

Method for computing angle constrained isoptic curves for surfaces*

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Submitted October 10, 2013 — Accepted December 13, 2013

Abstract

In computer graphics and geometric modeling for good quality displaying of an object it is required that the object must fit on the screen. It often happens, for example when we are using modeling software, that the object we would like to rotate around an axis, or edit from another point of view, is partly out of the screen, and thus some parts are not visible. In two dimensions the isoptic curve of a curve is constructed by involving lines with a given angle intersect each other at a certain point of the isoptic curve. In three dimensions the points of an isoptic curve may be the admissible positions of the camera. So from these points we can watch the object with respect to the given viewing angle. The purpose of this paper is to find a general method and computational algorithm that helps to locate the closest possible position of the camera, which positions form a closed curve around the surface. The developed algorithm produces this curve for a special case.

Keywords: isoptic curve, Bézier surface

MSC: 65D17, 68U07

1. Introduction

Let us overview briefly the planar case of the isoptic problem.

*This research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4.A/2-11-1-2012-0001 ‘National Excellence Program’.

Definition 1.1. For a given curve C , consider the locus of points from where the tangents to C meet at a fixed given angle.

The isoptic of quadratic curves can be determined from the definition by calculation based on elementary geometric results [5] and several further results are also known, mainly about classical curves [6, 7]. For freeform curves, the intersections of the appropriate tangents of the given curve determine the points of the isoptic curve. In this case we need to derivate the given curve to find the involving lines. With this algorithm we can find the set of points which form the isoptic curve of convex curves (see Figure 1).

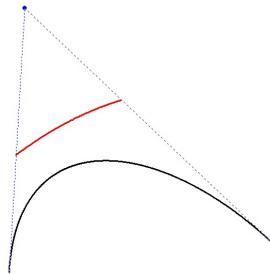


Figure 1: The isoptic curve with 90° (red one) of a Bézier curve

In case of Bézier curves there is a calculation for obtaining an exact formula of the isoptic curve based on [1]. This method defines the isoptic as the envelope of envelopes of families of isoptic circles over the chords of the Bézier curve. Although it cannot always be resolved exactly, the numerical algorithms can provide sufficient results as well.

In this paper we provide a way of generalization of isoptics in 3D and compute a curve around a convex surface from which the surface can be seen under a given angle.

2. Isoptic of surfaces

The first difficulty is that the generalization of the two-dimensional definition is not straightforward and not unique in the three dimensional space. It is not evident how to define the angle of view in 3D. One possibility is looking for points, from where we can draw tangent lines to the surface, which intersect each other at the given angle, but this computation is not uniquely determined. So we are looking for points in a special case for Bézier surfaces.

Figure 2 shows the special circumstances. At first, we suppose that the surface is convex and it is entirely above (but as close as possible to) the base plane which is given by the equation $Y = 0$ in the coordinate system. Moreover, we suppose that the origin of the coordinate system is in the orthogonal projection of the surface

onto the base plane. We will search for points of this plane around the surface, from which the viewing angle, that is the angle of the plane $Y = 0$ and the deepest tangent line to the surface from this point is a predefined angle. We will call these points isoptic points of the surface. To locate an isoptic point we use two vectors: \mathbf{m} and \mathbf{v} . The vector \mathbf{m} is parallel to the X -axis; the vector \mathbf{v} is rotated around the X -axis. This angle of the rotation needs to be selected as follows. If we look from one isoptic point to the origin, the surface should fit on the viewing screen on the top. For all points it is true that vector \mathbf{v} and one of the normal vectors of the surface are orthogonal. It is also true for the vector \mathbf{m} . There are two ways to do the scan: by rotating the surface and by rotating vectors \mathbf{m} and \mathbf{v} around the Y -axis. With this condition, we can find several points on the plane $Y = 0$, and based on these points we can produce a curve around the surface (see Figure 3) interpolating these points by a closed B-spline curve [2]. In the next subsections details of the two computational approach are provided.

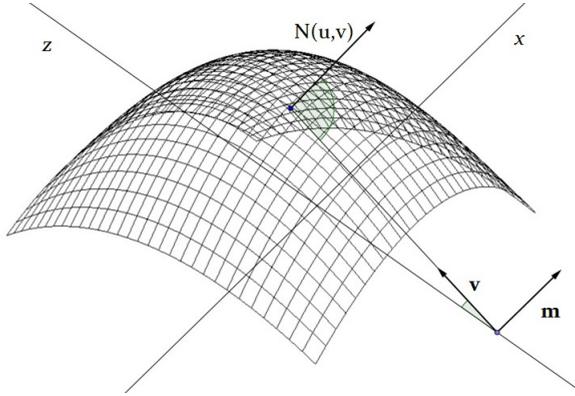


Figure 2: The special case for finding isoptic points of Bézier surface

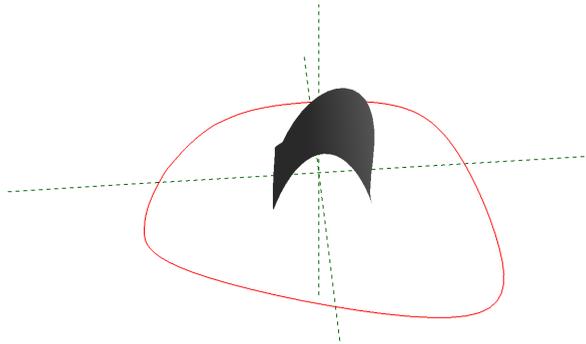


Figure 3: Result of the algorithm

2.1. Rotating the control points of the surface

In this algorithm we have to rotate the control points of the Bézier surface, which is given by the following equation:

$$S(u, v) = \sum_{i=0}^n \sum_{j=0}^n B_i^n(u) B_j^n(v) P_{i,j}.$$

We need to compute the normal vector by the partial derivatives of the surface, with respect to the parameters u and v : $U(u, v) = \frac{\partial S(u, v)}{\partial u}$ and $V(u, v) = \frac{\partial S(u, v)}{\partial v}$. The normal vector will be the cross product of the two partial derivatives: $N(u, v) = U(u, v) \times V(u, v)$.

We can write the equation of a plane that touches the Bézier surface in the point from where the normal vector perpendicular to vectors \mathbf{m} and \mathbf{v} :

$$N_x(u, v)(X - X_0) + N_z(u, v)(Z - Z_0) = N_y(u, v)Y_0,$$

where X_0, Y_0, Z_0 are the coordinates of the points of the Bézier surface. The isoptic point is lying on the line in which this plane and the base plane given by the equation $Y = 0$ intersect each other. By solving this equation with $X = 0$, we obtain the distance from the origin. This will be the Z coordinate of the isoptic point:

$$Z = \frac{N_y(u, v)Y_0 - N_x(u, v)X_0}{N_z(u, v)} + Z_0.$$

The X and Y coordinates of this point are equal to zero. We need to rotate this point with the same angle as we used to rotate the surface. While we rotate the control points around the Y -axis the following conditions need to be satisfied: $\langle N(u, v), \mathbf{m} \rangle = 0$ and $\langle N(u, v), \mathbf{v} \rangle = 0$, that means the following:

$$\begin{aligned} N_x(u, v) &= 0 \\ \sin(\alpha) \cdot N_y(u, v) &= \cos(\alpha) \cdot N_z(u, v). \end{aligned}$$

If we can solve this equation for u and v we obtain an exact formula to compute the normal vector and from this we can calculate exactly the position of each isoptic point. Unfortunately the solution cannot be given in closed form in most of the cases, but numerical methods work sufficiently.

2.2. Rotating vectors \mathbf{m} and \mathbf{v}

The other way to find the isoptic points is to rotate the vector m and v around the Y -axis. The coordinates of the vectors will be the following:

$$\mathbf{m} = (\cos(\beta), 0, -\sin(\beta)) \quad \text{and} \quad \mathbf{v} = (-\sin(\beta) \cdot \cos(\alpha), \sin(\alpha), -\cos(\beta) \cdot \cos(\alpha)),$$

where β is the angle of rotation around the Y -axis, in the interval $[0, 2\pi]$. The condition of the search is that $\langle N(u, v), \mathbf{m} \rangle = 0$ and $\langle N(u, v), \mathbf{v} \rangle = 0$ have to be fulfilled. From these we obtain the following:

$$N_x(u, v) = \tan \beta \cdot N_z(u, v)$$

$$N_y(u, v) = \cot \alpha \cdot \frac{1}{\cos \beta} \cdot N_z(u, v).$$

We also tried to resolve this equation for u and v , but it cannot be obtained in closed form as well. But there is no need to rotate the isoptic points so we can compute exactly the X and Y coordinates of these points.

Let e be the intersection line of the plane that touches the surface and the base plane given by the equation $Y = 0$, and let f be the line which has one point in the origin and its normal vector m . The isoptic point is the intersection of these lines:

$$X = \frac{\sin(\beta) \cdot c}{\cos(\beta) \cdot N_z(u, v) - \sin(\beta) \cdot N_x(u, v)}$$

$$Z = \frac{\cos(\beta) \cdot c}{\cos(\beta) \cdot N_z(u, v) - \sin(\beta) \cdot N_x(u, v)},$$

where $c = N_x(u, v)X_0 + N_y(u, v)Y_0 + N_z(u, v)Z_0$. Since $\langle N(u, v), \mathbf{m} \rangle = 0$, we can compute the coordinates by the following equations:

$$X = \frac{cN_x(u, v)}{\langle N_x(u, v), N_x(u, v) \rangle + \langle N_z(u, v), N_z(u, v) \rangle}$$

$$Z = \frac{cN_z(u, v)}{\langle N_x(u, v), N_x(u, v) \rangle + \langle N_z(u, v), N_z(u, v) \rangle}.$$

2.3. Calculation by linear combination

There is another way to compute the isoptic points. To find these points we still need to use the vectors \mathbf{m} and \mathbf{v} . If the normal vector of the surface is the cross product of the vector $U(u, v)$ and $V(u, v)$, and $N(u, v)$ is perpendicular to the vector \mathbf{m} and \mathbf{v} , then \mathbf{v} , \mathbf{m} , $U(u, v)$ and \mathbf{v} , \mathbf{m} , $V(u, v)$ are linearly dependent. Thus $U(u, v)$ and $V(u, v)$ can be obtained by linear combination of vectors \mathbf{m} and \mathbf{v} . This means the following:

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & \sin(\alpha) & -\cos(\alpha) \\ U_x(u, v) & U_y(u, v) & U_z(u, v) \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & \sin(\alpha) & -\cos(\alpha) \\ V_x(u, v) & V_y(u, v) & V_z(u, v) \end{vmatrix} = 0.$$

From this we obtain the following equations:

$$\sin(\alpha) \cdot U_z(u, v) + \cos(\alpha) \cdot U_y(u, v) = 0$$

$$\sin(\alpha) \cdot V_z(u, v) + \cos(\alpha) \cdot V_y(u, v) = 0.$$

By solving this system we acquire a complex fifth order equation for u and v , but it is not possible approximate the roots even by computer algebra systems.

3. Conclusions

The possibilities of 3 dimensional generalization of isoptic curves are considered. We provided a special scene where isoptic points of a surface can be computed, although in some cases only by numerical methods. This method can be applied in circumstances when a convex surface is above a given plane and the method provides a curve in this plane from the point of which one can see the surface under a given angle. Further investigations in terms of computational efficiency and generalization can be subject of future work.

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