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Orthogonal systems on local fields

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Dr. Schipp Ferenc, témavezető

Orthogonal systems on local fields

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban.

Írta: Simon Ilona okleveles matematikus.

Készült a Debreceni Egyetem Matematika- és számítástudományok doktori iskolája (Matematikai analízis, függvényegyenletek és egyenlőtlenségek programja) keretében.

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Contents

Chapter 1

Introduction

1.1 Motivation and historical background

Why are non-Archimedean local fields important? According to Volovich [47] some non-Archimedean normed fields have to be used for a global space-time theory in order to unify both microscopic and macroscopic physics. Some problems occurred with the practical applications of the classical fields $\mathbb{R}$ and $\mathbb{C}$, because in science there are absolute limitations on measurements like Plank time, Plank length, Plank mass. The use of real time and space-time coordinates in mathematical physics leads to some problems with the Archimedean axiom on the microscopic level. According to the Archimedian axiom, any given segment on the line can be surpassed by the successive addition of a smaller segment along the same line. This means, that we can measure arbitrary small distances. But a measurement of distances smaller than the Planck length is impossible. Volovich proposes to base physics on a coalition of non-Archimedean normed fields and classical fields as $\mathbb{R}$ or $\mathbb{C}$. Source claims, that the so-called p-series fields and p-adic fields are suitable non-Archimedean normed fields. As $p \to \infty$, many of the fundamental functions of p-adic analysis approach their counterparts in classical analysis. Thus p-adic analysis could provide a bridge from microscopic to macroscopic physics.

We deal with non-Archimedian normed fields, that is, the norm satisfies a stronger inequality than the triangle inequality: $\|a + b\| \leqq \max\{\|a\|, \|b\|\}$. The p-adic distance leads to interesting deviations from the classical real analysis, the geometry of these spaces is unlike the euclidian geometry based on real space

$\mathbb{R}$. In non-Archimedian geometry two different balls are either disjoint or the one is contained in the other one (splitting property). Furthermore the field of 2-adic and 2-series numbers have a hierarchical structure: every disc consists of two disjoint discs of smaller radius (tree property). Thus these fields are homeomorphic to a Cantor set on $\mathbb{R}$. Volovich[47] states, that the fractal-like structure of these fields enable their application not only for the description of geometry at small distances, but also for describing chaotic behavior of chaotic systems.

The simplest example of a p-adic field and a p-series field are the 2-adic (or arithmetic) field and the 2-series (or logical, dyadic) field used in this work. The 2-series addition is applied in numerous forms, it can be found for example in logic as XOR, or in the theory of games as the nim addition, a tool in the construction of the strategy for the nim-game.

A complete classification of locally compact, non-discrete fields results in two connected fields ($\mathbb{R}$ and $\mathbb{C}$) and a set of local fields (containing the p-adic fields among others). See Taibleson [45].

✠ On orthonormal systems:

After emphasizing the importance of the 2-adic and 2-series fields, let us address our attention to the several ways of construction of orthogonal systems and especially to the product systems of unitary dyadic martingale difference systems (UDMD systems).

There are several methods for constructing orthonormal and biorthogonal systems. The Schmidt-orthogonalization method in a Hilbert space for any linearly independent system results an orthonormed one. Eigenfunctions of several differential operators provide also such systems, used in mathematical physics. Using the tools of harmonic analysis, character systems of topological groups also result in orthonormal function systems. An other way of constructing such systems uses some concepts of the probability theory, mostly that of martingales.

Convergence problems of the orthogonal systems are connected to many other fields of mathematics, for example to probability theory. Alexits[2] stated, that many theorems related to orthogonal series and some corresponding statements of probability theory stand on the same mathematical fact. Fifteen years later Professor Ferenc Schipp introduced a new method for constructing orthogonal systems starting from some conditionally orthogonal functions. See [34], [35], [36]. Several classical and modern systems can be constructed by using this method. For example the trigonometric, the Walsh system, the Vilenkin system, UDMD and Walsh-similar systems can be obtained in this way. Besides the important theoretical properties, these systems have useful numerical applications, like the possibility to compute the Fourier-coefficients and partial sums with a fast algorithm similar to FFT. (Fast Fourier Transform)

✠ On Blaschke functions and orthogonal systems related to them:

Blaschke functions play an important role in complex analysis, in the theory of Hardy spaces, and in system and control theory. See Duren[5], Duren-Schuster[6], Chui-Chen[4], Schipp-Bokor[23], [24], and [25]. The congruence transforms in the Poincarè model of hyperbolic geometry can be described by means of Blaschke functions. See Schipp[26]. The Blaschke functions form a group with respect to the composition, and on the so-called Blaschke group a Voice transform was introduced by Schipp and Pap in [28], [29], and [31], and applied in signal and image processing in Schipp[30], Schipp-Bokor[32], and [33]. These results inspired the study of Blaschke functions on local fields.

The discrete Laguerre functions and their generalizations (Kautz-, and Malmquist-Takenaka systems) are widely applied in system and control theory. See [19], [20], [21], and [22].

Chebyshev polynomials play an important role in numerous fields of applications, for example in approximation theory (the resulting interpolation polynomial provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm).

These have motivated the author in construction of these systems on local fields.

1.2 Presentation overview

We construct some orthogonal systems related to the Blaschke functions and to the Walsh-Paley system or to the characters of the 2-adic field. Fourier-series with respect to these functions are examined. However, this work does not claim to be a complete treatment of the subject. We have chosen to use the methods of the product systems of UDMD systems.

This work is organized as follows: **Chapter 2** contains an introduction to the 2-series and 2-adic fields, especially concerning the algebraic and topological structure. This chapter follows the concepts, notations and proofs of Schipp-Wade[17]. We present in Paragraph 2.6 that if we consider the Fourier expansion with respect to a system given by the composition of the character system and a measure preserving transformation, then its partial sums and Cesaro means can be expressed by the original ones, that is by partial sums and Cesaro means of Fourier series with respect to the characters. This will be applied in the next chapters to discuss summability and convergence questions.

Chapter 3 is devoted to some useful tools applied in the following chapters. Paragraph 3.1 provides a description of the characters of the dyadic and 2-adic multiplicative groups based on [17] and using the notion of the product system. Based on the handbook of Schipp and Wade[17] we present the exponential function, with slightly different base and values, which is used in the next chapters.

Starting from Paragraph 3.3, this work contains the results of the author. Paragraph 3.3 contains the definitions and properties of the Blaschke functions on both fields. The logical Blaschke functions $B_a(x) = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x}$ $(x \in \mathbb{I}, a \in \mathbb{I}_1)$ defined on the dyadic field and the arithmetical Blaschke functions $B_a(x) = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x}$ $(x \in \mathbb{I}, a \in \mathbb{I}_1)$ defined on the 2-adic field form a commutative group with respect to the function composition. Although the classical Blaschke group is non-commutative, analogous thoughts result in commutative variants on local fields.

In **Chapter 4** we study transformations given by composition with a Blaschke function and in general with a dyadic martingale structure preserving transformation, or shortly a DMSP-transformation defined in this chapter, and we investigate questions related to the effect on special function classes of these transformations. We obtain, that composition with a DMSP-function preserves the classes of UDMD systems, that of $\mathcal{A}_n$-measurable functions, the dyadic function spaces $L^p(\mathbb{I})$, $H^p(\mathbb{I})$, and the Lipschitz classes $Lip(\alpha, \mathbb{I})$.

The idea of **Chapter 5** is given by the fact that the operation determined by the composition of Blaschke functions leads to the functional equation of the tangent function tan. Thus the characters of the 2-adic group are determined by means of a tangent-like function. We use the $(\tilde{\mathbb{S}}, \bullet)$-valued exponential function ζ, which was described in Paragraph 3.2. In order to construct the characters of the Blaschke group of the arithmetical field, we give a continuous isomorphism γ from the additive group $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$, which is the analogue of function tan. These thoughts can be interpreted as the solution of the functional equation of tan on the local field.

Chapter 6 is devoted to the construction of discrete Laguerre functions on both local fields. The role of the power function of the classical system is taken by the characters of the corresponding field, and their composition with Blaschke functions build the dyadic discrete Laguerre systems. After the model of the classical system, we introduce discrete Laguerre systems as the composition of the respective additive characters of the local fields and the Blaschke functions. We have shown in Paragraph 3.3, that the bits of the values of the Blaschke functions B_a can be obtained with recursion using the bits of the variable and the bits of the parameter a. As a consequence of this recursion follows, that the systems in question are UDMD-product systems, as well. As a consequence, results regarding UDMD-systems are valid for the discrete Laguerre systems. Paragraph 6.4 deals with the a.e. convergence and (C,1)-summability of the Fourier series with respect to these systems using some basic results of Schipp[15] and Gat[7] on the a.e. convergence and (C,1)-summability of the Fourier series with respect to the characters of the dyadic and 2-adic field.

Chapter 7 covers our investigations about the construction of the Malmquist-Takenaka systems on both studied local fields, which are a generalization of the discrete Laguerre systems. Being UDMD-product systems, Fourier series with respect to them fulfill a.e. convergence and summability statements.

In **Chapter 8** we construct several analogies of the Chebyshev polynomials on the 2-adic field. First, 2-adic cosine and sine functions are constructed in two ways: with the aim of the $\tilde{\mathbb{S}}$-valued exponential functions or with the characters v_n of the 2-adic additive group. Then follows the construction of some analogies of the Chebyshev polynomials using these cosine and sine functions. Orthogonality of these Chebyshev polynomials is also investigated.

Chapters 4,5, 6-7 and 8(based also on 4) can be read in optional order.

1.3 Credits

Chapter 4 is based on [42]:

Simon, I., *On transformations by dyadic martingale structure preserving functions*, Annales Univ. Sci. Budapest., Sect. Comp., 39 (2013), pp, 381-390.

Chapter 5 is based on [40]:

Simon, I., *The characters of the Blaschke-group*, Studia Univ. "Babes-Bolyai", Mathematica, 54(3)(2009), pp. 149-160.

Chapter 6, is based on [39]:

Simon, I., *Discrete Laguerre functions on the dyadic fields*, PU.M.A, 17(2006)(3-4), pp. 459-468.

Chapter 7 is based on [41]:

Simon, I. *Malmquist-Takenaka functions on local fields*, Acta Univ. Sapientiae Math., 3(2)(2011), pp. 135-143.

Chapter 8 is based on [43]:

Simon, I. *Construction of 2-adic Chebyshev polynomials*, submitted.

Chapter 2

Algebraic and topological structure

2.1 Non-Archimedean topology of the space of bytes $\mathbb{B}$

This chapter is an introduction to the 2-series and 2-adic fields, especially concerning the algebraic and topological structure. We follow the concepts, notations and proofs of Schipp-Wade[17]. The reason why we sometimes go into details, is to recall the techniques which we will use in the next chapters.

✠ Denote by $\mathbb{A} := \{0, 1\}$ *the set of bits*, and by

$$\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \mathbb{A} \text{ and } \lim_{j \to -\infty} a_j = 0\} \tag{2.1}$$

the set of bytes. The numbers a_j are called the additive digits of $a \in \mathbb{B}$. As each a_j is 0 or 1, the condition $\lim_{j \to -\infty} a_j = 0$ is equivalent with the existence of an integer $N \in \mathbb{Z}$ such that $a_j = 0$ for $j < N$.

The zero element of $\mathbb{B}$ is $\theta := (0, j \in \mathbb{Z})$, that is, $\theta = (\cdots, 0, 0, 0, \cdots)$.

The fundamental sequence of $\mathbb{B}$ is formed by the elements $e_k := (\delta_{jk}, j \in \mathbb{Z})$ defined for each $k \in \mathbb{Z}$, where δ_{jk} is the Kronecker-symbol. Thus e_k is the byte with k-th digit 1 and with other digits 0. The byte e_0 is denoted by e. We will denote the set $\mathbb{N} \setminus \{0\}$ by $\mathbb{P}$ and let $\mathbb{B}^* := \mathbb{B} \setminus \{\theta\}$.

✠ The *order* of a byte $x \in \mathbb{B}$ is defined in the following way: For $x \neq \theta$ let $\pi(x) := n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$, furthermore set

$\pi(\theta) := +\infty$. The norm of a byte x is introduced by the following rule:

$$\|x\| := 2^{-\pi(x)} \text{ for } x \in \mathbb{B}^*, \qquad \text{and} \quad \|\theta\| := 0. \tag{2.2}$$

We will see in Section 2.2 that this function possesses the properties of a norm with the corresponding operations even in a stronger form: instead of the triangle inequality takes place a stronger inequality.

For example, the order of the byte $x = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{0}, \overset{3}{1}, \overset{4}{0}, \cdots)$ is $\pi(x) = 1$, and its norm is $\|x\| = 2^{-1}$.

✠ A *metric* can be defined on $\mathbb{B}$ as follows.

$$\rho(x,y) := \begin{cases} 0, & \text{if } x = y, \\ 2^{-n}, & \text{if } x \neq y,\ n := \min\{k \in \mathbb{Z} : x_k \neq y_k\}, \end{cases} \tag{2.3}$$

that is, $n \in \mathbb{Z}$ is chosen so, that $x_j = y_j$ for $j < n$, but $x_n \neq y_n$. The mentioned minimum exists by the definition of $\mathbb{B}$. Clearly, $\rho(\theta, x) = \|x\|$.

It is clear, that ρ is a metric, as

$$\begin{aligned} &\rho(x,y) \geq 0; \text{ and } \rho(x,y) = 0 \Longleftrightarrow x = y, \\ &\rho(x,y) = \rho(y,x) \text{ for all } x, y \in \mathbb{B}, \\ &\rho(x,y) \leq \rho(x,z) + \rho(z,y) \text{ for all } x, y, z \in \mathbb{B}. \end{aligned}$$

In fact, the metric ρ satisfies a stronger condition then the last one:

$$\rho(x,y) \leq \max\{\rho(x,z), \rho(z,y)\} \text{ for all } x, y, z \in \mathbb{B}, \tag{2.4}$$

namely ρ is a **non-Archimedian metric** on $\mathbb{B}$.

✠ A sequence of bytes $(b_k)_{k\in\mathbb{N}}$ is said to *converge* to a byte $b \in \mathbb{B}$ if $\rho(b_k, b) \to 0$ as $k \to \infty$.

$(b_k)_{k\in\mathbb{N}}$ is said to be a *Cauchy sequence* if to any given $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\rho(b_k, b_l) < \varepsilon$ for all $k, l > N$. It is easy to see, that every Cauchy sequence in $\mathbb{B}$ is convergent, and consequently, $(\mathbb{B}, \rho)$ is a complete metric space.

A countable subset of $\mathbb{B}$ is the following:

$$\mathbb{B}^+ := \{a \in \mathbb{B} : \lim_{j\to\infty} a_j = 0\}.$$

Furthermore, for each $a \in \mathbb{B}$ and $n \in \mathbb{Z}$ define the n-th *truncation* of a by

$$(a_{\langle n\rangle})_j := \begin{cases} a_j, & \text{for } j < n, \\ 0, & \text{for } j \geq n. \end{cases} \tag{2.5}$$

We can see, that $\mathbb{B}^+$ is dense in $\mathbb{B}$. Indeed, for each $a \in \mathbb{B}$ we have $a_{\langle n\rangle} \in \mathbb{B}^+$ and $a_{\langle n\rangle} \to a$ as $n \to \infty$. Thus $(\mathbb{B}, \rho)$ **is a complete, separable metric space.**

✠ The sets $I_n(x) := \{y \in \mathbb{B} : y_k = x_k \text{ for } k < n\}$, the so-called *intervals* in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center x are of basic importance. Set $\mathbb{I}_n := I_n(\theta) = \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ for any $n \in \mathbb{Z}$. The *unit ball* $\mathbb{I} := \mathbb{I}_0$ can be identified with the set of sequences $\mathbb{I} = \{a = (a_j, j \in \mathbb{N}) |\ a_j \in \mathbb{A}\}$ via the map $(\ldots, 0, 0, a_0, a_1, \ldots) \mapsto (a_0, a_1, \ldots)$. Furthermore $\mathbb{S} := \{x \in \mathbb{B} : \|x\| = 1\} = \{x \in \mathbb{B} : \pi(x) = 0\} = \{x \in \mathbb{I} : x_0 = 1\}$ is *the unit sphere* of the field.

We can observe that $I_n(a) = \{x \in \mathbb{B} : \rho(x, a) \leq 2^{-n}\}$ for all $a \in \mathbb{B}$ and $n \in \mathbb{Z}$, that is, $I_n(a)$ is a disc of radius 2^{-n} with center at a. The boundary of $I_n(a)$ is $S_n(a) := \{x \in \mathbb{B} : x_k = a_k \text{ for } k < n, \text{ but } x_n \neq a_n\}$. Let us collect some properties concerning intervals:

$$\begin{aligned}
&\{x \in \mathbb{B} : \rho(x, a) < 2^{-n}\} = I_{n+1}(a) \subset I_n(a);\\
&I_n(a) \subset I_m(a) \qquad (n > m, a \in \mathbb{B});\\
&S_n(a) = I_n(a) \setminus I_{n+1}(a) \qquad (n \in \mathbb{Z}, a \in \mathbb{B});\\
&I_n(a) = \bigcup_{k \geq n} S_k(a) \qquad (n \in \mathbb{Z}, a \in \mathbb{B});\\
&\bigcap_{n \in \mathbb{Z}} I_n(a) = \{a\}; \ \bigcup_{n \in \mathbb{Z}} I_n(a) = \bigcup_{n \in \mathbb{Z}} S_n(a) = \mathbb{B}.
\end{aligned} \tag{2.6}$$

Easy consideration leads to the following lemma:

Lemma 1 *If* $b \in I_n(a)$*, then* $I_n(b) = I_n(a)$*.*

Denote the collection of intervals in $\mathbb{B}$ by $\mathcal{I}$. $\mathcal{I}$ is countable, satisfies the tree property and the splitting property. The tree property means that any two intervals in $\mathbb{B}$ either disjoint or one is contained in another, namely for $n \leq m$ and $a, b \in \mathbb{B}$ we have either $I_m(a) \subseteq I_n(b)$ or $I_m(a) \cap I_n(b) = \emptyset$. This is a simple consequence of Lemma 1. The splitting property is the feature to break every interval into disjoint intervals of higher rank, namely, if given $x \in \mathbb{B}$ and $m \in \mathbb{Z}$, there is an $y \in \mathbb{B}$ such that

$$I_m(x) = I_{m+1}(x) \cup I_{m+1}(y) \text{ and } I_{m+1}(x) \cap I_{m+1}(y) = \emptyset.$$

Indeed, the splitting property holds: if $x \in \mathbb{B}$ and $m \in \mathbb{Z}$, define $y = (y_j, j \in \mathbb{Z})$ by $y_j = x_j$ for $j \neq m$ and $y_m = 1 - x_m$. Thus, $I_m(x) = I_{m+1}(x) \cup I_{m+1}(y)$ and $I_{m+1}(x) \cap I_{m+1}(y) = \emptyset$. The countability of $\mathcal{I}$ follows from the fact that each algebraic digit a_j of a byte $a \in \mathbb{B}$ takes on only 2 values: 0 and 1. Thus $\mathbb{B}^+$ is countable, so $\mathcal{I} = \{I_n(a) : a \in \mathbb{B}, n \in \mathbb{Z}\} = \{I_n(a) : a \in \mathbb{B}^+, n \in \mathbb{Z}\} = \bigcup\limits_{n\in\mathbb{Z}} \bigcup\limits_{a\in\mathbb{B}^+} \{I_n(a)\}$ is also countable, as both unions are countable. Lemma 1 shows that every point of $I_n(a)$ is its center.

✠ We call a set $E \subseteq \mathbb{B}$ *open*, if for each $a \in E$ the set E includes a ball centered in a, namely there is an $r > 0$ such that $\{x \in \mathbb{B} : \rho(x, a) < r\} \subseteq E$; and *closed*, if its complement is open in $\mathbb{B}$.

The intervals $I_n(a) = \{x \in \mathbb{B} : \rho(x, a) \leq 2^{-n}\}$ are open in $\mathbb{B}$ as a simple consequence of Lemma 1 and (2.6).

By Lemma 1 (or directly by (2.4)) follows that $I_n(a)$ contains all its limit points, thus the interval $I_n(a)$ is closed. We have seen, that the intervals of $\mathbb{B}$ are both open and closed. Thus $\mathbb{B}$ is totally disconnected. This is one of the fundamental differences between the intervals of $\mathbb{B}$ and $\mathbb{R}$.

✠ The intervals form a base for the metric topology of $\mathbb{B}$, namely each open set in $\mathbb{B}$ is a union of intervals. Indeed, given an open set $E \subseteq \mathbb{B}$, for each $a \in E$ there is an $n \in \mathbb{Z}$ such that $I_n(a) \subseteq E$. Choose the smallest one: $n_a := \min\{n \in \mathbb{Z} : I_n(a) \subset E,\ n > \pi(a)\}$. (This minimum exists, because for each $a \in \mathbb{B}$ holds $\pi(a) > -\infty$.) Now, $E = \bigcup_{a\in E} I_{n_a}(a)$, thus the set E can be written as a union of intervals. Since $\mathcal{I}$ is countable and satisfies the tree property, it follows that each open set in $\mathbb{B}$ can be written as a countable union of pairwise disjoint intervals.

✠ $\mathbb{B}$ is a locally compact metric space, that is, every byte $x \in \mathbb{B}$ has a compact neighborhood. In fact, each interval is compact in $\mathbb{B}$:

Lemma 2 *A set $K \subset \mathbb{B}$ is compact if and only if it is closed and bounded.*

A consequence of Lemma 2 is that every interval $I_n(a)$ and sphere $S_n(a)$ is compact, thus the space $\mathbb{B}$ is locally compact. Using the tree property, we can see, that every compact set in $\mathbb{B}$ can be covered by a finite number of disjoint intervals of a fixed rank.

✠ A *measure* can be defined on $\mathbb{B}$ in the following way: for $n \in \mathbb{N}, a \in \mathbb{B}$ let

$$\mu(I_n(a)) := 2^{-n}. \tag{2.7}$$

Extend μ to the ring $\mathcal{R}$ of sets formed by finite unions of intervals so that μ is finitely additive. By the splitting property and the tree property it is clear that μ is countably additive on $\mathcal{R}$. The Caratheodory extension theorem gives, that there is a measure (denoted also by μ) defined on the σ-ring of Borel sets $\mathcal{B}_\mu$ which satisfies (2.7). Clearly, μ is normalized, and we will see that $\mathbb{B}$ is a normed field with the concerned operations and μ is a normalized Haar-measure on $\mathbb{B}$ with property $\mu(\mathbb{I}) = 1$. μ will be invariant with respect the additive operations of both studied fields, thus it will be a Haar-measure. (See the next chapters.)

✠ We have found some basic differences between the set of real numbers $\mathbb{R}$ and the non-Archimedian space of bytes $\mathbb{B}$. The intervals in $\mathbb{R}$ have the splitting property, but the tree property fails. Moreover, the intervals in the case of bytes are both open and closed sets, which property distinguishes the examined space essentially from $\mathbb{R}$.

In spite of these, there are close connections between $\mathbb{B}$ and $\mathbb{R}^+$. We will use the map β on $\mathbb{B}^+$ defined by

$$\beta(x) := \sum_{k=-\infty}^{\infty} x_k \cdot 2^k \quad (x = (x_k, k \in \mathbb{Z}) \in \mathbb{B}^+). \tag{2.8}$$

Let $\mathbb{Q}$ represent the set of *dyadic rationals* in $\mathbb{R}$: $\mathbb{Q} := \{p \cdot 2^m : p, m \in \mathbb{Z}\}$, and $\mathbb{Q}^+$ represent the set of nonnegative dyadic rationals, that is $\mathbb{Q}^+ := \mathbb{R}^+ \cap \mathbb{Q}$.

Clearly, β is a 1-1 map from $\mathbb{B}^+$ onto $\mathbb{Q}^+$, and its restriction is 1-1 from $\mathbb{B}^+ \cap \mathbb{I}$ onto $\mathbb{N}$.

2.2 The 2-series (or logical, dyadic) field

✠ Define the *2-series (or logical) sum* $a \overset{\circ}{+} b$ and *product* $a \circ b$ of elements $a, b \in \mathbb{B}$ by

$$\begin{aligned} &a \overset{\circ}{+} b := (a_n + b_n \pmod 2,\ n \in \mathbb{Z}) \\ &a \circ b := (c_n, n \in \mathbb{Z}), \quad \text{where} \quad c_n := \sum_{k \in \mathbb{Z}} a_k b_{n-k} \pmod 2 \quad (n \in \mathbb{Z}). \end{aligned} \tag{2.9}$$

For example, the logical sum and product of the bytes a and b,

$$a = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{0}, \overset{3}{1}, \overset{4}{0}, \overset{5}{1}, \cdots)$$

$$b = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{0}, \overset{2}{1}, \overset{3}{1}, \overset{4}{1}, \overset{5}{1}, \cdots)$$

are the following: $$a \overset{\circ}{+} b = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{1}, \overset{3}{0}, \overset{4}{1}, \overset{5}{0}, \cdots),$$

$$a \circ b = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{1}, \overset{4}{1}, \overset{5}{0}, \overset{6}{0}, \cdots).$$

✠ The operation $\overset{\circ}{+}$ is commutative and associative. The additive unit element is θ, and by $x \overset{\circ}{+} x = \theta$ $(x \in \mathbb{B})$ follows that the additive inverse element of $x \in \mathbb{B}$ is x itself. Thus $(\mathbb{B}, \overset{\circ}{+})$ is a commutative group.

The metric $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ defined in (2.3) can also be represented in form $\rho(a,b) = \|a \overset{\circ}{+} b\|$. The map $(a,b) \mapsto a \overset{\circ}{+} b$ is continuous with respect to this metric from $\mathbb{B} \times \mathbb{B}$ to $\mathbb{B}$. This is a simple consequence of the following:

$$\begin{aligned} I_n(a) &= a \overset{\circ}{+} \mathbb{I}_n := \{a \overset{\circ}{+} x \ : \ x \in \mathbb{I}_n\} \\ I_n(a \overset{\circ}{+} b) &= I_n(a) \overset{\circ}{+} I_n(b) := \{x \overset{\circ}{+} y \ : \ x \in \mathbb{I}_n(a), y \in \mathbb{I}_n(b)\}, \end{aligned} \tag{2.10}$$

where $a, b \in \mathbb{B}, n \in \mathbb{Z}$. By the continuity of $\overset{\circ}{+}$ follows that $(\mathbb{B}, \overset{\circ}{+})$ forms a topological group.

Note, that each $x \in \mathbb{B}$ can be written in form

$$x = \sum_{n \in \mathbb{Z}} x_n e_n \tag{2.11}$$

using its additive digits $x = (x_n, n \in \mathbb{Z})$ where the sum is considered with respect to the addition $\overset{\circ}{+}$.

✠ The logical multiplication $\circ$ is a convolution over the finite field $\mathbb{A}$, and it is associative and commutative on $\mathbb{B}$. We can observe, that $e_k \circ e_m = e_{k+m}$ for all $k, m \in \mathbb{Z}$. In general, multiplication by e_k shifts bytes: $e_k \circ a = (a_{n-k}, n \in \mathbb{Z})$. The multiplicative identity of $\mathbb{B}$ is the element $e = (\delta_{n0}, n \in \mathbb{N})$: indeed, $e \circ a = a$ holds for each $a \in \mathbb{B}$.

The existence of the multiplicative inverse element of each $a \in \mathbb{B}^* := \mathbb{B} \setminus \{\theta\}$ follows from the existence of the inverse element of any $b \in \mathbb{S}$. Let us show, that for each $b \in \mathbb{S}$ there is an $x \in \mathbb{S}$ such that:

$$b \circ x = e. \tag{2.12}$$

Observe that (2.12) holds if and only if the additive digits of x satisfy

$$\begin{aligned} x_0 &= 1 \\ x_n &\equiv \sum_{j=0}^{n-1} x_j b_{n-j} \qquad (\mod 2)(n \geq 1), \end{aligned}$$

which clearly defines $x_n \in \mathbb{A}$ recursively for all $n \in \mathbb{N}$. We will denote the multiplicative inverse of an element $b \in \mathbb{S}$ with respect to $\circ$ by b° or by b^{-1}.

Now, each $a \in \mathbb{B}^*$ can be uniquely written in the form:

$$a = e_n \circ b \text{ for some } n \in \mathbb{Z} \text{ and } b \in \mathbb{S}.$$

Hence, the inverse element of $a \in \mathbb{B}^*$ is found: $e_{-n} \circ b^\circ$; let us denote it with a° or a^{-1}. Thus $(\mathbb{B}^*, \circ)$ forms a commutative group.

The logical multiplication is continuous on $\mathbb{B}$. Indeed, given $a, b \in \mathbb{B}^*$ and $n \in \mathbb{N}$, $n > \pi(a) + \pi(b)$ for each $x \in I_{n-\pi(b)}(a)$ and $y \in I_{n-\pi(a)}(b)$ we have $x \circ y \in I_n(a \circ b)$. Thus the $(\mathbb{B}^*, \circ)$ is a **topological group**.

✠ Notice, that

$$\pi(a \circ b) = \pi(a) + \pi(b). \tag{2.13}$$

The rule of distributivity also holds, that is:

$$a \circ (b \overset{\circ}{+} c) = a \circ b \overset{\circ}{+} a \circ c \qquad (a, b, c \in \mathbb{B}).$$

Furthermore,

$$\|a \overset{\circ}{+} b\| \leqq \max\{\|a\|, \|b\|\}, \quad \|a \circ b\| = \|a\| \, \|b\| \quad (a, b \in \mathbb{B}). \tag{2.14}$$

Thus the set $\mathbb{B}$ with the operations $\overset{\circ}{+}$ and $\circ$ is **a non-Archimedian normed field**, i.e. Consider $nx := \underbrace{x \overset{\circ}{+} x \overset{\circ}{+} \ldots \overset{\circ}{+} x}_{n \text{ times}}$ for any integer $n \in \mathbb{Z}$ and $x \in \mathbb{B}$, which is either x or θ. The first rule of (2.14) gives the non-Archimedian property: $\|nx\| \leq \|x\|$ $(n \in \mathbb{Z}, x \in \mathbb{B})$.

2.3 The 2-adic (or arithmetical) field

✠ Consider *the 2-adic (or arithmetical) sum* $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z}), b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}$, defined by

$$a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z}),$$

where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$\begin{aligned} &q_n = s_n = 0 \quad \text{ for } \ n < m := \min\{\pi(a), \pi(b)\}, \\ &\text{and } \ a_n + b_n + q_{n-1} = 2q_n + s_n \quad \text{ for } \ n \geq m. \end{aligned} \tag{2.15}$$

The *2-adic (or arithmetical) product* of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by

$$\begin{aligned} & q_n = p_n = 0 \quad \text{ for } \; n < m := \pi(a) + \pi(b) \\ & \text{and } \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = 2q_n + p_n \quad \text{ for } \; n \geq m. \end{aligned} \tag{2.16}$$

The *reflection* x^- of a byte $x = (x_j, j \in \mathbb{Z})$ is defined by its additive digits:

$$(x^-)_j := \begin{cases} x_j, & \text{for } j \leqq \pi(x) \\ 1 - x_j, & \text{for } j > \pi(x). \end{cases} \tag{2.17}$$

For example, the arithmetical sum and product of the bytes a and b,

$$\begin{aligned} a &= (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{0}, \overset{3}{1}, \overset{4}{0}, \overset{5}{1}, \cdots) \\ b &= (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{0}, \overset{2}{1}, \overset{3}{1}, \overset{4}{1}, \overset{5}{1}, \cdots) \\ \text{are the following: } a \overset{\bullet}{+} b &= (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{1}, \overset{3}{0}, \overset{4}{0}, \overset{5}{1}, \cdots), \\ a \bullet b &= (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{1}, \overset{4}{1}, \overset{5}{0}, \overset{6}{1}, \cdots) \\ \text{and the reflection of } a: \; a^- &= (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{1}, \overset{3}{0}, \overset{4}{1}, \overset{5}{0}, \cdots). \end{aligned}$$

✠ The operation $\overset{\bullet}{+}$ is commutative. Note, that θ is the additive identity and x^- is the additive inverse of $x \in \mathbb{B}$: $x \overset{\bullet}{+} \theta = x$, and $x \overset{\bullet}{+} x^- = \theta$ $(x \in \mathbb{B})$. The arithmetic sum $\overset{\bullet}{+}$ is associative on $\mathbb{B}$ which is a corollary of the next lemma.

Lemma 3 *The map β is an isomorphism from the semigroup $(\mathbb{B}^+, \overset{\bullet}{+})$ onto $(\mathbb{Q}^+, +)$, that is:*

$$\beta(a \overset{\bullet}{+} b) = \beta(a) + \beta(b) \qquad (a, b \in \mathbb{B}^+). \tag{2.18}$$

(The proof of this lemma can be found in [17], pp.36 and here will be omitted.)

To see the associativity of $\overset{\bullet}{+}$, verify that $\beta((a \overset{\bullet}{+} b) \overset{\bullet}{+} c) = \beta(a \overset{\bullet}{+} (b \overset{\bullet}{+} c))$ $(a, b, c \in \mathbb{B}^+)$ results that $\overset{\bullet}{+}$ is associative on $\mathbb{B}^+$. Now, for each $a, b, c \in \mathbb{B}$, the truncations are elements of $\mathbb{B}^+$, thus $(a_{\langle n\rangle} \overset{\bullet}{+} b_{\langle n\rangle}) \overset{\bullet}{+} c_{\langle n\rangle} = a_{\langle n\rangle} \overset{\bullet}{+} (b_{\langle n\rangle} \overset{\bullet}{+} c_{\langle n\rangle})$ holds for every $n \in \mathbb{Z}$. Letting n tend to infinity, it follows, that $\overset{\bullet}{+}$ is associative on $\mathbb{B}$.

Hence $(\mathbb{B}, \overset{\bullet}{+})$ is a commutative group. Since

$$\|x \overset{\bullet}{+} y\| \leq \max\{\|x\|, \|y\|\}, \tag{2.19}$$

with equality if and only if $\|x\| \neq \|y\|$, this norm is non-Archimedean. ($\|nx\| \leq \|x\|$ for each $x \in \mathbb{B}$ and $n \in \mathbb{Z}$.)

The map $(a, b) \mapsto a \overset{\bullet}{+} b$ is continuous from $\mathbb{B} \times \mathbb{B}$ to $\mathbb{B}$; and the map $a \mapsto a^-$ is continuous from $\mathbb{B}$ to $\mathbb{B}$. This is a simple consequence of the following:

$$\begin{aligned} &I_n(a) = a \overset{\bullet}{+} \mathbb{I}_n := \{a \overset{\bullet}{+} x \;\; x \in \mathbb{I}_n\} \\ &I_n(a \overset{\bullet}{+} b) = I_n(a) \overset{\bullet}{+} I_n(b) := \{x \overset{\bullet}{+} y \;\; x \in I_n(a), y \in I_n(b)\} \\ &I_k^-(a) := \{x^- : x \in I_k(a)\} = I_k(a^-) \end{aligned} \tag{2.20}$$

where $a, b \in \mathbb{B}, n \in \mathbb{Z}$ and $k > \pi(a)$. Thus $(\mathbb{B}, \overset{\bullet}{+})$ is a topological group.

Note, that each $x \in \mathbb{B}$ can be written in form $x = \sum_{n\in\mathbb{Z}} x_n e_n$ using its additive digits $x = (x_n, n \in \mathbb{Z})$ where the sum is considered with respect to the addition $\overset{\bullet}{+}$.

✠ We will see in the following, that $(\mathbb{B}^*, \bullet)$ forms a commutative group. The arithmetic multiplication is commutative, and it is closely related to the usual multiplication of real numbers:

Lemma 4 *If $a, b \in \mathbb{B}^+$, then $a \bullet b \in \mathbb{B}^+$ and*

$$\beta(a \bullet b) = \beta(a)\beta(b). \tag{2.21}$$

(The proof can be found in [17], pp.38 and will be omitted here.)

An immediate consequence of (2.21) is, that multiplication $\bullet$ is associative on $\mathbb{B}^+$, thus $(a_{\langle n\rangle} \bullet b_{\langle n\rangle}) \bullet c_{\langle n\rangle} = a_{\langle n\rangle} \bullet (b_{\langle n\rangle} \bullet c_{\langle n\rangle})$ for all $n \in \mathbb{Z},\; a, b, c \in \mathbb{B}$. Letting $n \to \infty$, we find that $\bullet$ is associative on $\mathbb{B}$.

We can observe, that $e_k \bullet e_m = e_{k+m}$ for all $k, m \in \mathbb{Z}$. In general, multiplication by e_k shifts bytes: $e_k \bullet a = (a_{n-k}, n \in \mathbb{Z})$. The multiplicative identity of $\mathbb{B}$ is the element $e = e_0 = (\delta_{n0}, n \in \mathbb{Z})$, where δ_{nk} is the Kronecker-symbol.

The **existence of the multiplicative inverse element** of each $a \in \mathbb{B}^*$ follows from the existence of the inverse element of any $b \in \mathbb{S}$. Let us show, that for each $b \in \mathbb{S}$ there is an $x \in \mathbb{S}$ such that:

$$b \bullet x = e. \tag{2.22}$$

Observe, that (2.22) holds if and only if the additive digits of x satisfy

$$\begin{cases} q_0 = 0, x_0 = 1, \\ x_n + \displaystyle\sum_{j=0}^{n-1} x_j b_{n-j} + q_{n-1} = 0 + 2q_n \;\; (n \geq 1) \end{cases} \Leftrightarrow$$

$$\begin{cases} q_0 = 0, x_0 = 1, \\ x_n \equiv \displaystyle\sum_{j=0}^{n-1} x_j b_{n-j} + q_{n-1} \pmod 2 \\ q_n = \dfrac{1}{2}\left(x_n + \displaystyle\sum_{j=0}^{n-1} x_j b_{n-j} + q_{n-1} \right) \;\; (n \geq 1), \end{cases} \tag{2.23}$$

which defines $x_n \in \mathbb{A}$ and $q_n \in \mathbb{N}$ recursively for all $n \in \mathbb{N}$. The multiplicative inverse of an element $b \in \mathbb{S}$ with respect to $\bullet$ is denoted by $b^\bullet$ or b^{-1}.

Now, each $a \in \mathbb{B}^*$ can be uniquely written in the form:

$$a = e_n \bullet b \text{ for some } n \in \mathbb{Z} \text{ and } b \in \mathbb{S}.$$

We can easily see, that $e_{-n} \bullet b^\bullet$ is the inverse element of $a \in \mathbb{B}^*$; it is denoted by $a^\bullet$ or a^{-1}. Thus $(\mathbb{B}^*, \bullet)$ forms a **commutative group**.

The recursive form of inverse element of a byte $b \in \mathbb{S}_m$ $(m \in \mathbb{Z})$, which will be used in Section 5.2, can be given by the method of (2.23):

$$(b^{-1})_n = b_{n+m} + f_n(b_m, \cdots, b_{n+m-1}) \pmod 2 \tag{2.24}$$

for some $f : \mathbb{A}^{n-1} \to \mathbb{A}$.

✠ The operations $\overset{\bullet}{+}, \bullet$ are continuous with respect to the metric introduced by the norm (2.2).

The arithmetical multiplication is continuous on $\mathbb{B}^*$, because for given $a, b \in \mathbb{B}^*$ and $n > \pi(a) + \pi(b)$, and for each $x \in I_{n-\pi(b)}(a)$ and $y \in I_{n-\pi(a)}(b)$ we have $x \bullet y \in I_n(a \bullet b)$.

The rule of distributivity also holds, that is:

$$a \bullet (b \overset{\bullet}{+} c) = a \bullet b \overset{\bullet}{+} a \bullet c \qquad (a, b, c \in \mathbb{B}).$$

We conclude that $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is **a topological field**. $(\mathbb{S}, \bullet)$ is a subgroup of $(\mathbb{B}, \bullet)$.

Notice, that

$$\pi(a \bullet b) = \pi(a) + \pi(b). \tag{2.25}$$

hence, $\|x \bullet y\| = \|x\| \cdot \|y\|$. In addition to this,

$$\|x \overset{\bullet}{+} y\| \leq \max\{\|x\|, \|y\|\},$$

thus $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a **non-Archimedian normed field** with respect to the norm (2.2). We will use the following notation: $a \overset{\bullet}{-} b := a \overset{\bullet}{+} b^{-}$.

2.4 The Haar-measure

The measure μ defined in (2.7) is translation invariant with respect to $\overset{\circ}{+}$, that is, for $a \in \mathbb{B}$ and a Borel set $E \subseteq \mathbb{B}$ follows by (2.10) that the Borel set $E \overset{\circ}{+} a$ satisfies $\mu(E \overset{\circ}{+} a) = \mu(E)$. μ is also dilation preserving, that is, for each Borel set $E \subseteq \mathbb{B}$ and $b \in \mathbb{B}^*$ the Borel set $b \circ E := \{b \circ y,\ y \in E\}$ satisfies $\mu(b \circ E) = \|b\|\mu(E)$. Consequently μ is the normalized Haar-measure on the logical group $(\mathbb{B}, \overset{\circ}{+})$.

The measure μ is translation and reflection invariant with respect to $\overset{\bullet}{+}$, that is, if $a \in \mathbb{B}$ and E is a Borel set in $\mathbb{B}$, then by (2.20) the Borel sets E^{-} and $E \overset{\bullet}{+} a$ satisfy $\mu(E^{-}) = \mu(E)$, and $\mu(E \overset{\bullet}{+} a) = \mu(E)$. μ is also dilation preserving, that is, for each Borel set $E \subseteq \mathbb{B}$ and $b \in \mathbb{B}^*$ the Borel set $b \bullet E := \{b \bullet y,\ y \in E\}$ satisfies $\mu(b \bullet E) = \|b\|\mu(E)$. It follows, that μ is the normalized Haar-measure on the arithmetical group $(\mathbb{B}, \overset{\bullet}{+})$.

The Lebesgue measure is the Haar-measure on $(\mathbb{R}^+, +)$, moreover since the map

$$\alpha : \mathbb{B} \to \mathbb{R}^+, \quad \alpha(x) := \sum_{j=-\infty}^{\infty} x_j 2^{-j-1} \ (x \in \mathbb{B})$$

takes $I_n(a)$ to an interval $\left[\frac{p}{2^n}, \frac{p+1}{2^n}\right]$ with $p = \sum_{j=-\infty}^{n-1} a_j 2^{-j-1}$, where $n \in \mathbb{Z}, a_j \in \mathbb{A}$ $(j \in \mathbb{Z})$. Thus the map α is measure preserving from the measure space $(\mathbb{B}, \mathcal{B}_\mu, \mu)$ to the Lebesgue measure space $(\mathbb{R}^+, \mathcal{B}_L, \mu_L)$. (Here $\mathcal{B}_\mu$ denotes the σ-ring of Borel sets of $\mathbb{B}$ and $\mathcal{B}_L$ denotes the σ-ring of Borel sets of $\mathbb{R}^+$.) This represents a close connection between the considered measure spaces.

2.5 UDMD product systems

✠ Let $(\phi_n, n \in \mathbb{N})$ be a collection of complex valued functions defined on some common set. For each $m \in \mathbb{N}$ consider the following functions:

$$\psi_m := \prod_{j=0}^{\infty} \phi_j^{m_j} \qquad (m \in \mathbb{N}),$$

where m has the binary expansion $m = \sum_{j=0}^{\infty} m_j 2^j$ $(m_j \in \mathbb{A}, j \in \mathbb{N})$. The system $\psi = (\psi_m, m \in \mathbb{N})$ is called *the product system* generated by the system $(\phi_n, n \in \mathbb{N})$.

✠ Denote with $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. $\mathbb{I}, \mathcal{A}$, and the restriction of the measure μ on $\mathbb{I}$ gives a probability measure space $(\mathbb{I}, \mathcal{A}, \mu)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of $\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$ and $L^1(\mathbb{I})$ be the set of integrable functions $f : \mathbb{I} \to \mathbb{C}$. The *conditional expectation* of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) := \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f \, d\mu \qquad (x \in \mathbb{I}).$$

A sequence of functions $(f_n, n \in \mathbb{N}) \subset L^1(\mathbb{I})$ is called a *dyadic martingale* if each f_n is $\mathcal{A}_n$-measurable and

$$(\mathcal{E}_n f_{n+1}) = f_n \qquad (n \in \mathbb{N}).$$

The *sequence of martingale differences* of $(f_n, n \in \mathbb{N})$ is the sequence

$$\phi_n := f_{n+1} - f_n \qquad (n \in \mathbb{N}).$$

We notice that every dyadic martingale difference sequence has the form $\phi_n = r_n g_n$ $(n \in \mathbb{N})$ where $(g_n, n \in \mathbb{N})$ is a sequence of functions such that each g_n is $\mathcal{A}_n$-measurable and $(r_n, n \in \mathbb{N})$ denotes the Rademacher system on $\mathbb{I}$:

$$r_n(x) := (-1)^{x_n} \ (n \in \mathbb{N}).$$

The dyadic martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a *unitary dyadic martingale difference sequence* or a *UDMD sequence*, if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. Thus $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \ g_n \in L(\mathcal{A}_n), \ |g_n| = 1 \ \ (n \in \mathbb{N}). \tag{2.26}$$

Let us call a system $\psi = (\psi_m, m \in \mathbb{N})$ a *UDMD product system*, if it is a product system generated by a UDMD system, i.e., there is a UDMD system $(\phi_n, n \in \mathbb{N})$ such that for each $m \in \mathbb{N}$ with binary expansion $m = \sum_{j=0}^{\infty} m_j 2^j$ $(m_j \in \mathbb{A}, j \in \mathbb{N})$, the function ψ_m is obtained by:

$$\psi_m = \prod_{j=0}^{\infty} \phi_j^{m_j} \qquad (m \in \mathbb{N}).$$

The *dyadic maximal operator* and for $0 < p < \infty$ *the H_p norm* is defined by

$$\mathcal{E}^*(f) := \sup_{n \in \mathbb{N}} |\mathcal{E}_n f| \quad (f \in L^1(\mathbb{I})),$$
$$\|f\|_{H^p} := \|\mathcal{E}^* f\|_p \quad (f \in L^1(\mathbb{I})),$$

where $\|.\|_p$ denotes the $L^p(\mathbb{I})$ norm.

2.6 The transformation method

If we consider the Fourier expansion with respect to a system given by the composition of the character system and a measure preserving transformation, then its partial sums and Cesaro means can be expressed by the original ones, that is by partial sums and Cesaro means of Fourier series with respect to the characters. This will be applied in the next chapters to discuss summability and convergence questions.

Let now $\{\phi_n, n \in \mathbb{N}\}$ denote the character set of the studied additive group, and consider a measure-preserving variable transformation $T : \mathbb{I} \to \mathbb{I}$. Then,

$$\int_{\mathbb{I}} f \circ T \ d\mu = \int_{\mathbb{I}} f d\mu. \tag{2.27}$$

Definition 1 *Let us define the T-Fourier coefficients of an $f \in L^1(\mathbb{I})$ by*

$$\widehat{f^T}(n) := \int_{\mathbb{I}} f(x)\phi_n(T(x))d\mu(x) \qquad (n \in \mathbb{N}).$$

Furthermore the T-Fourier series $S^T f$ of f and the n-th partial sum $S_n^T f$ of the T-Fourier series $S^T f$ is defined by

$$S^T f := \sum_{k=0}^{\infty} \widehat{f^T}(k) \cdot \phi_k \circ T, \text{ and } S_n^T f := \sum_{k=0}^{n-1} \widehat{f^T}(k) \cdot \phi_k \circ T \ (n \in \mathbb{P}).$$

Let us define the T-Cesaro (or $(T-C,1)$) means of $S^T f$ by

$$\sigma_0^T f := 0 \text{ and } \quad \sigma_n^T f := \frac{1}{n} \sum_{k=1}^{n} S_k^T f \quad (n \in \mathbb{P}).$$

Proposition 1 *For any $f \in L^1(\mathbb{I})$, $n \in \mathbb{P}$ hold*

$$S_n^T f = \left[S_n(f \circ T^{-1})\right] \circ T, \text{ and} \tag{2.28}$$

$$\sigma_n^T f = \left[\sigma_n(f \circ T^{-1})\right] \circ T, \tag{2.29}$$

where S_n and σ_n stand for the corresponding notions with respect to the characters $\{\phi_n, n \in \mathbb{N}\}$ of the additive group.

Proof: If $\widehat{f}(n)$ denotes the Fourier coefficients with respect to the characters of an $f \in L^1(\mathbb{I})$ presented also in [16], we conclude by (2.27), that

$$\widehat{f^T}(n) = \widehat{f \circ T^{-1}}(n) \qquad (n \in \mathbb{N}).$$

Thus,

$$S_n^T f = \sum_{k=0}^{n-1} \widehat{f \circ T^{-1}}(n) \cdot \phi_k \circ T = \left[S_n(f \circ T^{-1})\right] \circ T,$$

which leads to

$$\sigma_n^T f(x) = \frac{1}{n} \sum_{k=1}^{n} \left[S_k(f \circ T^{-1})\right] (T(x)) = \sigma_n(f \circ T^{-1})(T(x)).$$

$\square$

Remark: On the complex field basically this method was used in terms of the scalar products in Bokor-Schipp [3]. On the studied fields the presented proposition enabled the author to handle a.e. convergence and summabilty questions of Fourier series with respect to the discrete Laguerre and $(v_n \circ \gamma, n \in \mathbb{N})$ systems in I. Simon [39] and I. Simon[40]. Professor F. Schipp claimed that the proposition is also true for general measure-preserving transformations. We will use the term "transformation method" in this work to ease the explanations.

Chapter 3

Some useful functions

This chapter is devoted to some useful tools which are used in the next chapters. Paragraph 3.1 provides a description of the characters of the dyadic and 2-adic multiplicative groups based on the handbook of Schipp and Wade [17] and using the notion of the product system. Paragraph 3.2 contains the notions and results regarding the $(\tilde{\mathbb{S}}, \bullet)$-valued exponential function ζ. Starting from Paragraph 3.3, this work contains the results of the author. Paragraph 3.3 contains the definitions and properties of the respective Blaschke functions, which is due to he author.

3.1 The characters of the additive groups

✠ A *character* of a topological group $(\mathbb{G}, *)$ is a continuous function $\phi : \mathbb{G} \to \mathbb{C}$ that satisfies

$$|\phi(x)| = 1 \quad \text{and} \quad \phi(x * y) = \phi(x)\phi(y) \text{ for all } x, y \in \mathbb{G}. \tag{3.1}$$

That is, the characters of a group are the continuous homomorphisms into the torus $(\mathbb{T}, \cdot)$. If ϕ is a character on $(\mathbb{G}, *)$, and θ represents the zero element of $\mathbb{G}$, then $\phi(\theta) = 1$. It can be easily seen, that the set of characters of a topological group $(\mathbb{G}, *)$ forms a group under pointwise multiplication; it is called the dual group of $(\mathbb{G}, *)$ and it is denoted by $\widehat{(\mathbb{G}, *)}$.

Consider a normed field $(\mathbb{F}, +, \cdot)$. Let u be a character of the additive group $(\mathbb{F}, +)$. Then for each $y \in \mathbb{F}$ the map $u_y(x) := u(x \cdot y)$ $(x \in \mathbb{F})$ is also a character of $(\mathbb{F}, +)$. If these functions exhaust the characters of the group, then u is called a *basic character* of $\mathbb{F}$. If $\mathbb{F}$ has a basic character u, then by the map $y \mapsto u_y$ it follows, that the group $(\mathbb{F},+)$ is isomorphic to $\widehat{(\mathbb{F},+)}$.

✠ **Characters of** $(\mathbb{B},\overset{\circ}{+})$ **and** $(\mathbb{I},\overset{\circ}{+})$

Let us consider $\epsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$ and the maps

$$\begin{aligned} w(x) &:= \epsilon\left(\frac{x_{-1}}{2}\right) \quad (x \in \mathbb{B}); \\ w_y(x) &:= w(x \circ y) \quad (x, y \in \mathbb{B}). \end{aligned} \tag{3.2}$$

w is a basic character of the group and $\widehat{(\mathbb{B},\overset{\circ}{+})} = \{w_y : y \in \mathbb{B}\}$. Easy computations show that $w(x) = (-1)^{x_{-1}}$ and $w(x \overset{\circ}{+} y) = w(x)w(y)$ for all $x, y \in \mathbb{B}$. Since w is constant on intervals with rank bigger than -1, thus w is continuous on $\mathbb{B}$, and it follows that w is a character of the group, and thus w_y is also a character for each $y \in \mathbb{B}$.

Moreover, it can be showed, that w is a basic character of $(\mathbb{B},\overset{\circ}{+})$: each character of $\mathbb{B}$ is a w_y for some $y \in \mathbb{B}$: $\widehat{(\mathbb{B},\overset{\circ}{+})} = \{w_y : y \in \mathbb{B}\}$. Furthermore,

$$w_{y \overset{\circ}{+} z}(x) = w_y(x) w_z(x) \qquad (x, y, z \in \mathbb{B}), \tag{3.3}$$

thus, by $(w_x(\theta) =) w_\theta(x) = 1$ it follows, that the map $y \mapsto w_y$ is an isomorphism from $(\mathbb{B},\overset{\circ}{+})$ onto $\widehat{(\mathbb{B},\overset{\circ}{+})}$, that is: $\widehat{(\mathbb{B},\overset{\circ}{+})} \cong (\mathbb{B},\overset{\circ}{+})$. (See [17], pp.63.)

Now, we will describe the characters of $(\mathbb{I},\overset{\circ}{+})$. For each $y \in \mathbb{B}$ let $[y] := y_{\langle 0 \rangle}$ represent the integer part of y, where we used the 0-th truncation defined in (2.5). If $x \in \mathbb{I}$, then

$$w_y(x) = (-1)^{(x \circ y)_{-1}} = (-1)^{\sum_{j=0}^{\infty} x_j y_{-j-1}} = (-1)^{(x \circ [y])_{-1}} = w_{[y]}(x).$$

Thus the characters of $(\mathbb{I},\overset{\circ}{+})$ are the restrictions of the $w_{[y]}$-s on $\mathbb{I}$. By identifying $[y]$ with the integer $n := \sum\limits_{j=0}^{\infty} y_{-j-1} 2^j \in \mathbb{N}$, we see that $w_{[y]}(x) = (-1)^{\sum_{j=0}^{\infty} x_j y_{-j-1}}$ can be written in the form $w_n(x) = (-1)^{\sum_{j=0}^{\infty} x_j n_j}$ $(x \in \mathbb{I})$ with dyadic expansion $n = \sum\limits_{j=0}^{\infty} n_j 2^j$. The functions $(w_n, n \in \mathbb{N})$ are the so-called *Walsh-Paley functions*.

The characters of $(\mathbb{I}, \overset{\circ}{+})$ can be expressed also with the so-called Rademacher functions $(r_n, n \in \mathbb{N})$ given by:

$$r_n(x) := (-1)^{x_n} \ \ (x \in \mathbb{I}).$$

The Walsh-Paley functions w_n are characters, being a finite product of characters:

$$w_n(x) = (-1)^{\sum_{j=0}^{+\infty} n_j x_j} = \prod_{j=0}^{\infty} r_j(x)^{n_j} \ \ (x \in \mathbb{I}) \tag{3.4}$$

where $n = \sum_{j=0}^{\infty} n_j 2^j \in \mathbb{N}$ $(n_j \in \mathbb{A})$. In particular, the Walsh-Paley functions form a product system generated by the Rademacher system $(r_n, n \in \mathbb{N})$.

✠ **Characters of** $(\mathbb{B}, \overset{\bullet}{+})$ **and** $(\mathbb{I}, \overset{\bullet}{+})$
Consider $\epsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$ and define the maps

$$\begin{aligned} v(x) &:= \epsilon\left(\frac{x_{-1}}{2} + \frac{x_{-2}}{2^2} + \dots\right) \quad (x \in \mathbb{B}); \\ v_y(x) &:= v(x \bullet y) \quad (x, y \in \mathbb{B}). \end{aligned} \tag{3.5}$$

v is a basic character of $(\mathbb{B}, \overset{\bullet}{+})$ and $\widehat{(\mathbb{B}, \overset{\bullet}{+})} = \{v_y : y \in \mathbb{B}\}$.

Since v is constant on intervals with rank bigger than -1, it follows that v is continuous on $\mathbb{B}$. Let us show that

$$v(x \overset{\bullet}{+} y) = v(x)v(y) \tag{3.6}$$

holds for all $x, y \in \mathbb{B}$. The definition of β gives, that $v(x) = \epsilon(\beta(x_{\langle 0 \rangle}))$ $(x \in \mathbb{B}^+)$, thus by (2.18) holds $v(x \overset{\bullet}{+} y) = v(x)v(y)$ for all $x, y \in \mathbb{B}^+$. Since v and the field operation $\overset{\bullet}{+}$ are continuous, and $\mathbb{B}^+$ is dense in $\mathbb{B}$, it follows that (3.6) holds for each $x, y \in \mathbb{B}$, thus v is a character of the group. By the distributivity of the field operations we have that (3.6) is valid also for v_z for each $z \in \mathbb{B}$: $v_z(x \overset{\bullet}{+} y) = v(z \bullet (x \overset{\bullet}{+} y)) = v(z \bullet x)v(z \bullet y) = v_z(x)v_z(y)$, and being continuous, v_z is also a character of the group for any $z \in \mathbb{B}$.

Furthermore, v is a basic character of $(\mathbb{B}, \overset{\bullet}{+})$: each character of $(\mathbb{B}, \overset{\bullet}{+})$ is one of the functions v_y for some $y \in \mathbb{B}$: $\widehat{(\mathbb{B}, \overset{\bullet}{+})} = \{v_y : y \in \mathbb{B}\}$. Now,

$$v_{y \overset{\bullet}{+} z}(x) = v_y(x)v_z(x) \qquad (x, y, z \in \mathbb{B}), \tag{3.7}$$

thus by $(v_x(\theta) =)v_\theta(x) = 1$ it follows, that the map $y \mapsto v_y$ is an isomorphism from $(\mathbb{B}, \overset{\bullet}{+})$ onto $\widehat{(\mathbb{B}, \overset{\bullet}{+})}$, that is: $\widehat{(\mathbb{B}, \overset{\bullet}{+})} \cong (\mathbb{B}, \overset{\bullet}{+})$. (See [17], pp.66.)

Now, we will describe the characters of $(\mathbb{I}, \overset{\bullet}{+})$. With the expansion $y = [y] \overset{\bullet}{+} y'$ $(y' \in \mathbb{I})$ for $x \in \mathbb{I}$ we have $x \bullet y = x \bullet [y] \overset{\bullet}{+} x \bullet y'$, thus

$$v_y(x) = v_{[y]}(x) \qquad (x \in \mathbb{I}).$$

Therefore, the characters of $(\mathbb{I}, \overset{\bullet}{+})$ are the restrictions of $v_{[y]}$-s on $\mathbb{I}$. By identifying $[y]$ with the integer $m := \sum_{j=0}^{\infty} y_{-j-1}2^j = \sum_{j=0}^{\infty} m_j 2^j \ \in \mathbb{N}$, we see that $v_{[y]}(x)$ can be written in the form

$$\begin{aligned} v_m(x) &= \prod_{j=0}^{\infty} v_{y_{-j-1}e_{-j-1}}(x) = \prod_{j=0}^{\infty} (v_{2^j}(x))^{m_j} \qquad \text{where} \\ v_{2^n}(x) &:= v_{e_{-n-1}} = \epsilon\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \ldots\right) \quad (x \in \mathbb{I}). \end{aligned} \tag{3.8}$$

Thus the character group of $(\mathbb{I}, \overset{\bullet}{+})$ is formed by the product system $(v_m, m \in \mathbb{N})$ generated by the functions $(v_{2^n}(x), n \in \mathbb{N})$.

3.2 The exponential function

✠ On some classical elementary functions

The exponential function on $\mathbb{C}$ is a nonzero continuous function satisfying the functional equation

$$\exp(x + y) = \exp(x)\exp(y) \qquad (x, y \in \mathbb{C}).$$

Consider the following classical functions expressed by the exp function:

$$\begin{aligned} \sin(x) &:= \frac{\exp(ix) - \exp(-ix)}{2i}; \quad \cos(x) := \frac{\exp(ix) + \exp(-ix)}{2}; \quad (x \in \mathbb{R}) \\ \tan(x) &:= \frac{\exp(ix) - \exp(-ix)}{i(\exp(ix) + \exp(-ix))} = \frac{\exp(2ix) - 1}{i(\exp(2ix) + 1)} \\ &\qquad\qquad \left(x \in \mathbb{R} \setminus \left\{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\right\}\right). \end{aligned}$$

The functional equation

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \tag{3.9}$$

of the function tan inspired the solution of the problem in Chapter 5.

✠ The exponential function on $\mathbb{I}_1$

A 2-adic exponential function is presented in Schipp [17], pp 59-60. We will use now a similar one determined by a slightly different base, starting from $b_1 = e \overset{\bullet}{+} e_2$ instead of $e \overset{\bullet}{+} e_1$. We will need in Chapter 5 the following exponential function. (As we use an exponential function on $\mathbb{I}$ in Chapter 8, we will investigate that version there.) Let us introduce the notation $\tilde{\mathbb{S}} := \{x \in \mathbb{S} : x_1 = 0\}$. By using the symbol $\prod$, we mean the arithmetical product.

Definition 2 *Consider the following base:*

$$b_1 := e \overset{\bullet}{+} e_2, \qquad b_n := b_{n-1} \bullet b_{n-1} \qquad (n \geq 2). \tag{3.10}$$

Definition 3 *Let us define the $(\tilde{\mathbb{S}}, \bullet)$-valued exponential function ζ on $\mathbb{I}_1$ by the following infinite product form:*

$$\zeta(x) := \prod_{j=1}^{\infty} b_j^{x_j} \qquad (x = (x_j, j \in \mathbb{Z}) \in \mathbb{I}_1). \tag{3.11}$$

As this function ζ is very similar to the exponential function presented in [17], the next proofs are the adaptations of the proofs of Theorem 2 in [17], pp.51-53 and of Proposition 4 in [17], pp.59.

An inductive argument shows, that $b_n = e \overset{\bullet}{+} c_n$ $(n \geq 1)$ with $\pi(c_n) = n+1$, thus the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} (e \overset{\bullet}{+} c_j)^{x_j} = \prod_{j=1}^{\infty} (e \overset{\bullet}{+} x_j c_j) \qquad (x \in \mathbb{I}_1). \tag{3.12}$$

Lemma 5 *The function ζ satisfies the functional equation*

$$\zeta(x \overset{\bullet}{+} y) = \zeta(x) \bullet \zeta(y) \qquad (x, y \in \mathbb{I}_1). \tag{3.13}$$

Proof: The proof is almost the same to that of Proposition 4 in [17], pp.59-60. By (2.15) and (3.10) we find that

$$\prod_{j=1}^{n} b_j^{x_j} \bullet \prod_{j=1}^{n} b_j^{y_j} \bullet \prod_{j=1}^{n} b_j^{q_{j-1}} = \prod_{j=1}^{n} b_j^{(x \overset{\bullet}{+} y)_j} \bullet \prod_{j=1}^{n} b_{j+1}^{q_j}.$$

Now, simplifying the product in the last terms and taking the limit as $n \to \infty$ we obtain (8.5) by using $\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty}(e \overset{\bullet}{+} e_{n+2} \overset{\bullet}{+} t_{n+1}) = e$ for $t_{n+1} \in \mathbb{I}_{n+3}$.

$\square$

The next lemma shows, that $\{\zeta(x) : x \in \mathbb{I}_1\} = \tilde{\mathbb{S}}$, and ζ is one-one and continuous from $\mathbb{I}_1$ onto $\tilde{\mathbb{S}}$.

Lemma 6 *The function ζ defined in (16) is a continuous isomorphism from $\mathbb{I}_1$ onto $\tilde{\mathbb{S}}$.*

Proof: The proof is similar to that of Theorem 2 in [17], pp.51-53. By the definition of b_j we get $b_j = e \overset{\bullet}{+} c_j$ $(j \geq 1)$ where $\pi(c_j) = j + 1$ $(j \geq 1)$, and so ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty}(e \overset{\bullet}{+} c_j)^{x_j} = \prod_{j=1}^{\infty}(e \overset{\bullet}{+} x_j c_j).$$

We begin by noticing that since $\pi(c_n) = n + 1$ $(n \geq 1)$, each c_n is of the form

$$\begin{aligned} c_1 &= e_2, \\ c_n &= e_{n+1} \overset{\bullet}{+} t_n, \text{ where } t_n \in \mathbb{I}_{n+2} \ (n \geq 2). \end{aligned}$$

Since $1 > \|c_1\| \geq \|c_2\| \geq \ldots \geq \|c_n\| \to 0$, the convergence of the modulus of continuity $\omega(\zeta, 2^{-n}) = \|c_n\| \to 0$ holds, thus ζ is continuous on $\mathbb{I}$. (See [17], pp.51)

To show that ζ is onto, it suffices to prove that to any given

$$y = e \overset{\bullet}{+} y_2 e_2 \overset{\bullet}{+} y_3 e_3 \overset{\bullet}{+} \ldots \in \tilde{\mathbb{S}}$$

there exists an $x \in \mathbb{I}_1$ and a sequence $T_n \in \mathbb{I}_{n+2}$ $(n \geq 1)$ such that

$$\begin{aligned}
p_1(x) &:= e \overset{\bullet}{+} x_1 c_1 = e \overset{\bullet}{+} y_2 e_2 \overset{\bullet}{+} T_1 \\
p_n(x) &:= \prod_{j=1}^{n} (e \overset{\bullet}{+} x_j c_j) = e \overset{\bullet}{+} y_2 e_2 \overset{\bullet}{+} y_3 e_3 \overset{\bullet}{+} \ldots \overset{\bullet}{+} y_{n+1} e_{n+1} \overset{\bullet}{+} T_n \ (n \geq 2)
\end{aligned} \tag{3.14}$$

We will establish (3.14) by induction on n. If $n = 1$, set $x_1 = y_2$ and $T_1 = 0$. If (3.14) holds for some $n \geq 1$, then write

$$T_n := \delta_{n+2} e_{n+2} \overset{\bullet}{+} T'_n \ (n \geq 2)$$

with some $\delta_{n+2} \in \mathbb{A}$ and $T'_n \in \mathbb{I}_{n+3}$. Thus (3.14) is satisfied for $n+1$ in place of n if and only if

$$\underbrace{\delta_{n+2} e_{n+2} \overset{\bullet}{+} T'_n}_{T_n} \overset{\bullet}{+} \underbrace{p_n(x)}_{e \overset{\bullet}{+} p'_n(x)} \ x_{n+1} \quad \underbrace{c_{n+1}}_{e_{n+2} \overset{\bullet}{+} t_{n+1}} \quad = y_{n+2} e_{n+2} \overset{\bullet}{+} \underbrace{T_{n+1}}_{\in \mathbb{I}_{n+3}}, \tag{3.15}$$

where $t_{n+1} \in \mathbb{I}_{n+3}$. Write $p_n(x)$ in the form

$$p_n(x) = e \overset{\bullet}{+} p'_n(x), \qquad \text{with some } \ p'_n(x) \in \mathbb{I}_1,$$

and define $q_{n+1}, l_{n+1} \in \mathbb{A}$ so that

$$\delta_{n+2} + x_{n+1} = 2q_{n+1} + l_{n+1}$$

holds. Clearly,

$$\delta_{n+2} e_{n+2} + x_{n+1} e_{n+2} = q_{n+1} e_{n+3} + l_{n+1} e_{n+2}.$$

Therefore, (3.15) is equivalent to

$$y_{n+2} = l_{n+1}, \qquad T_{n+1} = q_{n+1} e_{n+3} \overset{\bullet}{+} T'_n \overset{\bullet}{+} x_{n+1} e_{n+2} p'_n(x) \overset{\bullet}{+} x_{n+1} p_n(x) t_{n+1}.$$

In particular, (3.14) is satisfied if we set

$$\begin{aligned}
x_{n+1} &:= y_{n+2} + \delta_{n+2} \pmod 2 \\
q_{n+1} &:= \left[\frac{x_{n+1} + \delta_{n+2}}{2} \right] \\
\text{and } T_{n+1} &:= q_{n+1} e_{n+3} \overset{\bullet}{+} T'_n \overset{\bullet}{+} x_{n+1} (e_{n+2} p'_n(x) \overset{\bullet}{+} t_{n+1} p_n(x)).
\end{aligned}$$

Thus for every $y \in \tilde{\mathbb{S}}$ we get an $x \in \mathbb{I}_1$ such that $\zeta(x) = y$.

In order to prove that ζ is one-one, set

$$P_n(x) := \prod_{j=n}^{\infty} (e \overset{\bullet}{+} x_j c_j) \quad (n \geq 1),$$

and observe that $P_n(x)$ is of the form $P_n(x) = e \overset{\bullet}{+} x_n e_{n+1} \overset{\bullet}{+} \tilde{P}_n(x)$ for some $\tilde{P}_n(x) \in \mathbb{I}_{n+2}$ if $n \geq 1$. Thus for all $n \geq 1$, $P_n(x) = P_n(y)$ implies $x_n = y_n$. Since $\zeta(x) = \zeta(y)$ is equivalent to $P_1(x) = P_1(y) \Rightarrow x_1 = y_1 \Rightarrow P_2(x) = P_2(y)$ and so on, we conclude that $x_n = y_n$ for all $n \geq 1$. See [17] pp.53.

□

3.3 The Blaschke functions

We will present the logical and arithmetical Blaschke functions, which were introduced and studied in I. Simon[39] and I. Simon[40]. First let us sum up some properties of the Blaschke functions on $\mathbb{C}$.

✠ **The Blaschke functions on $\mathbb{C}$:** Consider the open unit disc and its boundary

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}; \ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\},$$

respectively, and $\bar{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}$ denotes the closure of $\mathbb{D}$. Let $\mathfrak{a}$ denote the disc algebra: $\mathfrak{a} := \{F : \bar{\mathbb{D}} \to \mathbb{C} : \ F$ is analytic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}\}$.

The Blaschke function on $\mathbb{C}$ associated to a complex parameter $a \in \mathbb{D}$ is defined by

$$B_a(z) := e^{i\gamma} \frac{z - a}{1 - \bar{a}z} \quad (z \in \mathbb{D}), \tag{3.16}$$

where $\gamma \in \mathbb{R}$ and $\bar{a}$ is the complex conjugate of $a \in \mathbb{D}$.

It is known, that $B_a \in \mathfrak{a}$ and B_a is a one-one map from $\mathbb{D}$ onto $\mathbb{D}$, and from $\mathbb{T}$ onto $\mathbb{T}$ for every $a \in \mathbb{D}$. The inverse of B_a is also a Blaschke function:

$$B_a^{-1}(z) = e^{-i\gamma} \frac{z + e^{i\gamma} a}{1 + e^{-i\gamma} \bar{a} z} \quad (z \in \mathbb{D}).$$

If $|z| = 1$ and a belongs to $\mathbb{D}$, then $|B_a(z)| = 1$, that is, B_a is a bijection on the unit circle $\mathbb{T}$. According to Bokor- Schipp[3], B_a can be written in the form

$$B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R}, a \in \mathbb{D}) \tag{3.17}$$

with the following bijection $\beta_a : [-\pi, \pi] \to [-\pi, \pi]$,

$$\beta_a(t) := \gamma + \varphi + 2\arctan\left(s \tan\left(\frac{t-\varphi}{2}\right)\right),$$

where $a = re^{i\varphi} \in \mathbb{C}$ and $s = \eta(r)$ is defined by means of the bijection $\eta : [0, \infty) \to [0, \infty)$:

$$\eta(r) := \begin{cases} \dfrac{1+r}{1-r} & \text{for } 0 \leq r < 1 \\ \dfrac{r-1}{r+1} & \text{for } 1 \leq r < \infty. \end{cases}$$

Furthermore, the composition of two Blaschke-functions, B_{a_1} and B_{a_2} is a Blaschke function. (See Bokor-Schipp[3] and Soummelidis-Bokor-Schipp[32].)

✠ The logical Blaschke function

Definition 4 *For $a \in \mathbb{I}_1$ define the (logical) Blaschke function on $(\mathbb{I}, \overset{\circ}{+}, \circ)$ by:*

$$B_a(x) := (x \overset{\circ}{+} a) \circ (e \overset{\circ}{+} a \circ x)^{-1} = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x} \qquad (x \in \mathbb{I}). \tag{3.18}$$

Since $\pi(a) \geqq 1$ and $\pi(x) \geqq 0$, we have $\pi(a \circ x) \geqq 1$, therefore by (2.9) we have $\pi(e \overset{\circ}{+} a \circ x) = 0$, hence $e \overset{\circ}{+} a \circ x \neq \theta$. Thus the function B_a is well-defined on $\mathbb{I}$.

Note, that by (2.13) it follows that $\pi(u \circ v^{-1}) = \pi(u) - \pi(v)$ $(u, v \in \mathbb{B})$, thus

$$\|B_a(x)\| \leqq 1 \text{ if } \|x\| \leqq 1, \text{ and } \|B_a(x)\| = 1 \text{ if } \|x\| = 1. \tag{3.19}$$

Since the additive inverse of a byte $a \in \mathbb{B}$ is the element a itself, we get that $B_a(x) = y$ implies $B_a(y) = x$, therefore B_a is a bijection on the unit ball $\mathbb{I}$ and on the unit sphere $\mathbb{S} := \mathbb{S}_0 = \{x \in \mathbb{B} \mid \|x\| = 1\}$. Moreover, for the inverse of B_a we have

$$B_a^{-1} = B_a.$$

It is easy to see for $a, b \in \mathbb{I}_1$, that

$$B_a(B_b(x)) = B_c(x) \quad (x \in \mathbb{I}), \text{ where } c = \frac{a \overset{\circ}{+} b}{e \overset{\circ}{+} a \circ b} = B_a(b) \in \mathbb{I}_1. \tag{3.20}$$

This implies that the maps B_a ($a \in \mathbb{I}_1$) form a commutative group with respect to the composition of functions and each element is of order 2.

✠ In the following we will establish **the recursive form of the byte** $B_a(x)$. Let $x \in \mathbb{I}, a \in \mathbb{I}_1$ and set $y = B_a(x)$. Then $\|y\| \leqq 1$ and by (3.18) we have

$$y = x \overset{\circ}{+} a \overset{\circ}{+} y \circ a \circ x$$

and consequently the n-th digit of y is

$$\begin{cases} y_n = 0, \text{ for } n < 0, \\ y_n = x_n + a_n + (y \circ a \circ x)_n \pmod 2, \text{ for } n \geqq 0. \end{cases}$$

Thus the bits of $y = B_a(x)$ can be obtained by recursion for any given $x \in \mathbb{I}$, since in order to compute $(y \circ a \circ x)_n$ we only need y_k-s with $k < n$. Indeed, let us verify that $(y \circ a \circ x)_n$ really depends only on $y_0, ..., y_{n-2}, y_{n-1}$. Use definition (2.9) of the product to get

$$(y \circ a \circ x)_n = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} a_{n-i-j} x_i y_j \pmod 2$$

and the x_i-s and y_j-s can be different from 0 only for $i, j \geq 0$ and the a_{n-i-j}-s for $n - i - j > 0$. And so,

$$(y \circ a \circ x)_n = \sum_{0 \leqq i,j,i+j<n} a_{n-i-j} x_i y_j \pmod 2 = \sum_{j=0}^{n-1} y_j \Big(\sum_{i=0}^{n-1-j} a_{n-i-j} x_i \Big) \pmod 2$$

and we obtain the following recursion:

$$\begin{aligned} y_0 &= x_0 + a_0 \pmod 2 \\ y_n &= x_n + a_n + \sum_{j=0}^{n-1} \Big(\sum_{i=0}^{n-j-1} a_{n-i-j} x_i \Big) y_j \pmod 2 \quad (n = 1, 2, \cdots). \end{aligned} \tag{3.21}$$

This implies that $y_n = (B_a(x))_n$ can be written in the form

$$y_n = x_n + a_n + f_n(x_0, \cdots, x_{n-1}) \pmod 2 \tag{3.22}$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ ($n = 1, 2, \cdots$) depend only on the bits of a.

✠ The arithmetical Blaschke function

Definition 5 *For $a \in \mathbb{I}_1$ define the (arithmetical) Blaschke function on $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$:*

$$B_a(x) := (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x} \qquad (x \in \mathbb{I}). \tag{3.23}$$

For $x \in \mathbb{I}$ and $a \in \mathbb{I}_1$ from (2.19) we have that $e \overset{\bullet}{-} a \bullet x \neq \theta$, thus $e \overset{\bullet}{-} a \bullet x$ has a multiplicative inverse in $\mathbb{B}$, and so the function is well-defined.

We state first, that

Proposition 2 *$B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$ on $\mathbb{I}$, and on $\mathbb{S} \subset \mathbb{I}$ as well.*

Proof: From (2.25) it follows that $\pi(u \bullet v^{-1}) = \pi(u) - \pi(v)$ $(u, v \in \mathbb{B})$, thus

$$\|B_a(x)\| \leqq 1 \ \text{ if } \|x\| \leqq 1, \ \text{ and } \ \|B_a(x)\| = 1 \text{ if } \|x\| = 1. \tag{3.24}$$

$B_a(x) = y$ implies

$$x = B_{a^-}(y) = \frac{y \overset{\bullet}{+} a}{e \overset{\bullet}{+} a \bullet y} \in \mathbb{I},$$

where a^- denotes the reflection of a, the additive inverse of a defined in (2.17), and clearly, $e \overset{\bullet}{+} a \bullet x \neq \theta$, as required. Therefore we have seen, that the Blaschke function $B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$ on $\mathbb{I}$ and on $\mathbb{S}$.

Moreover, if B_a^{-1} is the inverse of B_a, then the former argument results, that

$$B_a^{-1} = B_{a^-} \tag{3.25}$$

and (3.24) holds with "exactly when" instead of "if".

□

The composition of two Blaschke functions is also a Blaschke function:

$$B_a \circ B_b = B_c, \quad \text{where } c = \frac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b} \in \mathbb{I}_1 \quad (a, b \in \mathbb{I}_1). \tag{3.26}$$

We will use the notation $a \triangleleft b := \frac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b} \in \mathbb{I}_1 \quad (a, b \in \mathbb{I}_1)$ in Chapter 5. Now, $B_a \circ B_b = B_{a \triangleleft b}$ $(a, b \in \mathbb{I}_1)$ ensures that the maps B_a $(a \in \mathbb{I}_1)$ form a commutative group with respect to the composition of functions. The identity element is the identity map $B_\theta = \imath$, and the inverse element of B_a is B_{a^-}.

Definition 6 *We will call* $(\mathcal{B}, \circ)$ *the Blaschke-group of the field* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$*, where*

$$\mathcal{B} := \{B_a, a \in \mathbb{I}_1\} \tag{3.27}$$

and $\circ$ *denotes the composition of functions.*

✠ Now, we will mention **the recursive form of the byte** $B_a(x)$ in the arithmetic case. For $B_a(x) = y$ we have

$$y = x \overset{\bullet}{-} a \overset{\bullet}{+} a \bullet x \bullet y. \tag{3.28}$$

Thus we can give the byte $y = B_a(x)$ recursively by

$$\begin{cases} y_n = 0, \text{ for } n < 0 \\ y_n = x_n - a_n - q_{n-1} + 2q_n + (y \bullet a \bullet x)_n + Q_{n-1} - 2Q_n \quad \text{for } n \geqq 0. \end{cases} \tag{3.29}$$

Here q_n is the rest given in the definition of the 2-adic difference $(x \overset{\bullet}{-} a)_n$ by: $q_n = 0$ for $n < m := \min\{\pi(x), \pi(a)\}$ and $x_n - a_n - q_{n-1} + 2q_n = (x \overset{\bullet}{-} a)_n$ for $n \geq m$, and Q_n is the rest given in the definition of the 2-adic sum $(x \overset{\bullet}{-} a) \overset{\bullet}{+} (y \bullet a \bullet x)$: namely $Q_n = 0$ for $n < m_1 := \min\{\pi(x \overset{\bullet}{-} a), \pi(y \bullet a \bullet x)\}$ and $(x \overset{\bullet}{-} a)_n + (y \bullet a \bullet x)_n + Q_{n-1} - 2Q_n = \left[(x \overset{\bullet}{-} a) \overset{\bullet}{+} (y \bullet a \bullet x)\right]_n$ for $n \geq m_1$. To compute y_n we need Q_{n-1} computed in the previous step, and after y_n we get Q_n by the following integer part

$$Q_n := \left[\frac{(x \overset{\bullet}{-} a)_n + (y \bullet a \bullet x)_n + Q_{n-1} - y_n}{2}\right].$$

Thus the recursive form of $y = B_a(x)$ is well-defined for $x \in \mathbb{I}$, because for $(y \bullet a \bullet x)_n$ we only need y_k-s with $k < n$, which can be shown similarly to the 2-series case.

This implies that also in the arithmetic case, the digit $y_n = (B_a(x))_n$ can be written in the form

$$y_n = x_n + f_n(x_0, \cdots, x_{n-1}) \pmod 2$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ depend only on the bits of a.

Chapter 4

Dyadic martingale structure preserving transformations

This chapter is based on I. Simon[42], and with exception of Example 2 is completely due to the author.

Numerous results were published in the last century about the effect of the composition with a Blaschke function on the convergence of the power series of regular functions in a boundary point of the complex disc $\mathbb{D}$. First, Turán [46] showed, that to any $\zeta \in \mathbb{C}$ $(0 < |\zeta| < 1)$ there is a complex function $f_1(z) = \sum_{n=1}^{\infty} a_n z^n$, regular in $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, with convergent power-series for $z = 1$, but the power series of $f_2(z) := f_1(B_\zeta(z)) = \sum_{n=1}^{\infty} b_n z^n$ diverges for the corresponding point $z = B_\zeta^{-1}(1)$, where $B_\zeta(z)$ denotes the Blaschke function with parameter $\zeta \in \mathbb{C}$: $B_\zeta(z) = \frac{z-\zeta}{1-\overline{\zeta}z}$ $(z \in \overline{\mathbb{D}})$. After results of Clunie, Schwarz, Halász, Alpár and others, Indlekofer [13] constructed a function f, which is continuous on $\overline{\mathbb{D}}$, its power-series converges for $z = 1$, but the power series of $f^*(z) := f(B_\zeta(z)) = \sum_{n=1}^{\infty} b_n z^n$ diverges for the corresponding point $z = B_\zeta^{-1}(1)$, and with the condition on the modulus of continuity $\omega(f, h) = O\left(\left(log\frac{}{h/2\pi}\right)^{-1}\right)$ as $h \searrow 0+$. He solved hereby the primal conjecture of Turán.

In this chapter is concerned the argument transformation given by the composition with a Blaschke function, and in general, a dyadic martingale structure preserving transformation or shortly a DMSP-transformation, and we deal with questions related to the effect of a DMSP-tranformation on special function classes. We obtain, that composition with a DMSP-function preserves the classes of UDMD systems, that of $\mathcal{A}_n$-measurable functions, the dyadic function spaces $L^p(\mathbb{I})$, $H^p(\mathbb{I})$, and the Lipschitz classes $Lip(\alpha, \mathbb{I})$.

4.1 The effect of a DMSP-transformation

Definition 7 *We call a function $B : \mathbb{I} \to \mathbb{I}$ a dyadic martingale structure preserving function or shortly a DMSP-transformation if it is generated by a system of bijections $(\vartheta_n, n \in \mathbb{N})$, $\vartheta_n : \mathbb{A} \to \mathbb{A}$, and an arbitrary system $(\eta_n, n \in \mathbb{N}^*)$, $\eta_n : \mathbb{A}^n \to \mathbb{A}$ in the following way:*

$$
\begin{aligned}
&(B(x))_0 := \vartheta_0(x_0),\\
&(B(x))_n := \vartheta_n(x_n) + \eta_n(x_0, x_1, \ldots, x_{n-1}) \pmod 2 \quad (n \in \mathbb{N}^*).
\end{aligned}
$$

The notion of the DMSP-transformation refers mostly to the function, but at times to the composition with the given DMSP-transformation, which is obvious from the context.

An immediate inductive argument implies the next propositions:

Proposition 3 *For each generating systems $(\vartheta_n, n \in \mathbb{N})$ and $(\eta_n, n \in \mathbb{N}^*)$, the generated DMSP-transformation B is a bijection on $\mathbb{I}$ and its inverse function, B^{-1} is also a DMSP-transformation.*

Proposition 4 *Composition of DMSP-functions is also a DMSP-function.*

The question, which function systems can be transformed by a DMSP-transformation into a UDMD system, has a simple answer: exactly the UDMD systems. The following lemma is needed to see this.

Lemma 7 *[I. Simon[42]] a) Let $B : \mathbb{I} \to \mathbb{I}$ be a DMSP-transformation. Then, for each $n \in \mathbb{N}$ we have*

$$r_n \circ B = r_n \cdot h_n \text{ with some } h_n \in L(\mathcal{A}_n), |h_n| = 1. \tag{4.1}$$

b) $L(\mathcal{A}_n)$ is invariant under any DMSP-transformation.

Proof: a) When $\vartheta_n(z) = z$ $(z \in \mathbb{A})$, then $(-1)^{\vartheta_n(x_n)} = (-1)^{x_n}(-1)^0 = r_n(x)(-1)^{\vartheta_n(0)}$. In the other case, when $\vartheta_n(z) = 1 - z$ $(z \in \mathbb{A})$, then $(-1)^{\vartheta_n(x_n)} = (-1)^{1-x_n} = (-1)^{1+x_n} = (-1)^{\vartheta_n(0)} r_n(x)$.

By the definition of $y = B(x)$ we have

$$\begin{aligned} r_n(B(x)) =&(-1)^{y_n} = (-1)^{\vartheta_n(x_n)}(-1)^{\eta_n(x_0,\cdots,x_{n-1})} = \\ =&r_n(x)(-1)^{\vartheta_n(0)}(-1)^{\eta_n(x_0,\cdots,x_{n-1})} = r_n(x)h_n(x). \end{aligned}$$

Obviously, $h_n(x) := (-1)^{\vartheta_n(0)}(-1)^{\eta_n(x_0,\cdots,x_{n-1})} \in L(\mathcal{A}_n)$ and $|h_n| = 1$.

b) The statement is a simple consequence of the definitions. □

Theorem 1 *[I. Simon[42]] Let $B : \mathbb{I} \to \mathbb{I}$ be a DMSP-transformation. The function system $(f_n, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$, if and only if $(f_n \circ B, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$.*

Proof: Let B be a DMSP-transformation. If $(f_n, n \in \mathbb{N})$ is a UDMD system, then by (2.26) there are functions $g_n \in L(\mathcal{A}_n)$ with $|g_n| = 1$ so that $f_n(x) = r_n(x)g_n(x)$ $(x \in \mathbb{I})$. The previous lemma ensures the decomposition $r_n(B(x)) = r_n(x)h_n(x)$ for some $h_n \in L(\mathcal{A}_n)$, $|h_n| = 1$. As $g_n \in L(\mathcal{A}_n)$, follows by the second statement of the previous lemma, that $g_n \circ B \in L(\mathcal{A}_n)$. Consequently, $h_n\,(g_n \circ B) \in L(\mathcal{A}_n)$, $|h_n\,(g_n \circ B)\,| = 1$, and

$$f_n(B(x)) = r_n(B(x))g_n(B(x)) = r_n(x)\underbrace{h_n(x)g_n(B(x))}_{\in L(\mathcal{A}_n)} \qquad (x \in \mathbb{I}).$$

Thus $(f_n \circ B, n \in \mathbb{N})$ fulfills the requirements of a UDMD-system formulated in (2.26).

Because the inverse of a DMSP-transformation is also a DMSP-transformation, follows that if for any given system $(f_n, n \in \mathbb{N})$ the system $(g_n := f_n \circ B, n \in \mathbb{N})$ is a UDMD-system, then the original one $(f_n = g_n \circ B^{-1}, n \in \mathbb{N})$ is also a UDMD-system.

□

Similarly follows for different DMSP-transformations:

Theorem 2 *Let $(B_n : \mathbb{I} \to \mathbb{I}, n \in \mathbb{N})$ be a system of DMSP-transformations. The function system $(f_n, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$, if and only if $(f_n \circ B_n, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$.*

Remarks 1: *i)* As the Walsh-Paley functions $w_n (n \in \mathbb{N})$ and the functions v_n $(n \in \mathbb{N})$ are UDMD-product systems on $\mathbb{I}$, their DMSP-transformed results a UDMD-product system. For a precise statement see Remark 3.

ii) Gát[10, 11] constructed the Vilenkin-like systems, a generalization of the UDMD-systems on the more general space G_m. Extending the definition of the DMSP-transformations on the general space G_m, similar statement holds, which is a consequence of Lemma 7, b) and Remark 3.

iii) Schipp[35, 38] defined a general concept of systems, the adapted conditionally orthonormal systems or AC-ONS with respect to a regular sequence of weights. Specially, an AC-ONS on $\mathbb{I}$ is transformed under a DMSP-transformation into an AC-ONS, which is a consequence of Lemma 7 b) and (4.8), a later identity on the conditional expectations.

iv) As UDMD-systems are taken into UDMD-systems by a DMSP-transformation, follows by the so-called transformation method presented in Paragraph 2.6 that convergence and $(C,1)$-summation of UDMD-systems are also preserved by this kind of transformation.

The question is in the following, whether function classes $L^p(\mathbb{I})$ $(0 < p \leq \infty)$ and $H^p(\mathbb{I})$ $(0 < p < \infty)$ are invariant under a DMSP-transformation. For the answer it is essential that this kind of transformations are measure-preserving.

Lemma 8 *[I. Simon[42]] Let $B : \mathbb{I} \to \mathbb{I}$ be a DMSP-transformation and $n \in \mathbb{N}$. Then*

$$B(I_n(x)) = I_n(B(x)) \qquad (x \in \mathbb{I}). \tag{4.2}$$

Proof: If $t \in I_n(x)$, then $t_0 = x_0,\ t_1 = x_1, \ldots, t_{n-1} = x_{n-1}$. For $k < n$ we have $\vartheta_k(t_k) + \eta_k(t_0, t_1, \ldots, t_{k-1}) = \vartheta_k(x_k) + \eta_k(x_0, x_1, \ldots, x_{k-1})$, that is, $(B(t))_k = (B(x))_k$ $(k < n)$. Thus $B(t) \in I_n(B(x))$ $(t \in I_n(x))$, so

$$B(I_n(x)) \subseteq I_n(B(x)) \quad (x \in \mathbb{I}). \tag{4.3}$$

(4.3) holds specially for DMSP-function B^{-1} and $x = B(y)$, too. Thus by $B^{-1}(I_n(B(y))) \subseteq I_n(y)$ $(y \in \mathbb{I})$ follows $I_n(B(y)) \subseteq B(I_n(y))$ $(y \in \mathbb{I})$, which completes the proof together with (4.3).

□

From (4.2) follows that $\mu(B(I_n(x))) = \mu(I_n(B(x))) = 2^{-n} = \mu(I_n(x))$, so $\mu(B(E)) = \mu(E)$ holds for each $E \in \mathcal{A}_n$, thus

$$\mu(B(E)) = \mu(E) \qquad (E \in \mathcal{A}). \tag{4.4}$$

Proposition 5 *[I. Simon[42]] DMSP-transformations $B : \mathbb{I} \to \mathbb{I}$ are measure-preserving. Hence,*

$$\int_{\mathbb{I}} f \circ B \; d\mu = \int_{\mathbb{I}} f d\mu \qquad (f \in L^1(\mathbb{I})). \tag{4.5}$$

Theorem 3 *[I. Simon[42]] A DMSP-transformation preserves $L^p(\mathbb{I})(0 < p \leq \infty)$ and the dyadic Hardy space $H^p(\mathbb{I})$ $(0 < p < \infty)$. Moreover,*

$$\|f \circ B\|_p = \|f\|_p \;\; (0 < p \leq \infty), \tag{4.6}$$
$$\|f \circ B\|_{H^p} = \|f\|_{H^p} \;\; (0 < p < \infty). \tag{4.7}$$

Proof: For $0 < p < \infty$ and $f \in L^p(\mathbb{I})$, we have by (8.12) that $\|f \circ B\|_p = \|f\|_p < \infty$. Hence $f \circ B \in L^p(\mathbb{I})$, too.

If $f \in L^\infty(\mathbb{I})$, then for $M := \|f\|_\infty \in \mathbb{R}$, we have $|f(x)| \leq M$ for a.e. $x \in \mathbb{I}$, and by (4.4) follows that

$$\begin{aligned} \mu(\{x \in \mathbb{I} : \; |(f \circ B)(x)| > M \;\}) &= \mu(\{B(x) \in \mathbb{I} : \; |f(B(x))| > M \;\}) = \\ &= \mu(\{y \in \mathbb{I} : \; |f(y)| > M \;\}) = 0. \end{aligned}$$

Hence $f \circ B \in L^\infty(\mathbb{I})$ and $\|f \circ B\|_\infty \leq \|f\|_\infty$. As this holds specially for DMSP-function B^{-1} instead of B and $f \circ B$ instead of f, follows equality (4.6) for $p = \infty$.

For $f \in H^p(\mathbb{I})$ $(0 < p < \infty)$ we have by definition that $\|\mathcal{E}^* f\|_p < \infty$. By (4.2) follows for characteristic functions, that $1_{I_n(x)}(t) = 1_{I_n(B(x))}(B(t))$ $(t \in \mathbb{I})$, hence by (8.12)

$$\begin{aligned} \mathcal{E}_n\left(f \circ B\right)(x) &= \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f(B(t)) d\mu(t) = 2^n \int_{\mathbb{I}} f(B(t)) \cdot 1_{I_n(x)}(t) d\mu(t) = \\ &= 2^n \int_{\mathbb{I}} f(B(t)) \cdot 1_{I_n(B(x))}\left(B(t)\right) d\mu(t) = \\ &= \frac{1}{\mu(I_n(B(x)))} \int_{I_n(B(x))} f(t) d\mu(t) = \mathcal{E}_n\left(f\right)\left(B(x)\right). \end{aligned} \tag{4.8}$$

Thus

$$\mathcal{E}^*(f \circ B) := \sup_{n \in \mathbb{N}} |\mathcal{E}_n\left(f \circ B\right)| = \sup_{n \in \mathbb{N}} |\left(\mathcal{E}_n f\right) \circ B| = (\mathcal{E}^* f) \circ B.$$

This gives by (4.6) and by assumption, that

$$\|\mathcal{E}^*(f \circ B)\|_p = \|\left(\mathcal{E}^* f\right) \circ B\|_p = \|\mathcal{E}^* f\|_p < \infty.$$

Thus $f \circ B \in H^p(\mathbb{I})$ and $\|f \circ B\|_{H^p} = \|f\|_{H^p} \quad (0 < p < \infty)$.

$\square$

Remark 2. From (4.7) and (4.8) follows that

$$\|f \circ B\|_{BMO} = \sup_{n \in \mathbb{N}} \| \left(\mathcal{E}_n |f - \mathcal{E}_n f|^2\right)^{\frac{1}{2}} \circ B\|_\infty = \|f\|_{BMO}.$$

Thus the space of dyadic bounded mean oscillation (BMO) and the space of dyadic vanishing mean oscillation (VMO) are also preserved under a DMSP-transformation. For more on these spaces see Schipp [16].

Recall, that for $\alpha > 0$ the function class $Lip(\alpha, \mathbb{B})$ denotes the collection of functions $f : \mathbb{I} \to \mathbb{R}$ which satisfy

$$|f(y) - f(x)| \le c\ \rho(x, y)^\alpha \quad (x, y \in \mathbb{B})$$

for some constant $c \in \mathbb{R}$ which depends only on f.

Theorem 4 *[I. Simon[42]] A DMSP-transformation preserves $Lip(\alpha, \mathbb{I})$ ($\alpha > 0$).*

Proof: For $x, y \in \mathbb{I},\ x \neq y$ consider $m := \min\{n : x_n \neq y_n\}$. Now, $\rho(x, y) = 2^{-m}$ and m is the largest number in $\mathbb{N}$ so that $x \in I_m(y)$. By (4.2) follows, that $B(x) \in I_m(B(y))$ and m is the largest integer with this property. Thus

$$\rho(B(x), B(y)) = 2^{-m} = \rho(x, y) \quad (x, y \in \mathbb{I}).$$

For $f \in Lip(\alpha, \mathbb{I})$ follows

$$|f(B(y)) - f(B(x))| \le c\ \rho(B(x), B(y))^\alpha = c\ \rho(x, y)^\alpha$$

for some $c \in \mathbb{R}$. That is, $f \circ B \in Lip(\alpha, \mathbb{I})$.

$\square$

4.2 Examples of DMSP-functions

Some examples of DMSP-functions are presented on the 2-series (or logical) field $(\mathbb{B}, \overset{\circ}{+}, \circ)$ and the 2-adic (or arithmetical) field $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$, as the translations, dilatations, the function resulting the multiplicative inverse, a generalization of ζ and the Blaschke functions, as well.

1) The following functions are trivial DMSP-functions on $(\mathbb{I}, \overset{\circ}{+}, \circ)$ and $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$. (The last one is not trivial, and it is based on (2.24).)

$$
\begin{aligned}
&B(x) := x \overset{\circ}{+} a, \qquad B(x) := x \overset{\bullet}{+} a \qquad (x \in \mathbb{I})(a \in \mathbb{I}),\\
&B(x) := x \circ a, \qquad B(x) := x \bullet a \qquad (x \in \mathbb{I})(a \in \mathbb{S}),\\
&B(x) := x, \qquad B(x) := x^{-1} \qquad (x \in \mathbb{I}).
\end{aligned}
$$

2) If $c_n \in \mathbb{I}$ satisfies $\pi(c_n) = n$ $(n \in \mathbb{N}^*)$, then the function

$$
B(x) := \prod_{j=1}^{\infty} (e + c_j)^{x_j} = \prod_{j=1}^{\infty} (e + x_j c_j)
$$

can be obtained by a simple recursion, thus it is a DMSP-function from $\mathbb{I}_1$ to $\mathbb{S}$. See Schipp [17], pp 51-53. Its importance lies in the consequence, that the multiplicative digits of a given byte $y \in \mathbb{S}$ with respect to a sequence $(b_n = e + c_n, n \in \mathbb{N}^*)$, $\pi(c_n) = n$ can be obtained from its additive digits.

A further consequence of these is, that the $(\tilde{S}, \bullet)$-valued exponential function is a DMSP-function, too.

3) Both the logical and arithmetical Blaschke functions with parameter $a \in \mathbb{I}_1$

$$
\begin{aligned}
B_a(x) &= (x \overset{\circ}{+} a) \circ (e \overset{\circ}{+} a \circ x)^{-1} = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x} \qquad (x \in \mathbb{I}),\\
B_a(x) &= (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x} \qquad (x \in \mathbb{I})
\end{aligned}
$$

are also DMSP-functions, as they can be obtained by a simple recursion.

Remark 3. As the additive and multiplicative characters of $\mathbb{I}$ on both fields can be obtained recursively, a DMSP-transformation of them result a UDMD-product system. Furthermore, for $n \in \mathbb{N}^*$ let $j := \max\{k \in \mathbb{N} : \ n \geq 2^k\}$. Then,

$$
\begin{aligned}
w_n \circ B &= w_n \cdot g_j \text{ with some } g_j \in L(\mathcal{A}_j), |g_j| = 1,\\
v_n \circ B &= v_n \cdot g_j \text{ with some } g_j \in L(\mathcal{A}_j), |g_j| = 1.
\end{aligned}
$$

The statements hold obviously for $n = j = 0$, too.

Proof: We have $n = \sum_{i=0}^{j} n_i 2^i$ and by (4.1) follows

$$w_n(B(x)) = \prod_{i=0}^{j} r_i^{n_i}(B(x)) = \prod_{i=0}^{j} r_i^{n_i}(x) h_i^{n_i}(x) = w_n(x) g_j(x) \qquad (n \in \mathbb{N}^*),$$

where $h_i \in L(\mathcal{A}_i)$ and $|h_i| = 1$ $(i \in \{0, 1, \ldots, j\})$, thus $g_j := \prod_{i=0}^{j} h_i^{n_i} \in L(\mathcal{A}_j)$ and $|g_j| = 1$.

The statement for $(v_n, n \in \mathbb{N})$ follows analogously.

Chapter 5

The characters of the Blaschke group

In this chapter we will see that the Blaschke group $(\mathcal{B}, \circ)$ of the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ is a topological group and we will construct its characters. After determining the type of the recursion we will discuss summability and convergence questions. This chapter is based on I. Simon[40].

5.1 The construction of the characters of the Blaschke group

✠ To establish that *the Blaschke group* $(\mathcal{B}, \circ)$ of the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ is a topological group, recall first, that

$$\mathcal{B} := \{B_a, a \in \mathbb{I}_1\}$$

and $\circ$ denotes the composition of functions.

Consider the map $\|.\| : \mathcal{B} \to \mathbb{R}$,

$$\|B_a\| := \sup_{x \in \mathbb{I}} \|x \overset{\bullet}{-} B_a(x)\| \quad (B_a \in \mathcal{B}). \tag{5.1}$$

As inequality $\|e \overset{\bullet}{-} x^2\| \leq 1$ $(x \in \mathbb{I})$ holds with equality for each $x \in \mathbb{I}_1$, and the equation $\|e \overset{\bullet}{-} a \bullet x\| = 1$ is valid for any $a \in \mathbb{I}_1, x \in \mathbb{I}$, we have

$$\|B_a\| = \sup_{x\in\mathbb{I}} \|x - B_a(x)\| = \sup_{x\in\mathbb{I}} \left\| a \bullet \frac{e \overset{\bullet}{-} x^2}{e \overset{\bullet}{-} a \bullet x} \right\| = \|a\|. \tag{5.2}$$

Hence the map defined in (5.1) is a non-Archimedian norm: for $a, b \in \mathbb{I}_1$ from (3.25) and (3.26) results that:

$$\begin{aligned}
&\|B_a\| = \|a\| \geq 0, \\
&\|B_a\| = 0 \Leftrightarrow a = \theta \Leftrightarrow B_a = \imath, \text{(the identity map)}, \\
&\|B_a^{-1}\| = \|B_{a^-}\| = \|a^-\| = \|a\| = \|B_a\|, \\
&\|B_a \circ B_b\| = \|B_{a \lhd b}\| = \left\| \frac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b} \right\| = \|a \overset{\bullet}{+} b\| \leq \\
&\qquad\qquad \leq \max\{\|a\|, \|b\|\} = \max\{\|B_a\|, \|B_b\|\}.
\end{aligned}$$

As usual, set the map $d : \mathcal{B}^2 \to \mathbb{R}$

$$d(B_a, B_b) := \|B_a \circ B_b^{-1}\| \quad (B_a, B_b \in \mathcal{B}) \tag{5.3}$$

i.e. the **metric** induced by the norm (5.1). Consequently $(\mathcal{B}, \circ)$ is a **topological group** with the topology induced by the metric (5.3). By (5.2) follows that the map $a \mapsto B_a$ is an **isometry** on $\mathbb{I}_1$.

✠ The idea of the construction

Being a subgroup of a topological group (see Paragraph 2.3), $(\mathbb{I}_1, \overset{\bullet}{+})$ is also a topological group, and the topology is induced by the metric. With notation $x \lhd y := \frac{x \overset{\bullet}{+} y}{e \overset{\bullet}{+} x \bullet y}$ $(x, y \in \mathbb{I}_1)$ we find, that $(\mathbb{I}_1, \lhd)$ is a group. By the same argument as before, $(\mathbb{I}_1, \lhd)$ is a topological group.

Recall, that the characters of the group $(\mathbb{I}_1, \overset{\bullet}{+})$ are given by the product system $(v_m, m \in \mathbb{P})$ generated by the functions

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_1}{2^n}\right) \qquad (x = (0, x_1, x_2 \ldots) \in \mathbb{I}_1, n \in \mathbb{N})$$

presented in (3.8).

The map

$$B : (\mathbb{I}_1, \lhd) \to (\mathcal{B}, \circ), \ a \mapsto B_a$$

is a continuous isomorphism, hence in order to establish the characters of $(\mathcal{B}, \circ)$, it is sufficient if we define the character group of $(\mathbb{I}_1, \triangleleft)$.

As we already know the characters of $(\mathbb{I}_1, \overset{\bullet}{+})$, it is sufficient to find a continuous isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$, that is a function γ satisfying the equation

$$\gamma(x \overset{\bullet}{+} y) = \frac{\gamma(x) \overset{\bullet}{+} \gamma(y)}{e \overset{\bullet}{+} \gamma(x) \bullet \gamma(y)} \ (x, y \in \mathbb{I}_1). \tag{5.4}$$

This equation is the analogue of the functional equation of the classical tangent function, where the tangent function can be expressed by the exponential function in the following way:

$$\tan(x) = \frac{\exp(ix) - \exp(-ix)}{i(\exp(ix) + \exp(-ix))} = \frac{\exp(2ix) - 1}{i(\exp(2ix) + 1)}.$$
$$\left(x \in \mathbb{R} \setminus \left\{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\right\}\right)$$

✠ **The tangent-like function**

Recall, that with notation $\tilde{\mathbb{S}} := \{x \in \mathbb{S} : x_1 = 0\}$, the $(\tilde{\mathbb{S}}, \bullet)$-valued exponential function ζ on $\mathbb{I}_1$ was presented in Paragraph 3.2.

Definition 8 *Define tangent-like function on* $(\mathbb{I}_1, \overset{\bullet}{+})$ *by*

$$\gamma(x) := \frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \quad (x \in \mathbb{I}_1) \tag{5.5}$$

and the tangent function on $(\mathbb{I}_1, \overset{\bullet}{+})$ *by*

$$\tan(x) := \frac{\zeta^2(x) \overset{\bullet}{-} e}{\zeta^2(x) \overset{\bullet}{+} e} \quad (x \in \mathbb{I}_1), \tag{5.6}$$

where $\zeta^2(x) := \zeta(x) \bullet \zeta(x)$.

We collect in a lemma the properties that are needed for our subsequent study.

Lemma 9 *[I. Simon[40]] For any $a, b \in \mathbb{B}, x \in \mathbb{I}_1$ and $y \in \mathbb{I}_1$, the following holds:*

$$i)\ a \overset{\bullet}{+} a = e_1 \bullet a$$

$$ii)\ \frac{a \overset{\bullet}{+} a}{b \overset{\bullet}{+} b} = \frac{a}{b}$$

$$iii)\ \zeta^2(x) = \zeta(e_1 \bullet x)$$

$$iv)\ \frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y} \in \tilde{\mathbb{S}}.$$

Proof: i) Using the notations of the recursive definition of the addition $\overset{\bullet}{+}$, we find that $(a \overset{\bullet}{+} a)_n = 0$ if and only if $q_{n-1} = 0$. But $q_{n-1} = 0$ is equivalent to $a_{n-1} = 0$, which holds exactly when $(e_1 \bullet a)_n = 0$, because multiplication by e_1 shifts a.

ii) By the commutativity and distributivity of the operations we have $a \bullet (b \overset{\bullet}{+} b) = b \bullet (a \overset{\bullet}{+} a)$, thus the relation holds. The relation can be seen also by *i)*: $\frac{a \overset{\bullet}{+} a}{b \overset{\bullet}{+} b} = \frac{e_1 \bullet a}{e_1 \bullet b} = \frac{a}{b}$.

iii) It is a simple consequence of *i)* and the functional equation of ζ. In an other way, it follows directly by the definition of the base: $b_j \bullet b_j = b_{j+1}$ $(j \geq 1)$. Using the commutativity and associativity of the product $\bullet$, we get $\zeta^2(x) = \left(\prod_{j=1}^{\infty} b_j^{x_j}\right) \bullet \left(\prod_{j=1}^{\infty} b_j^{x_j}\right) = \prod_{j=1}^{\infty} b_{j+1}^{x_j} = \zeta(e_1 \bullet x)$ $(x \in \mathbb{I}_1)$.

iv) It can be easily established, that if $y = (0, y_1, y_2 \ldots) \in \mathbb{I}_1$, than

$$\begin{aligned} e \overset{\bullet}{+} y &= (1, y_1, y_2, y_3, \ldots) \\ \text{and} \quad e \overset{\bullet}{-} y &= (1, y_1, (y^-)_2, (y^-)_3, \ldots). \end{aligned} \tag{5.7}$$

Applying the notation

$$\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y} = z,$$

we can state first, that $\pi(z) = \pi(e \overset{\bullet}{+} y) - \pi(e \overset{\bullet}{-} y) = 0$, that is, $z \in \mathbb{S}$, thus $z_0 = 1$. Furthermore,

$$e \overset{\bullet}{+} y = z \bullet (e \overset{\bullet}{-} y).$$

Now, examining (the 0th and) the 1st digit of the right and the left side, we find with $z_0 = 1$ and by (5.7), that:

$$\begin{cases} (1 = z_0 \cdot 1) \\ y_1 = z_0 \cdot y_1 + z_1 \cdot 1 \quad (\mod 2) \end{cases}$$

which means, that ($z_0 = 1$ and) $z_1 = 0$, and so $z \in \tilde{\mathbb{S}} = \{z \in \mathbb{I} : \ z_0 = 1, z_1 = 0\}$.

$\square$

Lemma 9 *iii)* shows, that the tangent-like function γ is closely related to tan: namely $\gamma(x) = \tan(e_{-1} \bullet x)$ $(x \in \mathbb{I}_1)$.

Theorem 5 *[I. Simon[40]] The function γ defined in (5.5) is a continuous isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$.*

Proof: The continuity of γ follows from the continuity of ζ and of the field operations.

Using Lemma 9 *i)* and the functional equation of ζ, we get:

$$\gamma(x) \triangleleft \gamma(y) = \frac{\gamma(x) \overset{\bullet}{+} \gamma(y)}{e \overset{\bullet}{+} \gamma(x) \bullet \gamma(y)} = \frac{\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \overset{\bullet}{+} \frac{\zeta(y) \overset{\bullet}{-} e}{\zeta(y) \overset{\bullet}{+} e}}{e \overset{\bullet}{+} \frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \bullet \frac{\zeta(y) \overset{\bullet}{-} e}{\zeta(y) \overset{\bullet}{+} e}} =$$

$$= \frac{\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x) \bullet \zeta(y) \overset{\bullet}{-} e \overset{\bullet}{-} e}{\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x) \bullet \zeta(y) \overset{\bullet}{+} e \overset{\bullet}{+} e} = \frac{\zeta(x) \bullet \zeta(y) \overset{\bullet}{-} e}{\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} e} = \gamma(x \overset{\bullet}{+} y).$$

We will show, that γ is a one-one map from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$. Equation

$$\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} = \frac{\zeta(y) \overset{\bullet}{-} e}{\zeta(y) \overset{\bullet}{+} e}$$

results

$$\zeta(x) \overset{\bullet}{+} \zeta(x) = \zeta(y) \overset{\bullet}{+} \zeta(y).$$

Taking in consideration, that $f(a) := a \overset{\bullet}{+} a = e_1 \bullet a$ is a 1-1 function on $\mathbb{I}$, we have $\zeta(x) = \zeta(y)$, which gives by Lemma 6 that $x = y$.

In order to see, that for any $y \in \mathbb{I}_1$ there is an $x \in \mathbb{I}_1$ with $\gamma(x) = y$, we have to solve for x the equation

$$\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} = y,$$

that is,

$$\zeta(x) = \frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}.$$

Now, by Lemma 9 *iv)* and Lemma 6 follows

$$x = \zeta^{-1}\left(\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}\right) \qquad \text{and} \qquad \zeta^{-1}\left(\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}\right) \in \mathbb{I}_1,$$

and the proof is complete.

□

✠ Characters of the Blaschke group

Theorem 6 *[I. Simon[40]] The characters of the group* $(\mathbb{I}_1, \triangleleft)$ *are the functions*

$$v_n \circ \gamma^{-1} \quad (n \in \mathbb{N}).$$

Proof: As γ^{-1} is a continuous isomorphism from $(\mathbb{I}_1, \triangleleft)$ onto $(\mathbb{I}_1, \overset{\bullet}{+})$, and $(v_n, n \in \mathbb{N})$ forms the character group of $(\mathbb{I}_1, \overset{\bullet}{+})$, it follows that $(v_n \circ \gamma^{-1}, n \in \mathbb{P})$ are the characters of the group $(\mathbb{I}_1, \triangleleft)$.

□

Corollary 1 *[I. Simon[40]] The characters of the Blaschke group* $(\mathcal{B}, \circ)$ *are the functions*

$$v_n \circ \gamma^{-1} \circ B^{-1} \quad (n \in \mathbb{N}),$$

where $(\mathcal{B}, \circ)$ *denotes the Blaschke group of the arithmetic field* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$*, and* $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$ *represents the function* $a \mapsto B_a$*.*

5.2 Recursion

✠ The aim of this Paragraph is to show that $\gamma(x)$ can be obtained by a simple recursion, and therefore $(v_n \circ \gamma^{-1}, n \in \mathbb{N})$ is a UDMD product system.

Proposition 6 *[I. Simon[40]] The functions $v_n \circ \gamma^{-1} (n \in \mathbb{N})$, the characters of $(\mathbb{I}_1, \triangleleft)$ form a UDMD product system.*

Proof: Equation (3.12) shows, that the basic sequence of bytes $(b_n, n \in \mathbb{N})$ defined in (3.10) can be written in the form $b_n = e \overset{\bullet}{+} e_{n+1} \overset{\bullet}{+} d_n$ $(n \geq 1)$ where $\pi(d_n) \geq n+2$ and the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} (e \overset{\bullet}{+} e_{j+1} \overset{\bullet}{+} d_j)^{x_j} = \prod_{j=1}^{\infty} (e \overset{\bullet}{+} x_j e_{j+1} \overset{\bullet}{+} x_j d_j) \quad (d_j \in \mathbb{I}_{n+2}).$$

Similar to the proof of Lemma 6, (where to a given $y \in \tilde{\mathbb{S}}$ the recursion yields $x \in \mathbb{I}_1$ such that $\zeta(x) = y$, now conversely) an inductive argument leads to a so-called simple recursion of ζ:

$$\begin{aligned} &(\zeta(x))_0 = 1, \\ \text{and } &(\zeta(x))_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (\mod 2) \qquad (n \geq 1) \end{aligned} \tag{5.8}$$

with some function $f : \mathbb{A}^{n-2} \to \mathbb{A}$. With $y = \zeta(x)$, $z := \zeta(x) \overset{\bullet}{-} e = (\zeta(x) \overset{\bullet}{+} e^{-}) = (1, 0, y_2, y_3, \ldots) \overset{\bullet}{+} (1, 1, 1, 1, \ldots) = (0, 0, y_2, y_3, y_4, \ldots)$ can also be expressed as:

$$z_0 = z_1 = 0, \text{ and } z_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 2).$$

Similarly, $t := \zeta(x) \overset{\bullet}{+} e = (1, 0, y_2, y_3, \ldots) \overset{\bullet}{+} (1, 0, 0, 0, \ldots) = (0, 1, y_2, y_3, y_4, \ldots)$ can be expressed with a simple recursion formula:

$$t_0 = 0, t_1 = 1, \text{ and } t_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 2)$$

with the same function $f : \mathbb{A}^{n-2} \to \mathbb{A}$.

Recursive form (2.24) allows us to give the multiplicative inverse element of $t \in \mathbb{S}_1$ with a simple recursion:

$$\begin{aligned} &(t^{-1})_j = 0 \qquad (j < -1) \\ &(t^{-1})_{-1} = 1, \\ &(t^{-1})_n = t_{n+1} + f'(t_1, \ldots, t_n) \ (n \geq 0) \end{aligned}$$

with some function $f' : \mathbb{A}^n \to \mathbb{A}$.

With $z = \zeta(x) \overset{\bullet}{-} e$ and $t := \zeta(x) \overset{\bullet}{+} e$ we get by $\gamma(x) = z \bullet t^{-1}$ and by $z_0 = z_1 = 0$ that

$$(\gamma(x))_n = z_2(t^{-1})_{n-2} + \ldots + z_{n+1}(t^{-1})_{-1} + q_{n-1} \pmod 2.$$

Finally, we obtain by $(t^{-1})_{-1} = 1$ and the previous recursions for z and t^{-1}, that

$$(\gamma(x))_n = x_n + \widetilde{f}(x_1, \ldots, x_{n-1}) \ (n \geq 0) \tag{5.9}$$

for some function $\widetilde{f} : \mathbb{A}^{n-1} \to \mathbb{A}$, and we have our desired result.

By (5.9) the byte $\gamma^{-1}(x)$ can also be written by a simple recursion for any $x \in \mathbb{I}_1$, thus $v_{2^n}\left(\gamma^{-1}(x)\right) = \varepsilon\left(\frac{x_n}{2}\right) g(x_1, \ldots, x_{n-1}) = (-1)^{x_n} g(x_1, \ldots, x_{n-1})$, with some $g \in L(\mathcal{A}_n)$, and $|g(x_1, \ldots, x_{n-1})| = 1$. In consequence of (2.26) follows, that $(v_{2^n} \circ \gamma^{-1}, n \in \mathbb{N})$ satisfy the requirements of a UDMD system and $(v_n \circ \gamma^{-1}, n \in \mathbb{P})$ is a UDMD product system.

□

As $(v_n \circ \gamma^{-1}, n \in \mathbb{N})$ is a UDMD product system, the discrete Fourier coefficients with respect to this system can be computed with the Fast Fourier Algorithm. See Schipp-Wade[17], pp. 106-111 about the FFT Algorithms.

5.3 (C,1) summability and a.e.convergence of the Gamma-Fourier series

By (5.9), the function γ is a bijection on $I_n(x)$, $(x \in \mathbb{I}_1, n \in \mathbb{N})$, $\gamma(I_n(x)) = I_n(\gamma(x))$, thus for any dyadic interval E we have $\mu(t \in \mathbb{I}_1 : \gamma(t) \in E) = \mu(E)$, and this follows for any measurable set E as well. Therefore the variable transformation γ is measure preserving. This follows also by the fact, that γ is a DMSP-transformation presented in Chapter 4. Thus,

$$\int_{\mathbb{I}_1} f \circ \gamma \, d\mu = \int_{\mathbb{I}_1} f \, d\mu. \tag{5.10}$$

Definition 9 *The Gamma-Fourier coefficients of an $f \in L^1(\mathbb{I}_1)$ with respect to the system $(v_m \circ \gamma^{-1}, m \in \mathbb{N})$ are defined by*

$$\widehat{f^{\gamma}}(m) := \int_{\mathbb{I}_1} f(x) v_m(\gamma^{-1}(x)) d\mu(x) \qquad (m \in \mathbb{N}).$$

Define the Gamma-Fourier series of an $f \in L^1(\mathbb{I}_1)$ and the n-th partial sums of the Gamma-Fourier series S^γ by

$$S^\gamma f := \sum_{k=0}^{\infty} \widehat{f^\gamma}(k) \cdot v_k \circ \gamma^{-1},$$

$$S_n^\gamma f := \sum_{k=0}^{n-1} \widehat{f^\gamma}(k) \cdot v_k \circ \gamma^{-1} \ (n \in \mathbb{P})$$

Furthermore define the Gamma-Cesaro (or $(G{-}C,1)$) means of $S^\gamma f$ by $\sigma_0 f := 0$ and

$$\sigma_n^\gamma f := \frac{1}{n} \sum_{k=1}^{n} S_k^\gamma f \qquad (n \in \mathbb{P}).$$

The counterparts of the Carleson-Hunt theorem on the a.e. convergence of Fourier series of an $f \in L^p(\mathbb{R})$ $(p > 1)$ and of the Lebesgue's theorem about the (C,1)-summability for $f \in L^1(\mathbb{R})$ hold for the Gamma-Fourier series of an $f \in L^p(\mathbb{I}_1)$ $(p > 1)$ and $f \in L^1(\mathbb{I}_1)$, respectively. The first one is a direct consequence of the general result of Schipp [37](Theorem 4) on the a.e. convergence of Fourier series with respect to any UDMD-product system of an $f \in L^p(\mathbb{R})$ $(p > 1)$. The second one is a consequence of the general result of Gát[9](Theorem 15) for Vilenkin-like systems, a generalization of UDMD-product systems, thus also of $(v_n \circ \gamma^{-1}, n \in \mathbb{N})$. However, these can be obtained directly using results on expansion with respect to the character system $(v_n, n \in \mathbb{N})$ and applying the transformation method presented in Paragraph 2.6.

Theorem 7 *On the field $(\mathbb{I}_1, \overset{\bullet}{+}, \bullet)$ we have*

a) $S_n^\gamma f \to f$ a.e. as $n \to \infty$ for any $f \in L^p(\mathbb{I}_1)(p > 1)$;

b) $\sigma_n^\gamma f \to f$ a.e. as $n \to \infty$ for any $f \in L^1(\mathbb{I}_1)$.

Proof: The first step is to apply the transformation method for measure preserving function γ: Let $S_n f$ stand for the n-th partial sum of the Fourier series of f, and $\sigma_n f$ denotes the n-th Cesaro mean of Sf, both with respect to the characters. Then,

$$S_n^\gamma f = [S_n(f \circ \gamma)] \circ \gamma^{-1}, \tag{5.11}$$

$$\sigma_n^\gamma f(x) = \sigma_n(f \circ \gamma)(\gamma^{-1}(x)). \tag{5.12}$$

a) We use the theorem of the a.e. convergence of the Fourier series on the field $(\mathbb{I}, \dot{+}, \bullet)$ due to Schipp [36]: $\lim_{m\to\infty}(S_m f)(x) = f(x)$ a.e. for any $f \in L^p(\mathbb{I})(p > 1)$. Thus with (5.11) we get $\lim_{n\to\infty} S_n^\gamma f(x) = \lim_{n\to\infty} S_n(f\circ\gamma)(\gamma^{-1}(x)) = (f\circ\gamma\circ\gamma^{-1})(x) = f(x)$ a.e. for any $f \in L^p(\mathbb{I}_1)$ $(p > 1)$, as required.

b) We use the theorem of the $(C, 1)$-summability of the Walsh-Fourier series on the field $(\mathbb{I}, \dot{+}, \bullet)$ due to Gát [7]: $\lim_{m\to\infty}(\sigma_m f)(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I})$. Thus with (5.12) we get $\lim_{n\to\infty} \sigma_n^\gamma f(x) = \lim_{n\to\infty} \sigma_n(f\circ\gamma)(\gamma^{-1}(x)) = (f\circ\gamma\circ\gamma^{-1})(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I}_1)$, as required.

□

Remark 4. As $v_n \circ \gamma$ is a UDMD-product system, the general theorems for UDMD systems and Vilenkin-like systems imply also norm convergence of the Fourier series with respect to this system:

$$\lim_{n\to\infty} \|S_{2^n}^\gamma f - f\|_q = 0,\ (f \in L^q(\mathbb{I}_1), (1 \le q < \infty) \tag{5.13}$$

$$\lim_{m\to\infty} \|S_m^\gamma f - f\|_q = 0, (f \in \mathbb{L}^q(\mathbb{I}_1)), (1 < q < \infty) \tag{5.14}$$

$$\lim_{n\to\infty} \|\sigma_n^\gamma f - f\|_q = 0,\ (f \in L^1(\mathbb{I}_1). \tag{5.15}$$

Moreover, (5.14) and (5.15) holds for $q = \infty$ when f is continuous on $\mathbb{I}$.

Chapter 6

Discrete Laguerre functions on local fields

6.1 Introduction

In this chapter we will introduce the discrete Laguerre system on the dyadic (or 2-series) and 2-adic fields using the corresponding characters and the analogue of the Blaschke functions. The complex variants of these systems play an important role in the system identification. The discrete Laguerre functions and their generalizations (Malmquist-Takenaka, and Kautz systems) are often used in control theory to identify the transfer function of the system. (For more details see [3].) Some analogous properties and summability questions of Fourier expansion with respect to these systems are presented on these local fields.

Let us recall, that the discrete Laguerre functions $L_n^{(a)}$ $(n \in \mathbb{N})$ contain a complex parameter $a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and can be expressed by the Blaschke functions $B_a(z) := e^{i\gamma}\frac{z-a}{1-\bar{a}z}$ $(z \in \mathbb{C}), (a \in \mathbb{D},\ \gamma \in \mathbb{R})$. On $\mathbb{C}$, the discrete Laguerre functions $L_n^{(a)}$ associated to B_a are defined by

$$L_k^{(a)}(z) := m_a(z)B_a^k(z), \text{ where } m_a(z) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \quad (z \in \mathbb{C}, k \in \mathbb{Z}) \tag{6.1}$$

for $a \in \mathbb{D}$. The boundary of $\mathbb{D}$ is denoted by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

If a belongs to $\mathbb{D}$, then B_a is a bijection on $\mathbb{D}$ and on $\mathbb{T}$, respectively. Thus

B_a can be written in the form

$$B_a(e^{is}) = e^{i\beta_a(s)}(s \in \mathbb{R}, a \in \mathbb{D}) \tag{6.2}$$

with some bijection $\beta_a : [-\pi, \pi] \to [-\pi, \pi]$ mentioned in (3.17). (See [3].)

We can observe, that $L_k^{(0)}(z) = z^k$ $(k \in \mathbb{Z})$ coincides with the trigonometric system on $\mathbb{T}$. Thus by (6.1) the discrete Laguerre system except the factor m_a can be obtained on $\mathbb{T}$ from the trigonometric system by an argument transformation $T(z) = B_a(z)$ $(z \in \mathbb{T})$.

The discrete Laguerre system is orthogonal with respect to the scalar product

$$\langle F, G\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\overline{G}(e^{it})\,dt \quad (F, G \in L^1). \tag{6.3}$$

This is a consequence of the orthogonality of the trigonometric system. Indeed, by (6.2)

$$\langle L_n^{(a)}, L_m^{(a)}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\beta_a(s)}\beta_a'(s)\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}\,dt = \delta_{nm},$$

$(n, m \in \mathbb{N})$. Linear approximation algorithms proposed in the literature are based on weighted partial sums of the trigonometric Fourier series of the transfer functions. By applying an appropriate variable transform, the Laguerre and Kautz basis can be related to the trigonometric one. (See [3].) This can be used to transfer some summation results to Laguerre-Fourier series.

The name of the above presented discrete Laguerre functions comes from the fact, that the Fourier coefficients of them give the discrete analogues of the Laguerre functions.

This chapter is based on I. Simon [39], where the author constructed the analogue of discrete Laguerre functions starting from the characters of the additive group of the dyadic and 2-adic field, and using an argument transformation. Some convergence and summability properties of Fourier expansion with respect to these systems are examined.

6.2 Discrete Laguerre functions on the dyadic (or 2-series) field

The functions corresponding to the trigonometric system $(e^{ikt}, k \in \mathbb{Z})$ $(t \in \mathbb{R})$ will be now the characters of the group $(\mathbb{I}, \overset{\circ}{+})$, namely the Walsh-Paley functions $(w_k, k \in \mathbb{N})$ presented in (3.4).

Definition 10 *Define the logical discrete Laguerre functions associated to B_a with parameter $a \in \mathbb{I}_1$ by*

$$L_k^{(a)}(x) := w_k(B_a(x)) \quad (k \in \mathbb{N}, x \in \mathbb{I}). \tag{6.4}$$

For $a \in \mathbb{I}_1$ consider the functions $r_n \circ B_a \quad (x \in \mathbb{I},\ n \in \mathbb{N})$. (Here $\circ$ stands for function-composition.) The logical discrete Laguerre system $(L_k^{(a)}, k \in \mathbb{N})$ is the product system generated by $(r_n \circ B_a, n \in \mathbb{N})$:

$$L_k^{(a)}(x) = \prod_{n=0}^{\infty} \left[r_n(B_a(x))\right]^{k_n}.$$

Theorem 8 *[I. Simon[39]] For each $a \in \mathbb{I}_1$ the functions $(r_n \circ B_a,\ n \in \mathbb{N})$ form a UDMD-system on $\mathbb{I}$.*

Proof: We use recursion form (3.22) of $y = B_a(x)$ to get

$$r_n(B_a(x)) = (-1)^{y_n} = (-1)^{x_n}(-1)^{a_n + f_n(x_0,\cdots,x_{n-1})} = r_n(x)g_n(x)$$

with some function $g_n(x) = (-1)^{a_n + f_n(x_0,\cdots,x_{n-1})}$, that is constant on intervals of rank bigger than n. Thus $g_n \in L(\mathcal{A}_n)$. Clearly, $|g_n| = 1$. Hence $(r_n \circ B_a, n \in \mathbb{N})$ is a UDMD sequence on $\mathbb{I}$.

$\square$

Corollary 2 *[I. Simon[39]] The logical discrete Laguerre-system $(L_k^{(a)}, k \in \mathbb{N})$ is a UDMD-product system generated by $(r_n \circ B_a, n \in \mathbb{N})$, consequently it is complete and orthonormal.*

6.3 Discrete Laguerre functions on the 2-adic field

As before, the functions corresponding to the orthonormed system $(e^{ikt}, k \in \mathbb{Z}, t \in \mathbb{R})$ will be the characters of the group $(\mathbb{I}, \dot{+})$, namely the functions $(v_k, k \in \mathbb{N})$ presented in (3.8).

Definition 11 *Let us define the arithmetical discrete Laguerre functions associated to B_a in the following way:*

$$L_k^{(a)}(x) := v_k(B_a(x)) \qquad (k \in \mathbb{N}, x \in \mathbb{I}). \tag{6.5}$$

For $a \in \mathbb{I}_1$ and $n \in \mathbb{N}$ consider the functions $v_{2^n} \circ B_a$ on $\mathbb{I}$. The arithmetical discrete Laguerre system $(L_k^{(a)}, k \in \mathbb{N})$ is the product system generated by $(v_{2^n} \circ B_a, n \in \mathbb{N})$:

$$L_k^{(a)}(x) = \prod_{j=0}^{+\infty} [v_{2^j}(B_a(x))]^{k_j} \quad (x \in \mathbb{I}).$$

Theorem 9 *[I. Simon[39]] For each $a \in \mathbb{I}_1$ and $n \in \mathbb{N}$ the functions $v_{2^n} \circ B_a$ form a UDMD-system on $\mathbb{I}$.*

Proof: For $B_a(x) = y$ we have $y = x \overset{\bullet}{-} a \overset{\bullet}{+} a \bullet x \bullet y$, thus

$$\begin{aligned} v_{2^n}(B_a(x)) &= v_{2^n}(x \overset{\bullet}{-} a \overset{\bullet}{+} y \bullet a \bullet x) = \frac{v_{2^n}(x) v_{2^n}(y \bullet a \bullet x)}{v_{2^n}(a)} = \\ &= \varepsilon\left(\frac{x_n}{2}\right) \frac{\varepsilon\left(\frac{x_{n-1}}{2^2} + \cdots + \frac{x_0}{2^{n+1}}\right) v_{2^n}(y \bullet a \bullet x)}{v_{2^n}(a)} = \\ &= r_n(x) g_n(x), \end{aligned}$$

where $g_n(x)$ depends only on $x_0, \ldots, x_{n-1}$. This follows by the recursive computation of $(y \bullet a \bullet x)_1, (y \bullet a \bullet x)_1, \ldots, (y \bullet a \bullet x)_n$, because $v_{2^n}(y \bullet a \bullet x)$ depends only on x_k-s with $k < n$. Hence $g_n \in L(\mathcal{A}_n)$. Furthermore, it is clear, that $|v_{2^n}(B_a(x))| = 1$ $(x \in \mathbb{I})$. Thus these functions form a UDMD-system.

□

Corollary 3 *[I. Simon[39]] The discrete Laguerre-system $(L_m^{(a)}, m \in \mathbb{N})$ is a UDMD-product system generated by $(v_{2^n} \circ B_a, n \in \mathbb{N})$, consequently it is complete and orthonormal.*

6.4 (C,1)-summability and a.e. convergence of Laguerre-Fourier series

Let now B_a and $L_n^{(a)}$ denote the respective Blaschke-functions and discrete Laguerre functions on the studied fields $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ and $(\mathbb{I}, \overset{\circ}{+}, \circ)$ $(a \in \mathbb{I}_1)$. The variable transformation $T : \mathbb{I} \to \mathbb{I}$, $T(x) := B_a(x)$ is measure preserving, as it is a bijection on intervals: $T(I_n(x)) = I_n(T(x))$ $(x \in \mathbb{I}, n \in \mathbb{N})$. Thus for any dyadic interval E we have $\mu(t \in \mathbb{I} : T(t) \in E) = \mu(E)$, and this follows for any measurable set E, as well. Hence,

$$\int_{\mathbb{I}} f \circ B_a \, d\mu = \int_{\mathbb{I}} f d\mu. \tag{6.6}$$

Definition 12 *Let us define the Laguerre-Fourier coefficients of an $f \in L^1(\mathbb{I})$ by*

$$\widehat{f^{(a)}}(n) := \int_{\mathbb{I}} f(x) L_n^{(a)}(x) d\mu(x) \qquad (n \in \mathbb{N}).$$

Furthermore the Laguerre-Fourier series $S^{(a)}f$ of an $f \in L^1(\mathbb{I})$ and the n-th partial sum $S_n^{(a)}f$ of the Laguerre-Fourier series $S^{(a)}$ is defined by

$$S^{(a)}f := \sum_{k=0}^{\infty} \widehat{f^{(a)}}(k) L_k^{(a)},$$

$$S_n^{(a)}f := \sum_{k=0}^{n-1} \widehat{f^{(a)}}(k) L_k^{(a)} \qquad (n \in \mathbb{P}).$$

Let us define the Laguerre-Cesaro (or $(L-C,1)$) means of $S^{(a)}f$ by $\sigma_0^{(a)}f := 0$ and

$$\sigma_n^{(a)}f := \frac{1}{n} \sum_{k=1}^{n} S_k^{(a)}f \qquad (n \in \mathbb{P}).$$

The counterparts of the Carleson-Hunt theorem on the a.e. convergence of Fourier series of an $f \in L^p(\mathbb{R})$ $(p > 1)$ and of the Lebesgue's theorem about the (C,1)-summability for $f \in L^1(\mathbb{R})$ hold for the Laguerre-Fourier series of an $f \in L^p(\mathbb{I})$ $(p > 1)$ and $f \in L^1(\mathbb{I})$, respectively. The first one is a direct consequence of the general result of Schipp [37](Theorem 4) on the a.e. convergence of Fourier series with respect to any UDMD-product system of an $f \in L^p(\mathbb{R})$ $(p > 1)$. The second one is a consequence of the general result of Gát[9](Theorem 15) for Vilenkin-like systems, a generalization of UDMD-product systems, thus also of $(L_n^{(a)}, n \in \mathbb{N})$. However, these can be obtained using previous special results on expansion with respect to character systems $(w_n, n \in \mathbb{N})$ and $(v_n, n \in \mathbb{N})$, respectively, and using the transformation method presented in Paragraph 2.6.

Theorem 10 *On both fields $(\mathbb{I}, \overset{\circ}{+}, \circ)$ and $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ we have*

a) $S_n^{(a)}f \to f$ a.e. as $n \to \infty$ for any $f \in L^p(\mathbb{I}), p > 1$;

b) $\sigma_n^{(a)}f \to f$ a.e. as $n \to \infty$ for any $f \in L^1(\mathbb{I})$.

Proof: The first step is to apply the transformation method for measure preserving function B_a: let $S_n f$ stand for the n-th partial sum of the Fourier series of f, and $\sigma_n f$ denotes the n-th Cesaro mean of Sf, both with respect to the characters. Then,

$$S_n^{(a)} f = \left[S_n(f \circ B_a^{-1})\right] \circ B_a, \tag{6.7}$$

$$\sigma_n^{(a)} f = \left[\sigma_n(f \circ B_a^{-1})\right] \circ B_a. \tag{6.8}$$

Let us now consider the field $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

a) We use the theorem of the a.e. convergence of the Walsh-Fourier series due to Schipp [36]: $\lim\limits_{m\to\infty}(S_m f)(x) = f(x)$ a.e. for any $f \in L^p(\mathbb{I})(p > 1)$. Thus with (6.7) we have $\lim\limits_{n\to\infty} S_n^{(a)} f(x) = \lim\limits_{n\to\infty} S_n(f \circ B_a)(B_a^{-1}(x)) = (f \circ B_a \circ B_a^{-1})(x) = f(x)$ a.e. for any $f \in L^p(\mathbb{I}_1)$ $(p > 1)$, as required.

b) Now, we use the theorem of the $(C, 1)$-summability of the Walsh-Fourier series on the field $(\mathbb{I}, \overset{\circ}{+}, \circ)$ due to Fine and Schipp [15]: $\lim_{m\to\infty}(\sigma_m f)(x) = f(x)$ a.e. for any $f \in L^1$. Thus with (6.8) we have $\lim_{n\to\infty} \sigma_n^{(a)} f(x) = \lim_{n\to\infty} \sigma_n(f \circ B_a^{-1})(B_a(x)) = (f \circ B_a^{-1} \circ B_a)(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I})$, and this proves the theorem.

We can get the same result on the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ using the corresponding theorem of Gát [7].

$\square$

Remark 5. As discrete Laguerre systems $(L_n^{(a)}, n \in \mathbb{N})$ are UDMD-product systems, also the norm convergence of the Fourier series with respect to them is valid:

$$\lim_{n\to\infty} \|S_{2^n}^{(a)} f - f\|_q = 0, \ (f \in L^q(\mathbb{I}), (1 \le q < \infty) \tag{6.9}$$

$$\lim_{m\to\infty} \|S_m^{(a)} f - f\|_q = 0, (f \in \mathrm{L}^q(\mathbb{I})), (1 < q < \infty) \tag{6.10}$$

$$\lim_{n\to\infty} \|\sigma_n^{(a)} f - f\|_q = 0, \ (f \in L^1(\mathbb{I})). \tag{6.11}$$

Moreover, (6.10) and (6.11) holds for $q = \infty$ when f is continuous on $\mathbb{I}$.

Chapter 7

Malmquist-Takenaka functions on local fields

The complex variants of the Malmquist-Takenaka systems play an important role in system identification. In this chapter are presented the analogue of these functions on two local fields using the generator system of the corresponding characters and the Blaschke-functions. Properties of these systems, Fourier expansion and summability questions are presented. This chapter is based on I. Simon [41].

7.1 Introduction

The Malmquist-Takenaka functions $\Psi_n^{(p)}$ on $\mathbb{C}$ are defined by

$$\Psi_0^p(z) = \frac{\sqrt{1-|a_0|^2}}{1-\bar{a}_0 z}, \qquad \Psi_n^{(p)}(z) := \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} \prod_{j=0}^{n-1} B_{a_j}(z), \quad (z \in \mathbb{C}, k \in \mathbb{Z})$$

for $(a_j \in \mathbb{D}, j \in \mathbb{N})$ and $p = (a_0, a_1, a_2, \dots)$.

The Malmquist-Takenaka system is orthogonal with respect to the scalar product (6.3). Note, that using the same parameters $a_j = a$ $(j \in \mathbb{N})$, the functions $\Psi_n^{(p)}$ give the discrete Laguerre system $(L_n^{(a)}, n \in \mathbb{N})$. (For more details on these systems see [3].)

We will construct the analogue of the Malmquist-Takenaka functions starting from the generator systems of the characters of the dyadic and 2-adic additive groups, and using an argument transformation. This will be a UDMD product system, thus also a complete orthonormal system, which equals the discrete Laguerre system for identical parameters $a_n = a$ $(n \in \mathbb{N})$.

7.2 Malmquist-Takenaka systems on the dyadic and arithmetic field

✠ The logical Malmquist-Takenaka functions

Definition 13 *To any given system of bytes* $(a_i \in \mathbb{I}_1, i \in \mathbb{N})$ *define the logical Malmquist-Takenaka functions* $(\psi_k^{(p)}, k \in \mathbb{N})$ *with parameters* $p = (a_0, a_1, \ldots)$ *on the 2-series field* $(\mathbb{I}, \overset{\circ}{+}, \circ)$ *as the product system generated by*

$$(\varphi_{n,a_n} := r_n \circ B_{a_n}, n \in \mathbb{N}). \tag{7.1}$$

That is, $\psi_k^{(p)}(x) = \prod_{n=0}^{\infty} [r_n(B_{a_n}(x))]^{k_n} \quad (x \in \mathbb{I})$.

Remark 6. In the recursion form $y_n = x_n + f_n^m(x_0, \ldots, x_{n-1})$ of $y = B_{a_m}(x)$ presented in (3.22) the functions $f_n^m : \mathbb{A}^n \to \mathbb{A}$ depend on the parameter $a_m \in \mathbb{I}_1$. As we need here different bytes a_m in our construction, we use upper indices to indicate the applied byte. Now, from all these functions

$$\begin{array}{rccccc}
\text{belonging to } a_1 : & f_1^1 & f_2^1 & \ldots & f_n^1 & \ldots \\
\text{belonging to } a_2 : & f_1^2 & f_2^2 & \ldots & f_n^2 & \ldots \\
 & \vdots & \vdots & \ddots & \vdots & \\
\text{belonging to } a_m : & f_1^m & f_2^m & \ldots & f_n^m & \ldots \\
 & \vdots & \vdots & & \vdots & \ddots
\end{array}$$

we will use only the elements from the diagonal, the f_n^n-s, like in the Cantor's diagonal argument.

Theorem 11 *[I. Simon[41]] For every parameter-sequence* $(a_i \in \mathbb{I}_1, i \in \mathbb{N})$ *the functions* $(\varphi_{n,a_n},\ n \in \mathbb{N})$ *defined in (7.1) form a UDMD system on* $\mathbb{I}$.

Proof: Using the recursion form of $y = B_{a_n}(x)$ we get

$$\varphi_{n,a_n}(x) = (-1)^{y_n} = (-1)^{x_n}(-1)^{(a_n)_n + f_n^n(x_0,\cdots,x_{n-1})} = r_n(x)g_n(x)$$

where $g_n(x) = (-1)^{(a_n)_n + f_n^n(x_0,\cdots,x_{n-1})}$ is $\mathcal{A}_n$-measurable: $g_n \in L(\mathcal{A}_n)$. Clearly, $|g_n(x)| = 1$ $(x \in \mathbb{I})$, and the proof is complete.

□

Corollary 4 *[I. Simon[41]] The logical Malmquist-Takenaka system $(\psi_k^{(p)}, k \in \mathbb{N})$ is a UDMD product system, consequently it is a complete orthonormal system on $(\mathbb{I}, \overset{\circ}{+}, \circ)$.*

✠ The arithmetical Malmquist-Takenaka functions

We consider the functions $(v_{2^n}(x), n \in \mathbb{N})$ known as a generator system of the characters of the group $(\mathbb{I}, \overset{\bullet}{+})$ mentioned in (3.8).

Definition 14 *Let us define the arithmetical Malmquist-Takenaka functions $\left(\Psi_k^{(p)}, k \in \mathbb{N}\right)$ with parameters $p = (a_0, a_1, \ldots)$ $(a_n \in \mathbb{I}_1, n \in \mathbb{N})$ on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ in the following way: the system $\left(\Psi_k^{(p)}, k \in \mathbb{N}\right)$ is the product system generated by*

$$(\Phi_{n,a_n} := v_{2^n} \circ B_{a_n}, n \in \mathbb{N})\,. \tag{7.2}$$

That is, $\Psi_n^{(p)}(x) = \prod\limits_{j=0}^{\infty} \left[v_{2^j}(B_{a_j}(x))\right]^{n_j} \quad (x \in (\mathbb{I}, \overset{\bullet}{+}, \bullet))$.

Theorem 12 *[I. Simon[41]] For any $(a_n \in \mathbb{I}_1,\ n \in \mathbb{N})$ the functions $(\Phi_{n,a_n},\ n \in \mathbb{N})$ defined by (7.2) form a UDMD system on $\mathbb{I}$.*

Proof: For $B_{a_n}(x) = y$ we have $y = x \overset{\bullet}{-} a_n \overset{\bullet}{+} a_n \bullet x \bullet y$, thus similarly to the proof of Theorem 9 in Paragraph 6.3, we have

$$\begin{aligned}
v_{2^n}(B_{a_n}(x)) = v_{2^n}(x \overset{\bullet}{-} a_n \overset{\bullet}{+} y \bullet a_n \bullet x) &= \frac{v_{2^n}(x) v_{2^n}(y \bullet a_n \bullet x)}{v_{2^n}(a_n)} = \\
&= \varepsilon\left(\frac{x_n}{2}\right) \frac{\varepsilon\left(\frac{x_{n-1}}{2^2} + \cdots + \frac{x_0}{2^{n+1}}\right) v_{2^n}(y \bullet a_n \bullet x)}{v_{2^n}(a_n)} = \\
&= r_n(x) g_n(x),
\end{aligned}$$

where $g_n(x)$ depends only on $x_0, \ldots, x_{n-1}$ and on the parameters. Hence $g_n \in L(\mathcal{A}_n)$. Clearly, $|\Phi_{n,a}(x)| = 1$ $(x \in \mathbb{I})$, and the proof is complete.

□

Corollary 5 *[I. Simon[41]] The arithmetical Malmquist-Takenaka functions* $(\Psi_k^{(p)}, k \in \mathbb{N})$ *form a UDMD-product system, consequently it is a complete orthonormal system on* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

✠ In the following we consider the corresponding Malmquist-Takenaka-systems on fields $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ and $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

We will see in the next proposition, that the Malmquist-Takenaka system is a generalization of the discrete Laguerre system on both fields.

Proposition 7 *Particularly, using the same parameters* $a_n = a \in \mathbb{I}_1 \quad (n \in \mathbb{N})$ *the Malmquist-Takenaka functions* $\Psi_n^{(p)}(x)$ *equal the discrete Laguerre functions* $L_n^{(a)}(x)$ *on fields* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ *and* $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

Clearly, with the special parameters $a_n = \theta$ $(n \in \mathbb{N})$, this system is not else, than the character system of the corresponding field. That is, Malmquist-Takenaka systems are a generalization of the character system of the corresponding additive group, as well.

7.3 Summability and convergence questions

Definition 15 *Let* $a_n \in \mathbb{I}_1 (n \in \mathbb{N})$ *form a parameter sequence* $p = (a_0, a_1, \ldots)$. *The Malmquist-Takenaka-Fourier coefficients* $\widehat{f^{(p)}}$ *of an* $f \in L^1(\mathbb{I})$ *with parameter sequence* p, *the* n*-th partial sum* $S_n^{(p)} f$ *of the Malmquist-Takenaka-Fourier series* $S^{(p)} f$, *and the Malmquist-Takenaka-Cesaro (or* $MT - (C,1)$*) means of* $S^{(p)} f$ *are defined by*

$$\widehat{f^{(p)}}(n) := \int_{\mathbb{I}} f(x) \Psi_n^{(p)}(x) d\mu(x) \quad (n \in \mathbb{N}),$$

$$S_n^{(p)} f := \sum_{k=0}^{n-1} \widehat{f^{(p)}}(k) \Psi_k^{(p)} \ (n \in \mathbb{P}),$$

$$\sigma_0^{(p)} f := 0 \text{ and } \sigma_n^{(p)} f := \frac{1}{n} \sum_{k=1}^{n} S_k^{(p)} f \qquad (n \in \mathbb{P}).$$

The transformation method doesn't hold here because of the different parameters a_k of the transformation functions, but properties of UDMD product systems are valid for the Malmquist-Takenaka systems $(\Psi_k^{(p)}, k \in \mathbb{N})$ on the corresponding fields, thus applying general theorems on convergence and summability of Schipp [37](Theorem 4) and of Gát[9](Theorem 15), the following holds:
Remark: We have

$$\lim_{n\to\infty} \|S_{2^n}^{(p)} f - f\|_q = 0,\ (f \in L^q(\mathbb{I}), 1 \le q < \infty), \tag{7.3}$$

$$\lim_{m\to\infty} \|S_m^{(p)} f - f\|_q = 0, (f \in \mathbb{L}^q(\mathbb{I}), 1 < q < \infty), \tag{7.4}$$

$$\lim_{n\to\infty} \|\sigma_n^{(p)} f - f\|_q = 0,\ (f \in L^1(\mathbb{I})), \tag{7.5}$$

$$S_{2^n}^{(p)} f \to f \qquad a.e. \qquad (f \in \mathbb{L}^1(\mathbb{I})), \tag{7.6}$$

$$S_m^{(p)} f \to f \qquad a.e. \qquad (f \in \mathbb{L}^q(\mathbb{I}), q > 1), \tag{7.7}$$

$$\sigma_n^{(p)} f \to f \qquad a.e. \qquad (f \in \mathbb{L}^1(\mathbb{I})). \tag{7.8}$$

Moreover, (7.4) and (7.5) holds for $q = \infty$ when f is continuous on $\mathbb{I}$. (7.6) holds a.e. and also at every point of continuity of f.

Chapter 8

Construction of 2-adic Chebyshev polynomials

This chapter is based on I. Simon [43].

8.1 Introduction

This Chapter is based on [43]. Chebyshev polynomials play an important role for example in approximation theory (the resulting interpolation polynomial provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm) and other fields of applications. In classical analysis the Chebyshev polynomials of the first and second kind can be expressed through the identities

$$T_n(x) = \cos(n \arccos x) \qquad (x \in [-1, 1],\ n \geq 0);$$
$$U_n(x) = \frac{\sin\left[(n+1) \arccos x\right]}{\sin(\arccos x)} \qquad (x \in [-1, 1],\ n \geq 0),$$

where the cosine and sine functions can be given by means of the exponential function: $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. Each of the Chebyshev polynomials of the first and second kind form an orthogonal system with respect to the weight function $(1-x^2)^{-1/2}$ and $(1-x^2)^{1/2}$, respectively.

In this chapter we will construct some analogies of the Chebyshev polynomials on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ using several kinds of 2-adic cosine and sine functions. We present two opportunities to construct 2-adic trigonometric functions expressed by the additive characters $(v_n, n \in \mathbb{N})$ or by the $\tilde{\mathbb{S}}$-valued exponential functions, which is in connection with the multiplicative characters. In this way we will obtain first two dyadic martingale structure preserving transformations of $(v_n, n \in \mathbb{N})$, which will yield a UDMD-product system, thus complete and orthonormal. Then follows two further types of Chebyshev polynomials, which will also fulfil orhogonality.

Throughout this chapter for $x \in I$ let

$$n \cdot x := \underbrace{x \overset{\bullet}{+} x \overset{\bullet}{+} \ldots \overset{\bullet}{+} x}_{n \text{ times}} \quad \text{if } n \in \mathbb{N}^*, \text{ and let } 0 \cdot x := \theta. \tag{8.1}$$

Note, that $2 \cdot x = x \overset{\bullet}{+} x = e_1 \bullet x$ $(x \in \mathbb{I})$ and $2^n \cdot x = e_n \bullet x$ $(x \in \mathbb{I}, n \in \mathbb{N})$.

The notion of DMSP-functions and some properties of compositions with them were presented in Chapter 4. Here we will refer to some restrictions of DMSP-functions on dyadic intervals also as DMSP-functions, as they fulfill the same properties.

The $\tilde{\mathbb{S}}$-valued exponential function on $\mathbb{I}$: A 2-adic exponential function is presented in Schipp [17], pp 59-60. We will use now a similar one determined by a slightly different base, starting from $b_1 = e \overset{\bullet}{+} e_2$ instead of $e \overset{\bullet}{+} e_1$. Recall first the base defined in 3.10: $b_1 := e \overset{\bullet}{+} e_2$, $b_n := b_{n-1} \bullet b_{n-1}$ $(n \geq 2)$.

The structure of these bytes will be essential, and we will need the first 6 digits of the first four exactly, which can be calculated simply:

$$\begin{aligned} b_2 &= e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_4 = e \overset{\bullet}{+} e_3 \overset{\bullet}{+} d_3, \quad \pi(d_3) \geq 4, \\ b_3 &= e \overset{\bullet}{+} e_4 \overset{\bullet}{+} e_5 \overset{\bullet}{+} e_6 \overset{\bullet}{+} e_9 = e \overset{\bullet}{+} e_4 \overset{\bullet}{+} d_4, \quad \pi(d_4) \geq 5, \\ b_4 &= e \overset{\bullet}{+} e_5 \overset{\bullet}{+} e_6 \overset{\bullet}{+} e_7 \overset{\bullet}{+} e_8 \overset{\bullet}{+} \ldots = e \overset{\bullet}{+} e_5 \overset{\bullet}{+} d_5, \ \pi(d_5) \geq 6, \end{aligned} \tag{8.2}$$

where $d_3 := e_4$, $d_4 := e_5 \overset{\bullet}{+} e_6 \overset{\bullet}{+} e_9$ $d_5 := e_6 \overset{\bullet}{+} \ldots$

Recall, that in general,

$$b_n = e \overset{\bullet}{+} e_{n+1} \overset{\bullet}{+} d_{n+1} \ (n \geq 1) \text{ with } \pi(d_{n+1}) \geq n+2. \tag{8.3}$$

Definition 16 *Consider* $\tilde{\mathbb{S}} := \{x \in \mathbb{S} \ : \ x_1 = 1\} = I_2(e \overset{\bullet}{+} e_1)$. *Define the*

$\tilde{\mathbb{S}}$-valued exponential function on $\mathbb{I}$ by:

$$\zeta(x) := \prod_{j=1}^{\infty} b_j^{x_{j-1}} \qquad (x = (x_j, j \in \mathbb{N}) \in \mathbb{I}).$$

With the notations of (8.1), the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty}(e \overset{\bullet}{+} e_{j+1} \overset{\bullet}{+} d_{j+1})^{x_{j-1}} = \prod_{j=1}^{\infty}\left[e \overset{\bullet}{+} x_{j-1}(e_{j+1} \overset{\bullet}{+} d_{j+1})\right]. \tag{8.4}$$

This function is similar to those defined in Schipp[17], thus with similar arguments we have that ζ is a continuous function satisfying the functional-equation

$$\zeta(x \overset{\bullet}{+} y) = \zeta(x) \bullet \zeta(y) \qquad (x, y \in \mathbb{I}). \tag{8.5}$$

For more on ζ see Schipp [17], pp 59-60.

8.2 2-adic sine and cosine functions

In this section we present two ways of constructions of 2-adic trigonometric functions. The first one is expressed by the $\tilde{\mathbb{S}}$-valued exponential functions, which is in connection with the 2-adic multiplicative characters. See [17], pp.72-73. An other way of the construction is expressed by the additive characters and is a complex-valued function.

Definition 17 *Define* **the 2-adic cosine and sine function on** $\mathbb{I}$ *as follows:*

$$\cos x := (\zeta(x) \overset{\bullet}{+} \zeta(x^-)) \bullet e_{-1} \qquad (x \in \mathbb{I}),$$
$$\sin x := (\zeta(x) \overset{\bullet}{-} \zeta(x^-)) \bullet e_{-1} \qquad (x \in \mathbb{I}).$$

Definition 18 *To any $n \in \mathbb{N}$ define* **the 2-adic COS_n and SIN_n functions on** $\mathbb{I}$ *as follows:*

$$COS_n(x) := \frac{v_n(x) + v_n(x^-)}{2} \qquad (x \in \mathbb{I}),$$
$$SIN_n(x) := \frac{v_n(x) - v_n(x^-)}{2i} \qquad (x \in \mathbb{I}).$$

Addition formulas for 2-adic sine and cosine functions are a result of the functional equation (8.5) of the exponential function, and can be derived as in the real case but with slightly different coefficients. We state first that by $x^- = x \bullet e^-$ $(x \in \mathbb{B})$ and by the distributivity of the 2-adic operations we have $(x \overset{\bullet}{+} y)^- = x^- \overset{\bullet}{+} y^-$. Furthermore, $2a := a \overset{\bullet}{+} a = a \bullet e_1$, thus $a = (a \overset{\bullet}{+} a) \bullet e_{-1}$, and $e_{-1} \bullet e_{-1} = e_{-2}$. Now,

$$
\begin{aligned}
\cos(x \overset{\bullet}{+} y) &= \Big(\zeta(x \overset{\bullet}{+} y) \overset{\bullet}{+} \zeta(x^- \overset{\bullet}{+} y^-)\Big) \bullet e_{-1} = \\
&= \Big(\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x^-) \bullet \zeta(y^-)\Big) \bullet e_{-1} = \\
&= \Big([\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x^-) \bullet \zeta(y)] \overset{\bullet}{+} [\zeta(x^-) \bullet \zeta(y^-) \overset{\bullet}{+} \zeta(x) \bullet \zeta(y^-)] \overset{\bullet}{+} \\
&\quad [\zeta(x) \bullet \zeta(y) \overset{\bullet}{-} \zeta(x) \bullet \zeta(y^-)] \overset{\bullet}{+} [\zeta(x^-) \bullet \zeta(y^-) \overset{\bullet}{-} \zeta(x^-) \bullet \zeta(y)]\Big) \bullet e_{-2} = \\
&= \cos x \bullet \cos y \overset{\bullet}{+} \sin y \bullet \sin x.
\end{aligned}
$$

Similarly, $\sin(x \overset{\bullet}{+} y) = \sin x \bullet \cos y \overset{\bullet}{+} \cos x \bullet \sin y$ $(x, y \in \mathbb{I})$. Clearly, cosine is even and sine is odd, that is, $\cos(x^-) = \cos(x)$, and $\sin(x^-) = \overset{\bullet}{-} \sin(x)$ $(x \in \mathbb{I})$. Thus also holds $\cos(x \overset{\bullet}{-} y) = \cos x \bullet \cos y \overset{\bullet}{-} \sin x \bullet \sin y$, and so, by addition turns out, that

$$\cos(x \overset{\bullet}{+} y) \overset{\bullet}{+} \cos(x \overset{\bullet}{-} y) = \cos x \bullet \cos y \bullet e_1.$$

This means, that the 2-adic cosine and sine functions satisfy the so-called d'Alembert equation and sine-cosine functional equation investigated in Sahoo[14] and in Staetker[44].

Evidently, we have

$$
\begin{aligned}
&\cos 2x = \cos^2 x \overset{\bullet}{+} \sin^2 x, \qquad \sin 2x = \sin x \bullet \cos x \bullet e_1 \\
&e = \cos(\theta) = \cos^2 x \overset{\bullet}{-} \sin^2 x, \\
&\cos u \overset{\bullet}{+} \cos v = \cos\Big([u \overset{\bullet}{+} v] \bullet e_{-1}\Big) \bullet \cos\Big([u \overset{\bullet}{-} v] \bullet e_{-1}\Big) \bullet e_1.
\end{aligned}
$$

Clearly, COS_n is even and SIN_n is odd, that is $COS_n(x^-) = COS_n(x)$, and $SIN_n(x^-) = -SIN_n(x)$ $(x \in \mathbb{I}, n \in \mathbb{N})$. Addition formulas are in this case also a result of the functional equation $v_n(x \overset{\bullet}{+} y) = v_n(x)v_n(y)$ of the characters:

$$COS_n(x \overset{\bullet}{+} y) = COS_n(x)COS_n(y) - SIN_n(x)SIN_n(y),$$
$$COS_n(x \overset{\bullet}{-} y) = COS_n(x)COS_n(y) + SIN_n(x)SIN_n(y), \quad \text{thus}$$
$$COS_n(x \overset{\bullet}{+} y) + COS_n(x \overset{\bullet}{-} y) = COS_n(x)COS_n(y).$$

Ths COS_n and SIN_n satisfy the so-called d'Alembert equation and sine-cosine functional equation investigated for example in Sahoo[14] and in Staetker[44]. We have furthermore: $COS_n^2(x) + SIN_n^2(x) = 1$ $(x \in \mathbb{I},\ n \in \mathbb{N})$.

As the inverse function of cos is needed in the chosen construction of Chebyshev polynomials, we determine now a set $\tilde{\mathbb{S}}$, on which cos is bijective. It is not injective on the original domain $\mathbb{I}$, thus we consider its restriction on $\tilde{\mathbb{S}}$ and on its multiplicative shifts $\tilde{\mathbb{S}_l}$, and we determine the ranges also: $\mathbb{S}^\dagger$ and $\mathbb{S}^\dagger_{l,a_l}$, respectively.

Notation 1 *Consider the following sets of bytes*

$$\tilde{\mathbb{S}} := I_2(e \overset{\bullet}{+} e_1) = e \overset{\bullet}{+} e_1 \overset{\bullet}{+} \mathbb{I}_2 = \{x \in \mathbb{S} \ :\ x_1 = 1\},$$
$$\mathbb{S}^\natural := I_3(e) = e \overset{\bullet}{+} \mathbb{I}_3 = \{e \overset{\bullet}{+} t \ :\ t \in \mathbb{I}_3\} = \{x \in \mathbb{I} \ :\ x_0 = 1, x_1 = x_2 = 0\ \},$$
$$\mathbb{S}^\dagger := I_6(e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5) = \{x \in \mathbb{I} \ :\ x_0 = x_3 = x_5 = 1, x_1 = x_2 = x_4 = 0\ \} \subset \mathbb{S}^\natural,$$
$$\tilde{\mathbb{S}_l} := I_{l+2}(e_l \overset{\bullet}{+} e_{l+1}), \quad \mathbb{S}_l = e_l \overset{\bullet}{+} \mathbb{I}_{l+1} = I_{l+1}(e_1)\ (l \in \mathbb{N}),$$
$$\mathbb{S}^\dagger_{l,a_l} := a_l \overset{\bullet}{+} \mathbb{I}_{2l+6} \subset \mathbb{S}^\natural \quad \textit{with some } a_l \in \mathbb{S}\ (l \in \mathbb{N}).$$

Theorem 13 *[I. Simon[43]] a) The function* cos *takes* $\mathbb{S}$ *to* $\mathbb{S}^\dagger$*. Specially,* $\cos : \tilde{\mathbb{S}} \subset \mathbb{S} \to \mathbb{S}^\dagger$ *is a bijection.*
b) The function cos *takes* $\mathbb{I}$ *to* $\mathbb{S}^\natural$*.*

Proof: a) If $x \in \mathbb{S}$, then $x_0 = (x^-)_0 = 1$ and $(x^-)_j = 1 - x_j$ $(j \geq 1)$. Thus with the notations of (8.3) and representation (8.4) we have:

$$\cos(x) = b_1^{x_0} \bullet \left(\prod_{j=2}^{\infty} b_j^{x_{j-1}} \overset{\bullet}{+} \prod_{j=2}^{\infty} b_j^{1-x_{j-1}} \right) \bullet e_{-1} =$$

$$= b_1 \bullet e_{-1} \bullet \left(\prod_{j=2}^{\infty} \left[e \overset{\bullet}{+} x_{j-1}(e_{j+1} \overset{\bullet}{+} d_{j+1}) \right] \overset{\bullet}{+} \prod_{j=2}^{\infty} \left[e \overset{\bullet}{+} (1 - x_{j-1})(e_{j+1} \overset{\bullet}{+} d_{j+1}) \right] \right).$$

Now, set $z := (b_1)^{-1} \bullet e_1 \bullet \cos(x)$, which is the expression in the huge round brackets. Let us investigate the first digits of z: each of the products belongs to $\mathbb{S}$, thus the first terms are $e \overset{\bullet}{+} e = e_1$, and the next possibly nonzero digit is z_3. So, we compute the digits from the 3rd to the 8th using the structure (8.2) of the base and establishing also the rests q_i determined by the 2-adic sum:

$$
\begin{aligned}
&z_3 + 2q_3 = x_1 + (1 - x_1) = 1 \;\Rightarrow\; z_3 = 1,\; q_3 = 0 \\
&z_4 + 2q_4 = x_2 + (1 - x_2) + \underbrace{(d_3)_4}_{=1}(x_1 + (1 - x_1)) + q_3 = 2 \;\Rightarrow\; z_4 = 0,\; q_4 = 1 \\
&z_5 + 2q_5 = x_3 + (1 - x_3) + \underbrace{(d_3)_5}_{=0}(x_1 + (1 - x_1)) + \underbrace{(d_4)_5}_{=1}(x_2 + (1 - x_2)) + \underbrace{q_4}_{=1} \\
&\qquad\qquad = 3 \qquad \Rightarrow\; z_5 = q_5 = 1 \\
&z_6 + 2q_6 = x_4 + (1 - x_4) + \underbrace{(d_3)_6}_{=0} + \underbrace{(d_4)_6}_{=1} + \underbrace{(d_5)_6}_{=1} + \underbrace{q_5}_{=1} = 4 \;\Rightarrow\; z_6 = 0,\; q_6 = 2 \\
&z_7 + 2q_7 = \underbrace{x_5 + (1 - x_5)}_{\text{always}=1} + \underbrace{[x_1x_2 + (1 - x_1)(1 - x_2)]}_{\text{depends on } x_1, x_2} \underbrace{(e_3 \bullet e_4)_7}_{=1} + (d_3)_7 + (d_4)_7 \\
&\qquad\qquad + (d_5)_7 + (d_6)_7 + q_6 \\
&z_8 = 1 + \underbrace{[x_1x_3 + (1 - x_1)(1 - x_3)]}_{\text{depends on } x_1, x_3} + \varphi(x_1, x_2) \;(mod\ 2) \\
&\quad\vdots \\
&z_k = 1 + \underbrace{[x_1x_{k-5} + (1 - x_1)(1 - x_{k-5})]}_{\text{depends on } x_1, x_{k-5}} + \varphi(x_1, x_2, \ldots, x_{k-6}) \;(mod\ 2)\; (k \geq 7).
\end{aligned}
\tag{8.6}
$$

This computation resulted, that the 1st, 3rd and 5th digits of z were equal to 1, and the others were 0 until the 6th digit. Thus

$$\cos(x) = b_1 \bullet e_{-1} \bullet \left(e_1 \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5 \overset{\bullet}{+} \tilde{d}_6\right) = e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5 \overset{\bullet}{+} d_5'$$

with some $\tilde{d}_6 \in \mathbb{I}_7$, $d_5' \in \mathbb{I}_6$. Thus $y = cosx \in \mathbb{S}^\dagger$ and $\cos : \mathbb{S} \to \mathbb{S}^\dagger$.

Computation (8.6) also implies, that z_7 can take either 0 or 1 depending on x_1 and x_2, and so do the following digits, too, but depending on further digits of x. Thus setting condition $x_1 = 1$, which is the case for $x \in \tilde{\mathbb{S}}$, the 7th digit of z determines x_2, the 8th one determines x_3, the k-th digit of z determines x_{k-5}, and by an inductive argument follows the existence of a unique $x \in \tilde{\mathbb{S}}$ with the required property. Thus to any given $y \in \mathbb{S}^\dagger$ there exists an $x \in \tilde{\mathbb{S}}$ uniquely such that $\cos x = y$.

b) When $x \in \mathbb{I} \setminus \mathbb{S}$, then only base elements b_i of higher indexes ($i \geq 2$) will occur in $\cos x$, thus the nonzero coordinates except of the 0th are shifted to the right, so $\cos x \in \mathbb{S}$ and $(\cos x)_1 = (\cos x)_2 = 0$ holds in each case, thus the image of cos on $\mathbb{I}$ is a subset of $\mathbb{S}^\natural$.

$\square$

Remark: We have seen in Theorem 1 b) that $\cos : \mathbb{I} \to \mathbb{S}^\natural$. More exactly, the function cos takes $\mathbb{S}_l$ to $\mathbb{S}_l^\dagger$. Furthermore, $\cos : \tilde{\mathbb{S}_l} \to \mathbb{S}_l^\dagger$ is a bijection. Indeed, $\mathbb{I} = \bigcup_{l=0}^\infty \mathbb{S}_l$, and it turns out, that $\cos : \mathbb{S}_l \to \mathbb{S}_l^\dagger$ and $\cos : \tilde{\mathbb{S}_l} \to \mathbb{S}_l^\dagger$ is bijective. If $x \in \mathbb{S}_l$, than

$$\cos(x) = e_{-1} \bullet b_{l+1} \bullet \left(\prod_{j=l+2}^{\infty} b_j^{x_{j-1}} \stackrel{\bullet}{+} \prod_{j=l+2}^{\infty} b_j^{1-x_{j-1}} \right) =$$

$$= \underbrace{e_{-1} \bullet b_{l+1} \bullet \left(e_1 \stackrel{\bullet}{+} \sum_{j=l+3}^{2l+6} e_j \stackrel{\bullet}{+} \sum_{j=l+3}^{2l+6} d_j \right)}_{:=a_l} \stackrel{\bullet}{+}$$

$$\stackrel{\bullet}{+} e_{-1} \bullet b_{l+1} \bullet [x_{l+1}x_{l+2} + (1 - x_{l+1})(1 - x_{l+2})] e_{2l+7} \stackrel{\bullet}{+} \ldots,$$

thus $\cos x \in \mathbb{S}_{l,a_l}^\dagger = a_l \stackrel{\bullet}{+} \mathbb{I}_{2l+6}$. Furthermore, if $x_{l+1} = 1$ is given, than the digits x_j ($j > l + 1$) are determined uniquely by y, thus cos is bijective on $x \in \tilde{\mathbb{S}_l} = I_{l+2}(e_l \stackrel{\bullet}{+} e_{l+1})$.

Notation 2 *Let us denote the inverse of* $\cos : \tilde{\mathbb{S}} \to \mathbb{S}^\dagger$ *by* arccos, *which has domain* $\mathbb{S}^\dagger$*.*

We will use the following lemma in the next section.

Lemma 10 *[I. Simon[43]]* $f(t) := \cos(e_{-4} \bullet t)$ *is a DMSP-function on* $\tilde{\mathbb{S}_4} = I_6(e_4 \stackrel{\bullet}{+} e_5)$*, and also on* $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4} = I_6(e_4)$*.*

Proof: Computation (8.6) implies that for $x \in \tilde{\mathbb{S}}$ we have recursion

$$z_k = x_{k-5} + \varphi(x_2, x_3, \ldots, x_{k-6}) \pmod 2 \quad (k \geq 6)$$

with an arbitrary $\varphi : \mathbb{A}^{k-7} \to \mathbb{A}$. As $b_1 \in \mathbb{S}$, $z \bullet b_1 \in \mathbb{S}$ has the same type of recursion, furthermore follows for $y = \cos x = e_{-1} \bullet b_1 \bullet z$ the recursion form

$$y_k = x_{k-4} + \varphi(x_2, x_3, \ldots, x_{k-5}) \qquad (\mod 2)\ (k \geq 5)$$

with some $\varphi : \mathbb{A}^{k-6} \to \mathbb{A}$. We have $(x \bullet e_{-4})_k = x_{k+4}$ $(k \in \mathbb{Z})$. Thus $f(x) = \cos(e_{-4} \bullet x)$ is a DMSP-function from $\tilde{\mathbb{S}_4} = I_6(e_4 \overset{\bullet}{+} e_5)$ onto $\mathbb{S}^\dagger$.

Computation (8.6) also implies that for $x \in \mathbb{S} \setminus \tilde{\mathbb{S}}$ we have recursion

$$z_k = 1 - x_{k-5} + \varphi(x_2, x_3, \ldots, x_{k-6}) \qquad (\mod 2)\ (k \geq 6)$$

with an arbitrary $\varphi : \mathbb{A}^{k-7} \to \mathbb{A}$. Thus follows that $f(x) = \cos(e_{-4} \bullet x)$ is also a DMSP-function from $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4} = I_6(e_4)$ onto $\mathbb{S}^\dagger$.

□

Remark: It turns out similarly, that $\sin : \mathbb{S} \to I_3(e + e_2)$ is a bijection, and a simple recursion yields the digits of bytes $y = \sin x \bullet e_{-2}$, thus $x \mapsto \sin(x) \bullet e_{-2}$ is a DMSP-function on $\mathbb{S}$.

Theorem 14 *[I. Simon[43]] The systems $(\sqrt{2}COS_n, n \in \mathbb{N})$, $(\sqrt{2}SIN_n, n \in \mathbb{N})$ are orthogonal and for $n \in \mathbb{N}^*$ also orthonormal.*

Proof: We will investigate first the Rademacher functions on reflections:

$$\begin{aligned} r_n(x^-) = (-1)^{(x^-)_n} &= \begin{cases} (-1)^{x_n}, & \text{for } n \leqq \pi(x) \\ (-1)^{1-x_n}, & \text{for } n > \pi(x) \end{cases} = \\ &= r_n(x) \begin{cases} 1, & \text{if } x \in \mathbb{I}_n \\ -1, & \text{if } x \in \mathbb{I} \setminus \mathbb{I}_n \end{cases} \\ &= r_n(x)[-1 + 2\chi_{\mathbb{I}_n}(x)] \quad (x \in \mathbb{I}). \end{aligned} \tag{8.7}$$

Let $n, m \in \mathbb{N}$.

$$\int_{\mathbb{I}} COS_n(x)\overline{COS_m(x)}d\mu(x) = \frac{1}{4}\int_{\mathbb{I}} v_n(x)\overline{v_m(x)}d\mu(x) + \frac{1}{4}\int_{\mathbb{I}} v_n(x)\overline{v_m(x^-)}d\mu(x) +$$
$$+\frac{1}{4}\int_{\mathbb{I}} v_n(x^-)\overline{v_m(x)}d\mu(x) + \frac{1}{4}\int_{\mathbb{I}} v_n(x^-)\overline{v_m(x^-)}d\mu(x) =: \frac{1}{4}(I_1 + I_2 + I_3 + I_4).$$

Since $x \mapsto x^-$ is measure-preserving, I_4 is as simply as $I_1 = \delta_{n,m}$. For $n \neq m$ let $i := \min\{j \in \mathbb{N} : n_j \neq m_j\}$. Then by definitions $v_{2^i} = r_i g_i$ with some $g_i \in L(\mathcal{A}_i)$, and by (8.7) follows that the same holds for reflections, too: $v_{2^i}(x^-) = r_i(x)h_i(x)$ $(x \in \mathbb{I})$ with some $h_i \in L(\mathcal{A}_i)$. Hence,

$$v_n(x)\overline{v_m(x^-)} = \prod_{k=0}^{\infty} (v_{2^k}(x))^{n_k} \left(v_{2^k}(x^-)\right)^{-m_k} = r_i(x)g_i(x)$$

where $g_i \in L(\mathcal{A}_i)$. This implies as usual the statement: the properties of conditional expectations (see [17],pp.89) imply

$$\mathcal{E}_0\left(v_n(x)\overline{v_m(x^-)}\right) = \mathcal{E}_0\left(\mathcal{E}_i(r_i g_i)\right) = \mathcal{E}_0\left(g_i \mathcal{E}_i(r_i)\right) = 0 \ (n \neq m). \tag{8.8}$$

Now, for I_2 and I_3 results 0 by (8.8). In case of $n = m = 0$, by $\mu(\mathbb{I}) = 1$ follows $\int_{\mathbb{I}} COS_0(x)\overline{COS_0(x)} = 1$. For $n = m > 0$ we have $v_n(x)\overline{v_n(x^-)} = v_n(x)v_n(x) = v_n(2x) = v_{n-1}(x)$ $(n \geq 1)$, thus $\int_{\mathbb{I}} COS_n(x)\overline{COS_n(x)} = \frac{1}{2}$. The statement for $(SIN_n, n \in \mathbb{N})$ follows similarly.

□

8.3 The 2-adic Chebyshev polynomials

It seems at first sight to have exaggerated in the following definitions by using k twice, but the first one ensures that the system will be a UDMD-product system, and the second one belongs to the nature of Chebyshev polynomials.

Definition 19 *Define* **the 2-adic Chebyshev polynomials of the first kind** *as the product system of* $t_k(x) := v_{2^{k+6}}\left(\cos[(2k+1)\arccos(x)]\right)$ $(x \in \mathbb{S}^\dagger, k \in \mathbb{N})$*, that is,*

$$T_n(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+6}}\left(\cos[(2k+1)\arccos(x)]\right)\right]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}). \tag{8.9}$$

Definition 20 *Define* **the 2-adic Chebyshev polynomials of the second kind** *as the product system of* $u_k(x) := v_{2^{k+3}}\left(\sin[(2k+1)\arccos(x)]\right)$ $(x \in \mathbb{S}^\dagger, k \in \mathbb{N})$*, that is*

$$U_n(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+3}}\left(\sin[(2k+1)\arccos(x)]\right)\right]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}). \tag{8.10}$$

In order to see the orthogonality, we need first to examine the functions $x \mapsto \cos(n \arccos x)$ and $x \mapsto \sin(n \arccos x)$ $(x \in \mathbb{S}^\dagger)$.

Lemma 11 *[I. Simon[43]] The functions* $x \mapsto \cos((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ *and* $x \mapsto e_3 \bullet \sin((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ *are DMSP-functions on* $\mathbb{S}^\dagger$ *for any* $n \in \mathbb{Z}$.

Proof: The first function in question is obtained by a composition of functions

$$\begin{aligned} f_1(x) &:= e_4 \bullet \arccos(x), \qquad f_1 : \mathbb{S}^\dagger \to \tilde{\mathbb{S}_4} \\ f_2(x) &:= (2n+1)\cdot x = \underbrace{x \stackrel{\bullet}{+} x \stackrel{\bullet}{+} \ldots \stackrel{\bullet}{+} x}_{2n+1 \text{ times}}, \qquad f_2 : \tilde{\mathbb{S}_4} \to \mathbb{S}_4 \\ f_3(x) &:= \cos(x \bullet e_{-4}), \qquad f_3 : \mathbb{S}_4 \to \mathbb{S}^\dagger. \end{aligned}$$

The distributivity implies that $(2n+1)\cdot(e_4 \bullet x) = e_4 \bullet [(2n+1)\cdot x]$, thus $(f_3 \circ f_2 \circ f_1)(x) = \cos(n \arccos x)$ $(x \in \mathbb{S}^\dagger)$.

We have already seen in Lemma 10, that $f_3(x)$ is a DMSP-function on $\tilde{\mathbb{S}_4}$ and on $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4}$, too. Thus proposition4.1 of DMSP-functions results that f_1 is also a DMSP-function on $\mathbb{S}^\dagger$.

Let us examine f_2. With the dyadic expansion $n = \sum_{i=0}^\infty n_i 2^i$ we have $n \cdot x = \sum_{i=0}^\infty n_i (2^i \cdot x) = \sum_{i=0}^\infty n_i (e_i \bullet x)$, where the sum is taken in sense $\stackrel{\bullet}{+}$. Thus $(n\cdot x)_k = \sum_{i=0}^k n_i x_{k-i}$ $(k \in \mathbb{N},\ x \in \mathbb{I})$, which contains x_k if and only if $n_0 = 1$, that is, if n is odd. Thus $f_2(x) = (2n+1)\cdot x$ is a DMSP-function on $\tilde{\mathbb{S}_4}$.

□

Theorem 15 *[I. Simon[43]] The 2-adic Chebyshev polynomials of the first and second kind* $(T_n, n \in \mathbb{N})$ *and* $(U_n, n \in \mathbb{N})$ *are complete and orthonormal systems.*

Proof: As for each $c \in \mathbb{I}$ the system $(v_{2^{k+6}}, k \in \mathbb{N})$ is a UDMD-system on $\mathbb{I}_6(c)$, we have by Proposition 4.1 on DMSP-transformations that $(t_n, n \in \mathbb{N})$ is a UDMD-system on $\mathbb{S}^\dagger$, which results that $(T_n, n \in \mathbb{N})$ is a UDMD-product system on $\mathbb{S}^\dagger$, thus complete and orthonormal. (See Schipp[17], pp. 92-94.) The proof is similarly for the second kind Chebyshev polynomials.

□

Remarks: 1) Like for any UDMD-product systems, Fourier series of any $f \in L^p(\mathbb{I})$ $(p > 1)$ with respect to systems $(T_n, n \in \mathbb{N})$ and $(U_n, n \in \mathbb{N})$ converges a.e. to f, which is a consequence of Theorem 4 in Schipp [37]. Furthermore (C,1)-summability of any $f \in L^1(\mathbb{I})$ with respect to these systems also holds, which is a consequence of Theorem 15 in Gát[9] stated for Vilenkin-like systems, a generalization of UDMD-product systems.

2) The constructions and statements for the Chebyshev polynomials are valid if we use any proper UDMD-systems in place of $v_{2^{k+6}}$ and $v_{2^{k+3}}$ $(k \in \mathbb{N})$.

3) The 2-adic Chebyshev polynomials of the first and second kind can be defined also on $\mathbb{I}$ by establishing a proper shift operation: $S : \mathbb{I} \to \mathbb{S}^\dagger = I_6(e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5)$, $S(x) := x \bullet e_6 \overset{\bullet}{+} e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5$. Now,

$$\widetilde{T_n}(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+6}}\left(\cos[(2k+1)\arccos(S(x))]\right)\right]^{n_k} \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$
$$\widetilde{U_n}(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+3}}\left(\sin[(2k+1)\arccos(S(x))]\right)\right]^{n_k} \qquad (x \in \mathbb{I}, n \in \mathbb{N}).$$

Notation 3 *Consider shift operations:*

$$S : \mathbb{I} \to \mathbb{S}^\dagger, \ S(x) := x \bullet e_6 \overset{\bullet}{+} e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5,$$
$$S' : \tilde{\mathbb{S}} \to \mathbb{I}, \ S'(x) := [x \overset{\bullet}{-} e \overset{\bullet}{-} e_1] \bullet e_{-2}.$$

Definition 21 *Define* **the 2-adic Chebyshev polynomials of the third and fourth kind** *by*

$$\begin{aligned} \overline{T_n}(x) &:= COS_n[S'(\arccos(S(x))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}), \\ \overline{U_n}(x) &:= SIN_n[S'(\arccos(S(x))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}). \end{aligned} \tag{8.11}$$

Theorem 16 *[I. Simon[43]] The 2-adic Chebyshev polynomials of the third and fourth kind $(\overline{T_n}, n \in \mathbb{N}), (\overline{U_n}, n \in \mathbb{N})$ are orthogonal systems in $L^2(\mathbb{I})$.*

Proof: The variable transformation $B : x \mapsto S'(\arccos(S(x))$ is a DMSP-transformation on $\mathbb{I}$, thus it is measure-preserving. Hence,

$$\int_{\mathbb{I}} f \circ B \, d\mu = \int_{\mathbb{I}} f d\mu \qquad (f \in L^1(\mathbb{I})). \tag{8.12}$$

Let $n, m \in \mathbb{N}^*$. By (8.12) and by the orthogonality of the systems $(COS_n, n \in \mathbb{N})$, $(SIN_n, n \in \mathbb{N})$ follows the statement:

$$\int_{\mathbb{I}} \overline{T_n}(x)\overline{T_m}(x) d\mu(x) = \int_{\mathbb{I}} COS_n(y) COS_m(y) d\mu(y) = \frac{1}{2}\delta_{n,m}. \quad \square$$

Theorem 17 *[I. Simon[43]] The subsystems of 2-adic Chebyshev polynomials of the third and fourth kind* $(\overline{T_{2^n}}, n \in \mathbb{N}), (\overline{U_{2^n}}, n \in \mathbb{N})$ *form UDMD systems on* $\mathbb{I}$.

Proof: Recall, that $r_n(x^-) = r_n(x)[-1 + 2\chi_{\mathbb{I}_n}(x)] \quad (x \in \mathbb{I})$. By requirement (2.26) of UDMD systems we have for $(v_{2^n}, n \in \mathbb{N})$, that there exist $\mathcal{A}_n$-measurable functions $(g_n, n \in \mathbb{N})$ on $\mathbb{I}$, such that $v_{2^n} = r_n g_n$. Thus,

$$COS_{2^n}(x) = r_n(x) \frac{g_n(x) + g_n(x^-)[-1 + 2\chi_{\mathbb{I}_n}(x)]}{2}, \quad (x \in \mathbb{I})$$

and the function $h_n(x) := \frac{g_n(x)+g_n(x^-)[-1+2\chi_{\mathbb{I}_n}(x)]}{2} \in L(\mathcal{A}_n)$, thus $(COS_{2^n}, n \in \mathbb{N})$ fulfils the criteria (2.26) of UDMD systems. Similarly, $(SIN_{2^n}, n \in \mathbb{N})$ is a UDMD-system.

Since $x \mapsto S'(\arccos[S(x)])$ is a DMSP-transformation on $\mathbb{I}$, Theorem 4.1 on DMSP-transformations implies that $(\overline{T_{2^n}}, n \in \mathbb{N}), (\overline{U_{2^n}}, n \in \mathbb{N})$ are UDMD-systems.

$\square$

Summary

The present work consists of four main topics related to the Blaschke functions defined on two special locally compact totally disconnected non-Archimedian normed fields: on the 2-adic (or arithmetic) field and on the 2-series (or logical, dyadic) field. First, we investigate the effect of dyadic martingale structure preserving transformations, or shortly DMSP-transformations on function classes like the classes of UDMD-systems, that of $\mathcal{A}_n$-measurable functions, the dyadic function spaces $L^p(\mathbb{I})$, $H^p(\mathbb{I})$, and the Lipschitz classes $Lip(\alpha, \mathbb{I})$. Secondly, we establish the character system of the Blaschke-group on the arithmetic field. Then, we introduce the discrete Laguerre and the Malmquist-Takenaka systems on these fields, that are constructed by the Blaschke functions and the characters of the corresponding field. Both of the last mentioned are UDMD-product systems, thus complete and orthonormal, while in the second topic $v_n \circ \gamma$ possesses these properties. At last, 2-adic Chebyshev polynomials are constructed with several 2-adic trigonometric functions investigated in this work. All these are connected to DMSP-transformations, as they share essentially the type of the recursion.

Chapter 2 contains an introduction to the 2-series and 2-adic fields, especially concerning the algebraic and topological structure. This chapter follows the concepts, notations and propositions of Schipp and Wade[17]. The *set of bytes* is defined by: $\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \{0, 1\} \text{ and } \lim_{j \to -\infty} a_j = 0\}$. We present the 2-adic/arithmetical and 2-series/logical/dyadic operations, the order and the norm of a byte. We recall, that $(\mathbb{B}, \overset{\circ}{+}, \circ)$ and $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ are non-Archimedian normed fields. We use furthermore the intervals $I_n := \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ for any $n \in \mathbb{Z}$ and the unit ball $\mathbb{I} := \mathbb{I}_0 = \{a = (a_j, j \in \mathbb{N}) | \, a_j \in \{0, 1\}\}$ to construct dyadic martingale structure. We consider a normalized Haar measure μ with property $\mu(\mathbb{I}) = 1$, and the concept of UDMD-systems is summerized.

In Paragraph 2.6 we consider a measure-preserving variable transformation $T : \mathbb{I} \to \mathbb{I}$, and we mention that the n-th partial sum $S_n^T f$ of the T-Fourier series $S^T f$ and the T-Cesaro means $\sigma_n^T f$ of $S^T f$ with respect to $\phi_n \circ T$ can be expressed by the n-th partial sum $S_n f$ of the Fourier series and the Cesaro-means $\sigma_n f$ with respect to the characters $\{\phi_n, n \in \mathbb{N}\}$ of the corresponding additive group as follows:

$$S_n^T f = \left[S_n(f \circ T^{-1})\right] \circ T$$
$$\sigma_n^T f = \left[\sigma_n(f \circ T^{-1})\right] \circ T.$$

Based on the handbook of Schipp and Wade[17] in **Chapter 3** we first summarize the notions and results regarding the characters of the additive groups of these local fields and the exponential functions, which are used in the next chapters.

Paragraph 3.1 provides a description of the characters of the 2-series/dyadic and 2-adic additive groups using the notion of the product system.

In Paragraph 3.2 we use the notations $\mathbb{S} := \{x \in \mathbb{B} \mid \|x\| = 1\}$ and $\tilde{\mathbb{S}} := \{x \in \mathbb{S} : x_1 = 0\}$. The $(\tilde{\mathbb{S}}, \bullet)$-valued **exponential function** ζ on $\mathbb{I}_1$ is defined by the following infinite product form:

$$\zeta(x) := \prod_{j=1}^{\infty} b_j^{x_j} \qquad (x = (x_j, j \in \mathbb{Z}) \in \mathbb{I}_1),$$

where $b_1 := e \overset{\bullet}{+} e_2,\ b_n := b_{n-1} \bullet b_{n-1}\ (n \geq 2)$. Function ζ is a simple adaptation of the $(\mathbb{S}, \bullet)$-valued exponential function presented in [17], as we have defined ζ with a slightly different base. The function ζ satisfies the functional equation $\zeta(x \overset{\bullet}{+} y) = \zeta(x) \bullet \zeta(y)\ (x, y \in \mathbb{I}_1)$, and it is a continuous isomorphism from $\mathbb{I}_1$ onto $\tilde{\mathbb{S}}$.

Starting from Paragraph 3.3, this work contains the results of the author. We define the Blaschke functions on the studied fields and we investigate some properties of them. The logical Blaschke-functions $B_a(x) = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x}\ (x \in \mathbb{I}, a \in \mathbb{I}_1)$ defined on the dyadic field and the arithmetical Blaschke-functions $B_a(x) = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x}\ (x \in \mathbb{I}, a \in \mathbb{I}_1)$ defined on the 2-adic field are isometries on the unit ball $\mathbb{I}$ and on the unit sphere $\mathbb{S}$. Furthermore, they form a commutative group with respect to the function composition. We show, that the byte $y = B_a(x)$ can be computed by a recursion:

$$y_n = x_n + a_n + f_n(x_0, \cdots, x_{n-1}) \pmod 2 \qquad (n > 0),$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ depend on the bits of a.

In **Chapter 4** is concerned the argument transformation given by the composition with a Blaschke function, and in general, the dyadic martingale structure preserving transformation or shortly the DMSP-transformation, and we deal with questions related to the effect of a DMSP-transformation on special function classes.

We call a function $B : \mathbb{I} \to \mathbb{I}$ a DMSP-transformation if it is generated by a system of bijections $(\vartheta_n, n \in \mathbb{N})$, $\vartheta_n : \mathbb{A} \to \mathbb{A}$, and an arbitrary system $(\eta_n, n \in \mathbb{N}^*)$, $\eta_n : \mathbb{A}^n \to \mathbb{A}$ in the following way:

$$\begin{aligned} &(B(x))_0 := \vartheta_0(x_0), \\ &(B(x))_n := \vartheta_n(x_n) + \eta_n(x_0, x_1, \ldots, x_{n-1}) \pmod 2 \quad (n \in \mathbb{N}^*). \end{aligned}$$

For each generating systems $(\vartheta_n, n \in \mathbb{N})$ and $(\eta_n, n \in \mathbb{N}^*)$, the generated DMSP-transformation B is a bijection on $\mathbb{I}$ and its inverse function, B^{-1} is also a DMSP-transformation. B is also measure-preserving. A DMSP-transformation preserves the classes of UDMD systems, that of $\mathcal{A}_n$-measurable functions, the dyadic function spaces $L^p(\mathbb{I})$, $H^p(\mathbb{I})$, and the Lipschitz classes $Lip(\alpha, \mathbb{I})$. Furthermore, some examples of DMSP-functions are presented, mentioned the translations, dilatations, a generalization of ζ, and the Blaschke functions, as well.

Then, in **Chapter 5** we show, that the group of the Blaschke functions, the so-called Blashke-group $(\mathcal{B}, \circ)$ of the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ is a topological group, and we determine its characters. The operation $x \triangleleft y := \frac{x \overset{\bullet}{+} y}{e \overset{\bullet}{+} x \bullet y}$ $(x, y \in \mathbb{I}_1)$ determined by the composition $B_a \circ B_b = B_{a \triangleleft b}$ leads to the functional equation of the tangent function tan. This gives the idea of this chapter, where the characters of the Blaschke group of the 2-adic group are constructed by means of a tangent-like function.

The map $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$, $a \mapsto B_a$ is a continuous isomorphism, hence in order to establish the characters of $(\mathcal{B}, \circ)$, it is sufficient to define the character group of $(\mathbb{I}_1, \triangleleft)$. Furthermore, the characters of $(\mathbb{I}_1, \overset{\bullet}{+})$ are already known: the functions $(v_n, n \in \mathbb{N})$.

Thus we give a continuous isomorphism from the additive group $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$, that is a function γ satisfying the equation

$$\gamma(x \overset{\bullet}{+} y) = \frac{\gamma(x) \overset{\bullet}{+} \gamma(y)}{e \overset{\bullet}{+} \gamma(x) \bullet \gamma(y)} \ (x, y \in \mathbb{I}_1).$$

These thoughts can be interpreted as the solution of the functional equation of tan on the local field.

The *tangent-like function* on $(\mathbb{I}_1, \overset{\bullet}{+})$ is introduced as:

$$\gamma(x) := \frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \quad (x \in \mathbb{I}_1).$$

We show, that γ is a continuous isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$. This implies, that the characters of the group $(\mathbb{I}_1, \triangleleft)$ are the functions

$$(v_n \circ \gamma^{-1},\ n \in \mathbb{P}),$$

which allows us to conclude, that the characters of the Blaschke group $(\mathcal{B}, \circ)$ are the functions

$$(v_n \circ \gamma^{-1} \circ B^{-1},\ n \in \mathbb{P}),$$

where $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$ represents the function $a \mapsto B_a$.

A simple recursion yields the proposition, that the functions $v_n \circ \gamma^{-1}(n \in \mathbb{P})$, the characters of $(\mathbb{I}_1, \triangleleft)$ form a UDMD product system. Thus the discrete Fourier coefficients with respect to this system can be computed with the Fast Fourier Algorithm.

As the variable transformation γ is measure preserving, for the partial sums $S_n^\gamma f$ and the Gamma-Cesaro means $\sigma_n^\gamma f$ of the Gamma-Fourier series $S^\gamma f$ with respect to the system $(v_n \circ \gamma^{-1}, n \in \mathbb{N})$ follows the convergence $\lim_{n\to\infty} \sigma_n^\gamma f(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I}_1)$ and $\lim_{n\to\infty} S_n^\gamma f(x) = f(x)$ a.e. $(f \in L^p(\mathbb{I}_1),\ p > 1)$.

Chapter 6 is devoted to the construction of the discrete Laguerre functions on both local fields. The power functions on the torus $\mathbb{T}$ coincide with the classical characters, and the discrete Laguerre systems are given by their composition with the complex Blaschke functions. After the model of the classical system, we introduce the discrete Laguerre system as the composition of the additive characters of the studied local fields and the Blaschke functions.

For $a \in \mathbb{I}_1$ we introduce the *logical discrete Laguerre functions* on $(\mathbb{I}, \overset{\circ}{+}, \circ)$ associated to B_a in the following way:

$$L_k^{(a)}(x) := w_k(B_a(x)) \ (k \in \mathbb{N}, x \in \mathbb{I}),$$

which form the product system generated by $(r_n \circ B_a, n \in \mathbb{N})$, that is, $L_k^{(a)}(x) = \prod_{n=0}^{\infty} [r_n(B_a(x))]^{k_n}$.

For $a \in \mathbb{I}_1$ we introduce the *arithmetical discrete Laguerre functions* on $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ associated to B_a in the following way:

$$L_k^{(a)}(x) := v_k(B_a(x)) \ (k \in \mathbb{N}, x \in \mathbb{I}),$$

which build the product system generated by $(v_{2^n} \circ B_a, n \in \mathbb{N})$, that is, $L_k^{(a)}(x) = \prod_{j=0}^{+\infty} [v_{2^j}(B_a(x))]^{k_j} \quad (x \in \mathbb{I})$.

The discrete Laguerre-system $(L_k^{(a)}, k \in \mathbb{N})$ defined on the respective field is a UDMD-product system, thus it is complete and orthonormal.

Paragraph 6.4 is devoted to the (C,1)-summability of the Fourier series with respect to these systems using the basic results of Schipp[15] and Gat[7] on the a.e. convergence and (C,1)-summability of the Fourier series with respect to the characters of the dyadic and 2-adic field. We consider the Laguerre-Cesaro means $\sigma_n^{(a)} f$ and n-th partial sum $S_n^{(a)} f$ of the Laguerre-Fourier series $S^{(a)} f$ of an $f \in L^1(\mathbb{I})$ with respect to the corresponding discrete Laguerre functions. We show on both fields that $\lim_{n\to\infty} \sigma_n^{(a)} f(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I})$ and $\lim\limits_{n\to\infty} S_n^{(a)} f(x) = f(x)$ a.e. $(f \in L^p(\mathbb{I}_1),\ p > 1)$.

Chapter 7 covers our investigations on the Malmquist-Takenaka systems on both studied local fields. The *logical/arithmetical Malmquist-Takenaka functions* $(\Psi_k^{(p)}, k \in \mathbb{N})$ with parameters $p = (a_0, a_1, \ldots)$ $(a_i \in \mathbb{I}_1, i \in \mathbb{N})$ are defined in the following way: $(\Psi_k^{(p)}, k \in \mathbb{N})$ is the product system generated by

$$(\varphi_{n,a_n} := r_n \circ B_{a_n}, n \in \mathbb{N}) \quad \text{on} \quad (\mathbb{I}, \overset{\circ}{+}, \circ), \text{ and by}$$
$$(\Phi_{n,a_n} := v_{2^n} \circ B_{a_n}, n \in \mathbb{N}) \quad \text{on} \quad (\mathbb{I}, \overset{\bullet}{+}, \bullet),$$

respectively. Clearly, the Malmquist-Takenaka system is a generalization of the discrete Laguerre system: using identical parameters $a_n = a \in \mathbb{I}_1$ $(n \in \mathbb{N})$, the Malmquist-Takenaka functions $\Psi_n^{(p)}(x)$ equal the discrete Laguerre functions $L_n^{(a)}(x)$.

Being a UDMD-product system, we have a complete orthonormal system on both fields. As an other consequence of being UDMD-product systems, a.e. convergence and summability properties of Fourier series with respect to these systems hold.

In **Chapter 8** several 2-adic cosine and sine functions are constructed on the 2-adic field expressed by the $\tilde{\mathbb{S}}$-valued exponential functions and the characters v_n of the 2-adic additive group:

$$\cos x := (\zeta(x) \overset{\bullet}{+} \zeta(x^-)) \bullet e_{-1}, \quad \sin x := (\zeta(x) \overset{\bullet}{-} \zeta(x^-)) \bullet e_{-1} \quad (x \in \mathbb{I});$$
$$COS_n(x) := \frac{v_n(x) + v_n(x^-)}{2} \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$
$$SIN_n(x) := \frac{v_n(x) - v_n(x^-)}{2i} \qquad (x \in \mathbb{I}, n \in \mathbb{N}).$$

Addition formulas for both constructions hold, and we determine a set, on which cos bijective is: $\cos : \tilde{\mathbb{S}} \subset \mathbb{S} \to \mathbb{S}^\dagger$ is a bijection. We prove, that the systems $(COS_n, n \in \mathbb{N})$, $(SIN_n, n \in \mathbb{N})$ are orthogonal. The functions $x \mapsto \cos((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ and $x \mapsto e_3 \bullet \sin((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ are DMSP-functions on $\mathbb{S}^\dagger$ for any $n \in \mathbb{Z}$.

Then follows the construction of some analogies of the Chebyshev polynomials on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ using these cosine and sine functions. The 2-adic Chebyshev polynomials of the first and second kind are defined as the product system of $t_k(x) := v_{2^{k+6}}(\cos[(2k+1)\arccos(x)])$ and $u_k(x) := v_{2^{k+3}}(\sin[(2k+1)\arccos(x)])$ $(x \in \mathbb{S}^\dagger, k \in \mathbb{N})$, that is,

$$T_n(x) := \prod_{k=0}^{\infty} [v_{2^{k+6}}(\cos[(2k+1)\arccos(x)])]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}),$$
$$U_n(x) := \prod_{k=0}^{\infty} [v_{2^{k+3}}(\sin[(2k+1)\arccos(x)])]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}).$$

We prove, that $(T_n, n \in \mathbb{N})$ and $(U_n, n \in \mathbb{N})$ are UDMD-product systems, thus complete and orthonormal systems.

The 2-adic Chebyshev polynomials of the third and fourth kind are defined by

$$\overline{T_n}(x) := COS_n[S'(\arccos(S(x))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$
$$\overline{U_n}(x) := SIN_n[S'(\arccos(S(x))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The 2-adic Chebyshev polynomials of the third and fourth kind $(\overline{T_n}, n \in \mathbb{N}), (\overline{U_n}, n \in \mathbb{N})$ are orthogonal systems in $L^2(\mathbb{I})$.

Összefoglaló (Hungarian summary)

Ez a dolgozat négy fő témát ölel fel, melyek a Blaschke függvények két lokálisan kompakt nem-Archimédeszi normált testen értelmezett változatával kapcsolatosak: a 2-adikus (vagy aritmetikai) és a 2-soros (vagy logikai, diadikus) testen. Először a diadikus martingál struktúrát megőrző, azaz DMSP-transzfomációk hatását vizsgáljuk olyan függvényosztályokra, mint az UDMD rendszereké, az $\mathcal{A}_n$-mérhető függvényeké, a diadikus $L^p(\mathbb{I})$, $H^p(\mathbb{I})$ függvényosztályok, illetve a $Lip(\alpha, \mathbb{I})$ Lipschitz-osztály. Majd meghatározzuk a Blaschke csoport karakterrendszerét, és végül bevezetjük a diszkrét Laguerre és a Malmquist-Takenaka függvényeket a Blaschke függvények és a megfelelő additív csoportok karakterei segítségével. Ez utóbbi két rendszer UDMD szorzatrendszer és ortonormált, míg a második téma esetén a konstrukcióban fellépő $v_n \circ \gamma$-ról mondhatjuk el ugyanezt. Végül pedig 2-adikus Chebyshev polinomokat konstruálunk különböző 2-adikus trigonometrikus függvény segítségével, melyeket ugyancsak értelmezünk és vizsgálunk. Mindezek kapcsolatosak a diadikus martingál struktúrát megőrző transzformációkkal, hiszen ezek rekurziós előállításainak lényegében azonos a típusa.

A **2. Fejezet** bevezetést tartalmaz a 2-adikus és 2-soros testek elméletébe, különösen az algebrai és topológiai struktúrát illetően. Ebben a fejezetben a Schipp-Wade[17] fogalmait, jelöléseit, és állításait használjuk az áttekintés végett. Legyen a *bájtok halmaza* a következő: $\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \{0,1\}$ és $\lim_{j\to-\infty} a_j = 0\}$. Bemutatjuk a 2-adikus és 2-soros műveleteket, egy bájt rendjének, normájának, és a metrikának az értelmezését. Felidézzük, hogy $(\mathbb{B}, \overset{\circ}{+}, \circ)$ és $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ nem-Archimédeszi normált testeket alkot. Tekintjük továbbá a következő intervallumokat: $I_n := \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ minden $n \in \mathbb{Z}$-

re és az $\mathbb{I} := \mathbb{I}_0 = \{a = (a_j, j \in \mathbb{N}) |\ a_j \in \{0,1\}\}$ egység-gömböt. Tekintjük a $\mu(\mathbb{I}) = 1$ azonossággal normalizált μ Haar mértéket, és bemutatjuk az UDMD rendszer fogalmát is.

A 2.6. Részfejezetben tekintünk egy $T : \mathbb{I} \to \mathbb{I}$ mértéktartó argumentum-transzformációt, és áttekintjük, hogy a $\phi_n \circ T$ rendszer szerinti $S^T f$-el jelölt T-Fourier sor $S_n^T f$-el jelölt n-edik részletösszege és a $\sigma_n^T f$ T-Cesaro közepe kifejezhető a $\{\phi_n, n \in \mathbb{N}\}$ karakterrendszer szerinti Fourier sor $S_n f$-el jelölt n-edik részletösszegével, illetve a $\sigma_n f$ Cesaro/Fejér-közepével a következőképpen:

$$S_n^T f = \left[S_n(f \circ T^{-1})\right] \circ T$$
$$\sigma_n^T f = \left[\sigma_n(f \circ T^{-1})\right] \circ T.$$

A Schipp-Wade[17] kézikönyvre támaszkodva a **3. Fejezetben** összefoglaljuk ezen lokális testek additív csoportjának karaktereivel és az exponenciális fügvénnyel kapcsolatos azon fogalmakat és eredményeket, melyeket a következő fejezetekben alkalmazunk.

A 3.1. Részfejezet a diadikus és 2-adikus additív csoport karaktereinek leírását tartalmazza a szorzatrendszer fogalmára alapozva.

A 3.2. Részfejezetben használjuk a következő jelöléseket: $\mathbb{S} := \{x \in \mathbb{B} |\ \|x\| = 1\}$ és $\tilde{\mathbb{S}} := \{x \in \mathbb{S} : x_1 = 0\}$. Az $(\tilde{\mathbb{S}}, \bullet)$-értékű ζ **exponenciális függvény**t az $\mathbb{I}_1$-en a következő végtelenszorzatformában adjuk meg:

$$\zeta(x) := \prod_{j=1}^{\infty} b_j^{x_j} \qquad (x = (x_j, j \in \mathbb{Z}) \in \mathbb{I}_1),$$

ahol $b_1 := e \overset{\bullet}{+} e_2$, $b_n := b_{n-1} \bullet b_{n-1}$ $(n \geq 2)$. A ζ függvény a [17]-ban bemutatott $(\mathbb{S}, \bullet)$-értékű exponenciális függvénytől némileg különbözik, egy kissé módosított bázisra épül. A ζ függvény eleget tesz a $\zeta(x \overset{\bullet}{+} y) = \zeta(x) \bullet \zeta(y)$ $(x, y \in \mathbb{I}_1)$ függvény-egyenletnek, és egy folytonos izomorfizmus az $\mathbb{I}_1$-ről az $\tilde{\mathbb{S}}$-ra.

A 3.3. Részfejezettől kezdődően, a dolgozat a szerző eredményeit tartalmazza. Értelmezzük a Blaschke függvényeket a vizsgált testeken, és megállapítjuk azok néhány fontos tulajdonságát. A diadikus testen értelmezett $B_a(x) = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x}$ $(x \in \mathbb{I}, a \in \mathbb{I}_1)$ logikai Blaschke-függvények és a 2-adikus testen értelmezett $B_a(x) = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x}$ $(x \in \mathbb{I}, a \in \mathbb{I}_1)$ aritmetikai Blaschke-függvények izometriák az $\mathbb{I}$ egységgömbön és annak határán, az $\mathbb{S}$-en. Továbbá, kommutatív csoportot alkotnak a függvény-kompozícióra nézve. Megmutatjuk, hogy az $y = B_a(x)$ bájt a következő rekurzióval rendelkezik:

$$y_n = x_n + a_n + f_n(x_0, \cdots, x_{n-1}) \pmod 2 \qquad (n > 0)$$

ahol az $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ függvények az a paraméter bitjeitől is függenek.

A **4. Fejezetben** olyan argumentum-transzformációval foglalkozunk, melyet a Blaschke-függvénnyel, sőt, általánosabban a diadikus martingál struktúrát megőrző transzformáció, azaz a DMSP-transformációval való függvény-kompozíció ad meg, és a DMSP-transzformáció hatását is vizsgáljuk speciális függvényosztályokra.

Egy $B : \mathbb{I} \to \mathbb{I}$ függvényt DMSP-transzformációnak nevezünk, ha egy $(\vartheta_n, n \in \mathbb{N})$, $\vartheta_n : \mathbb{A} \to \mathbb{A}$ bijektív függvényrendszer és egy tetszőleges $(\eta_n, n \in \mathbb{N}^*)$, $\eta_n : \mathbb{A}^n \to \mathbb{A}$ függvényrendszer generálja a következőképpen:

$$\begin{aligned} &(B(x))_0 := \vartheta_0(x_0), \\ &(B(x))_n := \vartheta_n(x_n) + \eta_n(x_0, x_1, \ldots, x_{n-1}) \pmod 2 \quad (n \in \mathbb{N}^*). \end{aligned}$$

Bármely $(\vartheta_n, n \in \mathbb{N})$ és $(\eta_n, n \in \mathbb{N}^*)$ rendszer esetén a származtatott B DMSP-transzformáció egy bijekció $\mathbb{I}$-n és inverz függvénye, B^{-1} is egy DMSP-transzformáció. Továbbá B mértéktartó. Egy DMSP-transzformáció során megőrződik az UDMD-rendszerek osztálya, az $\mathcal{A}_n$-mérhető függvényeké, a diadikus $L^p(\mathbb{I})$, $H^p(\mathbb{I})$ osztályok és a $Lip(\alpha, \mathbb{I})$ Lipschitz osztály. Továbbá, bemutatunk néhány példát DMSP-függvényre, melyek között megemlítjük a transzlációt, a dilatációt, a bájtokhoz a multiplikatív inverzüket rendelő $\frac{1}{x}$ függvényt, a ζ egy általánosítását, és a Blaschke-függvényeket is.

Az **5. Fejezetben** bemutatjuk az $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ 2-adikus testen értelmezett Blaschke függvények csoportját, az úgynevezett $(\mathcal{B}, \circ)$ Blashke-csoportot, melyről miután beláttuk, hogy topológikus csoport, meghatározzuk annak karakter-csoportját. A $B_a \circ B_b = B_{a \triangleleft b}$ kompozíció által meghatározott művelet, az $x \triangleleft y := \frac{x \overset{\bullet}{+} y}{e \overset{\bullet}{+} x \bullet y}$ $(x, y \in \mathbb{I}_1)$ a tan függvény függvény-egyenletéhez vezet. Ez ihlette a keresett karakterek konstrukcióját, ahol a 2-adikus test Blaschke-csoportjának karaktereit egy tangens-szerű függvény segítségével értelmezzük. A $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$, $a \mapsto B_a$ leképezés egy folytonos izomorfizmus, ezért a $(\mathcal{B}, \circ)$ karaktereinek meghatározásához elegendő, ha meghatározzuk a $(\mathbb{I}_1, \triangleleft)$ csoport karakter-rendszerét. Továbbá, a $(\mathbb{I}_1, \overset{\bullet}{+})$ csoport karakter-rendszere már ismert: $(v_n, n \in \mathbb{N})$. A keresett karakterek megadásához tehát egy folytonos izomorfizmust keresünk az $(\mathbb{I}_1, \overset{\bullet}{+})$-ről az $(\mathbb{I}_1, \triangleleft)$-re, azaz egy olyan γ függvényt, mely eleget tesz a

$$\gamma(x \overset{\bullet}{+} y) = \frac{\gamma(x) \overset{\bullet}{+} \gamma(y)}{e \overset{\bullet}{+} \gamma(x) \bullet \gamma(y)} \qquad (x, y \in \mathbb{I}_1)$$

egyenletnek. Ez a megoldási út úgy fogható fel, hogy a 2-adikus test tangens függvényének függvény-egyenletét oldjuk meg.

A *tangens-szerű függvény*t az $(\mathbb{I}_1, \overset{\bullet}{+})$-en a klasszikus esethez hasonlóan az exponenciális függvény felhasználásával vezetjük be:

$$\gamma(x) := \frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \qquad (x \in \mathbb{I}_1).$$

Megmutattuk, hogy a γ függvény egy folytonos izomorfizmus az $(\mathbb{I}_1, \overset{\bullet}{+})$-ről az $(\mathbb{I}_1, \triangleleft)$-re. Ebből következik, hogy az $(\mathbb{I}_1, \triangleleft)$ karakterei az alábbi függvények:

$$v_n \circ \gamma^{-1} \quad (n \in \mathbb{P}).$$

Következésképpen a $(\mathcal{B}, \circ)$ Blaschke csoport karakterei a

$$v_n \circ \gamma^{-1} \circ B^{-1} \quad (n \in \mathbb{P})$$

függvények, ahol $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$ az $a \mapsto B_a$ függvényt takarja.

Egy (végeredményében) egyszerűnek nevezett rekurzió szolgáltatja az állítást, hogy a $v_n \circ \gamma^{-1}(n \in \mathbb{P})$ függvények, az $(\mathbb{I}_1, \triangleleft)$ karakterei egy UDMD-szorzatrendszert alkotnak. Ezért ezen rendszerekre vonatkozó Fourier együtthatók az úgynevezett FFT, azaz a Gyors Fourier Algoritmussal (Fast Fourier Algorithm) szamolhatók.

Mivel a γ függvény (változócsere) méréktartó, ezért a $(v_n \circ \gamma^{-1}, n \in \mathbb{N})$ rendszerre vonatkozó $S^\gamma f$ Gamma-Fourier sor $S_n^\gamma f$ részletösszegeire és $\sigma_n^\gamma f$ Gamma-Cesaro közepeire fennállnak a következők: $\lim_{n\to\infty} \sigma_n^\gamma f(x) = f(x)$ m.m., ahol $f \in L^1(\mathbb{I}_1)$, és $\lim_{n\to\infty} S_n^\gamma f(x) = f(x) m.m.$, ahol $f \in L^p(\mathbb{I}_1),\ p > 1$.

A **6. Fejezet** a két vizsgált lokális testen értelmezett diszkrét Laguerre függvényeknek van szentelve. A hatványfüggvények a $\mathbb{T}$ tóruszon megegyeznek a klasszikus karakterekel, és az összetételük a komplex Blaschke függvényekkel éppen a diszkrét Laguerre rendszert származtatják. A klasszikus rendszer modellje alapján bevezetjük a vizsgált lokális testeken a diszkrét Laguerre rendszert, mint a megfelelő additív csoport karakterének és a Blaschke függvényeknek az összetétele.

Az $a \in \mathbb{I}_1$ esetén az $(\mathbb{I}, \overset{\circ}{+}, \circ)$-n értelmezzük a B_a-hoz rendelt *logikai diszkrét Laguerre függvényeket* a következőképpen:

$$L_k^{(a)}(x) := w_k(B_a(x)) \ (k \in \mathbb{N}, x \in \mathbb{I}),$$

amelyek az $(r_n \circ B_a, n \in \mathbb{N})$ által generált szorzat-rendszert alkotják, azaz: $L_k^{(a)}(x) = \prod_{n=0}^{\infty} [r_n(B_a(x))]^{k_n} \quad (x \in \mathbb{I},\ k \in \mathbb{N})$.

Az $a \in \mathbb{I}_1$ esetén az $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$-n értelmezzük a B_a-hoz rendelt *aritmetikai diszkrét Laguerre függvényeket* a következőképpen:

$$L_k^{(a)}(x) := v_k(B_a(x)) \ (k \in \mathbb{N}, x \in \mathbb{I}),$$

amelyek a $(v_{2^n} \circ B_a, n \in \mathbb{N})$ által generált szorzat-rendszert alkotják, azaz: $L_k^{(a)}(x) = \prod_{j=0}^{+\infty} [v_{2^j}(B_a(x))]^{k_j} \quad (x \in \mathbb{I},\ k \in \mathbb{N})$.

A megfelelő testen értelmezett $(L_k^{(a)}, k \in \mathbb{N})$ diszkrét Laguerre rendszerek UDMD- szorzatrendszerek, ezért azok teljesek és ortonormáltak.

A 6.4. Részfejezetben az ezen rendszerek szerinti Fourier sorok $(C, 1)$-szummabilitási kérdésére térünk ki. A Schipp[15] és Gát[7] 2-adikus és diadikus klasszikus karakterek szerinti Fourier sorok (C,1)-szummabilitására vonatkozó alapvető eredményeire támaszkodunk. Tekintjük egy $f \in L^1(\mathbb{I})$ függvény diszkrét Laguerre rendszerre vonatkozó $S^{(a)}f$ Laguerre-Fourier sorának $S_n^{(a)}f$ részletösszegeit és a $\sigma_n^{(a)}f$ Laguerre-Cesaro/Fejér közepeit. A karakterek szerinti Fourier sorokra vonatkozó eredményekre támaszkodva megmutattuk mindkét testen, hogy $\lim_{n\to\infty} \sigma_n^{(a)}f(x) = f(x)$ m.m. teljesül minden $f \in L^1(\mathbb{I})$ esetén és $\lim\limits_{n\to\infty} S_n^{(a)}f(x) = f(x)$ *m.m.* teljesül $f \in L^p(\mathbb{I}_1),\ p > 1$ esetén.

A **7. Fejezet** a Malmquist-Takenaka rendszerekkel kapcsolatos eredményeket tartalmazza a vizsgált lokális testeken. Az $(a_i \in \mathbb{I}_1, i \in \mathbb{N})$ bájtokhoz tartozó $p = (a_0, a_1, \ldots)$ paraméterű $(\Psi_k^{(p)}, k \in \mathbb{N})$ *logikai/aritmetikai Malmquist-Takenaka függvényeke*t a következő függvények által generált szorzatrendszereként értelmezzük:

$$(\varphi_{n,a_n} := r_n \circ B_{a_n}, n \in \mathbb{N}) \quad \text{az} \quad (\mathbb{I}, \overset{\circ}{+}, \circ)\text{-en, illetve}$$

$$(\Phi_{n,a_n} := v_{2^n} \circ B_{a_n}, n \in \mathbb{N}) \quad \text{az} \quad (\mathbb{I}, \overset{\bullet}{+}, \bullet)\text{-en.}$$

Világos, hogy a Malmquist-Takenaka rendszerek általánosításai a diszkrét Laguerre rendszereknek: az $a_n = a \in \mathbb{I}_1$ $(n \in \mathbb{N})$ azonos paramétereket használva a $\Psi_n^{(p)}(x)$ Malmquist-Takenaka függvények az $L_n^{(a)}(x)$ diszkrét Laguerre függvényekkel egyenlőek. Mivel ezek UDMD-szorzatrendszerek, teljes ortonormált rendszert alkotnak mindkét testen. Az előbbinek egy fontos következménye, hogy a szerintük vett Fourier sorokra m.m. konvergencia és összegezhetőségi tulajdonságok teljesülnek.

A **8. Fejezet**ben különböző 2-adikus koszinusz és szinusz függvényeket konstruálunk a 2-adikus testen: előbb az $\tilde{\mathbb{S}}$-értékű exponenciális függvények segítségével, majd a 2-adikus additív csoport v_n karakterei felhasználásával:

$$\begin{aligned}
&\cos x := (\zeta(x) \overset{\bullet}{+} \zeta(x^-)) \bullet e_{-1}, \quad \sin x := (\zeta(x) \overset{\bullet}{-} \zeta(x^-)) \bullet e_{-1} \quad (x \in \mathbb{I});\\
&COS_n(x) := \frac{v_n(x) + v_n(x^-)}{2} \qquad (x \in \mathbb{I}, n \in \mathbb{N}),\\
&SIN_n(x) := \frac{v_n(x) - v_n(x^-)}{2i} \qquad (x \in \mathbb{I}, n \in \mathbb{N}).
\end{aligned}$$

Addíciós formulák teljesülnek mindkét féle értelmezés esetén. Meghatározzuk a legbővebb halmazt $\mathbb{S}$-ben, amin a koszinusz függvény bijektív: $\cos : \tilde{\mathbb{S}} \subset \mathbb{S} \to \mathbb{S}^\dagger$. Belátjuk, hogy a $(COS_n, n \in \mathbb{N})$, $(SIN_n, n \in \mathbb{N})$ rendszerek ortogonálisak. Az $x \mapsto \cos((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ és $x \mapsto e_3 \bullet \sin((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ leképezések DMSP-függvények a $\mathbb{S}^\dagger$ halmazon minden $n \in \mathbb{Z}$ esetén. Ezután a Chebyshev polinomok néhány analogonjának értelmezése következik ezen koszinusz és szinusz függvények felhasználásával. Az első- és másodfajú 2-adikus Chebyshev polinomokat a $t_k(x) := v_{2^{k+6}}(\cos[(2k+1)\arccos(x)])$ $(x \in \mathbb{S}^\dagger, k \in \mathbb{N})$ és $u_k(x) := v_{2^{k+3}}(\sin[(2k+1)\arccos(x)])$ $(x \in \mathbb{S}^\dagger, k \in \mathbb{N})$ szorzatrendszereként értelmezzük, vagyis,

$$\begin{aligned}
T_n(x) &:= \prod_{k=0}^{\infty} \left[v_{2^{k+6}}(\cos[(2k+1)\arccos(x)])\right]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}),\\
U_n(x) &:= \prod_{k=0}^{\infty} \left[v_{2^{k+3}}(\sin[(2k+1)\arccos(x)])\right]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}).
\end{aligned}$$

Ekkor $(T_n, n \in \mathbb{N})$ és $(U_n, n \in \mathbb{N})$ UDMD-szorzatrendszerek, tehát teljes ortonormált rendszerek.

A harmadik- és negyedik fajú 2-adikus Chebyshev polinomokat a következőképpen értelmezzük:

$$\overline{T_n}(x) := COS_n[S'(\arccos(S(x))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$
$$\overline{U_n}(x) := SIN_n[S'(\arccos(S(x))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$

majd belátjuk, hogy $(\overline{T_n}, n \in \mathbb{N}), (\overline{U_n}, n \in \mathbb{N})$ ortogonális rendszerek $L^2(\mathbb{I})$-ben.

Appendix

On a generalization

The space $\mathbb{B}$ and its algebraic structure related to the 2-adic (or arithmetical) and 2-series (or logical) addition has some reasonable generalizations: the p-adic field, or the Vilenkin group (presented in [1] and by Hewitt and Ross in [12] pp.106-116 and in its most general form by Gát in [9], see also [16], Appendices 0.7). In this most general case the system and the algebraic structure is generated by a sequence of positive integers $m := (m_k, k \in \mathbb{N})$ such that $m_k \geq 2$. The character system of the Vilenkin group (see [1]), the Vilenkin-like system given in [9], is a common generalization of the presented character systems of the corresponding additive groups. Summability theorems of these systems hold in this most general case. These thoughts would inspire a wide generalization of the Blaschke function and the discrete Laguerre and Malmquist-Takenaka systems, but the above mentioned space with the multiplication (presented in [12], pp.112) yields a field only in the following special cases: the r-adic field if r is a prime power, and the r-series field if r is a prime.

In this case, we consider *the set of bits* $\mathbb{A}_r := \{0, 1, \ldots, r-1\}$, and *the set of bytes*

$$\mathbb{B}_r := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \mathbb{A}_r \text{ and } \lim_{j \to -\infty} a_j = 0\}.$$

Let $\theta = (\cdots, 0, 0, 0, \cdots)$. The *order* of a byte $x \in \mathbb{B}_r$ is defined in the following way: For $x \neq \theta$ let $\pi(x) := n$ if and only if $x_n \neq 0$ and $x_j = 0$ for all $j < n$, furthermore set $\pi(\theta) := +\infty$. The *norm* of a byte x can be introduced by the following rule:

$$\|x\| := r^{-\pi(x)} \text{ for } x \in \mathbb{B}_r \setminus \{\theta\}, \qquad \text{and } \ \|\theta\| := 0.$$

Consider *the r-adic sum* $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z}), b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}_r$, defined by

$$a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$$

where the bits $q_n, s_n \in \mathbb{A}_r$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$\begin{aligned} &q_n = s_n = 0 \quad \text{ for } n < m := \min\{\pi(a), \pi(b)\}, \\ &\text{and } \ a_n + b_n + q_{n-1} = r \cdot q_n + s_n \quad \text{ for } n \geq m. \end{aligned} \tag{8.13}$$

The *r-adic product* of $a, b \in \mathbb{B}_r$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}_r$ $(n \in \mathbb{Z})$ are defined recursively by

$$\begin{aligned} &q_n = p_n = 0 \quad (n < m := \pi(a) + \pi(b)) \\ &\text{and } \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = r \cdot q_n + p_n \quad (n \geq m). \end{aligned} \tag{8.14}$$

Define the *r-series sum* $a \overset{\circ}{+} b$ and *r-series product* $a \circ b$ of elements $a, b \in \mathbb{B}_r$ by

$$\begin{aligned} &a \overset{\circ}{+} b := (a_n + b_n \pmod r), \ n \in \mathbb{Z}) \\ &a \circ b := (c_n, n \in \mathbb{Z}), \quad \text{where } \ c_n := \sum_{k \in \mathbb{Z}} a_k b_{n-k} \pmod r \ \ (n \in \mathbb{Z}). \end{aligned} \tag{8.15}$$

Now, $(\mathbb{B}_r, \overset{\bullet}{+}, \bullet)$ is a non-Archimedian normed field for a prime power r and $(\mathbb{B}_r, \overset{\circ}{+}, \circ)$ is a non-Archimedian normed field for a prime r. For more details see [12], pp.112-113.

The product system of the collection of the function systems

$$\Phi_n := \{\phi_n^k : 0 \leq k < r\}$$

is the set of functions $\{\psi_m : m \in \mathbb{N})$, where to a given $m \in \mathbb{N}$ we have expansion

$$m = \sum_{n=0}^{\infty} m_n r^n \quad (m_n \in \{0, 1, 2, \cdots, r-1\})$$

and the function ψ_m is defined by

$$\psi_m := \prod_{n=0}^{\infty} \phi_n^{m_n}.$$

The topology and the Haar measure is given in the same way, like in the case of 2-adic and 2-series field, and so is the conditional expectation E_n with respect to the σ-algebra $\mathcal{A}_n$ generated by the intervals of rank n for any $n \in \mathbb{N}$.

Let $\mathbb{I} := \{a = (a_j, j \in \mathbb{N}) | \ a_j \in \mathbb{A}_r\}$. The character system $(\Upsilon_m, m \in \mathbb{N})$ of $(\mathbb{I}, \overset{\bullet}{+})$ is now formed by the product system generated by

$$\phi_n(x) := \epsilon\left(\frac{x_n}{r} + \frac{x_{n-1}}{r^2} + \cdots + \frac{x_0}{r^{n+1}}\right) \ (n \in \mathbb{N}),$$

namely to $m \in \mathbb{N}$

$$\Upsilon_m(x) := \prod_{n=0}^{\infty} (\phi_n(x))^{m_n} \quad (x \in \mathbb{I}).$$

The characters of $(\mathbb{I}, \overset{\circ}{+})$ are now the functions of the product system generated by the so-called *generalized Rademacher functions*

$$\phi_n(x) := \epsilon\left(\frac{x_n}{r}\right) \ (n \in \mathbb{N}),$$

namely the system

$$\Upsilon_m^{\circ}(x) = \prod_{n=0}^{\infty} \epsilon\left(\frac{m_n x_n}{r}\right) \ (m \in \mathbb{N}).$$

When the system $(\Upsilon_m, m \in \mathbb{N})$ takes the role of character system $(v_n, n \in \mathbb{N})$ in Paragraphs 6.3 and 7.2, and replacing the Walsh-Paley functions $(w_n, n \in \mathbb{N})$ with the system $(\Upsilon_m^{\circ}, m \in \mathbb{N})$ in Paragraphs 6.2 and 7.2, we obtain the generalized discrete Laguerre and Malmquist-Takenaka systems.

As all the techniques and results show a simple analogy with the presented case, we do not go into details in this subject.

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5. Simon, I.: *Construction of 2-adic Chebyshev polynomials*, submitted.

List of talks of the author

I participated and gave a talk in the following international conferences with the following titles:

1. Discrete Laguerre Functions on Local Fields, *6th Joint Conference on Mathematics and Computer Science,* Pécs, Hungary, July 2006.

2. Characters of the Blaschke-group on the arithmetic field, *7th Joint Conference on Mathematics and Computer Science,* Cluj, Romania, July 2008.

3. Orthogonal series on the arithmetic field, *Workshop on Dyadic Analysis and Related Areas with Applications,* Dobogókő, Hungary, June, 2009.

4. Malmquist-Takenaka functions on local fields, *8th Joint Conference on Mathematics and Computer Science,* Komarno, Slovakia, July, 2010.

5. Transformation with a Blaschke Function, *9th Joint Conference on Mathematics and Computer Science,* Siófok, Hungary, February 2012.

6. Some orthogonal systems on the dyadic and 2-adic field, Workshop titled "Walsh-Fourier sorok approximációs kérdései", Nyíregyháza, Hungary, 27 November 2012.

Lectures were given by the author at workshops of the following universities:

1. Universität Paderborn (Germany), January 2010.
2. University of Camerino (Italy), May 2011.
3. HTW- Hochhscule für Technik und Wirtshaft, Dresden (Germany), May 2012.