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Spectral analysis and moment functions on hypergroups

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Vajday László

Témavezető: Dr. Székelyhidi László

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Spectral analysis and moment functions on hypergroups

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Dr. ....

Dr. ....

Dr. ....

A bírálóbizottság:

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Dr. ....

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Contents

CHAPTER 1

Introduction

The theory of functional equations is one of the classical fields of mathematics. Functional equation problems arose in different areas from the ancient times both in theory and in applications. In 1966 J. Aczél published his book “Lectures on functional equations and their applications” (see [**1**]), which is considered the bible of this theory. There are other important contributions by J. Aczél and J. Dhombres in “Functional Equations Containing Several Variables” and also a basic reference book is due to M. Kuczma [**13**]. A pioneer work of A. Járai (see [**9**]) led to the observation that the strong algebraic character of a functional equation implies important consequences for the analytic behaviour of the solutions. An other stream started in the 90’s with the monograph of László Székelyhidi (see [**25**]) emphasizing and introducing the fundamental role of spectral analysis and spectral synthesis in the theory of a special type of functional equations. In the monograph [**25**] the author offers a general method for the solution of convolution type systems of functional equations. The essence of the method is that first the “basic building blocks” of the solution space of the functional equation should be found – these are the so-called “exponential monomials” – and then – in case of spectral synthesis – the linear combinations of these basic solutions will form a dense set in the solution space that is, they characterize the solution space. It happens the exponential monomial solutions play a very special and important role in the solution process.

It turns out that several ideas of this type can be adopted to a more exciting situation: to the situation of hypergroups. The concept of *DJS-hypergroup* which we shall use here (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) is due to R. Lasser (see e.g. [**5**], [**10**], [**23**]). One can realize a hypergroup like the convolution structure of some measure algebra over a group, but the group structure has been neglected. One can introduce translation operators on hypergroups, which makes it possible to set up a theory of harmonic analysis. Using translation operators on hypergroups a wide range of machinery can be adopted from the group-case. Nevertheless, the classical group-methods can be applied only restrictively: the special situation does not make it possible to "copy" the well-known classical methods. However, there are some function classes, like additive functions, exponential functions and generalized moment functions (or generalized moment function sequences), which play a central role on hypergroups.

Now, we present the heuristic meaning of the concept of hypergroup. This is the following: suppose that a locally compact topological group is given. We consider all complex Radon-measures defined on this topological group. These measures form a $*$-algebra with the convolution as multiplication. Since this mapping is obviously injective, the group can be embedded into the measure algebra as a sub-semigroup. We get the concept of *hypergroup* in the way that at this time we "forget about" the group, we "throw away" the group from the measure-algebra and "forget" that convolution of measures is defined by the multiplication defined on the group. Only the basic properties of the convolution remain and the fact that the measures are defined on a locally compact Haussdorff-space. When we collect these facts in the form of axioms we arrive at the definition of the hypergroup. The theory of hypergroups has interesting and useful several fields of mathematics. Such areas are harmonic analysis, probability theory, orthogonal polynomials, differential equations and boundary value problems. The detailed definition of hypergroups can be found in [**4**].

The appearance of translation operators enables us to utilize an effective method of studying functional equations and systems of functional equations on hypergroups. Namely, it turns out that some of the methods of spectral analysis and spectral synthesis can be adopted and used in the hypergroup-situation.

In the second and third chapter we present the notation, terminology, the definition of hypergroups and basic properties, together with some examples which are in the focus of this PhD dissertation.

In the fourth chapter we present the general form of exponentials and additive functions on some hypergroups. These characterization problems will be presented on the $SU(2)$-hypergroup by a detailed investigation (see [**31**]).

In the fifth chapter we introduce the generalized moment function sequences and prove that on any commutative hypergroup the generalized moment functions are linearly independent. Moreover we present the form of these funcions on the $SU(2)$-hypergroup (see [**35**] and [**31**]).

The sixth chapter contains a detailed investigation related to the linear independence and the translation property of exponential monomials on the Sturm–Liouville hypergroups (see [**36**]).

In the seventh chapter we formulate the problem of spectral analysis on commutative hypergroups and solve it for finite dimensional varieties. This investigation is followed by a useful tool, namely spectral analysis using moment functions (see [**33**] and [**35**]).

The last two chapters contain particular applications. We formulate the uniqueness of a moment problem in the cases of polynomial and Sturm–Liouville hypergroups. Last, but not least, we solve some conditional functional equations and use these results to give the general form of exponentials and additive functions on some two-point support hypergroups (see [**34**] and [**32**]).

The results presented in this PhD dissertation are based on the papers [**36**], [**33**], [**37**], [**34**], [**35**], [**31**] and [**32**] which have been prepared in collaboration with my supervisor, professor László Székelyhidi.

CHAPTER 2

Notation and terminology

In this chapter we present the basic notations and concepts, the definition of hypergroups and some elements of harmonic analysis which can be applied on hypergroups.

1. Terminology

Let X be a locally compact Hausdorff-space and let $\mathcal{C}(X)$ be the set of all complex valued continuous function on X. Let $\mathcal{C}_c(X)$ be the set of compactly supported functions from $\mathcal{C}(X)$. The space $\mathcal{C}_c(X)$ is the union of subspaces

$$\mathcal{C}_c(X,K) = \{f \in \mathcal{C}_c(X) | \operatorname{supp} f \subset K\}\,,$$

where K is a compact subset of X. If we equip these spaces with the uniform convergence topology on compact sets, we get a locally convex topology on $\mathcal{C}_c(X)$, which is the inductive limit of the topologies on the spaces $\mathcal{C}_c(X)$. We say that μ is a *Radon-measure* on X if it is a continuous linear functional of the space $\mathcal{C}_c(X)$. We use the notation

$$\mu(f) = \int_X f(x)\,d\mu(x) = \int_X f\,d\mu,$$

where f is an element of $\mathcal{C}_c(X)$. We denote the set of all complex Radon-measures on X by $\mathcal{M}(X)$ and if μ is an element of $\mathcal{M}(X)$, then

$$||\mu|| = \sup\{\ |\mu(f)|\ \ |f \in \mathcal{C}_c(X),\ ||f||_\infty \leq 1\}.$$

The measure μ in $\mathcal{M}(X)$ is a *bounded measure*, if $||\mu||$ is finite; furthermore μ is a *probability measure* if $\mu \geq 0$ and $||\mu|| = 1$. The set of all bounded and probability measures on X are denoted by $\mathcal{M}^b(X)$, $\mathcal{M}^1(X)$, respectively.

The measure δ_x defined by

$$\delta_x(f) = f(x) \qquad (f \in \mathcal{C}_c(X))$$

for each f in $\mathcal{C}(X)$ and x in X is called the *Dirac measure*, or *point mass* corresponding to x.

Let $\mathcal{K}(X)$ be the set of all nonvoid compact subsets of X and

$$\mathcal{K}_A(B) = \{K \in \mathcal{K}(X) \,|\, K \cap A \neq \emptyset \quad \text{and} \quad K \subset B\} \qquad (A, B \subset X).$$

If we equip the space $\mathcal{K}(X)$ with the *Michael-topology* **[15]**, which has the subbase

$$\{\mathcal{K}_A(B) \,|\, A, B \text{ open subsets of } X\},$$

then the space $\mathcal{K}(X)$ will be a locally compact Hausdorff space. This topology has the property that if the set X is compact, then the space $\mathcal{K}(X)$ is compact, and if the space (X, ρ) is a metric space, then the Michael-topology and the Hausdorff-topology are equivalent **[12]**, the latter being generated by the metric

$$\rho(A, B) = \inf\{r \mid A \subset V_r(B) \quad \text{and} \quad B \subset V_r(A)\},$$

where

$$V_r(A) = \{y \in X \,|\, \exists x \in A : \; \rho(x, y) < r\}.$$

2. The measure algebra of hypergroups

Let K be a locally compact Hausdorff space and we suppose the following:

- H_1 There is a binary operation $*$ on the vector space $\mathcal{M}^b(K)$, with the property that $(\mathcal{M}^b(K), +, *)$ is an algebra (*convolution*).
- H_2 If x, y are in K, then $\delta_x * \delta_y$ is in $\mathcal{M}^1(K)$ and $\operatorname{supp}(\delta_x * \delta_y)$ is compact.
- H_3 The mapping $(x, y) \mapsto \delta_x * \delta_y$ is continuous, where the topology of $\mathcal{M}^1(K)$ is the weak topology induced by $\mathcal{C}_c(K)$.

H_4 The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is continuous, where the topology of $\mathcal{K}(K)$ is the Michael-topology.

H_5 There exists a unique element e in K that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ holds for any x in K (*identity*).

H_6 There exists a homeomorphism $^\vee : K \to K$, such that $(x^\vee)^\vee = x$ holds for any x in K and for x, y in K the element e is in $\operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = y^\vee$ (*involution*).

H_7 For any x, y in K, the property $(\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee}$ holds, where $\mu^\vee$ is defined by

$$\int_K f(x)\, d\mu^\vee(x) = \int_K f(x^\vee)\, d\mu(x) \qquad (f \in \mathcal{C}_c(K)),$$

whenever μ is in $\mathcal{M}^b(K)$.

We say that the quadruple $(K, *, ^\vee, e)$ is a *hypergroup*, the operation $*$ is the *convolution* and the operation $^\vee$ is the *involution*. Sometimes we use the notation $(K, *)$, or simply K for hypergroups. It is important to note that K has no algebraical structure, all properties come from the measure algebra in the way that we identify the members of K by the appropriate Dirac measures.

The set of all boundedly supported measures is dense in the space $\mathcal{M}^b(K)$, hence by the continuity of convolution for every μ, ν in $\mathcal{M}^b(K)$ the convolution of μ and ν is defined by

$$(\mu * \nu)(f) = \int_K \int_K \left(\int_K f(z)\, d(\delta_x * \delta_y)(z) \right) d\mu(x) d\nu(y),$$

which means that the convolution of Dirac measures determines the whole hypergroup.

For instance, let K be a locally compact topological group, $x^\vee = x^{-1}$ and $\delta_x * \delta_y = \delta_{xy}$, then $(K, *, ^\vee)$ is a hypergroup. Indeed, by the associativity of the group operation, the axiom H_1 holds, the identity of the group K is the identity of the hypergroup $(K, *, ^\vee)$, and e is in $\operatorname{supp}(\delta_x * \delta_y) = \{xy\}$ if and only if $x = y^{-1}$, furthermore

$$(\delta_x * \delta_y)^\vee = \delta_{(xy)^{-1}} = \delta_{y^{-1}x^{-1}} = \delta_{y^\vee} * \delta_{x^\vee}.$$

This observation shows that the concept of hypergroup can be considered as a generalization of locally compact topological groups, where the convolution structure of bounded measures is similar to the Banach algebra of measures defined on groups.

The hypergroup K is *commutative* if $(\mathcal{M}^b(K), +, *)$ is a commutative algebra. The hypergroup K is *Hermitian* if the involution on it is the identity. Clearly, any Hermitian hypergroup is commutative, since

$$\delta_x * \delta_y = (\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee} = \delta_y * \delta_x .$$

If K is a discrete Hermitian hypergroup, then the axiom systems of K is the following:

H_1^* There is a binary operation $*$ on the vector space $\mathcal{M}^b(K)$, with the property that $(\mathcal{M}^b(K), +, *)$ is an algebra.

H_2^* If x, y is in K, then $\delta_x * \delta_y$ is in $\mathcal{M}^1(K)$.

H_3^* There exists uniquely such an element e in K that

$$\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$$

holds for any x in K.

H_4^* $\operatorname{supp}(\delta_x * \delta_y)$ is finite for any x, y belongs to K.

H_5^* For x, y in K, the property e is in $\operatorname{supp}(\delta_x * \delta_y)$ holds, if and only if $x = y$.

With the help of convolution we can introduce the notion of translation. For f in $\mathcal{C}(K)$ and x, y in K the *left (right) translate* of f by y is defined by

$$\tau_L^y f(x) = \int_K f(t)\ d(\delta_x * \delta_y)(t), \qquad \tau_R^y f(x) = \int_K f(t)\ d(\delta_y * \delta_x)(t),$$

respectively. We shall use the following suggestive notation for the *translation operator* τ:

$$f(x * y) = \tau_y\ f(x) = \int_K f(t)\, d(\delta_x * \delta_y)(t)\,.$$

Let f in $\mathcal{C}(K)$ and μ in $\mathcal{M}^b(K)$, then the *convolution of f and μ* is defined by

$$(f * \mu)(x) = \int_K f(x * y^\vee)\ d\mu(y) \qquad (x \in K),$$

and

$$(\mu * f)(x) = \int_K f(y^\vee * x)\, d\mu(y) \qquad (x \in K).$$

We say that the positive measure ω is a *left invariant measure* or, in other words, *left invariant Haar measure*, if

$$\int_K \tau_y\, f(x)\, d\omega(x) = \int_K f(x)\, d\omega(x) \qquad (y \in K),$$

holds for any f in $\mathcal{C}_c(K)$, which means that $\omega(\delta_y * f) = \omega(f)$ for any y in K. The right invariant (Haar) measures can be defined in a similar way.

There is no existence result for (left) invariant measures on hypergroups, but in some special important cases, the existence of these measures has been proved. If there exist Haar measures on a hypergroup, then they are unique up to a constant factor [**10**]. For every commutative hypergroup there exists Haar measure [**23**]. There exists Haar measure on every commutative and on every compact hypergroup [**10**].

On discrete hypergroups there exist left and right invariant measures [**10**], but in contrast with the case of groups not every discrete hypergroup is unimodular [**11**] that is, they are not necessarily identical. Adetailed investigation of invariant measures can be found in [**4**].

If there exists an invariant measure ω on a hypergroup, we can define the convolution of ω-integrable functions f and g, where f, g are in $\mathcal{C}(K)$, with the formula

$$(f * g)(y) = \int_K \tau_y\, f(x) g(x^\vee)\, d\omega(x) \qquad (y \in K).$$

We say that the function χ in $\mathcal{C}(K)$ is a *semi-character* on the hypergroup $(K, *, ^\vee)$, if it is not identically zero and

$$\chi(x * y) = \chi(x)\chi(y), \qquad \chi(x^\vee) = \overline{\chi(x)} \qquad (x, y \in K)$$

hold. The set of all bounded semi-charecters of the hypergroup K is denoted by $K^\wedge$. The members of $K^\wedge$ are the *characters*.

The *Fourier-transform of the measure* μ from $\mathcal{M}_b(K)$ is the function $\overset{\wedge}{\mu} : K^\wedge \to \mathbb{C}$ for which

$$\overset{\wedge}{\mu}(\chi) = \int_K \overline{\chi(x)}\, d\mu(x),$$

where χ is in $K^\wedge$ is an arbitrary character. The property

$$(\mu * \nu)^\wedge(\chi) = \overset{\wedge}{\mu}(\chi)\overset{\wedge}{\nu}(\chi) \qquad (\chi \in K^\wedge)$$

holds for any μ, ν in $\mathcal{M}_b(K)$.

In the case of compactly supported measures the Fourier-transformation can be extended to the set of semi-characters, this is the *Fourier–Laplace-transformation.*

Let the function f be integrable with respect to the Haar measure ω. The Fourier transform of the function f is $\hat{f} : K^\wedge \to \mathbb{C}$, where

$$\hat{f}(\chi) = \int_K f(x)\overline{\chi(x)}\, d\omega(x) \qquad (\chi \in K^\wedge),$$

furthermore, if the functions f and g are ω-integrable, then

$$(f * g)^\wedge(\chi) = \hat{f}(\chi)\hat{g}(\chi) \qquad (\chi \in K^\wedge).$$

The generalization of harmonic analysis to hypergroups can be found in [**10**] and [**4**].

CHAPTER 3

Hypergroups and examples

1. Polynomial hypergroups

An important special class of Hermitian hypergroups is closely related to polynomials.

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be real sequences with the following properties: $c_n > 0$, $b_n \geq 0$, $a_{n+1} > 0$ for all n in $\mathbb{N}$, moreover $a_0 = 0$, and $a_n + b_n + c_n = 1$ for all n in $\mathbb{N}$. We define the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ by $P_0(x) = 1$, $P_1(x) = x$, and by the recursive formula

$$xP_n(x) = a_n P_{n-1}(x) + b_n P_n(x) + c_n P_{n+1}(x)$$

for all $n \geq 1$ and x in $\mathbb{R}$. The following theorem holds.

THEOREM 3.1. *If the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ satisfies the conditions above, then there exist constants $c(n, m, k)$ for all n, m, k in $\mathbb{N}$ such that*

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k$$

holds for all n, m in $\mathbb{N}$.

PROOF. By the theorem of Favard (see **[6]**) the conditions on the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ imply that there exists a probability measure μ on $[-1, 1]$ such that $(P_n)_{n\in\mathbb{N}}$ forms an orthogonal system on

$[-1, 1]$ with respect to μ. As P_n has degree n, we have

$$P_n P_m = \sum_{k=0}^{n+m} c(n, m, k) P_k$$

for all n, m in $\mathbb{N}$, where

$$c(n, m, k) = \frac{\int_{-1}^{1} P_k P_n P_m \, d\mu}{\int_{-1}^{1} P_k^2 \, d\mu}$$

holds for all n, m, k in $\mathbb{N}$. The orthogonality of $(P_n)_{n\in\mathbb{N}}$ with respect to μ implies $c(n, m, k) = 0$ for $k > n + m$ or $n > m + k$ or $m > n + k$. Hence our statement is proved. □

The formula in the theorem is called *linearization formula*, and the coefficients $c(n, m, k)$ are called *linearization coefficients*. The recursive formula for the sequence $(P_n)_{n\in\mathbb{N}}$ implies $P_n(1) = 1$ for all n in $\mathbb{N}$, hence we have

$$\sum_{k=|n-m|}^{n+m} c(n, m, k) = 1$$

for all n in $\mathbb{N}$. It may or may not happen that $c(n, m, k) \geq 0$ for all n, m, k in $\mathbb{N}$. If it happens, then we can define a hypergroup structure on $\mathbb{N}$ by the following rule:

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) \delta_k$$

for all n, m in $\mathbb{N}$, with involution as the identity mapping and with e as 0. The resulting discrete Hermitian (hence commutative) hypergroup is called *the polynomial hypergroup associated with the sequence* $(P_n)_{n\in\mathbb{N}}$ (see **[17]**).

As an example we consider the hypergroup associated with the Legendre-polynomials. The corresponding recurrence relation is

$$xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$$

for all $n \geq 1$ and x in $\mathbb{R}$. It can be seen easily that the linearization coefficients are nonnegative, and the resulting hypergroup associated with the Legendre-polynomials is the *Legendre-hypergroup.*

2. Sturm–Liouville hypergroups

Sturm–Liouville hypergroups represent another important class of hypergroups, which arise from Sturm–Liouville boundary value problems on nonnegative reals. In order to build up the Sturm–Liouville operator basic to the construction of hypergroups, one introduces the Sturm–Liouville functions. Let $\mathbb{R}_0 = [0, +\infty[$.

The continuous function $A : \mathbb{R}_0 \to \mathbb{R}$ is called a *Sturm–Liouville function*, if it is positive and continuously differentiable on the positive reals. For a given Sturm–Liouville function A one defines the *Sturm–Liouville operator* L_A by

$$L_A\, f = -f'' - \frac{A'}{A} f',$$

where f is a twice continuously differentiable real function on the positive reals. Using L_A one introduces the differential operator l by

$$l[u](x,y) = (L_A)_x u(x,y) - (L_A)_y u(x,y) =$$

$$= -\partial_1^2 u(x,y) - \frac{A'(x)}{A(x)}\, \partial_1 u(x,y) + \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\, \partial_2 u(x,y),$$

where u is twice continuously differentiable on positive reals.

A hypergroup on $\mathbb{R}_0$ is called *Sturm–Liouville hypergroup*, if there exists a Sturm–Liouville function A such that given any real-valued C^∞-function f on $\mathbb{R}_0$ the function u_f defined by

$$u_f(x,y) = f(x * y) = \int_{\mathbb{R}_0} f\, d(\delta_x * \delta_y)$$

for all positive x, y is twice continuously differentiable and satisfies the partial differential equation

$$l[u_f] = 0$$

with $\partial_2 u_f(x,0) = 0$ for all positive x. Hence u_f is a solution of the Cauchy-problem

$$\partial_1^2 u(x,y) + \frac{A'(x)}{A(x)}\,\partial_1 u(x,y) = \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\,\partial_2 u(x,y),$$
$$\partial_2 u_f(x,0) = 0$$

for all positive x, y. In other words, u_f is the unique solution of the boundary value problem

$$\partial_1^2 u_f(x,y) + \frac{A'(x)}{A(x)}\,\partial_1 u_f(x,y) = \partial_2^2 u_f(x,y) + \frac{A'(y)}{A(y)}\,\partial_2 u_f(x,y),$$
$$\partial_1 u_f(0,y) = 0, \qquad \partial_2 u_f(x,0) = 0,$$
$$u_f(x,0) = f(x), \qquad u_f(0,y) = f(y),$$

for all positive x, y. As this boundary value problem uniquely defines u_f for any f, we may consider it the boundary value problem defining the Sturm–Liouville hypergroup (see [**4**] and [**29**]).

3. The $SU(2)$-hypergroup

In this section we present the definion of the $SU(2)$-hypergroup which is related to the set of continuous unitary irreducible representations of the group $G = SU(2)$, the *special linear group* in two dimensions. The definition of the underlying hypergroup is taken from [**4**].

If G is a compact topological group, then its dual object $\widehat{G}$ consists of equivalence classes of continuous irreducible representations of G. For any two classes U, V of this type their tensor product can be decomposed into its irreducible components $U_1, U_2, \dots, U_n$ with the respective multiplicities $m_1, m_2, \dots, m_n$. We define convolution on $\widehat{G}$ by

$$\delta_U * \delta_V = \sum_{i=1}^{n} \frac{m_i\, d(U_i)}{d(U)\, d(V)}\, \delta_{U_i} \tag{3.1}$$

where $d(U)$ denotes the dimension of U and δ_U is the Dirac measure concentrated at U. Then $\widehat{G}$ with this convolution and with the discrete topology is a commutative hypergroup.

In the special case of $G = SU(2)$ the dual object $\widehat{G}$ can be identified with the set $\mathbb{N}$ of natural numbers as it is indicated in [**4**]: the set of equivalence classes of continuous unitary irreducible representations of $SU(2)$ is given by $\{T^{(0)}, T^{(1)}, T^{(2)}, \dots\}$, where $T^{(n)}$ has dimension $n+1$, and we identify this set with $\mathbb{N}$.

For every m, n in $\mathbb{N}$ the tensor product of $T^{(m)}$ and $T^{(n)}$ is unitary equivalent to

$$T^{(|m-n|)} \bigoplus T^{(|m-n|+2)} \bigoplus \cdots \bigoplus T^{(m+n)} . \tag{3.2}$$

The convolution is given by

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)} \delta_k , \tag{3.3}$$

where the dash denotes that every second term appears in the sum, only. With this convolution $\mathbb{N}$ becomes a discrete commutative hypergroup, and since all the $T^{(n)}$ are self-conjugate, the hypergroup is in fact Hermitian. We call this hypergroup the $SU(2)$-*hypergroup.*

4. Two-point support hypergroups

Here we present some other examples for hypergroups.

4.1. The $K_1 = ([0,1], *)$ hypergroup.

Let K_1 be the hypergroup on the interval $[0,1]$ with the convolution defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}.$$

This is a one-dimensional compact hypergroup (see [**4**], Example 3.4.6 on p.191.).

4.2. The $K_2 = ([0, +\infty[, *)$ hypergroup.

The hypergroup K_2 is defined on the nonnegative reals $[0, +\infty[$ and the convolution is defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y} \qquad (0 \le y < x).$$

This hypergroup is a noncompact one-dimensional hypergroup (see [**4**], Example 3.4.5 on p. 191.).

4.3. The cosh-hypergroup.

Using the function cosh, we can build up a Sturm–Liouville hypergroup on the nonnegative reals, called the *cosh-hypergroup*. In this case the Sturm–Liouville function will be the function $x \mapsto \cosh^2(x)$.

Another way to introduce the cosh-hypergroup is the following. We consider the nonnegative reals as a base set and we introduce the convolution with the formula

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}\,.$$

This hypergroup is also a special two-point support hypergroup, which is actually identical with the cosh-hypergroup (see [**4**]). We denote this hypergroup by $K_3 = (\mathbb{R}_0, *, \cosh)$.

CHAPTER 4

Exponential and additive functions on hypergroups

In the case of commutative groups exponential polynomials play a fundamental role in several problems concerning functional equations. As exponential polynomials are built up from additive and exponential functions, which are closely related to translation operators, the presence of translation operators on hypergroups makes it possible to define these basic functions on hypergroups too.

Let K be a hypergroup with convolution $*$, involution $^\vee$ and identity$e$. For any y in K let τ_y denote the right translation operator on the space of all complex valued functions on K which are integrable with respect to $\delta_x * \delta_y$ for any x, y in K. In particular, any continuous complex valued function belongs to this class. We call the continuous complex valued function a on K *additive*, if it satisfies

$$\tau_y\, a(x) = a(x) + a(y)$$

for all x, y in K. In more details this means that

$$\int_K a(t)\, d(\delta_x * \delta_y)(t) = a(x) + a(y)$$

holds for any x, y in K. The continuous complex valued function m on K is called an *exponential*, if it is not identically zero, and

$$\tau_y\, m(x) = m(x)m(y)$$

holds for all x, y in K. In other words m satisfies the functional equation

$$\int_K m(t)\, d(\delta_x * \delta_y)(t) = m(x)m(y).$$

It is obvious that any linear combination of additive functions is additive again. However, in contrast with the case of groups, the product of exponentials is not necessarily an exponential. The bounded exponential m is called a *character*, if $m(x^\vee) = \overline{m(x)}$ holds for any x in K. Obviously $a(e) = 0$ for any additive function a, and $m(e) = 1$ for any exponential m.

An *exponential monomial* on a locally compact Abelian group G is a function of in the form

$$x \mapsto P[a_1(x), a_2(x), \ldots, a_n(x)]m(x),$$

where n is a nonnegative integer, $a_1, a_2, \ldots, a_n : G \to \mathbb{C}$ are additive functions, m is an exponential and P is a complex polynomial. In the case $n = 0$ we consider this function to be identically equal to m. A linear combination of exponential monomials is called an *exponential polynomial.* A product of additive functions is called a *monomial*, and a product of monomials we call a *polynomial.*

If we want to introduce these concepts on commutative or arbitrary hypergroups, we have to remember the fact that product of exponentials is not necessarily an exponential. On the other hand, in case of commutative groups exponential polynomials can be characterized by the fact that the linear space of functions spanned by the translates is finite dimensional. This property is of fundamental importance from the point of view of spectral synthesis. Hence it seems to be reasonable to define exponential polynomials by this property, even on arbitrary - not necessarily commutative - hypergroups. Anyway, additive and exponential functions on hypergroups obviously have this property and it seems to be interesting to describe these function classes on different hypergroups.

In what follows, we present the general form of exponentials and additive functions on polynomial hypergroups in a single variable and on Sturm–Liouville hypergroups.

1. The case of polynomial hypergroups

Let $K = (\mathbb{N}, P_n)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. Now we describe all exponential functions defined on K (see also [**4**]).

THEOREM 4.1. *Let $K = (\mathbb{N}, P_n)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $\varphi : \mathbb{N} \to \mathbb{C}$ is an exponential on K if and only if there exists a complex number λ such that*

$$\varphi(n) = P_n(\lambda) \tag{4.1}$$

holds for all n in $\mathbb{N}$.

Next theorem describes exponentials on the hypergroup $K = (\mathbb{N}, P_n)$.

THEOREM 4.2. *Let $K = (\mathbb{N}, P_n)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $a : \mathbb{N} \to \mathbb{C}$ is an additive function on K if and only if there exists a complex number c such that*

$$a(n) = cP_n'(1) \tag{4.2}$$

holds for all n in $\mathbb{N}$.

2. The case of Sturm–Liouville hypergroups

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup. Now we describe all exponentials and additive functions defined on K (see also [**29**]).

THEOREM 4.3. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. Then the continuous function $m : \mathbb{R}_0 \to \mathbb{C}$ is an exponential on K if and only if it is C^∞ and there exists a complex number λ such that*

$$m''(x) + \frac{A'(x)}{A(x)}\, m'(x) = \lambda m(x), \qquad m(0) = 1, \quad m'(0) = 0, \tag{4.3}$$

holds for any positive x.

Hence any exponential function on a Sturm–Liouville hypergroup is an eigenfunction of the Sturm–Liouville operator corresponding to the given hypergroup. Each complex number is an eigenvalue and there is a one-to-one correspondence between complex numbers and exponentials. For any fixed complex λ we shall denote by $x \mapsto \varphi(x, \lambda)$ the unique solution of the boundary value problem. Then the function

$$\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$$

represents a one-parameter family of exponentials of the Sturm-Liouville hypergroup K, which is called *exponential family* of K. We obviously have

$$\partial_1^2 \varphi(x, \lambda) + \frac{A'(x)}{A(x)}\, \partial_1\, \varphi(x, \lambda) = \lambda \varphi(x, \lambda)$$
$$\varphi(0, \lambda) = 1, \qquad \partial_1\, \varphi(0, \lambda) = 0$$

that holds for each positive x.

For instance, the complex number $\lambda = 0$ corresponds to the eigenvalue problem

$$m''(x) + \frac{A'(x)}{A(x)}\, m'(x) = 0, \quad m(0) = 1, \quad m'(0) = 0,$$

which obviously has the unique solution $m \equiv 1$, hence $\varphi(x, 0) = 1$ for each x in $\mathbb{R}_0$.

The next theorem describes additive functions on Sturm–Liouville hypergroups.

THEOREM 4.4. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. Then the continuous function $a : \mathbb{R}_0 \to \mathbb{C}$ is an additive function on K if and only if it is C^∞ and there exists a complex number λ such that*

$$a''(x) + \frac{A'(x)}{A(x)}\, a'(x) = \lambda, \qquad a(0) = 0, \quad m'(0) = 0, \tag{4.4}$$

holds for any positive x.

We get easily that the unique solution a_λ of the boundary value problem (4.4) is λa_1, where a_1 is the uniqe solution of (4.4) with $\lambda = 1$. This implies that all additive functions of a Sturm–Liouville hypergroup are constant multiples of a fixed nonzero additive function. The function

a_1 is called *the generating additive function* of the given Sturm–Liouville hypergroup. The next theorem shows that the boundary value problem (4.4) can be solved explicitely.

THEOREM 4.5. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. Then the generating additive function of the hypergroup K is given by*

$$a_1(x) = \int_0^x \int_0^y \frac{A(t)}{A(y)}\, dt\ dy \tag{4.5}$$

for each nonnegative x. Hence any additive function of the hypergroup K is given by

$$a_\lambda(x) = \lambda \int_0^x \int_0^y \frac{A(t)}{A(y)}\, dt\ dy \tag{4.6}$$

for each nonnegative x, where λ is an arbitrary complex number.

3. The case of the $SU(2)$-hypergroup

In this section we describe the exponential and additive functions on the $SU(2)$-hypergroup. We recall that the function $M : \mathbb{N} \to \mathbb{C}$ is an exponential if and only if it satisfies

$$M(m)M(n) = M(m * n) = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)} M(k) \tag{4.7}$$

for all natural numbers m, n. Here and everywhere "dash" means that each second term appears in the sum, only.

THEOREM 4.6. *The function $M : \mathbb{N} \to \mathbb{C}$ is an exponential on the $SU(2)$-hypergroup if and only if there exists a complex number λ such that*

$$M(n) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda} \tag{4.8}$$

holds for each natural number n. (Here $\lambda = 0$ corresponds to the exponential $M = 1$.)

Proof. Let $M:\mathbb{N}\to\mathbb{C}$ be a solution of (4.7) and let

$$f(n)=(n+1)M(n)$$

for each n in $\mathbb{N}$. Then we have

$$f(m)f(n)=\sum_{k=|m-n|}^{m+n}{}'\,f(k)$$

for each m,n in $\mathbb{N}$. With $m=1$ it follows that f satisfies the following second order homogeneous linear difference equation

$$f(n+2)-f(1)f(n+1)+f(n)=0 \tag{4.9}$$

for each n in $\mathbb{N}$ with $f(0)=1$.

Suppose that $f(1)=2$. Then from (4.9) we infer that $f(n)=n+1$ and $M=1$ which corresponds to the case $\lambda=1$ in (4.8). Otherwise $f(1)\neq 2$ and let $\lambda\neq 0$ be a complex number with $f(1)=2\cosh\lambda$. Then we have that

$$f(n)=\alpha e^{n\lambda}+\beta e^{-n\lambda}$$

holds for any n in $\mathbb{N}$ with some complex numbers α,β satisfying $\alpha+\beta=1$. It is easy to see that in this case

$$f(n)=\frac{\sinh[(n+1)\lambda]}{\sinh\lambda}$$

holds for each n in $\mathbb{N}$. Finally, we have

$$M(n)=\frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda}\,.$$

Conversely, it is easy to check that any function M of the given form is an exponential on the $SU(2)$-hypergroup, hence the theorem is proved. □

Now we describe the additive functions on the $SU(2)$-hypergroup. We recall that the function $A:\mathbb{N}\to\mathbb{C}$ is an additive function if and only if it satisfies

$$A(m)+A(n)=A(m*n)=\sum_{k=|m-n|}^{m+n}{}'\,\frac{k+1}{(m+1)(n+1)}\,A(k) \tag{4.10}$$

for all natural numbers m,n.

THEOREM 4.7. *The function $A:\mathbb{N}\to\mathbb{C}$ is an additive function on the $SU(2)$-hypergroup if and only if there exists a complex number c such that*

$$A(n)=\frac{c}{3}n(n+2) \tag{4.11}$$

holds for each natural number n.

PROOF. Let $A:\mathbb{N}\to\mathbb{C}$ be a solution of (4.10) and

$$f(n)=(n+1)A(n)$$

for each n in $\mathbb{N}$. Then we have

$$(n+1)f(m)+(m+1)f(n)=\sum_{k=|m-n|}^{m+n}{}'\, f(k)$$

for each m,n in $\mathbb{N}$. With $m=1$ it follows that f satisfies the following second order homogeneous linear difference equation

$$f(n+2)-2f(n+1)+f(n)=2c(n+2)$$

for each n in $\mathbb{N}$ with $f(0)=0$ and $f(1)=2c$. As the second difference of f is linear it follows that f is a cubic polynomial and simple computation gives that A has the desired form.

Conversely, it is easy to check that any function A of the given form is an additive function on the $SU(2)$-hypergroup, hence the theorem is proved. □

CHAPTER 5

Generalized moment function sequences

After the classical function classes the next important class of functions is the class of moment functions. For any nonnegative integer n the complex valued continuous function φ on the hypergroup K is called a *generalized moment function of order* n, if there are complex valued continuous functions $\varphi_k : K \to \mathbb{C}$ for $k = 0, 1, \ldots, n$ such that $\varphi_0 \neq 0$, $\varphi_n = \varphi$ and

$$\varphi_k(x * y) = \sum_{i=0}^{k} \binom{k}{i} \varphi_i(x)\varphi_{k-i}(y) \tag{5.1}$$

holds for $k = 0, 1, \ldots, n$ and for all x, y in K. In this case we say that the functions φ_k ($k = 0, 1, \ldots, n$) form a *generalized moment function sequence of order* n. For more about generalized moment function sequences see **[38]**, **[17]**, **[16]**, **[18]**.

For instance, if $\varphi_0 = 1$, then the moment functions of order 1 are the additive functions.

In the following subsection we show that nonzero generalized moment functions are linearly independent on any commutative hypergroups.

1. Linear independence of generalized moment functions

THEOREM 5.1. *Let K be a commutative hypergroup, $n \geq 1$ an integer and $(\varphi_k)_{k=0}^n$ a sequence of generalized moment functions with $\varphi_1 \neq 0$. Then the generalized moment function φ_n is not the linear combination of the generalized moment functions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$.*

PROOF. We prove our statement by induction on n. Let first $n = 1$ and suppose that $\varphi_1 = \lambda \varphi_0$ with some nonzero complex λ. Then, by (5.1), it follows

$$\varphi_0(x)\varphi_0(y) = \varphi_0(x * y) = \frac{1}{\lambda}\,\varphi_1(x * y) =$$

$$\frac{1}{\lambda}\,\varphi_1(x)\varphi_0(y) + \frac{1}{\lambda}\,\varphi_0(x)\varphi_1(y) = 2\,\varphi_0(x)\varphi_0(y)\,,$$

a contradiction.

Now let $n \geq 2$ be any integer and suppose that we have proved our statement for all integers not greater than n. For $n+1$ we suppose the contrary that is, that there are complex numbers c_i $(i = 0, 1, \ldots, n)$ such that

$$\varphi_{n+1}(x * y) = \sum_{i=0}^{n} c_i \varphi_i(x * y) \tag{5.2}$$

holds for each x, y in K. By (5.1) we have

$$\sum_{j=0}^{n+1} \binom{n+1}{j} \varphi_j(x)\varphi_{n+1-j}(y) = \sum_{i=0}^{n}\sum_{j=0}^{i} c_i \binom{i}{j} \varphi_j(x)\varphi_{i-j}(y) \tag{5.3}$$

for each x, y from K. Using (5.1), (5.2) and reordering the sum on the right hand side after simplification we get

$$\sum_{j=0}^{n} \Big[\binom{n+1}{j} \varphi_{n+1-j}(y) + c_j \varphi_0(y) - \sum_{i=j}^{n} c_i \binom{i}{j} \varphi_{i-j}(y)\Big] \varphi_j(x) = 0\,. \tag{5.4}$$

By our assumption, the coefficient of φ_n must be zero for each y in K that is

$$(n+1)\varphi_1(y) = 0\,,$$

which is impossible. The theorem is proved. □

This result has the following consequence.

THEOREM 5.2. *Let K be a commutative hypergroup, $n \geq 1$ an integer and $(\varphi_k)_{k=0}^n$ a sequence of generalized moment functions with $\varphi_1 \neq 0$. Then the functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ are linearly independent. In particular, none of them is identically zero.*

2. Generalized moment functions on the $SU(2)$-hypergroup

Here we describe the generalized moment functions on the $SU(2)$-hypergroup. Let N be a nonnegative integer. Making use of the results in Section 3 we introduce the function

$$\Phi(n, \lambda) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda} \tag{5.5}$$

for each n in $\mathbb{N}$ and $\lambda \neq 0$ in $\mathbb{C}$, while $\Phi(n, 0) = 1$ for each n in $\mathbb{N}$. The function $\Phi : \mathbb{N} \times \mathbb{C} \to \mathbb{C}$ is an *exponential family* for the $SU(2)$-hypergroup: each exponential on this hypergroup has the form

$$n \mapsto \Phi(n, \lambda)$$

with some unique λ in $\mathbb{C}$, and, conversely, the function $n \mapsto \Phi(n, \lambda)$ is an exponential on the $SU(2)$-hypergroup for every complex λ.

THEOREM 5.3. *Let K denote the $SU(2)$-hypergroup and Φ the exponential family given by (5.5). The functions $\varphi_0, \varphi_1, ..., \varphi_N : K \to \mathbb{C}$ form a generalized moment sequence of order N on K if and only if there exist complex numbers c_j for $j = 1, 2, \ldots, N$ such that*

$$\varphi_k(n) = \frac{d^k}{dt^k}\Phi(n, f(t))(0) \tag{5.6}$$

holds for each n in $\mathbb{N}$ and for $k = 0, 1, \ldots, N$, where

$$f(t) = \sum_{j=0}^{N} \frac{c_j}{j!} t^j$$

for each t in $\mathbb{C}$.

PROOF. First we note that, by (3.3), we have for $n \geq 1$

$$\delta_n * \delta_1 = {\sum_{k=n-1}^{n+1}}' \frac{k+1}{2(n+1)} \delta_k = \frac{n}{2(n+1)} \delta_{n-1} + \frac{n+2}{2(n+1)} \delta_{n+1}\,, \tag{5.7}$$

hence, by 3.2.1 Proposition in [**4**], K is a polynomial hypergroup that is, there exists a sequence $(P_n)_{n \in \mathbb{N}}$ of polynomials such that $\deg P_n = n$ for $n = 0, 1, \ldots,$ there exists an x_0 in $\mathbb{R}$ such that $P_n(x_0) = 1$ for $n = 0, 1, \ldots,$ and

$$P_n(x) P_m(x) = \sum_{k=0}^{\infty} c(m,n,k) P_k(x) \tag{5.8}$$

holds for each x in $\mathbb{R}$ and m, n in $\mathbb{N}$ with some nonnegative numbers $c(m,n,k)$, further we have

$$\delta_m * \delta_n = \sum_{k=0}^{\infty} c(m,n,k) \delta_k \tag{5.9}$$

for each m, n in $\mathbb{N}$. Here we shall determine this sequence of polynomials.

Our basic observation is that the function $\lambda \mapsto \Phi(n, \lambda)$ is a polynomial of $\cosh \lambda$ of degree n for each n in $\mathbb{N}$. We prove by induction. For $n = 0$ and $n = 1$ we have by (5.5)

$$\Phi(0, \lambda) = \frac{\sinh \lambda}{\sinh \lambda} = 1\,,$$

$$\Phi(1, \lambda) = \frac{\sinh(2\lambda)}{2 \sinh \lambda} = \cosh \lambda\,.$$

Suppose that for $k = 0, 1, \ldots, n$ there exists a polynomial P_k of degree k such that

$$\Phi(k, \lambda) = P_k(\cosh \lambda) \tag{5.10}$$

holds. Clearly $P_0(x) = 1$ and $P_1(x) = x$. Then, by equation (5.7), we have

$$P_n(\cosh \lambda) \cosh \lambda = \frac{n}{2(n+1)} P_{n-1}(\cosh \lambda) + \frac{n+2}{2(n+1)} \Phi(n+1, \lambda)\,, \tag{5.11}$$

that is,

$$\Phi(n+1, \lambda) = \frac{2(n+1)}{n+2} P_n(\cosh \lambda) \cosh \lambda - \frac{n}{n+2} P_{n-1}(\cosh \lambda)\,, \tag{5.12}$$

and here the right hand side is a polynomial of degree $n+1$ in $\cosh\lambda$:

$$P_{n+1}(x) = \frac{2(n+1)}{n+2} x P_n(x) - \frac{n}{n+2} P_{n-1}(x)\,,$$

hence

$$\Phi(n+1,\lambda) = P_{n+1}(\cosh\lambda)\,,$$

which was to be proved.

Finally, we have for all m, n in $\mathbb{N}$ and λ in $\mathbb{C}$

$$P_n(\cosh\lambda)P_m(\cosh\lambda) = \Phi(n,\lambda)\Phi(m,\lambda) = \Phi(n*m,\lambda) =$$

$$= \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}\,\Phi(k,\lambda) = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}\,P_k(\cosh\lambda)\,,$$

which implies

$$P_n(x)P_m(x) = \sum_{k=|m-n|}^{m+n}{}' \frac{k+1}{(m+1)(n+1)}\,P_k(x)$$

for each x in $\mathbb{R}$ and m, n in $\mathbb{N}$. This means that K is the polynomial hypergroup associated to the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. Then, by Theorem 4. in [**17**], our statement follows. □

Actually, we have proved the important fact that the $SU(2)$-hypergroup can be transformed into a polynomial hypergroup.

CHAPTER 6

Exponential monomials on Sturm–Liouville hypergroups

Here we define exponential monomials on Sturm–Liouville hypergroups and we prove a special linear independence property of them.

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup. We recall that the continuous function $m : \mathbb{R}_0 \to \mathbb{C}$ is an exponential on K if and only if it is C^∞ on the positive reals and there exists a complex number λ such that

$$m''(x) + \frac{A'(x)}{A(x)} m'(x) = \lambda m(x), \qquad m(0) = 1, \qquad m'(0) = 0 \tag{6.1}$$

holds for any positive x. Exponential functions satisfy Cauchy's functional equation

$$m(x * y) = m(x)m(y) \tag{6.2}$$

for all x, y in K. It is obvious that we can define an exponential family $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ with the property that the function $x \mapsto \varphi(x, \lambda)$ is an exponential of K for each complex λ, and for each exponential m of K there exists a unique complex λ such that $m(x) = \varphi(x, \lambda)$ holds for every x in $\mathbb{R}_0$. Hence the exponential family satisfies

$$\partial_1^2 \varphi(x, \lambda) + p(x) \partial_1 \varphi(x, \lambda) = \lambda \varphi(x, \lambda), \tag{6.3}$$

$$\varphi(0, \lambda) = 1, \qquad \partial_1 \varphi(0, \lambda) = 0$$

for each x in $\mathbb{R}_+$ and complex number λ, where $p(x) = \frac{A'(x)}{A(x)}$. Actually, (6.3) characterizes the exponential family. Clearly φ is C^∞ on $\mathbb{R}_0$ in x and entire in λ.

Using the exponential family we define *exponential monomials* on K as functions of the form $x \mapsto P(\partial_2)\varphi(x,\lambda)$, where P is a complex polynomial and λ is a complex number. The meaning of $P(\partial_2)$ is obvious. In particular, if $P \equiv 1$, then we have that any exponential function is an exponential monomial. Observe that this is an analoguous concept to the "exponential monomial" on polynomial hypergroups in several variables in [**27**] and [**30**]. Sums of exponential monomials are called *exponential polynomials.*

1. Linear independence of exponential monomials

A particular subclass of exponential monomials is formed by the functions of the type $x \mapsto \partial_2^k\varphi(x,\lambda)$, where k is a nonnegative integer and λ is a complex number. Here we note that if $\lambda = 0$, then $\varphi(x,0) = 1$ for each x in $\mathbb{R}_0$, hence the corresponding function $x \mapsto \partial_2^k\varphi(x,0)$ is identically 1 for $k = 0$, and it is identically 0 for $k > 0$. For the sake of simplicity we will call the functions $x \mapsto \partial_2^k\varphi(x,\lambda)$ *special exponential monomials* if k is a nonnegative integer and λ is a complex number, supposing that if $\lambda = 0$, then $k = 0$. Our aim is to show that different special exponential monomials are linearly independent.

First we show that different exponential functions are linearly independent.

THEOREM 6.1. *On any hypergroup different exponentials are linearly independent.*

PROOF. Let $m_1, m_2, \dots, m_n$ be different exponentials on the hypergroup K. We prove by induction on n. For $n = 1$ the statement is trivial. Suppose that $n > 1$ and

$$c_1m_1(t) + c_2m_2(t) + \cdots + c_{n-1}m_{n-1}(t) + c_nm_n(t) = 0 \tag{6.4}$$

holds for each t in K. Let x, y be arbitrary in K and we integrate both sides of equation (6.4) with respect to the measure $\delta_x * \delta_y$:

$$c_1m_1(x*y)+c_2m_2(x*y)+\cdots+c_{n-1}m_{n-1}(x*y)+c_nm_n(x*y) = 0\,. \tag{6.5}$$

Using the exponential properties of the m's we have

$$c_1 m_1(x)m_1(y)+\cdots+c_{n-1}m_{n-1}(x)m_{n-1}(y)+c_n m_n(x)m_n(y)=0\,. \tag{6.6}$$

Now we write $t=x$ in (6.4) and multiply the equation obtained by $m_n(y)$:

$$c_1 m_1(x)m_n(y)+\cdots+c_{n-1}m_{n-1}(x)m_n(y)+c_n m_n(x)m_n(y)=0\,. \tag{6.7}$$

We subtract (6.7) from (6.6) to get

$$c_1 m_1(x)[m_1(y)-m_n(y)]+\cdots+c_{n-1}m_{n-1}(x)[m_{n-1}(y)-m_n(y)]=0\,. \tag{6.8}$$

By assumption the exponentials $m_1, m_2, \ldots, m_{n-1}$ are linearly independent, hence

$$c_i[m_i(y)-m_n(y)]=0 \tag{6.9}$$

for $i=1,2,\ldots,n-1$. As $m_n \neq m_1$ we can choose a y in K such that $m_n(y)\neq m_1(y)$; it follows that $c_1=0$. Continuing this argument we get $c_i=0$ for $i=1,2,\ldots,n-1$, which also implies $c_n=0$. The proof is complete. □

We shall also need the following result in the sequel.

THEOREM 6.2. *Let K be a Sturm–Liouville hypergroup with the exponential family $\varphi : \mathbb{R}_0\times\mathbb{C}\to\mathbb{C}$, n a nonnegative integer and $\lambda_0\neq 0$ a complex number. Then the special exponential monomials*

$$x\mapsto\varphi(x,\lambda_0), x\mapsto\partial_2\varphi(x,\lambda_0),\ldots,x\mapsto\partial_2^n\varphi(x,\lambda_0) \tag{6.10}$$

are linearly independent.

PROOF. We prove the statement by induction on n, which is obviously true for $n=0$. Suppose that we have proved it for n, and we prove it for $n+1$, where n is some nonnegative integer. Proving our statement by contradiction we suppose that the function $x\mapsto\partial_2^{n+1}\varphi(x,\lambda_0)$ is a linear combination of the functions

$$x\mapsto\partial_2^k\varphi(x,\lambda_0) \qquad k=0,1,\ldots,n\,,$$

that is there are complex numbers c_k for $k=0,1,\ldots,n$ such that

$$\partial_2^{n+1}\varphi(x,\lambda_0)=\sum_{k=0}^{n} c_k\partial_2^k\varphi(x,\lambda_0) \tag{6.11}$$

holds for each x in K. By the definition of the exponential family we have

$$\partial_1^2\varphi(x,\lambda)+p(x)\partial_1\varphi(x,\lambda)=\lambda\varphi(x,\lambda) \tag{6.12}$$

for each $x>0$ and λ in $\mathbb{C}$. We differentiate both sides k times with respect to λ for $k=0,1,\dots,n+1$. We obtain

$$\partial_1^2\partial_2^k\varphi(x,\lambda)+p(x)\partial_1\partial_2^k\varphi(x,\lambda)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{(j)}\cdot\partial_2^{k-j}\varphi(x,\lambda)\,, \tag{6.13}$$

or, equivalently

$$\partial_1^2\partial_2^k\varphi(x,\lambda)+p(x)\partial_1\partial_2^k\varphi(x,\lambda)=\lambda\partial_2^k\varphi(x,\lambda)+k\partial_2^{k-1}\varphi(x,\lambda) \tag{6.14}$$

for each $x>0$ and λ in $\mathbb{C}$ and for $k=0,1,\dots,n+1$.

Here $\partial_2^{-1}\varphi(x,\lambda)=0$. We shall use this equation several times in the sequel.

Differentiating equation (6.11) two times with respect to x we have the equations

$$\partial_1\partial_2^{n+1}\varphi(x,\lambda_0)=\sum_{k=0}^{n}c_k\partial_1\partial_2^k\varphi(x,\lambda_0) \tag{6.15}$$

and

$$\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_0)=\sum_{k=0}^{n}c_k\partial_1^2\partial_2^k\varphi(x,\lambda_0) \tag{6.16}$$

for each $x>0$. From these equations by (6.14), we have

$$\sum_{k=0}^{n}c_k\partial_1^2\partial_2^k\varphi(x,\lambda_0)+\sum_{k=0}^{n}c_kp(x)\partial_1\partial_2^k\varphi(x,\lambda_0)=$$

$$=\lambda_0\partial_2^{n+1}\varphi(x,\lambda_0)+(n+1)\partial_2^n\varphi(x,\lambda_0)=$$

$$=\sum_{k=0}^{n}\lambda_0c_k\partial_2^k\varphi(x,\lambda_0)+(n+1)\partial_2^n\varphi(x,\lambda_0)\,.$$

We can reorder the terms in this equation to obtain

$$\sum_{k=0}^{n}c_k[\partial_1^2\partial_2^k\varphi(x,\lambda_0)+p(x)\partial_1\partial_2^k\varphi(x,\lambda_0)-\lambda_0\partial_2^k\varphi(x,\lambda_0)]=$$

$$= (n+1)\partial_2^n \varphi(x,\lambda_0)\,,$$

or, equivalently, using again (6.14)

$$\sum_{k=1}^{n} kc_k \partial_2^{k-1}\varphi(x,\lambda_0) - (n+1)\partial_2^n\varphi(x,\lambda_0) = 0\,. \tag{6.17}$$

But this is a contradiction, because equation (6.17) presents a nontrivial linear combination of linearly independent functions, which has the value zero. Hence the proof is complete. □

Now we are in the position to prove linear independence of the special exponential monomials.

THEOREM 6.3. *On any Sturm–Liouville hypergroup different special exponential monomials are linearly independent.*

PROOF. We have to show that any finite set of special exponential monomials is linearly independent. First we suppose that this set does not include the special exponential monomial 1. We may suppose that this set consists of special exponential monomials of the form

$$x \mapsto \partial_2^l \varphi(x,\lambda_j)$$

for $l = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, k$ with some restrictions on the nonnegative integer n and the positive integer k. Actually, we shall consider two cases: in the first case we suppose that we have proved the linear independence of the functions

$$x \mapsto \partial_2^l \varphi(x,\lambda_j)$$

for $l = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, k$, where n is a nonnegative integer and k is a positive integer, and we show that the function

$$x \mapsto \partial_2^{n+1} \varphi(x,\lambda_1)$$

is not a linear combination of them, and in the second case we suppose that we have proved the linear independence of the functions

$$x \mapsto \partial_2^l \varphi(x,\lambda_s), x \mapsto \partial_2^{n+1} \varphi(x,\lambda_t)$$

for $l = 0, 1, \ldots, n$, $s = 1, 2, \ldots, k$ and $t = 1, 2, \ldots, j$, where n is a nonnegative integer, $k \geq 2$ is a positive integer and j is a positive integer with $j \leq k-1$, and we show that the function $x \mapsto \partial_2^{n+1}\varphi(x,\lambda_{j+1})$ is not a linear combination of them. It is easy to see that any other case can be reduced to these two cases (eventually, by renumbering the λ's). We apply induction again: in the first case the statement is clearly true for $n = 0$ and $k = 1$. Also, if $n = 0$ and k is arbitrary, then the statement follows from Theorem 6.1, and if $k = 1$ and n is arbitrary, then the statement follows from Theorem 6.2. Hence we can consider the first case and prove by contradiction: suppose that the function $x \mapsto \partial_2^{n+1}\varphi(x,\lambda_1)$ is a linear combination of the functions

$$x \mapsto \partial_2^l \varphi(x,\lambda_j)$$

for $l = 0,1,\ldots,n$ and $j = 1,2,\ldots,k$, where n is a nonnegative integer and k is a positive integer. This means that there are complex numbers $c_{l,j}$ for $l = 0,1,\ldots,n$ and $j = 1,2,\ldots,k$ such that

$$\partial_2^{n+1}\varphi(x,\lambda_1) = \sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_2^l\varphi(x,\lambda_j) \tag{6.18}$$

holds for each $x > 0$. Differentiating two times with respect to x we get the equations

$$\partial_1\partial_2^{n+1}\varphi(x,\lambda_1) = \sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_1\partial_2^l\varphi(x,\lambda_j) \tag{6.19}$$

and

$$\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_1) = \sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_1^2\partial_2^l\varphi(x,\lambda_j) \tag{6.20}$$

for each $x > 0$. From these equations by (6.14) we have

$$\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_1^2\partial_2^l\varphi(x,\lambda_j) + \sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}p(x)\partial_1\partial_2^l\varphi(x,\lambda_j) =$$
$$= \lambda_1\partial_2^{n+1}\varphi(x,\lambda_1) + (n+1)\partial_2^n\varphi(x,\lambda_1) =$$
$$\sum_{l=0}^{n}\sum_{j=1}^{k} \lambda_1 c_{l,j}\partial_2^l\varphi(x,\lambda_j) + (n+1)\partial_2^n\varphi(x,\lambda_1)\,.$$

We can reorder the terms in this equation to obtain

$$\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}[\partial_1^2\partial_2^l\varphi(x,\lambda_j) + p(x)\partial_1\partial_2^l\varphi(x,\lambda_j) - \lambda_1\partial_2^l\varphi(x,\lambda_j)] =$$

$$= (n+1)\partial_2^n \varphi(x, \lambda_1)\,,$$

or, equivalently, using again (6.14)

$$\sum_{l=1}^{n}\sum_{j=1}^{k} l c_{l,j}\partial_2^{l-1}\varphi(x,\lambda_j) - (n+1)\partial_2^n\varphi(x,\lambda_1) = 0\,. \tag{6.21}$$

But this is a contradiction, because equation (6.21) presents a nontrivial linear combination of linearly independent functions, which has the value zero. Hence the proof of our statement in the first case is complete.

Now we consider the second case and we prove again by contradiction: we suppose that we have proved the linear independence of the functions

$$x \mapsto \partial_2^l\varphi(x,\lambda_s), x \mapsto \partial_2^{n+1}\varphi(x,\lambda_t)$$

for $l = 0, 1, \ldots, n$, $s = 1, 2, \ldots, k$ and $t = 1, 2, \ldots, j$, where n is a nonnegative integer, $k \geq 2$ is a positive integer and j is a positive integer with $j \leq k-1$, and we show that the function $x \mapsto \partial_2^{n+1}\varphi(x, \lambda_{j+1})$ is a linear combination of them. This means that there are complex numbers $c_{l,s}, d_t$ for $l = 0, 1, \ldots, n$ and $s = 1, 2, \ldots, k$, $t = 1, 2, \ldots, j$ such that

$$\partial_2^{n+1}\varphi(x,\lambda_{j+1}) = \sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_2^{n+1}\varphi(x,\lambda_t) \tag{6.22}$$

holds for each $x > 0$. Differentiating two times with respect to x we get the equations

$$\partial_1\partial_2^{n+1}\varphi(x,\lambda_{j+1}) = \sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_1\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_1\partial_2^{n+1}\varphi(x,\lambda_t) \tag{6.23}$$

and

$$\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_{j+1}) = \sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_1^2\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_t) \tag{6.24}$$

for each $x > 0$. From these equations by (6.14), we have

$$\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_1^2\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_t) +$$

$$+\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}p(x)\partial_1\partial_2^l\varphi(x,\lambda_s)+\sum_{t=1}^{j} d_t p(x)\partial_1\partial_2^{n+1}\varphi(x,\lambda_t)=$$

$$=\lambda_{j+1}\partial_2^{n+1}\varphi(x,\lambda_{j+1})+(n+1)\partial_2^n\varphi(x,\lambda_{j+1})=$$

$$=\sum_{l=0}^{n}\sum_{s=1}^{k}\lambda_{j+1}c_{l,s}\partial_2^l\varphi(x,\lambda_s)+$$

$$+\sum_{t=1}^{j} d_t\lambda_{j+1}\partial_2^{n+1}\varphi(x,\lambda_t)+(n+1)\partial_2^n\varphi(x,\lambda_{j+1})\,.$$

We can reorder the terms in this equation to obtain

$$\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}[\partial_1^2\partial_2^l\varphi(x,\lambda_s)+p(x)\partial_1\partial_2^l\varphi(x,\lambda_s)-\lambda_{j+1}\partial_2^l\varphi(x,\lambda_s)]+$$

$$+\sum_{t=1}^{j} d_t[\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_t)+p(x)\partial_1\partial_2^{n+1}\varphi(x,\lambda_t)-\lambda_{j+1}\partial_2^{n+1}\varphi(x,\lambda_t)]=$$

$$=(n+1)\partial_2^n\varphi(x,\lambda_{j+1})\,,$$

or, equivalently, using again (6.14)

$$\sum_{l=1}^{n}\sum_{s=1}^{k} lc_{l,s}\partial_2^{l-1}\varphi(x,\lambda_s)+\sum_{t=1}^{j} d_t(n+1)\partial_2^n\varphi(x,\lambda_t)+$$

$$+\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}(\lambda_s-\lambda_{j+1})\partial_2^l\varphi(x,\lambda_s)+\sum_{t=1}^{j} d_t(\lambda_t-\lambda_{j+1})\partial_2^{n+1}\varphi(x,\lambda_t)-$$

$$-(n+1)\partial_2^n\varphi(x,\lambda_{j+1})=0\,. \tag{6.25}$$

The term containing $\partial_2^n\varphi(x,\lambda_{j+1})$ does not appear in the first two sums, it appears with zero coefficient in the third sum, it does not appear in the fourth sum, hence its coefficient on the left hand side is $-(n+1)\neq 0$. This is a contradiction and the proof of our statement also in the second case is complete.

To finish the proof we have to consider the case where the special exponential monomial 1 is in the set of the exponential monomials. We prove by contradiction again: suppose that there are nonzero complex numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ and there is a nonnegative integer n such that

$$1 = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} \partial_2^l \varphi(x, \lambda_j) \tag{6.26}$$

holds for each $x > 0$. Differentiating equation two times with respect to x we obtain

$$0 = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} \partial_1 \partial_2^l \varphi(x, \lambda_j) \tag{6.27}$$

and

$$0 = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} \partial_1^2 \partial_2^l \varphi(x, \lambda_j) \tag{6.28}$$

for each $x > 0$. Adding equations (6.27) and (6.28) we get

$$0 = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} [\partial_1^2 \partial_2^l \varphi(x, \lambda_j) + p(x) \partial_1 \partial_2^l \varphi(x, \lambda_j)] = \tag{6.29}$$

$$= \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} [\lambda_j \partial_2^l \varphi(x, \lambda_j) + l \partial_2^{l-1} \varphi(x \lambda_j)]$$

for each $x > 0$. On the right hand side we have a linear combination of linearly independent functions. The coefficient of $\partial_2^n \varphi(x, \lambda_j)$ is $c_{n,j} \lambda_j$, which must be zero, hence $c_{n,j} = 0$ for $j = 1, 2, \dots, k$. Continuing recursively we get that $c_{n-1,j} = c_{n-2,j} = \cdots = c_{0,j} = 0$ for $j = 1, 2, \dots, k$, which contradicts to equation (6.26). Now the proof of the theorem is complete. □

2. Translates of exponential monomials

The main result of this section is that the translates of exponential monomials have the characteristic property of moment functions. This means that there is a close connection between exponential monomials and moment functions. The idea of the proof is based on the exchangeability of translation and derivation. We give the details in the following.

Let us turn to describe what we get if we consider the equation of the N^{th} monomial and substitute $x * y$ for x. We easily get on one hand the following equation

$$\partial_1^2\partial_2^N\Phi(x*y,\lambda)+\frac{A'(x)}{A(x)}\partial_1\partial_2^N\Phi(x*y,\lambda)= \\ =N\partial_2^{N-1}\Phi(x*y,\lambda)+\lambda\ \partial_2^N\Phi(x*y,\lambda). \tag{6.30}$$

On the other hand, if we use the multiplicative property of the exponential function, then we have that equation

$$\partial_1^2\Phi(x*y,\lambda)+\frac{A'(x)}{A(x)}\partial_1\Phi(x*y,\lambda)=\lambda\ \Phi(x*y,\lambda)=\lambda\Phi(x,\lambda)\Phi(y,\lambda) \tag{6.31}$$

holds. If we differentiate equation (6.31) N-times with respect to λ, then we have

$$\partial_1^2\partial_2^N\Phi(x*y,\lambda)+\frac{A'(x)}{A(x)}\partial_1\partial_2^N\Phi(x*y,\lambda)=\partial_2^N\big(\lambda\Phi(x,\lambda)\Phi(y,\lambda)\big), \tag{6.32}$$

where ∂_2^N clearly stands for $\frac{d^N}{d\lambda^N}$. We introduce the function

$$\nu:\mathbb{R}_0\times\mathbb{C}\to\mathbb{C}$$

of the form $\nu(x,\lambda)=1\cdot\lambda=\lambda$ so in the equation (6.32) the second order partial derivate can be written in the form

$$\partial_2^N\big(\lambda\Phi(x,\lambda)\Phi(y,\lambda)\big)=\partial_2^N\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]. \tag{6.33}$$

From equation (6.30), (6.32) and (6.33) we get

$$N\partial_2^{N-1}\Phi(x*y,\lambda)+\lambda\ \partial_2^N\Phi(x*y,\lambda)=\partial_2^N\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]. \tag{6.34}$$

This makes the connection between exponential monomials and their translates clear. The following theorem presents the exact form of the N^{th} derivatives of the function $(x,\lambda)\mapsto\nu(x,\lambda)\ \Phi(x*y,\lambda)$ with respect to λ.

Theorem 6.4. *Let $K=(\mathbb{R}_0,A)$ be a Sturm-Liouville hypergroup, λ a fixed complex number, N a nonnegative integer, y an arbitrary nonnegtive real number, $\Phi:\mathbb{R}_0\times\mathbb{C}\to\mathbb{C}$ an exponential function and*

$\nu : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ *a function with the the property* $\nu(x,\lambda) = 1 \cdot \lambda = \lambda$. *Then the second order partial derivatives of the function* $\nu(x,\lambda)\ \Phi(x*y,\lambda)$ *can be written in the form*

$$\partial_2^N \big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] = \lambda \partial_2^N \Phi(x,\lambda)\Phi(y,\lambda) +$$
$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x,\lambda)\big(\lambda \binom{N}{i} \partial_2^{N-i}\Phi(y,\lambda) + N\binom{N-1}{i}\partial_2^{N-i-1}\Phi(y,\lambda)\big).$$

Proof. We use induction by N, first we consider the case $N = 1$. We get

$$\frac{d}{d\lambda}(\lambda\ \Phi(x*y,\lambda)) = \frac{d}{d\lambda}(\lambda\ \Phi(x,\lambda)\Phi(y,\lambda)) =$$
$$= \Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \Phi(x,\lambda)\partial_2\Phi(y,\lambda) =$$
$$= \lambda\ \partial_2\Phi(x,\lambda)\Phi(y,\lambda) + \Phi(x,\lambda)\big(\lambda\ \partial_2\Phi(y,\lambda) + \Phi(y,\lambda)\big),$$

which shows that the theorem holds for $N = 1$. Let us suppose that the formula is true for N, and we will show that it is valid for $N + 1$. First we differentiate the equation with respect to λ. We get the following:

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] =$$
$$= \partial_2^N\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda) +$$
$$+ \sum_{i=0}^{N-1} \partial_2^{i+1}\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda) + N\binom{N-1}{i}\partial_2^{N-i-1}\Phi(y,\lambda)\big) +$$
$$+ \sum_{i=0}^{N-1} \partial_2^i\Phi(x,\lambda)\big(\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda) + \lambda\binom{N}{i}\partial_2^{N-i+1}\Phi(y,\lambda)\big) +$$
$$+ \sum_{i=0}^{N-1} \partial_2^i\Phi(x,\lambda)\big(N\binom{N-1}{i}\partial_2^{N-i}\Phi(y,\lambda)\big).$$

In the next step we will focus on the index range of the sums

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] =$$
$$= \partial_2^N\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda) +$$
$$+ \partial_2^N\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda) + N\Phi(y,\lambda)\big) +$$
$$+ \sum_{i=0}^{N-2} \partial_2^{i+1}\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda) + N\binom{N-1}{i}\partial_2^{N-i-1}\Phi(y,\lambda)\big) +$$

$$+\sum_{i=0}^{N-1}\partial_2^i\Phi(x,\lambda)\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)+\Phi(x,\lambda)\big(\lambda\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^N\Phi(y,\lambda)\big)+$$
$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i+1}\Phi(y,\lambda)+N\binom{N-1}{i}\partial_2^{N-i}\Phi(y,\lambda)\big).$$

To change the index range of the first sum we apply a substitution

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]=$$
$$=\partial_2^N\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda)+$$
$$+\partial_2^N\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda)+N\Phi(y,\lambda)\big)+$$
$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\binom{N}{i-1}\partial_2^{N-i+1}\Phi(y,\lambda)+N\binom{N-1}{i-1}\partial_2^{N-i}\Phi(y,\lambda)\big)+$$
$$+\sum_{i=0}^{N-1}\partial_2^i\Phi(x,\lambda)\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)+\Phi(x,\lambda)\big(\lambda\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^N\Phi(y,\lambda)\big)+$$
$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i+1}\Phi(y,\lambda)+N\binom{N-1}{i}\partial_2^{N-i}\Phi(y,\lambda)\big).$$

We simplify the expression

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]=$$
$$=\partial_2^N\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda)+$$
$$+\sum_{i=0}^{N-1}\binom{N}{i}\partial_2^i\Phi(x,\lambda)\partial_2^{N-i}\Phi(y,\lambda)+$$
$$+\binom{N}{N}\partial_2^N\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda)+N\Phi(y,\lambda)\big)+$$
$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\Big[\binom{N}{i-1}+\binom{N}{i}\Big]\partial_2^{N-i+1}\Phi(y,\lambda)+$$
$$\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(N\Big[\binom{N-1}{i-1}+\binom{N-1}{i}\Big]\partial_2^{N-i}\Phi(y,\lambda)\big)+$$
$$+\binom{N}{0}\Phi(x,\lambda)\big(\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^N\Phi(y,\lambda)\big),$$

and using the identity for binomial coefficients, we have

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] =$$
$$= \partial_2^{N}\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\ \partial_2^{N}\Phi(x,\lambda)\partial_2\Phi(y,\lambda) +$$
$$+\sum_{i=0}^{N-1}\binom{N}{i}\partial_2^{i}\Phi(x,\lambda)\partial_2^{N-i}\Phi(y,\lambda)+$$
$$+\binom{N}{N}\partial_2^{N}\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda)+N\Phi(y,\lambda)\big)+$$
$$+\sum_{i=1}^{N-1}\partial_2^{i}\Phi(x,\lambda)\big(\lambda\binom{N+1}{i}\partial_2^{N-i+1}\Phi(y,\lambda)+$$
$$\sum_{i=1}^{N-1}\partial_2^{i}\Phi(x,\lambda)\big(N\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)\big)+$$
$$+\binom{N}{0}\Phi(x,\lambda)\big(\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^{N}\Phi(y,\lambda)\big).$$

All this can be written as

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] =$$
$$= \lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+$$
$$+\sum_{i=0}^{N}\binom{N}{i}\partial_2^{i}\Phi(x,\lambda)\partial_2^{N-i}\Phi(y,\lambda)+$$
$$+N\sum_{i=0}^{N}\binom{N}{i}\partial_2^{i}\Phi(x,\lambda)\partial_2^{N-i}\Phi(y,\lambda)+$$
$$+\sum_{i=0}^{N}\partial_2^{i}\Phi(x,\lambda)\big(\lambda\binom{N+1}{i}\partial_2^{N-i+1}\Phi(y,\lambda)\big).$$

It is easy to see that the previous and the following equations are equal

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] =$$
$$= \lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+$$
$$+\ (N+1)\sum_{i=0}^{N}\binom{N}{i}\partial_2^{i}\Phi(x,\lambda)\partial_2^{N-i}\Phi(y,\lambda)+$$

$$+\sum_{i=0}^{N} \partial_2^i \Phi(x,\lambda)\big(\lambda \binom{N+1}{i} \partial_2^{N-i+1}\Phi(y,\lambda)\big).$$

This final step completes the proof of the expected formula

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big] = \lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+$$

$$+\sum_{i=0}^{N} \partial_2^i \Phi(x,\lambda)\big(\lambda \binom{N+1}{i} \partial_2^{N-i+1}\Phi(y,\lambda) + (N+1)\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)\big),$$

which means that the statement holds and the theorem is true. □

Now we will show that the translates of exponential monomials can be written in a compact form. The previous theorem will be utilized to prove the statement.

THEOREM 6.5. *Let $K = (\mathbb{R}_0, A)$ be a Sturm-Liouville hypergroup, λ a complex number, N a nonnegative integer and the function*

$$x \mapsto \partial_2^N \Phi(x,\lambda)$$

is a special exponential monomial. Then

$$\tau_y\ \partial_2^N \Phi(x,\lambda) = \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i}\Phi(y,\lambda) \tag{6.35}$$

for any y in $\mathbb{R}_0$.

PROOF. The proof is based on induction by N. In the first step, if $N = 1$ on one hand we get that

$$\partial_1^2\partial_2\Phi(x*y,\lambda) + p(x)\partial_1\partial_2\Phi(x*y,\lambda) = \Phi(x*y,\lambda) + \lambda\partial_2\Phi(x*y,\lambda)$$

and on the other hand the following is valid

$$\partial_1^2\partial_2\Phi(x*y,\lambda) + p(x)\partial_1\partial_2\Phi(x*y,\lambda) =$$

$$= \Phi(x,\lambda)\Phi(y,\lambda) + \lambda\partial_2\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\Phi(x,\lambda)\partial_2\Phi(y,\lambda).$$

Since the left hand side of these equations are equal we have

$$\Phi(x*y,\lambda) + \lambda\partial_2\Phi(x*y,\lambda) =$$

$$\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\partial_2\Phi(x,\lambda)\Phi(y,\lambda) + \lambda\Phi(x,\lambda)\partial_2\Phi(y,\lambda)$$

and if we use the multiplicative property of exponential functions and simplify the equation we get

$$\tau_y \partial_2 \Phi(x,\lambda) = \partial_2 \Phi(x,\lambda)\Phi(y,\lambda) + \Phi(x,\lambda)\partial_2 \Phi(y,\lambda).$$

It means that the statement holds for the case of $N = 1$. Now we can suppose that

$$\tau_y \; \partial_2^{N-1} \Phi(x,\lambda) = \sum_{i=0}^{N-1} \binom{N-1}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i-1} \Phi(y,\lambda)$$

holds. In the final step of proof we use the characterization equation of exponential monomials and the previous statement. Let us consider the characterization equation which is the following

$$\partial_1^2 \partial_2^N \Phi(x*y,\lambda) + p(x)\partial_1 \partial_2^N \Phi(x*y,\lambda) = N \partial_2^{N-1} \Phi(x*y,\lambda) + \lambda \partial_2^N \Phi(x*y,\lambda)$$

and according to the previous theorem

$$\partial_1^2 \partial_2^N \Phi(x*y,\lambda) + p(x)\partial_1 \partial_2^N \Phi(x*y,\lambda) = \lambda \partial_2^N \Phi(x,\lambda)\Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x,\lambda) \big(\lambda \binom{N}{i} \partial_2^{N-i} \Phi(y,\lambda) + N \binom{N-1}{i} \partial_2^{N-i-1} \Phi(y,\lambda)\big).$$

This is similar to the previous one, in particular, from the equality of left hand sides of the previous two equations we get

$$N \partial_2^{N-1} \Phi(x*y,\lambda) + \lambda \partial_2^N \Phi(x*y,\lambda) =$$

$$= \lambda \partial_2^N \Phi(x,\lambda)\Phi(y,\lambda) + \sum_{i=0}^{N-1} \lambda \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N-1} N \binom{N-1}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i-1} \Phi(y,\lambda) =$$

$$= \lambda \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ N \sum_{i=0}^{N-1} \binom{N-1}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i-1} \Phi(y,\lambda).$$

At this point using the assumption and simplifying the equation we can see that

$$\tau_y \, \partial_2^N \Phi(x,\lambda) = \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) \tag{6.36}$$

holds, which means that our statement follows. □

Summarizing the results of this section we see that the translate of an arbitrary exponential monomial presents a special linear combination, where the basis functions are the lower order exponential monomials. This is a very useful property when considering spectral synthesis. These exponential monomials are linearly independent which implies the question, whether spectral synthesis holds for varieties of the actual Sturm–Liouville hypergroup. Unfortunately we can not prove the complete result, but some partial results will be examined in the following section.

CHAPTER 7

Spectral analysis

1. Spectral analysis for commutative hypergroups

Spectral synthesis deals with the description of translation invariant function spaces over topological groups. Suppose that a locally compact Abelian group is given and consider the set of all continuous complex valued functions on it, equipped with the pointwise linear opertions and with the topology of uniform convergence on compact sets. The problem of spectral analysis and spectral synthesis can be formulated:

(1) Is it true that any nonzero, closed, translation invariant linear subspace of the space mentioned above (in other words a *variety*) contains an exponential function (*spectral analysis*)?

(2) Is it true that any variety is equal to the closed linear span of all exponential monomials in it (*spectral synthesis*)?

The study of spectral analysis and spectral synthesis problems is based on the concept of exponential monomials. Here we deal with spectral analysis only and we prove it for finite dimensional varieties on commutative hypergroups, hence we need exclusively the concept of exponentials.

The following statement presents a result related to the problem of spectral analysis on commutative hypergroups.

THEOREM 7.1. *Spectral analysis holds for nonzero finite dimensional varieties on every commutative hypergroup.*

PROOF. Suppose that K is a commutative hypergroup and $V \neq \{0\}$ is a finite dimensional variety in $\mathcal{C}(K)$. We have to show that V contains an exponential. Let $f_1, f_2, \dots, f_n$ be a basis of V, then there exist complex valued funcions $c_{i,j}$ for $i, j = 1, 2, \dots, n$ such that

$$f_i(x * y) = \sum_{j=1}^{n} c_{i,j}(y) f_j(x) \tag{7.1}$$

holds for every x, y in K and $i = 1, 2, \dots, n$. As the functions $f_1, f_2, \dots, f_n$ are linearly independent, hence there are elements x_k for $k = 1, 2, \dots, n$ in K such that the matrix $\big(f_j(x_k)\big)_{j,k=1}^{n}$ is regular. We have

$$f_i(x_k * y) = \sum_{j=1}^{n} c_{i,j}(y) f_j(x_k)$$

for each y in K and $k = 1, 2, \dots, n$. For any fixed i this is an inhomogeneous system of linear equations for the unknowns

$$c_{i,j}(y) \qquad (j = 1, 2, \dots, n)$$

with regular fundamental matrix, hence, by Cramer's rule, $c_{i,j}$ is a linear combination of translates of f_i, hence $c_{i,j}$ belongs to V $(i, j = 1, 2, \dots, n)$.

Going back to equation (7.1) and using the associativity of convolution we infer that

$$\sum_{j=1}^{n} c_{i,j}(z) f_j(x * y) = \sum_{j=1}^{n} c_{i,j}(y * z) f_j(x)\,, \tag{7.2}$$

or

$$\sum_{j=1}^{n} c_{i,j}(z) \sum_{l=1}^{n} c_{j,l}(y) f_l(x) = \sum_{j=1}^{n} c_{i,j}(y * z) f_j(x) \tag{7.3}$$

holds for each x, y, z in K. This is equivalent to

$$\sum_{k=1}^{n} \sum_{j=1}^{n} c_{i,j}(z) c_{j,k}(y) f_k(x) = \sum_{k=1}^{n} c_{i,k}(y * z) f_k(x)\,. \tag{7.4}$$

By the linear independence of the f_k's we have

$$\sum_{j=1}^{n} c_{i,j}(z)c_{j,k}(y) = c_{i,k}(y * z) = c_{i,k}(z * y) \tag{7.5}$$

for each y, z in K. Let $C(x)$ be the matrix $\big(c_{i,j}(x)\big)_{i,j=1}^{n}$, then from (7.5) it follows

$$C(x * y) = C(x) \cdot C(y) \tag{7.6}$$

for each x, y in K. In particular, the matrices $C(x)$ are commuting for different x's. It follows that there exists a nonsingular matrix S such that the matrix $T(x)$ defined by

$$T(x) = S^{-1} \cdot C(x) \cdot S \tag{7.7}$$

is lower triangular for each x in K. On the other hand, the entries of $T(x)$ are linear combinations of the $c_{i,j}$'s, hence they belong to V. Further we have for each x, y in K:

$$T(x * y) = S^{-1} \cdot C(x * y) \cdot S = S^{-1} \cdot C(x) \cdot C(y) \cdot S =$$

$$= S^{-1} \cdot C(x) \cdot S \cdot S^{-1} \cdot C(y) \cdot S = T(x) \cdot T(y)\,.$$

Suppose that $T(x) = \big(t_{i,j}(x)\big)_{i,j=1}^{n}$, then using the fact that $T(x)$ is lower triangular we have that

$$t_{i,j}(x * y) = \sum_{k=j}^{i} t_{i,k}(x) \cdot t_{k,j}(y)$$

holds for $j = 1, 2, \dots, i$ and for each x, y in K. If we put $j = i$ we get

$$t_{i,i}(x * y) = t_{i,i}(x) \cdot t_{i,i}(y) \tag{7.8}$$

for $i = 1, 2, \dots, n$ and for each x, y in K, which means that the functions $t_{i,i}$ $(i = 1, 2, \dots, n)$ are exponentials in V and the theorem is proved. □

2. Spectral analysis using moment functions

Here we show that if a variety contains certain nonzero generalized moment functions, then spectral analysis holds for this variety.

Theorem 7.2. *Let K be a commutative hypergroup, n a nonnegative integer and $\varphi_0, \varphi_1, \dots, \varphi_n$ a generalized moment function sequence with $\varphi_1 \neq 0$. If V is a variety in $\mathcal{C}(K)$ and φ_n belongs to V, then φ_k belongs to V for $k = 0, 1, \dots, n$. In particular, spectral analysis holds for V.*

Proof. By the previous results the functions $\varphi_0, \varphi_1, \dots, \varphi_n$ are linearly independent. Hence there are elements $y_0, \dots, y_n$ in K for which the matrix $A = \big(\varphi_{n-i}(y_j)\big)_{i,j=0}^{n}$ is regular. Now we fix the element x from K to get the following inhomogeneous system of linear equations for unknowns $\varphi_i(x)$ $(i = 0, 1, \dots, n)$:

$$\varphi_n(x * y_j) = \sum_{i=0}^{n} \binom{n}{i} \varphi_i(x) \varphi_{n-i}(y_j) \qquad (j = 0, 1, \dots, n)\,.$$

As the fundamental matrix of the previous system is the matrix A, the system has a unique solution. By Cramer's rule, the function $\binom{n}{i}\varphi_i(x)$, hence also the function $\varphi_i(x)$ $(i = 0, 1, \dots, n)$ is a linear combination of translates of $\varphi_n(x)$. As V is translation invariant, hence the translates of φ_k belong to V, which implies that $\varphi_0, \varphi_1, \dots, \varphi_n$ are also in V.

Since $\varphi_0(x)$ is an exponential, our theorem is proved. □

CHAPTER 8

A moment problem

The classical moment problem published in 1894 by Thomas Jan Stieltjes (see [**24**]) is the following: Given a sequence $s_0, s_1, \ldots$ of real numbers. Find necessary and sufficient conditions for the existence of a measure μ on $[0, \infty[$ so that

$$s_n = \int_0^\infty x^n \, d\mu(x)$$

holds for $n = 0, 1, \ldots$. Another form of the moment problem, also called "Hausdorff's moment problem" or the "little moment problem," may be stated as follows: Given a sequence of numbers $(s_n)_{n=0}^\infty$, under what conditions is it possible to determine a function α of bounded variation in the interval $]0, 1[$ such that

$$s_n = \int_0^1 x^n \, d\alpha(x)$$

for $n = 0, 1, \ldots$. Such a sequence is called a *moment sequence*, and Felix Hausdorff (see [**7**], [**8**]) was the first to obtain necessary and sufficient conditions for a sequence to be a moment sequence. In both cases the question of uniqueness of μ, resp. α arise. For a detailed discussion on classical moment problems see e.g. [**3**].

For more about generalized moment function sequences see [**38**], [**17**], [**16**], [**18**].

Let μ be a compactly supported measure on K and let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence. Then for each natural number

n the complex number

$$m_n = \int_K \varphi_n \, d\mu$$

is called the *n-th generalized moment of* μ with respect to the given generalized moment function sequence. In this case the sequence $(m_n)_{n=0}^{\infty}$ is called the *generalized moment sequence of the measure* μ with respect to the given generalized moment function sequence.

In this setting we can formulate the problem of existence: Let the generalized moment function sequence $(\varphi_k)_{k=0}^{\infty}$ and the sequence of complex numbers $(m_n)_{n=0}^{\infty}$ be given. Under what conditions is there a compactly supported measure μ on K such that $(m_n)_{n=0}^{\infty}$ is the generalized moment sequence of the measure μ with respect to the given generalized moment function sequence? The other basic question is about the uniqueness: if the compactly supported measures μ and ν have the same generalized moment sequences with respect to the given generalized moment function sequence, then does it follow $\mu = \nu$?

We study the problem of uniqueness and solve it in the case of polynomial hypergroups in a single variable and in the case of Sturm–Liouville hypergroups.

1. The case of polynomial hypergroups

In this section we shall use the results in [**17**] on the representation of generalized moment functions on polynomial hypergroups in the following form.

THEOREM 8.1. *Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. Then there exists a sequence $(c_k)_{k=0}^{\infty}$ such that for each natural number N we have*

$$\varphi_k(n) = \big(P_n \circ f\big)^{(k)}(0) \;\; (k = 0, 1, \ldots, N)\,, \tag{8.1}$$

where

$$f(t) = \sum_{j=0}^{N} c_j \frac{t^j}{j!}$$

for each t in $\mathbb{R}$.

THEOREM 8.2. *Let Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ a finitely supported measure on $\mathbb{N}$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{N}} \varphi_k(n)\, d\mu(n) = 0 \tag{8.2}$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

PROOF. First we remark that

$$\int_{\mathbb{N}} \varphi_k(n)\, d\mu(n) = \sum_{n=0}^{N} \varphi_k(n)\mu_n. \tag{8.3}$$

Hence, by assumption, we have the following system of equations

$$\sum_{n=0}^{N} \varphi_k(n)\mu_n = 0 \tag{8.4}$$

for $k = 0, 1, 2, \ldots, N$.

On the other hand, by the result (8.1), we have that

$$\varphi_k(n) = \big(P_n \circ f\big)^{(k)}(0) \tag{8.5}$$

for $k = 0, 1, 2, \ldots, N,\;\; n = 0, 1, 2, \ldots, N$, where

$$f(t) = \sum_{i=0}^{N} c_i \frac{t^i}{i!}$$

is a polynomial. Let $\lambda = f(0)$. From (8.5) we have for $k = 1$

$$\varphi_1(n) = P_n'(\lambda)\, c_1\,,$$

which implies $c_1 \neq 0$.

Let n be a fixed nonnegative integer and we let for $k = 0, 1, 2, \dots, N$ and for each t in $\mathbb{R}$:

$$F_k(t) = \big(P_n \circ f\big)^{(k)}(t)\,. \tag{8.6}$$

We show that for $k = 0, 1, 2, \dots, N$ and for each t in $\mathbb{R}$

$$F_k(t) = \sum_{j=0}^{k} p_{k,j}(t) P_n^{(j)}\big(f(t)\big)\,, \tag{8.7}$$

where $p_{k,j}$ is a polynomial and $p_{k,k}(t) = f'(t)^k$.

We prove (8.7) by induction on k. For $k = 0$ the statement is trivial with $p_{0,0}(t) = 1$. Supposing that (8.7) is proved we show it for $k+1$ instead of k. We have

$$F_{k+1}(t) = F_k'(t) = \sum_{j=0}^{k} p'_{k,j}(t) P_n^{(j)}\big(f(t)\big) + \sum_{j=0}^{k} p_{k,j}(t) P_n^{(j+1)}\big(f(t)\big)\, f'(t)$$

and this is the form (8.7) with $k+1$ for k. Moreover,

$$p_{k+1,k+1}(t) = p_{k,k}(t) \cdot f'(t) = f'(t)^{k+1}.$$

Then, using (8.5), we have

$$\varphi_k(n) = \sum_{j=0}^{k} c_{k,j}\, P_n^{(j)}(\lambda) \qquad\qquad k = 0, 1, 2, \dots, N\,, \tag{8.8}$$

where $c_{k,k} = f'(0)^k \neq 0$, $c_{0,0} = 1$. By (8.4), it follows

$$\sum_{n=0}^{N} \sum_{j=0}^{k} c_{k,j}\, P_n^{(j)}(\lambda)\mu_n = 0 \tag{8.9}$$

for $k = 0, 1, 2, \dots, N$. For $k = 0$ this means

$$\sum_{n=0}^{N} P_n(\lambda)\mu_n = 0\,. \tag{8.10}$$

Now let $k = 1$ in (8.9), then we have by (8.10)

$$\sum_{n=0}^{N} c_{1,0}\, P_n(\lambda)\mu_n + c_{1,1}\, P_n'(\lambda)\mu_n = c_{1,0} \sum_{n=0}^{N} P_n(\lambda)\mu_n + c_{1,1} \sum_{n=0}^{N} P_n'(\lambda)\mu_n =$$

$$= c_{1,1} \sum_{n=0}^{N} P_n'(\lambda)\mu_n = 0\,.$$

As $c_{1,1} \neq 0$, then it follows:

$$\sum_{n=0}^{N} P_n'(\lambda)\mu_n = 0\,.$$

Continuing this process we get the system of equations

$$\sum_{n=0}^{N} P_n^{(k)}(\lambda)\mu_n = 0\,, \tag{8.11}$$

for $k = 0, 1, 2, \dots, N$. Observe that the degree of P_n is exactly n, hence we can rewrite (8.11) in the form

$$\sum_{n=k}^{N} P_n^{(k)}(\lambda)\mu_n = 0\,, \tag{8.12}$$

for $k = 0, 1, 2, \dots, N$. This is a homogeneous system of linear equations for the unknowns μ_n, $n = 0, 1, \dots, N$. The fundamental matrix of this system is an $N \times N$ upper triangular matrix with the nonzero numbers $P_k^{(k)}(\lambda)$ in the main diagonal, hence this matrix is regular, which means that the system has only trivial solution: $\mu_n = 0$ for $n = 0, 1, 2, \dots, N$. This means $\mu = 0$ and the proof is complete. □

This result obviously implies the following uniqueness theorem.

THEOREM 8.3. *Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ, ν finitely supported measures on $\mathbb{N}$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu = \nu$.*

2. The case of Sturm–Liouville hypergroups

Following the previous ideas in this subsection we consider the same problem on Sturm–Liouville hypergroups. We shall use the results in [**18**] on the representation of generalized moment functions on polynomial hypergroups in the following form.

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, and let Φ the exponential family of the hypergroups K (see [**18**]). This means that for each z in $\mathbb{C}$ and x in $\mathbb{R}_+$ the function Φ satisfies

$$\partial_1^2 \Phi(x,z) + \frac{A'(x)}{A(x)}\, \partial_1 \Phi(x,z) = z\, \Phi(x,z)\,, \tag{8.13}$$

further $\Phi(0,z) = 1$ and $\partial_1 \Phi(0,z) = 0$. It follows that the function $z \mapsto \Phi(x,z)$ is entire for each x in $\mathbb{R}_0$.

THEOREM 8.4. *Let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence on K. Then there exists a sequence $(c_k)_{k=0}^\infty$ such that for each natural number N we have*

$$\varphi_k(x) = \frac{d^k}{dt^k} \Phi\bigl(x, f(t)\bigr)(0) \tag{8.14}$$

for $k = 0, 1, 2, \ldots, N$, x in $\mathbb{R}_0$, t in $\mathbb{C}$, where

$$f(t) = \sum_{i=0}^N c_i\, \frac{t^i}{i!}$$

is a polynomial.

THEOREM 8.5. *Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, μ a compactly supported measure on $\mathbb{R}_0$ and let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{R}_0} \varphi_k(x)\, d\mu(x) = 0 \tag{8.15}$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

PROOF. We show that if

$$\int_{\mathbb{R}_0} \varphi_k(x)\, d\mu(x) = 0 \tag{8.16}$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

Let N be a fixed positive integer. By the result (8.4), we have

$$\varphi_k(x) = \frac{d^k}{dt^k}\Phi\big(x, f(t)\big)(0) \tag{8.17}$$

for $k = 0, 1, 2, \dots, N$ x in $\mathbb{R}_0$, t in $\mathbb{C}$, where

$$f(t) = \sum_{i=0}^{N} c_i \, \frac{t^i}{i!}$$

is a polynomial. Let $\lambda = f(0)$. From (8.17) we have for $k = 1$

$$\varphi_1(x) = \frac{d}{dt})\Phi\big((x, f(t)\big)(0) = \partial_2\Phi(x, \lambda)\, c_1 \,,$$

which implies $c_1 \neq 0$.

Let x be a fixed nonnegative real number and we let for each t in $\mathbb{R}$ and for $k = 0, 1, 2, \dots, N$:

$$F_k(t) = \frac{d^k}{dt^k}\Phi\big(x, f(t)\big)\,. \tag{8.18}$$

We show that for $k = 0, 1, 2, \dots, N$ and for each t in $\mathbb{R}$

$$F_k(t) = \sum_{j=0}^{k} p_{k,j}(t)\partial_2^{(j)}\Phi\big(x, f(t)\big)\,, \tag{8.19}$$

where $p_{k,j}$ is a polynomial, and $p_{k,k}(t) = f'(t)^k$.

We prove (8.19) by induction on k. For $k = 0$ the statement is trivial. Supposing that (8.19) is proved we show it for $k+1$ instead of k. We have

$$F_{k+1}(t) = F_k'(t) = \sum_{j=0}^{k} p_{k,j}'(t)\partial_2^{(j)}\Phi\big(x, f(t)\big) +$$

$$+\sum_{j=0}^{k} p_{k,j}(t)\partial_2^{(j+1)}\Phi\big(x, f(t)\big)\, f'(t)$$

and this is the form (8.19) with $k+1$ for k. Moreover,

$$p_{k+1,k+1}(t) = p_{k,k}(t)\cdot f'(t) = f'(t)^k.$$

Then, using (8.17), we have

$$\varphi_k(x) = \sum_{j=0}^{k} c_{k,j}\partial_2^{(j)}\Phi\big(x,\lambda\big) \qquad k = 0,1,2,\ldots,N\,, \tag{8.20}$$

where $c_{k,k} \neq 0$, $c_{0,0} = 1$.

By (8.16), it follows

$$\sum_{j=0}^{k} c_{k,j} \int_{\mathbb{R}_0} \partial_2^{(j)}\Phi\big(x,\lambda\big)\,d\mu(x) = 0 \tag{8.21}$$

for $k = 0,1,2,\ldots,$. For $k = 0$ this gives

$$\int_{\mathbb{R}_0} \Phi(x,\lambda)\,d\mu(x) = 0\,. \tag{8.22}$$

For $k = 1$ we have

$$c_{1,0}\int_{\mathbb{R}_0} \Phi(x,\lambda)\,d\mu(x) + c_{1,1}\int_{\mathbb{R}_0} \partial_2\Phi\big(x,\lambda\big)\,d\mu(x) = 0\,. \tag{8.23}$$

By (8.22) and $c_{1,1} \neq 0$, this implies

$$\int_{\mathbb{R}_0} \partial_2\Phi(x,\lambda)\,d\mu(x) = 0\,. \tag{8.24}$$

Continuing this process we arrive at

$$\int_{\mathbb{R}_0} \partial_2^{(k)}\Phi(x,\lambda)\,d\mu(x) = 0 \tag{8.25}$$

for $k = 0,1,2,\ldots,N$. As N is arbitrary, we actually have that (8.25) holds for $k = 0,1,\ldots$.

We recall that the function $\hat{\mu} : \mathbb{C} \to \mathbb{C}$ defined for each complex z by

$$\hat{\mu}(z) = \int_{\mathbb{R}_0} \Phi(x,z)\,d\mu(x) \tag{8.26}$$

is the Fourier–Laplace transform of the measure μ. As μ is compactly supported, $\hat{\mu}$ is an entire function. On the other hand, as the integration in (8.26) is performed on the compact support of μ and $z \mapsto \Phi(x,z)$ is an entire function, hence the differentiation and the integration in (8.26) can be interchanged. This means that we have

$$\hat{\mu}^{(k)}(z)\frac{d^k}{dz^k}\int_{\mathbb{R}_0}\Phi(x,z)\,d\mu(x)=0 \tag{8.27}$$

holds for $k=0,1,2,\dots$, and for all z in $\mathbb{C}$. In particular, for $z=\lambda$

$$\hat{\mu}^{(k)}(\lambda)\frac{d^k}{dz^k}\int_{\mathbb{R}_0}\Phi(x,\lambda)\,d\mu(x)=0\,. \tag{8.28}$$

As $\hat{\mu}$ is an entire function, it follows $\hat{\mu}=0$. Then $\mu=0$ and our statement is proved. □

Similarly as above, we have the corresponding uniqueness result.

Theorem 8.6. *Let $K=(\mathbb{R}_0,A)$ be a Sturm–Liouville hypergroup, μ,ν compactly supported measures on $\mathbb{R}_0$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1\neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu=\nu$.*

CHAPTER 9

Conditional functional equations and applications

In this section we give the description of exponential and additive functions on some types of two-point support hypergroups. These hypergroups are studied in [**4**]. Some concerning results can be found in [**25**], [**28**], [**27**], [**30**], [**33**], [**35**] and [**26**]. Our problem leads us to the study of some conditional functional equations. Conditional functional equations play an important role in several applications of functional equations. For example, a recent volume (see [**14**]) is devoted to characterization problems in probability theory and statistics, where the results heavily depend on the solution of different conditional functional equations. In dealing with regular solutions of conditional functional equations the most powerful tools are the regularity results of [**9**], which play a fundamental role in this chapter, too.

We remark that there are several different types of investigations on the topic, which is called "conditional functional equations". The common feature is that the equation, which may have sense for all elements of some given algebraic structure, like group, ring, linear space, etc., is not supposed to hold for all values taken from this structure, but rather for some particularly chosen subset of it. The literature on such type of investigations is huge, see e.g. [**2**], [**21**], [**20**], [**19**] and [**22**].

However, it is not our purpose here to give a survey on the existing results on conditional functional equations, neither are we going to present "best possible" results from the point of view of simplicity of the proofs, or using most elementary tools — here we simply need results of this type for d'Alembert's functional equation and for the square-norm functional equation for our later purposes. We shall consider these equations on the nonnegative reals and on the $[0,1]$ interval and we are looking for continuous solutions, only. In this respect, in dealing with regular solutions of conditional functional equations the most powerful tools are the regularity results of Járai in **[9]**, which we shall use in this section, too.

1. Conditional d'Alembert-type functional equations

THEOREM 9.1. *Let $f : [0,1] \to \mathbb{C}$ be a continuous function satisfying $f(0) = 1$ and*

$$f(x+y) + f(x-y) = 2f(x)f(y)\,, \tag{9.1}$$

whenever $0 \le y \le x$ and $x + y \le 1$. Then there exists a complex number λ such that f has the form:

$$f(x) = \cosh \lambda x \tag{9.2}$$

for each x in $[0,1]$.

PROOF. Let $\varphi = Ref$ and $\psi = Imf$, then $\varphi, \psi : [0,1] \to \mathbb{R}$ are continuous functions and we have

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x)\varphi(y) - 2\psi(x)\psi(y) \tag{9.3}$$

$$\psi(x+y) + \psi(x-y) = 2\varphi(x)\psi(y) + 2\psi(x)\varphi(y) \tag{9.4}$$

whenever $0 \le y \le x$ and $x + y \le 1$, further $\varphi(0) = 1$ and $\psi(0) = 0$. Let $0 < a \le 1$ such that $\varphi(x) > 0$ for $0 \le x \le a$.

Suppose first that φ and ψ are linearly dependent that is, $\psi = c\varphi$ holds on $[0,1]$ for some real c. By $\varphi(0) = 1$ and $\psi(0) = 0$, it follows $c = 0$, hence $\psi = 0$, which implies that $f = \varphi$ is real valued. Let $A = \frac{a}{4}, B = \frac{a}{2}, C = 0, D = \frac{a}{4}$, then we can apply Remark 22.12. in **[9]** for the functional equation (9.1) on the intervals $]A,B[$ and $]C,D[$ to infer that $f = \varphi$ is $\mathcal{C}^\infty$ on the interval $]0, \frac{a}{4}[$.

If φ and ψ are linearly independent, then with the same choice of A, B, C, D we can apply the same result as above for the functional equation (9.3) on the intervals $]A,B[$ and $]C,D[$ to infer that φ and ψ

are $\mathcal{C}^\infty$ on the interval $]0, \frac{a}{4}[$. It follows that in any case f is $\mathcal{C}^\infty$ on some interval $]0, K[$, where $0 < K < 1$.

Let $m = \min\Big\{\dfrac{5K}{4}, 1\Big\}$. Suppose that $\dfrac{3K}{4} < t < m$, then, by (9.1), the substitution $x = t - \dfrac{K}{4}, y = \dfrac{K}{4}$ gives $0 \le y \le x \le 1$, $0 \le x + y \le 1$ and

$$f(t) = 2f\Big(t - \frac{K}{4}\Big)f\Big(\frac{K}{4}\Big) - f\Big(t - \frac{K}{2}\Big)\,. \tag{9.5}$$

As $t - \dfrac{K}{4}$ and $t - \dfrac{K}{2}$ is in $]0, K[$, the right hand side is $\mathcal{C}^\infty$ on $\Big]\dfrac{3K}{4}, m\Big[$. It follows that f is $\mathcal{C}^\infty$ on $\Big]0, m\Big[$. If $\dfrac{5K}{4} \ge 1$, then we have that f is $\mathcal{C}^\infty$ on $]0, 1[$. If $\dfrac{5K}{4} < 1$, then replacing K by $\dfrac{5K}{4}$ and repeating the above argument after some steps, we get that f is $\mathcal{C}^\infty$ on $]0, 1[$.

Differentiating (9.1) twice with respect to y and then substituting $y = 0$ we obtain that

$$f''(x) = cf(x) \tag{9.6}$$

for each x in $]0, 1[$ with $c = f''(0)$. As $f(0) = 1$, our statement follows. □

Theorem 9.2. *Let $f : [0, +\infty[\to \mathbb{C}$ be a continuous function satisfying $f(0) = 1$ and*

$$f(x + y) + f(x - y) = 2f(x)f(y)\,, \tag{9.7}$$

whenever $0 \le y \le x$. Then there exists a complex number λ such that f has the form:

$$f(x) = \cosh \lambda x \tag{9.8}$$

for each $x \ge 0$.

Proof. The proof is similar to that of the previous theorem. □

The following corollaries are easy consequences.

Corollary 9.1. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $f(0) = 1$ and*

$$f(x + y) + f(x - y) = 2f(x)f(y)\,, \tag{9.9}$$

whenever $0 \leq y \leq x$ *and* $x + y \leq 1$. *Then there exists a real number* λ *such that* f *has the form:*

$$f(x) = \cosh \lambda x \quad or \quad f(x) = \cos \lambda x \tag{9.10}$$

for each x *in* $[0, 1]$.

COROLLARY 9.2. *Let* $f : [0, +\infty[\to \mathbb{R}$ *be a continuous function satisfying* $f(0) = 1$ *and*

$$f(x + y) + f(x - y) = 2f(x)f(y)\,, \tag{9.11}$$

whenever $0 \leq y \leq x$. *Then there exists a real number* λ *such that* f *has the form:*

$$f(x) = \cosh \lambda x \quad or \quad f(x) = \cos \lambda x \tag{9.12}$$

for each $x \geq 0$.

2. Conditional square-norm equations

THEOREM 9.3. *Let* $f : [0, 1] \to \mathbb{C}$ *be a continuous function satisfying*

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)\,, \tag{9.13}$$

whenever $0 \leq y \leq x$ *and* $x + y \leq 1$. *Then there exists a complex number* λ *such that* f *has the form:*

$$f(x) = \lambda x^2 \tag{9.14}$$

for each $0 \leq x \leq 1$. *Moreover,* f *is real if and only if* λ *is real.*

PROOF. Clearly $f(0) = 0$. Let $\varphi = Ref$ and $\psi = Imf$, then

$$\varphi, \psi : [0, 1] \to \mathbb{R}$$

are continuous functions and we have

$$\varphi(x + y) + \varphi(x - y) = 2\varphi(x) + 2\varphi(y) \tag{9.15}$$

$$\psi(x + y) + \psi(x - y) = 2\psi(x) + 2\psi(y) \tag{9.16}$$

whenever $0 \leq y \leq x$ and $x + y \leq 1$, further $\varphi(0) = \psi(0) = 0$. This means that the real and imaginary parts of f satisfy the same functional equation (9.13), hence we may suppose that f itself is real valued.

If f and 1 are linearly dependent, then f is constant, $f = 0$, hence our statement follows.

However, if f and 1 are linearly independent, then with the same choice of A, B, C, D applying the same result as in the previous subsection for the functional equation (9.13) on the intervals $]A, B[$ and $]C, D[$ to infer that f is $\mathcal{C}^\infty$ on some interval $]0, K[$, where $0 < K < 1$. Then, using the same argument as above, we infer that f is $\mathcal{C}^\infty$ on $]0, 1[$.

Differentiating (9.13) twice with respect to y, then substituting $y = 0$ and differentiating again we obtain that

$$f'''(x) = 0 \tag{9.17}$$

for each x in $]0, 1[$, which implies that f is a quadratic polynomial on $[0, 1]$. Substituting into (9.13) our statement follows. The last assertion is obvious. □

THEOREM 9.4. *Let $f : [0, +\infty[\to \mathbb{C}$ be a continuous function satisfying*

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)\,, \tag{9.18}$$

whenever $0 \leq y \leq x$. Then there exists a complex number λ such that f has the form:

$$f(x) = \lambda x^2 \tag{9.19}$$

for each $x \geq 0$. Moreover, f is real if and only if λ is real.

PROOF. The proof is similar to that of the previous theorem. □

3. Exponentials and additive functions on two-point support hypergroups

3.1. THE CASE OF THE HYPERGROUP $K_1 = ([0, 1], *)$.

Let K_1 be the hypergroup on the interval $[0, 1]$ with the convolution defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}.$$

This is a one-dimensional compact hypergroup (see [**4**], Example 3.4.6 on p.191.). The characterizing equation of additive functions has the form

$$a(x+y)+a(|x-y|)=2a(x)+2a(y) \qquad (0\le x,y\le 1)\,. \tag{9.20}$$

Our next theorem describes the additive functions on K_1.

THEOREM 9.5. *Let K_1 be the hypergroup defined above. Then the continuous function $a:[0,1]\to\mathbb{C}$ is an additive function on K_1 if and only if there exists a complex number λ such that*

$$a(x)=\lambda x^2 \tag{9.21}$$

holds for each x in $[0,1]$.

PROOF. If a is additive on K_1, then it is continuous and satisfies equation (9.20), hence also equation (9.13). By Theorem 9.3 it has the given form.

Conversely, it is easy to check that any continuous function a of the given form is an additive function on the hypergroup K_1, hence the theorem is proved. □

Using the convolution and the definition of exponential functions we have that the continuous function $m:[0,1]\to\mathbb{C}$ is an exponential on K_1 if and only if it satisfies

$$m(x+y)+m(x-y)=2m(x)m(y) \qquad (0\le y\le x,\ x+y\le 1)\,. \tag{9.22}$$

Using the above results we obtain the following statement.

THEOREM 9.6. *Let K_1 be the two-point support hypergroup defined above. The continuous function $m:[0,1]\to\mathbb{C}$ is an exponential function on K_1 if and only if there exists a complex number λ such that*

$$m(x)=\cosh\lambda x \tag{9.23}$$

holds for each x in $[0,1]$.

PROOF. If m is an exponential on K_1, then it is continuous and satisfies equation (9.22), hence also equation (9.1). By Theorem 9.1, it has the given form.

Conversely, it is easy to check that any continuous function m of the given form is an exponential function on the K_1-hypergroup, hence the theorem is proved. □

3.2. The case of the hypergroup $K_2 = ([0, +\infty[, *)$.

The hypergroup K_2 is defined on the nonnegative reals $[0, +\infty[$ and the convolution is defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y} \qquad (0 \leq y < x).$$

This hypergroup $K_2 = ([0, +\infty[, *)$ is a noncompact one-dimensional hypergroup (see [**4**], Example 3.4.5 on p. 191.). On K_2 the characterizing equation of additive functions has the form

$$a(x+y) + a(x-y) = 2a(x) + 2a(y) \qquad (0 \leq y < x)\,. \tag{9.24}$$

Now we exhibit the general form of additive functions on the hypergroup K_2.

Theorem 9.7. *Let K_2 be the two-point support hypergroup defined above. The continuous function $a : [0, +\infty[\to \mathbb{C}$ is an additive function on K_2 if and only if there exists a complex number λ such that*

$$a(x) = \lambda x^2 \tag{9.25}$$

holds for each x in $[0, +\infty[$.

Proof. If a is additive on K_2, then it is continuous and satisfies equation (9.24), hence also equation (9.13). By Theorem 9.4 it has the given form.

Conversely, it is easy to check that any function a of the given form is an additive function on the hypergroup K_2, hence the theorem is proved. □

Using similar arguments and Theorem 9.2 we get the general form of exponential functions on K_2.

Theorem 9.8. *Let K_2 be the two-point support hypergroup defined above. The continuous function $m : [0, +\infty[\to \mathbb{C}$ is an exponential function on K_2 if and only if there exists a complex number λ such that*

$$m(x) = \cosh \lambda x \tag{9.26}$$

holds for each x in $[0, +\infty[$.

3.3. The case of the cosh-hypergroup.

As we mentioned before Sturm–Liouville hypergroups are generated by a Sturm–Liouville function. This function is continuous on the nonnegative reals and differentiable on the positive reals. Using the function cosh, we can build up a Sturm–Liouville hypergroup on the nonnegative reals, called the *cosh-hypergroup*.

Another way to introduce the cosh-hypergroup is the following. We consider the nonnegative reals as a base set and we introduce the convolution with the formula

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}\,.$$

This hypergroup is also a special two-point support hypergroup, which is actually identical with the cosh-hypergroup (see [**4**]). We denote this hypergroup by $K_3 = ([0, +\infty[, *)$. Exponentials on this hypergroup satisfy the following equation:

$$\cosh(x+y)f(x+y) + \cosh(|x-y|)f(|x-y|) = 2\cosh x\, f(x) \cosh y\, f(y)\,.$$

The substitution $g(t) = \cosh t f(t)$ gives

$$g(x+y) + g(x-y) = 2g(x)g(y)$$

for $0 \le y \le x$, which shows the relation to the hypergroup K_2. Hence the following result is a consequence of the previous theorems.

Theorem 9.9. *Let* K_3 *be the* cosh-*hypergroup. Then the continuous function* $m : [0, +\infty[\to \mathbb{C}$ *is an exponential on* K_3 *if and only if there exists a complex number* λ *such that*

$$m(x) = \frac{\cosh \lambda x}{\cosh x} \tag{9.27}$$

holds for each x *in* $[0, +\infty[$.

The case of additive functions is a bit more complicated. We consider the equation of additive functions

$$\cosh(x+y)\, a(x+y) + \cosh(x-y)\, a(x-y) =$$
$$= 2\cosh x \cosh y\, a(x) + 2\cosh x \cosh y\, a(y) \quad (0 \leq y \leq x).$$

Differentiating this equation twice with respect to the variable y, then substituting $y = 0$ and $\lambda = a''(0)$ and using the properties $a(0) = 0$ and $a'(0) = 0$ we have

$$a''(x) + 2\frac{\sinh x}{\cosh x}\, a'(x) = \lambda\,.$$

This means that on the cosh-hypergroup an additive function is the solution of the previous equation. The next theorem describes the additive functions on K_3.

Theorem 9.10. *Let* K_3 *be the* cosh-*hypergroup. Then the continuous function* $a : [0, +\infty[\to \mathbb{C}$ *is an additive function on* K_3 *if and only if there exists a complex number* λ *such that*

$$a''(x) + \frac{2\sinh x}{\cosh x} a'(x) = \lambda \tag{9.28}$$

holds for each x *in* $[0, +\infty[$.

We remark that the solutions of equation (9.28) are special Bessel-functions.

CHAPTER 10

Summary

This PhD dissertation deals with the study of classical functional equations on different types of hypergroups, characterizing additive, exponential and generalized moment functions on special hypergroups, describing linear independence and translation properties of exponential monomials and generalized moment sequences with applications to spectral analysis and spectral synthesis. We extend the classical moment problem to generalized moment sequences and solve its uniqueness on polynomial and Sturm-Liouville hypergroups.

In the first section we give the historical background of functional equations and the concept of hypergroups. In the second section we present the necessary notations and terminology. This section is followed by the third section, where we give the detailed definition of hypergroup and its basic properties. Now we give a brief overview of the concept of hypergroups. We start with a locally compact Hausdorff space K, the space of all compactly supported probability measures $\mathcal{M}_c^1(K)$, and the one-point support measures. After this we suppose that there exists a continuous mapping from $K \times K$ into $\mathcal{M}_c^1(K)$ (called convolution) and an involutive homeomorphism from K to K (called involution), furthermore there is a fixed element in K (called identity). If we introduce five axioms related to the previous mappings and identity, we get the concept of hypergroup denoted by $(K, *, \vee, e)$ or $(K, *)$. We also define commutative and Hermitian hypergroups in the obvious way. The translation plays a key role in the theory of topological groups. Since the hypergroups can be considered as the generalization of locally compact topological groups, we can introduce the translation based on the convolution. If f is a complex valued continuous function on K, and x, y are arbitrary from K, then the *left* and *right translation operator*

are defined by

$$\tau_L^y f(x) = \int_K f(t)\ d(\delta_x * \delta_y)(t), \qquad \tau_R^y f(x) = \int_K f(t)\ d(\delta_y * \delta_x)(t),$$

respectively. For the sake of simplicity we use the suggestive notation

$$f(x * y) = \tau_y f(x).$$

The following examples present several types of hypergroups. These examples show the wide applicability of hypergroups. Let $\mathbb{N}$ be the set of naturals, $(P_n)_{n\in\mathbb{N}}$ be an arbitrary orhogonal polynomial sequence with the property $P_n(1) = 1$. We say that the structure $K = (\mathbb{N}, P_n, *)$ is a *polynomial hypergroup* with convolution defined by

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n, m, k)\delta_k,$$

where $c(n, m, k)$ are nonnegative constants such that

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k.$$

The *Sturm-Liouville hypergroups* arise from Sturm-Liouville boundary value problems on nonnegative reals. A function positive and continuously differentiable on the positive reals, is called *Sturm-Liouville function.* The *Sturm-Liouville operator* L_A defined by the Sturm-Liouville function A by

$$L_A\ f = -f" - \frac{A'}{A} f',$$

where f is in $\mathcal{C}^2(\mathbb{R}_+)$. Using L_A one introduces the differential operator l by

$$l[u](x, y) = (L_A)_x u(x, y) - (L_A)_y u(x, y) =$$

$$= -\partial_1^2 u(x, y) - \frac{A'(x)}{A(x)}\ \partial_1 u(x, y) + \partial_2^2 u(x, y) + \frac{A'(y)}{A(y)}\ \partial_2 u(x, y),$$

where u is twice continuously differentiable for all positive reals x, y.

A hypergroup on $\mathbb{R}_0$ is called *Sturm-Liouville hypergroup* if there exists a Sturm-Liouville function A such that given any real-valued C^∞-function f on $\mathbb{R}_0$, the function u_f defined by

$$u_f(x,y) = f(x*y) = \int_{\mathbb{R}_0} f \; d(\delta_x * \delta_y)$$

for all positive x, y is twice continuously differentiable and satisfies the partial differential equation

$$l[u_f] = 0$$

with $\partial_2 u_f(x,0) = 0$ for all positive x. Hence u_f is a solution of the Cauchy-problem

$$\partial_1^2 u(x,y) + \frac{A'(x)}{A(x)} \, \partial_1 u(x,y) = \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)} \, \partial_2 u(x,y),$$

$$\partial_2 u_f(x,0) = 0$$

for all positive x, y.

The $SU(2)$-hypergroup is related to the set of continuous unitary irreducible representations of the group $G = SU(2)$, the *special linear group* in two dimensions. The dual object $\widehat{G}$ can be identified with the set $\mathbb{N}$ of natural numbers as it is indicated in [**4**]: the set of equivalence classes of continuous unitary irreducible representations of $SU(2)$ are given by $\{T^{(0)}, T^{(1)}, T^{(2)}, \dots\}$, where $T^{(n)}$ has dimension $n+1$, and we identify this set with $\mathbb{N}$. The convolution is given by

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n}{}' \, \frac{k+1}{(m+1)(n+1)} \delta_k \,,$$

where the dash denotes that every second term appears in the sum, only. With this convolution $\mathbb{N}$ becomes a discrete commutative hypergroup, and since all the $T^{(n)}$ are self-conjugate, the hypergroup is in fact Hermitian. We call this hypergroup the $SU(2)$*-hypergroup*.

Last but not least we introduce some special hypergroups, these will be the *two-point support hypergroups*.

Let $K_1 = ([0,1], *)$ be a hypergroup on the interval $[0,1]$ with convolution defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}.$$

This is a one-dimensional compact hypergroup (see [**4**], Example 3.4.6 on p.191.).

Let $K_2 = ([0,+\infty[, *)$ be a hypergroup on nonnegative reals convolution defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y} \qquad (0 \le y < x).$$

This hypergroup $K_2 = ([0,+\infty[, *)$ is a noncompact one-dimensional hypergroup (see [**4**], Example 3.4.5 on p. 191.).

Let K be the cosh-hypergroup on the nonnegative reals with convolution defined by

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}\,.$$

This hypergroup is also a special two-point support hypergroup, see [**4**], moreover the cosh-hypergroup can also be obtained using the Sturm–Liouville function $A(t) = \cosh^2(t)$ on the nonnegative reals as a Sturm-Liouville hypergroup.

The basic functional equations can be investigated on commutative hypergroups. Exponentials and additive functions are the solutions of the following functional equations on a fixed commutative hypergroup K

$$m(x*y) = \tau_y m(x) = \int_K m(t)\ d\ (\delta_x * \delta_y)(t) = m(x)m(y) \qquad (x, y \in K),$$

$$a(x*y) = \tau_y a(x) = \int_K a(t)\ d\ (\delta_x * \delta_y)(t) = a(x) + a(y) \qquad (x, y \in K),$$

respectively.

In the case of polynomial hypergroups exponentials have the form $m(n) = P_n(\lambda)$ and the additive functions have the form $a(n) = cP'_n(1)$

with some complex number λ and c, respectively. The case of Sturm–Liouville hypergroups is a bit more complicated, exponentials are the solutions of the boundary value problem

$$m''(x) + \frac{A'(x)}{A(x)}\, m'(x) = \lambda m(x), \qquad m(0) = 1, \quad m'(0) = 0,$$

where λ is a complex number, and additive functions are the solutions of the boundary value problem

$$a''(x) + \frac{A'(x)}{A(x)}\, a'(x) = \lambda, \qquad a(0) = 0, \quad a'(0) = 0,$$

respectively, where λ is a complex number.

In the fourth section we give the general form of exponentials (Theorem 4.6) and additive functions (Theorem 4.7) on the $SU(2)$-hypergroup.

Theorem. *The function $M : \mathbb{N} \to \mathbb{C}$ is an exponential on the $SU(2)$-hypergroup if and only if there exists a complex number λ such that*

$$M(n) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda}$$

holds for each natural number n. (Here $\lambda = 0$ corresponds to the exponential $M = 1$.)

Theorem. *The function $A : \mathbb{N} \to \mathbb{C}$ is an additive function on the $SU(2)$-hypergroup if and only if there exists a complex number c such that*

$$A(n) = \frac{c}{3} n(n+2)$$

holds for each natural number n.

The proofs of these theorems depend on solving second order homogeneous linear difference equations.

In the fifth section we prove that on any commutative hypergroup the generalized moment functions are linearly independent and we present the general form of these function sequences on the $SU(2)$-hypergroup. Let K be a commutative hypergroup. For any nonnegative integer n the complex valued continuous function φ on K is called a *generalized moment function of order n*, if there are complex valued continuous functions $\varphi_k : K \to \mathbb{C}$ for $k = 0, 1, \ldots, n$ such that $\varphi_0 \neq 0$, $\varphi_n = \varphi$ and

$$\varphi_k(x * y) = \sum_{i=0}^{k} \binom{k}{i} \varphi_i(x)\varphi_{k-i}(y)$$

holds for $k = 0, 1, \ldots, n$ and for all x, y in K.

The fact that generalized moment functions are linearly independent is proved (Theorem 5.1 and 5.2).

Theorem. *Let K be a commutative hypergroup, $n \geq 1$ an integer and $(\varphi_k)_{k=0}^n$ a sequence of generalized moment functions with $\varphi_1 \neq 0$. Then the generalized moment function φ_n is not the linear combination of the generalized moment functions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$.*

Theorem. *Let K be a commutative hypergroup, $n \geq 1$ be an integer and $(\varphi_k)_{k=0}^n$ be a sequence of generalized moment functions with $\varphi_1 \neq 0$. Then the functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ are linearly independent. In particular, none of them are identically zero.*

On the $SU(2)$-hypergroup generalized moment functions can be characterized. Using the concept of exponential family

$$\Phi(n, \lambda) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda}$$

we get the following statement (Theorem 5.3).

Theorem. *Let K denote the $SU(2)$-hypergroup and Φ the exponential family. The functions $\varphi_0, \varphi_1, ..., \varphi_N : K \to \mathbb{C}$ form a generalized moment sequence of order N on K if and only if there exist complex numbers c_j for $j = 1, 2, \ldots, N$ such that*

$$\varphi_k(n) = \frac{d^k}{dt^k} \Phi(n, f(t))(0)$$

holds for each n in $\mathbb{N}$ and for $k = 0, 1, \ldots, N$, where

$$f(t) = \sum_{j=0}^{N} \frac{c_j}{j!} t^j$$

for each t in $\mathbb{C}$.

The proof basically depends on the observation that the function $\lambda \mapsto \Phi(n, \lambda)$ is a polynomial of $\cosh(\lambda)$ of degree n.

In the sixth section we study the linear independence and the translation property of exponential monomials on the Sturm–Liouville hypergroups.

We define an exponential family $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ with the property that the function $x \mapsto \varphi(x, \lambda)$ is an exponential of K for each complex λ, and for each exponential m of K there exists a unique complex λ such that $m(x) = \varphi(x, \lambda)$ holds for every x in $\mathbb{R}_0$. Using the exponential family we define *exponential monomials* on K as functions of the form $x \mapsto P(\partial_2)\varphi(x, \lambda)$, where P is a complex polynomial and λ is a complex number. Related to exponential monomials our first result is the following (Theorem 6.1).

Theorem. *On any hypergroup different exponentials are linearly independent.*

The next result shows that a system of the second order partial derivates of an exponential family with respect to a fixed complex number also have the linear independence property (Theorem 6.2), furthermore, an arbitrary system of special exponential monomials is also linearly independent (Theorem 6.3).

Theorem. *Let K be a Sturm–Liouville hypergroup with the exponential family $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$, n a nonnegative integer and $\lambda_0 \neq 0$ a complex number. Then the special exponential monomials*

$$x \mapsto \varphi(x, \lambda_0), x \mapsto \partial_2\varphi(x, \lambda_0), \ldots, x \mapsto \partial_2^n\varphi(x, \lambda_0)$$

are linearly independent.

Theorem. *On any Sturm–Liouville hypergroup different special exponential monomials are linearly independent.*

Translation invariant function spaces play a central role in spectral synthesis. Translates of exponential monomials on Sturm–Liouville hypergroups are studied in (Theorem 6.4 and 6.5).

Theorem. *Let $K = (\mathbb{R}_0, A)$ be a Sturm-Liouville hypergroup, λ a fixed complex number, N a nonnegative integer, y an arbitrary nonnegative real number, $\Phi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ an exponential function and $\nu : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ a function with the the property $\nu(x, \lambda) = 1 \cdot \lambda = \lambda$. Then the second order partial derivatives of the function $\nu(x, \lambda)\ \Phi(x * y, \lambda)$ can be written in the form*

$$\partial_2^N \big[\nu(x, \lambda)\ \Phi(x * y, \lambda)\big] = \lambda \partial_2^N \Phi(x, \lambda)\Phi(y, \lambda) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x, \lambda)\big(\lambda \binom{N}{i} \partial_2^{N-i} \Phi(y, \lambda) + N \binom{N-1}{i} \partial_2^{N-i-1} \Phi(y, \lambda)\big).$$

Theorem. *Let $K = (\mathbb{R}_0, A)$ be a Sturm-Liouville hypergroup, λ a complex number, N a nonnegative integer and the function*

$$x \mapsto \partial_2^N \Phi(x, \lambda)$$

is a special exponential monomial. Then

$$\tau_y\ \partial_2^N \Phi(x, \lambda) = \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x, \lambda) \partial_2^{N-i} \Phi(y, \lambda)$$

for any y in $\mathbb{R}_0$.

In the seventh section we formulate the problem of spectral analysis on commutative hypergroups and solve it. Let K be an arbitrary commutative hypergroup. Spectral analysis means that any nonzero, closed, translation invariant linear subspace (variety) of $\mathcal{C}(K)$ contains exponential function. Our result can be applied to arbitrary commutative hypergroups (Theorem 7.1).

Theorem. *Spectral analysis holds for nonzero finite dimensional varieties on every commutative hypergroup.*

A similar statement can be formulated related to spectral analysis using generalized moment function (Theorem 7.2).

Theorem. *Let K be a commutative hypergroup, n a nonnegative integer and $\varphi_0, \varphi_1, \ldots, \varphi_n$ a generalized moment function sequence with $\varphi_1 \neq 0$. If V is a variety in $\mathcal{C}(K)$ and φ_n belongs to V, then φ_k belongs to V for $k = 0, 1, \ldots, n$. In particular, spectral analysis holds for V.*

In the eighth section we show an application closely related to approximation theory. Let μ be a compactly supported measure on the commutative hypergroup K and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence. Then for each natural number n the complex number

$$m_n = \int_K \varphi_n \, d\mu$$

is called the *n-th generalized moment of μ* with respect to the given generalized moment function sequence. In this case the sequence $(m_n)_{n=0}^{\infty}$ is called the *generalized moment sequence of the measure μ* with respect to the given generalized moment function sequence. Let the generalized moment function sequence $(\varphi_k)_{k=0}^{\infty}$ and the sequence of complex numbers $(m_n)_{n=0}^{\infty}$ be given. Under what conditions is there a compactly supported measure μ on K such that $(m_n)_{n=0}^{\infty}$ is the generalized moment sequence of the measure μ with respect to the given generalized moment function sequence? The other basic question is about the uniqueness: if the compactly supported measures μ and ν have the same generalized moment sequences with respect to the given generalized moment function sequence, then does it follow $\mu = \nu$? Here we deal the problem of uniqueness and solve it in the case of polynomial hypergroups in a single variable (Theorem 8.2 and 8.3) and in the case of Sturm–Liouville hypergroups (Theorem 8.5 and 8.6).

Theorem. *Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ a finitely supported measure on $\mathbb{N}$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{N}} \varphi_k(n) \, d\mu(n) = 0$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

Theorem. *Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ, ν finitely supported measures on $\mathbb{N}$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu = \nu$.*

We have similar statements for the case of Sturm–Liouville hypergroups (Theorem 8.5 and 8.6). These are the following.

Theorem. *Let* $K = (\mathbb{R}_0, A)$ *be a Sturm–Liouville hypergroup,* μ *a compactly supported measure on* $\mathbb{R}_0$ *and let* $(\varphi_k)_{k=0}^{\infty}$ *be a generalized moment function sequence on* K*. If* $\varphi_1 \neq 0$ *and*

$$\int_{\mathbb{R}_0} \varphi_k(x)\, d\mu(x) = 0$$

for $k = 0, 1, 2, \dots,$ *then* $\mu = 0$*.*

Theorem. *Let* $K = (\mathbb{R}_0, A)$ *be a Sturm–Liouville hypergroup,* μ, ν *compactly supported measures on* $\mathbb{R}_0$ *and let* $(\varphi_k)_{k=0}^{\infty}$ *be a generalized moment function sequence on* K*. If* $\varphi_1 \neq 0$ *and the generalized moment sequences of* μ *and* ν *with respect to the given generalized moment function sequence are the same, then* $\mu = \nu$*.*

In the last section we characterize exponentials and additive functions on some two-point support hypergroups. These investigations can be considered as solving conditional functional equations. Our first observation deals with the solution of a d'Alambert-type functional equation with some conditions (Theorem 9.1 and 9.2). We will utilize these results to characterize exponentials.

Theorem. *Let* $f : [0, 1] \to \mathbb{C}$ *be a continuous function satisfying* $f(0) = 1$ *and*

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

whenever $0 \leq y \leq x$ *and* $x + y \leq 1$*. Then there exists a complex number* λ *such that* f *has the form:*

$$f(x) = \cosh \lambda x$$

for each $x \geq 0$*.*

Now we reduce the conditions which is motivated by the definition of a special two-point support hypergroup.

Theorem. *Let* $f : [0, +\infty[\to \mathbb{C}$ *be a continuous function satisfying* $f(0) = 1$ *and*

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

whenever $0 \leq y \leq x$*. Then there exists a complex number* λ *such that* f *has the form:*

$$f(x) = \cosh \lambda x$$

for each $x \geq 0$*.*

In the case of real valued funtions, the d'Alembert-type functional equations with considered conditions implies the following theorems (Corollary 9.1 and 9.2).

Theorem. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying $f(0) = 1$ and*

$$f(x+y) + f(x-y) = 2f(x)f(y)\,,$$

whenever $0 \le y \le x$ and $x + y \le 1$. Then there exists a real number λ such that f has the form:

$$f(x) = \cosh \lambda x \quad \text{or} \quad f(x) = \cos \lambda x$$

for each $x \ge 0$.

Theorem. *Let $f : [0,+\infty[\to \mathbb{R}$ be a continuous function satisfying $f(0) = 1$ and*

$$f(x+y) + f(x-y) = 2f(x)f(y)\,,$$

whenever $0 \le y \le x$. Then there exists a real number λ such that f has the form:

$$f(x) = \cosh \lambda x \quad \text{or} \quad f(x) = \cos \lambda x$$

for each $0 \le x \le 1$.

Now we turn to the the conditional square-norm equations. We solve these functional equations with conditions mentioned above. Our results are the following (Theorem 9.3 and 9.4).

Theorem. *Let $f : [0,1] \to \mathbb{C}$ be a continuous function satisfying*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,,$$

whenever $0 \le y \le x$ and $x + y \le 1$. Then there exists a complex number λ such that f has the form:

$$f(x) = \lambda x^2$$

for each $0 \le x \le 1$. Moreover, f is real if and only if λ is real.

Theorem. *Let $f : [0,+\infty[\to \mathbb{C}$ be a continuous function satisfying*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,,$$

whenever $0 \le y \le x$. Then there exists a complex number λ such that f has the form:

$$f(x) = \lambda x^2$$

for each $x \geq 0$*. Moreover,* f *is real if and only if* λ *is real.*

We utilize these results to characterize exponentials and additive functions on some two-point support hypergroups. In the first case, let K_1 be the hypergroup on the interval $[0,1]$ with the convolution defined by $\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}$. The next theorem describes additive functions (Theorem 9.5) and exponentials (Theorem 9.6).

Theorem. *Let* K_1 *be the hypergroup defined above. Then the continuous function* $a : [0,1] \to \mathbb{C}$ *is an additive function on* K_1 *if and only if there exists a complex number* λ *such that*

$$a(x) = \lambda x^2$$

holds for each x *in* $[0,1]$.

Theorem. *Let* K_1 *be the two-point support hypergroup defined above. The continuous function* $m : [0,1] \to \mathbb{C}$ *is an exponential function on* K_1 *if and only if there exists a complex number* λ *such that*

$$m(x) = \cosh \lambda x$$

holds for each x *in* $[0,1]$.

In the next case let K_2 be the hypergroup defined on the nonnegative reals and the convolution is defined by $\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y}$ $(0 \leq y < x)$. Now we describes additive functions (Theorem 9.7) and exponentials (Theorem 9.8).

Theorem. *Let* K_2 *be the two-point support hypergroup defined above. The continuous function* $a : [0,+\infty[\to \mathbb{C}$ *is an additive function on* K_2 *if and only if there exists a complex number* λ *such that*

$$a(x) = \lambda x^2$$

holds for each x *in* $[0,+\infty[$.

Theorem. *Let* K_2 *be the two-point support hypergroup defined above. The continuous function* $m : [0,+\infty[\to \mathbb{C}$ *is an exponential function on* K_2 *if and only if there exists a complex number* λ *such that*

$$m(x) = \cosh \lambda x$$

holds for each x *in* $[0,+\infty[$.

In the final case let K_3 be the hypergroup defined on nonegative reals with convolution

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}\,.$$

This hypergroup is also a special two-point support hypergroup, which is actually identical with the cosh-hypergroup. We present the form of exponentials (Theorem 9.9) and additive functions (Theorem 9.10) on the cosh-hypergroup.

Theorem. *Let K_3 be the* cosh-*hypergroup. Then the continuous function $m : [0,+\infty[\to \mathbb{C}$ is an exponential on K_3 if and only if there exists a complex number λ such that*

$$m(x) = \frac{\cosh \lambda x}{\cosh x}$$

holds for each x in $[0,+\infty[$.

Theorem. *Let K_3 be the* cosh-*hypergroup. Then the continuous function $a : [0,+\infty[\to \mathbb{C}$ is an additive function on K_3 if and only if there exists a complex number λ such that*

$$a''(x) + \frac{2\sinh x}{\cosh x}a'(x) = \lambda$$

holds for each x in $[0,+\infty[$.

CHAPTER 11

Összefoglaló

Ebben a PhD disszertációban megvizsgáljuk a klasszikus függvényegyenleteket különböző típusú hipercsoportokon, megadjuk az additív, exponenciális és általánosított momentumfüggvények alakját speciális hipercsoportokon, leírjuk az exponenciális monomok lineáris függetlenségét és eltolással való kapcsolatukat, valamint általánosított momentumfüggvények felhasználásával más oldalról megközelítve igazoljuk a spektrálanalízist kommutatív hipercsoportokra. Kiterjesztjük a klasszikus momentumproblémát általánosított momentumfüggvény sorozatokra és megoldjuk az egyértelműség kérdését polinomiális és Sturm–Liouville hipercsoportokon.

Az első fejezetben bemutatjuk a függvényegyenletek történelmi hátterét és megadjuk a hipercsoportok definícióját. A második fejezet a szükséges matematikai analízis fogalmait és jelöléseit tartalmazza, valamint a hipercsoportok axiómarendszerét. Röviden áttekintjük a hipercsoportok fogalmát. Legyen K egy lokálisan kompakt Hausdorff tér és jelölje $\mathcal{M}_c^1(K)$ a K-n értelmezett kompakt tartójú valószínűségi mértékek terét, továbbá δ_x az egy pontban centrált Dirac-mértéket. Ezek után tegyük fel, hogy létezik egy $K \times K$-n értelmezett folytonos leképezés az $\mathcal{M}_c^1(K)$ térbe, nevezzük konvolúciónak, tegyük fel továbbá, hogy létezik egy folytonos leképezés K-ból K-ba, nevezzük involúciónak és tegyük fel, hogy létezik K-ban egy rögzített elem, nevezzük egységelemnek, amely egyfajta egységelem szerepet tölt be az ebben a pontban koncentrált Dirac-mértékre nézve. Ha megadjuk a szükséges axiómákat, melyekben a konvolúció, involúció és egységelem tulajdonságai rögzítettek, akkor a $(K, *, \vee, e)$ négyes, illetve rövidítve $(K, *)$ egy hipercsoportot definiál. A kommutatív és Hermite-hipercsoportok definícióját további axiómák segítségével kapjuk. Az eltolás központi szerepet játszik a topológikus csoportok elméletében és mivel a hipercsoportok bizonyos értelemben a lokálisan kompakt topológikus csoportok általánosításának tekinthetők, így hipercsoporton is bevezethetünk eltolást, mégpedig a konvolúció segítségével. Ehhez legyen f egy komplex értékű folytonos függvény K-n, legyenek x, y tetszőleges különböző K-beli elemek. Ekkor a *bal (jobb) eltolás operátora* az alábbiak szerint definiált

$$\tau_L^y f(x) = \int_K f(t)\ d(\delta_x * \delta_y)(t), \qquad \tau_R^y f(x) = \int_K f(t)\ d(\delta_y * \delta_x)(t).$$

Az átláthatóság könnyítésére bevezetjük az alábbi szuggesztív jelölést

$$f(x * y) = \tau_y f(x).$$

A harmadik fejezetben a disszertációban vizsgált konkrét hipercsoportok definícióit adjuk meg. Ezek a példák jól mutatják a hipercsoportok szeleskörű alkalmazhatóságának természetét. Elsőként legyen $\mathbb{N}$ a természetes számok halmaza, $(P_n)_{n\in\mathbb{N}}$ egy tetszőleges ortogonális polinomsorozat a $P_n(1) = 1$ feltétellel. Ekkor a $K = (\mathbb{N}, P_n, *)$ egy *polinomiális hipercsoport*, melyen a konvolúció művelet az alábbiak szerint definiált

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)\delta_k,$$

ahol $c(n, m, k)$ nemnegatív konstansok, melyekre

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n,m,k) P_k.$$

A *Sturm–Liouville hipercsoport*okat a Sturm–Liouville peremérték probléma segítségével definiáljuk a nemnegatív valós számok halmazán. Egy pozitív, a pozitív valós számokon folytonosan differenciálható A függvényt *Sturm–Liouville függvény*nek nevezünk. Az L_A Sturm–Liouville operátor az előbbi A függvény segítségével definiálható

$$L_A\ f = -f" - \frac{A'}{A} f',$$

ahol $f \in \mathcal{C}^2(\mathbb{R}_+)$. Az L_A operátor segítségével bevezetünk egy l differenciáloperátort

$$l[u](x,y) = (L_A)_x u(x,y) - (L_A)_y u(x,y) =$$
$$= -\partial_1^2 u(x,y) - \frac{A'(x)}{A(x)}\,\partial_1 u(x,y) + \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\,\partial_2 u(x,y),$$

ahol u egy kétszer folytonosan differenciálható függvény minden pozitív valós x, y-ra.

Egy hipercsoportot $\mathbb{R}_0$-on *Sturm–Liouville hipercsoport*nak nevezünk, ha létezik egy A Sturm–Liouville függvény, úgy hogy bármely, $\mathbb{R}_0$-on adott f valós értékű C^∞ függvény esetén u_f az alábbi módon definiált

$$u_f(x,y) = f(x * y) = \int_{\mathbb{R}_0} f \; d(\delta_x * \delta_y)$$

minden pozitív x, y-ra, amely kétszer folytonosan differenciálható és teljesíti a parciális differenciálegyenletet

$$l[u_f] = 0$$

a $\partial_2 u_f(x,0) = 0$ feltétellel minden pozitív x-re. Ekkor u_f megoldása a

$$\partial_1^2 u(x,y) + \frac{A'(x)}{A(x)}\,\partial_1 u(x,y) = \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\,\partial_2 u(x,y),$$
$$\partial_2 u_f(x,0) = 0$$

Cauchy-problémának minden pozitív x, y-ra.

Az $SU(2)$-hipercsoportot a $G = SU(2)$ csoport folytonos unitér irreducibilis reprezentációi vezetik be két dimenzióban. A G duálisa azonosítható a természetes számok halmazával. Részletes leírást [**4**] tartalmaz, heurisztikusan az $SU(2)$ folytonos unitér irreducibilis reprezentációi ekvivalencia osztályainak halmaza megadható a

$$\{T^{(0)}, T^{(1)}, T^{(2)}, \dots\}$$

alakban, ahol $T^{(n)}$ dimenziója $n+1$, így azonosíthatjuk $\mathbb{N}$-nel. A konvolúció az alábbi módon definiált

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n}{}' \,\frac{k+1}{(m+1)(n+1)}\delta_k\,,$$

ahol az összegzést csupán minden második tagra végezzük. Ezzel a konvolúció művelettel $\mathbb{N}$-en egy diszkrét kommutatív hipercsoportot definiálhatunk, és mivel minden $T^{(n)}$ önadjungált, így ez a hipercsoport egy Hermite-hipercsoport melyet $SU(2)$-*hipercsoport*nak nevezzük.

A következőkben bevezetünk néhány speciális, úgynevezett *két-pont tartójú hipercsoport*ot.

Legyen $K_1 = ([0,1],*)$ a $[0,1]$ kompakt intervallumon definiált hipercsoport az alábbi konvolúcióval

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}.$$

Ekkor K_1 egy kompakt, egydimenziós hipercsoport ([**4**], Example 3.4.6 on p.191.).

Legyen $K_2 = ([0,+\infty[,*)$ nemnegatív valós számokon definiált hipercsoport az alábbi konvolúcióval

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y} \qquad (0 \le y < x).$$

Ekkor K_2 egy nem kompakt egydimenziós hipercsoport ([**4**], Example 3.4.5 on p. 191.).

Legyen K a nemnegatív valós számokon definiált cosh-*hipercsoport* az alábbi konvolúcióval

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}.$$

Ez a hipercsoport is egy speciális két-pont tartójú hipercsoport [**4**], habár ez a hipercsoport egy más módon is származtatható. Ha tekintjük a $A(t) = \cosh^2(t)$ Sturm–Lioville függvényt a nemnegatív valósokon, akkor Sturm–Liouville hipercsoportként megkaphatjuk a cosh-hipercsoportot.

A klasszikus függvényegyenletek kommutatív hipercsoportokon is vizsgálhatók. Az exponenciális és additív függvények a következő függvényegyenletek megoldásai egy rögzített K kommutatív hipercsoporton

$$m(x*y) = \tau_y m(x) = \int_K m(t)\; d\,(\delta_x * \delta_y)(t) = m(x)m(y) \qquad (x,y \in K),$$

$$a(x*y) = \tau_y a(x) = \int_K a(t)\; d\,(\delta_x * \delta_y)(t) = a(x) + a(y) \qquad (x,y \in K).$$

Polinomiális hipercsoportok esetén az exponenciálisok $m(n) = P_n(\lambda)$ alakban írhatók, az additív függvények pedig $a(n) = cP_n'(1)$ alakban, ahol λ, c komplex számok. Sturm–Liouville hipercsoportok esetén a helyzet kissé komplikáltabb, az exponenciálisok a

$$m''(x) + \frac{A'(x)}{A(x)}\, m'(x) = \lambda m(x), \qquad m(0) = 1, \quad m'(0) = 0,$$

peremérték probléma megoldásai, ahol λ komplex szám, az additív függvények pedig az

$$a''(x) + \frac{A'(x)}{A(x)}\, a'(x) = \lambda, \qquad a(0) = 0, \quad a'(0) = 0,$$

peremérték probléma megoldásai, ahol λ komplex szám.

A negyedik fejezetben bizonyítással együtt megadjuk az $SU(2)$-hipercsoporton az exponenciális (4.6 Tétel) és az additív függvények (4.7 Tétel) alakját.

Tétel. *Az $M : \mathbb{N} \to \mathbb{C}$ függvény pontosan akkor exponenciális az $SU(2)$-hipercsoporton, ha létezik olyan λ komplex szám, hogy*

$$M(n) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda}$$

teljesül minden n természetes számra. (Itt a $\lambda = 0$ esetben a megfelelő exponenciális az $M = 1$.)

Tétel. *Az $A : \mathbb{N} \to \mathbb{C}$ függvény pontosn akkor additív függvény az $SU(2)$-hipercsoporton, ha létezik olyan c komplex szám, hogy*

$$A(n) = \frac{c}{3} n(n+2)$$

teljesül minden n természetes számra.

Az előbbi tételek bizonyításaiban másodrendű homogén lineáris differenciaegyenleteket oldunk meg.

Az ötödik fejezetben megmutatjuk, hogy bármely kommutatív hipercsoporton az általánosított momentumfüggvények lineárisan függetlenek és megadjuk ezen függvények konkrét alakját az $SU(2)$-hipercsoporton. Legyen K egy kommutatív hipercsoport. Minden n természetes számra a φ komplex értékű folytonos függvényt K-n *általánosított n-ed rendű* *momentumfüggvény*nek nevezzük, ha léteznek olyan $\varphi_k : K \to \mathbb{C}$ komplex értékű folytonos függvények, hogy $\varphi_0 \neq 0$, $\varphi_n = \varphi$ és

$$\varphi_k(x * y) = \sum_{i=0}^{k} \binom{k}{i} \varphi_i(x)\varphi_{k-i}(y)$$

teljesül $k = 0, 1, \ldots, n$ és minden K-beli x, y esetén.

A 5.1 és 5.2 Tételekben igazoljuk, hogy az általánosított momentumfüggvények lineárisan függetlenek.

Tétel. *Legyen K egy kommutatív hipercsoport, $n \geq 1$ egész szám és $(\varphi_k)_{k=0}^n$ egy általánosított momentumfüggvény sorozat, melyre $\varphi_1 \neq 0$. Ekkor a φ_n általánosított momentumfüggvény nem lineáris kombinációja a $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ általánosított momentumfüggvényeknek.*

Tétel. *Legyen K egy kommutatív hipercsoport, $n \geq 1$ egész szám és $(\varphi_k)_{k=0}^n$ egy általánosított momentumfüggvény sorozat, melyre $\varphi_1 \neq 0$. Ekkor a $\varphi_0, \varphi_1, \ldots, \varphi_n$ függvények lineárisan függetlenek. Speciálisan, egyikük sem azonosan zéró függvény.*

Az $SU(2)$-hipercsoporton az általánosított momentumfüggvények konkrét alakja megadható. Felhasználva az *exponenciális család*

$$\Phi(n, \lambda) = \frac{\sinh[(n+1)\lambda]}{(n+1)\sinh\lambda}$$

alakját, a 5.3 Tétel kapható.

Tétel. *Legyen K az $SU(2)$-hipercsoport és Φ legyen egy exponenciális család. A $\varphi_0, \varphi_1, ..., \varphi_N : K \to \mathbb{C}$ függvények pontosan akkor alkotnak N-ed rendű általánosított momentumfüggvény sorozatot K-n, ha léteznek olyan c_j komplex számok $j = 1, 2, \ldots, N$ esetén, melyekre*

$$\varphi_k(n) = \frac{d^k}{dt^k}\Phi(n, f(t))(0)$$

teljesül minden n természetes számra, $k = 0, 1, \ldots, N$ esetén, ahol

$$f(t) = \sum_{j=0}^{N} \frac{c_j}{j!} t^j$$

teljesül minden t komplex számra.

A bizonyítás azon az észrevételen alapúl, mely szerint a $\lambda \mapsto \Phi(n, \lambda)$ függvény $\cosh(\lambda)$-nak n-ed fokú polinomja.

A hatodik fejezetben megmutatjuk, hogy Sturm–Liouville hipercsoportokon az exponenciális monomok lineárisan függetlenek, valamint meghatározzuk ezen függvények eltoltjait.

Azt mondjuk, hogy a $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ exponenciális család, ha az $x \mapsto \varphi(x, \lambda)$ exponeciális függvény tetszőleges λ komplex szám esetén, valamint minden m exponenciális esetén K-n egyértelműen létezik olyan λ komplex szám, hogy $m(x) = \varphi(x, \lambda)$ teljesül minden x nemnegatív való szám esetén. Az exponenciális családot felhasználva definiálhatjuk az exponenciális monomokat K-n az $x \mapsto P(\partial_2)\varphi(x, \lambda)$alakban, ahol P egy komplex polinom és λ egy komplex szám. Az első lineáris függetlenséggel kapcsolatos eredményünk az alábbi (6.1 Tétel).

Tétel. *Bármely hipercsoporton az exponenciálisok lineárisan függetlenek.*

A következő eredményekben egy exponenciális család – egy rögzített komplex szám melletti – második változója szerinti parciális deriváltjainak lineáris függetlenségét bizonyítjuk (6.2 Tétel), valamint tetszőleges speciális exponenciális monomok rendszerének a lineáris függetlenségét (6.3 Tétel).

Tétel. *Legyen K egy Sturm–Liouville hipercsoport az*

$$\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$$

exponenciális családdal, n egy nemnegatív egész és $\lambda_0 \neq 0$ egy komplex szám. Ekkor a

$$x \mapsto \varphi(x, \lambda_0), x \mapsto \partial_2\varphi(x, \lambda_0), \ldots, x \mapsto \partial_2^n\varphi(x, \lambda_0)$$

speciális exponenciális monomok lineárisan függetlenek.

Tétel. *Bármely Sturm–Liouville hipercsoporton a különböző speciális exponenciális monomok lineárisan függetlenek.*

Az eltolásinvariáns függvényterek központi szerepet töltenek be a spektrálszintézisben. Sturm–Liouville hipercsoportokon az exponenciális monomok eltoltjait zárt alakban írhatjuk fel. Erről szólnak a következő eredmények (6.4 és 6.5 Tétel).

Tétel. *Legyen $K = (\mathbb{R}_0, A)$ egy Sturm–Liouville hipercsoport, λ egy rögzített komplex szám, N egy nemnegatív egész, y tetszőleges nemnegatív valós szám, $\Phi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ egy exponenciális függvény és $\nu : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ egy függvény a $\nu(x,\lambda) = 1 \cdot \lambda = \lambda$ tulajdonsággal. Ekkor a*

$$\nu(x,\lambda)\ \Phi(x * y, \lambda)$$

függvény másodrendű parciális deriváltjai az alábbi alakban írható

$$\partial_2^N \big[\nu(x,\lambda)\ \Phi(x * y,\lambda)\big] = \lambda \partial_2^N \Phi(x,\lambda)\Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x,\lambda)\big(\lambda \binom{N}{i} \partial_2^{N-i}\Phi(y,\lambda) + N\binom{N-1}{i}\partial_2^{N-i-1}\Phi(y,\lambda)\big).$$

Tétel. *Legyen $K = (\mathbb{R}_0, A)$ egy Sturm-Liouville hipercsoport, λ egy komplex szám, N egy nemnegatív egész szám és a $x \mapsto \partial_2^N \Phi(x,\lambda)$ függvény egy speciális exponenciális monom. Ekkor*

$$\tau_y\ \partial_2^N \Phi(x,\lambda) = \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i}\Phi(y,\lambda)$$

teljesül minden y nemnegatív valós szám esetén.

A hetedik fejezetben megfoglmazzuk a spektrálanalízis kérdéskörét kommutatív hipercsoportokon. Legyen K egy tetszőleges kommutatív hipercsoport. A spektrálanalízis azt vizsgálja, hogy egy nemüres, zárt, eltolás invariáns lineáris altere a $\mathcal{C}(K)$ térnek, tartalmaz-e exponenciális függvényt. Az előbbi speciális altereket szokás *varietás*nak nevezni. A következő eredményünk a spektrálanalízis teljesülését állítja véges dimenziós varietásokon (7.1 Tétel).

Tétel. *Kommutatív hipercsoportok véges dimenziós varietásaira a spektrálanalízis teljesül.*

Ezt az eredményt az általánosított momentumfüggvények segítségével is megfogalmazhatjuk (7.2 Tétel).

Tétel. *Legyen K egy kommutatív hipercsoport, n egy nemnegatív egész szám és a $\varphi_0, \varphi_1, \ldots, \varphi_n$ egy általánosított momentumfüggvény sorozat, ahol $\varphi_1 \neq 0$. Ha V egy varietás és φ_n a V altérhez tartozik, akkor*

φ_k is a V altérhez tartozik minden $k = 0, 1, \ldots, n$ esetén. Speciálisan, a spektrálanalízis teljesül a V altéren.

A nyolcadik fejezetben egy konkrét alkalmazást mutatunk be, amely az approximációelmélethez kapcsolódik. Legyen μ egy kompakt tartójú mérték a K kommutatív hipercsoporton és legyen $(\varphi_k)_{k=0}^{\infty}$ egy általánosított momentumfüggvény sorozat. Ekkor minden n természetes szám esetén a *μ mérték n-edik momentuma* a φ_k általánosított momentumfüggvény sorozatra vonatkozóan az alábbi

$$m_n = \int_K \varphi_n \, d\mu.$$

Ebben a megfogalmazásban a $(m_n)_{n=0}^{\infty}$ sorozatot a *μ mérték általánosított momentum sorozatának* nevezzük az adott általánosított momentumfüggvény sorozatra vonatkozóan. Legyen $(\varphi_k)_{k=0}^{\infty}$ egy általánosított momentumfüggvény sorozat és legyen adott egy $(m_n)_{n=0}^{\infty}$ komplex sorozat. Azt kérdezzük, hogy milyen feltételek mellett létezik egy μ mérték a K hipercsoporton úgy hogy a $(m_n)_{n=0}^{\infty}$ komplex sorozat éppen a μ mérték általánosított momentum sorozata az előre adott általánosított momentumfüggvény sorozatra vonatkozóan? A következő kérdés az egyértelműség problémáját veti fel: ha a μ és ν kompakt tartójú mértékek esetén az általánosított momentumok rendre megegyeznek egy adott általánosított momentumfüggvény sorozatra vonatkozóan, akkor következik-e, hogy a két mérték megegyezik, $\mu = \nu$? Az egyértelműség problémáját a következő eredmények tartalmazzák egyváltozós polinomiális (8.2 és 8.3 Tétel) és Sturm–Liouville hipercsoportokon (8.5 és 8.6 Tétel).

Tétel. *Legyen $K = (\mathbb{N}, P_n)$ egy polinomiális hipercsoport, μ egy véges tartójú mérték $\mathbb{N}$-en és legyen $(\varphi_k)_{k=0}^{\infty}$ egy általánosított momentumfüggvény sorozat K-n. Ha $\varphi_1 \neq 0$ és*

$$\int_{\mathbb{N}} \varphi_k(n) \, d\mu(n) = 0$$

teljesül minden $k = 0, 1, 2, \ldots,$ esetén, akkor $\mu = 0$.

Tétel. *Legyen* $K = (\mathbb{N}, P_n)$ *egy polinomiális hipercsoport,* μ, ν *véges tartójú mértékek* $\mathbb{N}$*-en és legyen* $(\varphi_k)_{k=0}^{\infty}$ *egy általánosított momentumfüggvény sorozat* K*-n. Ha* $\varphi_1 \neq 0$ *és a* μ *és* ν *mértékek általánosított momentumai az adott általánosított momentumfüggvény sorozatra vonatkozóan rendre megegyeznek, akkor* $\mu = \nu$*.*

Hasonló állításokat fogalmazhatunk meg Sturm–Liouville hipercsoportok esetén (8.5 és 8.6 Tétel).

Tétel. *Legyen* $K = (\mathbb{R}_0, A)$ *egy Sturm–Liouville hipercsoport,* μ *egy kompakt tartójú mérték* $\mathbb{R}_0$*-on és legyen* $(\varphi_k)_{k=0}^{\infty}$ *egy általánosított momentumfüggvény sorozat* K*-n. Ha* $\varphi_1 \neq 0$ *és*

$$\int_{\mathbb{R}_0} \varphi_k(x)\, d\mu(x) = 0$$

teljesül minden $k = 0, 1, 2, \ldots,$ *esetén, akkor* $\mu = 0$*.*

Tétel. *Legyen* $K = (\mathbb{R}_0, A)$ *egy Sturm–Liouville hipercsoport,* μ, ν *kompakt tartójú mértékek* $\mathbb{R}_0$*-on és legyen* $(\varphi_k)_{k=0}^{\infty}$ *egy általánosított momentumfüggvény sorozat* K*-n. Ha* $\varphi_1 \neq 0$*, valamint a* μ *és a* ν *mértékek általánosít momentumai az adott általánosított momentumfüggvény sorozatra vonatkozóan rendre megegyeznek, akkor* $\mu = \nu$*.*

Az utolsó fejezetben megadjuk az exponenciális és additív függvények alakját néhány két-pont tartójú hipercsoporton. Ezek a vizsgálatok feltételes függvényegyenletek megoldásán alapulnak. Elsőként a d'Alembert-típusú függvényegyenleteket oldjuk meg néhány plusz feltétel alkalmazásával (9.1 és 9.2 Tétel). Ezeket a plusz feltételeket a két-pont tartójú hipercsoportok definíciói igénylik. Ezeket az eredményeket az exponenciálisok meghatározásához használjuk.

Tétel. *Legyen* $f : [0, 1] \to \mathbb{C}$ *folytonos függvény az* $f(0) = 1$ *feltétellel és*

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

ahol $0 \leq y \leq x$ *és* $x + y \leq 1$*. Ekkor létezik olyan* λ *komplex szám, hogy* f *az alábbi alakban írható*

$$f(x) = \cosh \lambda x$$

minden $0 \leq x \leq 1$ *esetén.*

Tétel. *Legyen* $f : [0, +\infty[\to \mathbb{C}$ *folytonos függvény az* $f(0) = 1$ *feltétellel és*

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

ahol $0 \le y \le x$. *Ekkor létezik olyan* λ *komplex szám, hogy* f *az alábbi alakban írható*

$$f(x) = \cosh \lambda x$$

minden $x \ge 0$ *esetén.*

Valós értékű függvények esetén a d'Alembert-típusú függvényegyenletekre a szóban forgó feltételekkel a következő eredmények adódnak (9.1 és 9.2 Következmény).

Tétel. *Legyen* $f : [0, 1] \to \mathbb{R}$ *folytonos függvény az* $f(0) = 1$ *feltétellel és*

$$f(x+y) + f(x-y) = 2f(x)f(y) \, ,$$

ahol $0 \le y \le x$ *és* $x + y \le 1$ *teljesül. Ekkor létezik olyan* λ *valós szám, hogy* f *az alábbi alakban írható*

$$f(x) = \cosh \lambda x \quad \textit{vagy} \quad f(x) = \cos \lambda x$$

minden $0 \le x \le 1$ *esetén.*

Tétel. *Legyen* $f : [0, +\infty[\to \mathbb{R}$ *folytonos függvény az* $f(0) = 1$ *feltétellel és*

$$f(x+y) + f(x-y) = 2f(x)f(y) \, ,$$

ahol $0 \le y \le x$. *Ekkor létezik olyan* λ *valós szám, hogy* f *az alábbi alakban írhtó*

$$f(x) = \cosh \lambda x \quad \textit{vagy} \quad f(x) = \cos \lambda x$$

minden $0 \le x \le 1$ *esetén.*

Most rátérünk a feltételes norma-négyzet egyenletekre. Ezeket az egyenleteket az előbb tárgyalt feltételek mellett oldjuk meg a következőkben (9.3 és 9.4 Tétel).

Tétel. *Legyen* $f : [0, 1] \to \mathbb{C}$ *folytonos függvény és*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \, ,$$

ahol $0 \le y \le x$ *és* $x + y \le 1$. *Ekkor létezk olyan* λ *komplex szám, hogy* f *az alábbi alakban írható*

$$f(x) = \lambda x^2$$

minden $0 \le x \le 1$ *esetén. Továbbá,* f *valós függvény pontosan akkor, ha* λ *valós szám.*

Tétel. *Legyen* $f : [0, +\infty[\to \mathbb{C}$ *folytonos függvény és*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,,$$

ahol $0 \le y \le x$*. Ekkor létezik olyan* λ *komplex szám, hogy* f *az alábbi alakban írható*

$$f(x) = \lambda x^2$$

minden $x \ge 0$ *esetén. Továbbá,* f *valós függvény pontosan akkor, ha* λ *valós szám.*

Ezeket az eredményeket felhasználva megadhatjuk az exponenciális és additív függvények lakjait néhány két-pont tartójú hipercsoporton. Elsőként legyen K_1 a $[0,1]$ kompakt intervallumon definiált hipecsoport a $\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}$ konvolúcióval. Az additív (9.5 Tétel) és exponenciális (9.5 Tétel) függvények az alábbiak.

Tétel. *Legyen* K_1 *a fent definiált hipercsoport. Az* $a : [0,1] \to \mathbb{C}$ *folytonos függvény* K_1*-en additív függvény pontosan akkor, ha létezik olyan* λ *komplex szám, hogy*

$$a(x) = \lambda x^2$$

teljesül minden $0 \le x \le 1$ *esetén.*

Tétel. *Legyen* K_1 *a fent definiált hipercsoport. Az* $m : [0,1] \to \mathbb{C}$ *folytonos függvény* K_1*-en exponenciális pontosan akkor, ha létezik olyan* λ *komplex szám, hogy*

$$m(x) = \cosh \lambda x$$

teljesül $0 \le x \le 1$ *esetén.*

A következő hipercsoport legyen K_2, amely a nemnegatív valós számokon értelmezett a $\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y} \quad (0 \le y < x)$ konvolúcióval. K_2 additív függvényei (9.7 Tétel) és exponenciálisai (9.8 Tétel) az alábbiak.

Tétel. *Legyen* K_2 *a fent definiált hipercsoport. Az* $a : [0, +\infty[\to \mathbb{C}$ *folytonos függvény* K_2*-n pontosan akkor additív függvény, ha létezik olyan* λ *komplex szám, hogy*

$$a(x) = \lambda x^2$$

teljesül minden $x \geq 0$ esetén.

Tétel. *Legyen K_2 a fent definiált hipercsoport. Az $m : [0, +\infty[\to \mathbb{C}$ folytonos függvény K_2-n pontosan akkor exponenciális, ha létezik olyan λ komplex szám, hogy*

$$m(x) = \cosh \lambda x$$

teljesül minden $x \geq 0$ esetén.

A következő hipercsoportot jelölje K_3, amely a nemnegatív valós számokon értelmezett az alábbi konvolúcióval

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}\,.$$

Ez a hipercsoport is egy speciális két-pont tartójú hipercsoport és a cosh-hipercsoport nevet viseli. Az exponenciálisok (9.9 Tétel) és additív függvények (9.10 Tétel) az alábbi eredményekben tárgyalhatók.

Tétel. *Legyen K_3 a* cosh-*hipercsoport. Ekkor az $m : [0, +\infty[\to \mathbb{C}$ folytonos függvény K_3-on pontosan akkor exponenciális, ha létezik olyan λ komplex szám, hogy*

$$m(x) = \frac{\cosh \lambda x}{\cosh x}$$

teljesül minden $x \geq 0$ esetén.

Tétel. *Legyen K_3 a* cosh-*hipercsoport. Ekkor az $a : [0, +\infty[\to \mathbb{C}$ folytonos függvény K_3-on pontosan akkor additív függvény, ha létezik olyan λ komplex szám, hogy*

$$a''(x) + \frac{2\sinh x}{\cosh x}a'(x) = \lambda$$

teljesül minden $x \geq 0$ esetén.

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3. *A Moment Problem on Hypergroups II.*, The Eleventh Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities,
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