS

The authors thank C. Simon for discussions, and acknowledge financial support from the UK EPSRC, the Swiss National Science Foundation (Grant P000P2_138917), the EU DIQIP, the Hungarian National Research Fund OTKA (PD101461), a János Bolyai Grant of the Hungarian Academy of Sciences, and the TÁMOP-4.2.2.C-11/1/KONV-2012-0001 project. The project has also been supported by the European Union, co-financed by the European Social Fund.

APPENDIX

Here we give more details concerning the choice of parameters $x$ and $y$, defining our Bell inequalities (1), such that the local bound is 0. As explained in the main text, one must check a finite set of conditions, of the form (9), which define straight lines in the plane with coordinates $x$ and $y$. For small values of $n$ and $m$ the complete set of lines can be found. Figure 3 illustrates the situation for the case of $n = 5$ observers, and up to $m = 13$ measurements. Note that although the total number of lines is large, only a few turn out to be relevant. Also the optimal choice of $x$ and $y$, which may depend on the visibility of the state $v$ [see Eq. (7)], is always one of the intersections of two (or more) lines (marked by dots in Fig. 3). The $(x, y)$ pairs, along with the ranges of visibilities where they are optimal choice, are shown in Ref. [26]. These values have been used to generate the pieces of the curves shown in Fig. 1.

CLOSING THE DETECTION LOOPHOLE IN . . .

PHYSICAL REVIEW A 86, 062111 (2012)


[18] Note that in an asymmetric setup, where parties may have different efficiencies (e.g., in atom-photon entanglement), lower efficiencies were reported, see [19].


