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Mathematical structure of Positive Operator Valued Measures and Applications

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Mathematical structure of Positive Operator Valued Measures and Applications

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a matematika tudományágban.

Írta: Beneduci Roberto fizikus.

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Contents

Chapter 1

Introduction

The present dissertation is devoted to the study of positive operator valued measures (POVMs) which were introduced in the 40's [73, 74, 71] in order to study self-adjoint extensions of symmetric operators.

A POVM is a σ-additive map $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ from the Borel σ-algebra of a topological space X to the space $\mathcal{F}(\mathcal{H})$ of positive operators less than the identity (effects). This generalizes the concept of spectral resolution of the identity, which is a σ-additive map $E : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$ from the Borel σ-algebra of the reals to the space of orthogonal projections $\mathcal{E}(\mathcal{H})$. It is then apparent that the generalization from spectral measures to POVMs is based on the replacement of $\mathbb{R}$ and $\mathcal{E}(\mathcal{H})$ by a topological space X and the space of effects $\mathcal{F}(\mathcal{H})$ respectively.

In the 70's several scholars [32, 45, 48, 50, 49, 66, 2, 78, 27] used POVMs as the main tool in the description of the quantum measurement process and to formulate a general theory of statistical decision. Moreover POVMs suggested an extension of the concept of quantum observable which was previously encoded in the mathematical structure of a spectral measure or, equivalently, of a self-adjoint operator. Such an extension turned out to be very fruitful since it permitted a mathematical representation of the time observable, the photon localization observable and the phase observable which resulted to be impossible in the old framework (spectral measures). Another improvement allowed by POVMs consists in the possibility of building a representation of standard quantum mechanics in a phase space by mapping density operators to positive definite density distribution functions on a symplectic space [78, 83, 27, 85]. Nowdays, POVMs are a standard tool in quantum information theory and quantum optics [50, 87, 90].

It was then natural both from the mathematical and the physical viewpoint to ask what are the relationships between POVMs and spectral measures. Four possible answers have been given each one corresponding to a different characterizations of POVMs [73, 48, 2, 31, 10, 11, 22, 53]. Although the answer given by Naimark [73, 71, 1, 83] is the most powerful since it refers to general POVMs and is not limited to the commutative case, it is to be confronted with the problem of the physical interpretation of the extended Hilbert space it introduces. As we shall see, if one avoids the commitment with an extended Hilbert space, a clear answer to our question can be given in the commutative case [48, 2, 31, 10, 11, 12, 53], the commutative POVMs being the most similar to the spectral measures.

The present dissertation is based on the author's contribution to the formulation of one of the possible characterizations of commutative POVMs (chapter 2). The analysis of the relationships between such characterization and Naimark's theorem is the topic of chapter 3. Chapter 4 is devoted to the analysis of its relevance to the concept of "informational content" of an observable. The last chapter is devoted to the characterization of the uniform continuity of a general POVM (not necessarily commutative), to the analysis of the POVMs with the norm-1 property and to the analysis of the relevance of norm-1 property and uniform continuity to the localization problem in relativistic quantum mechanics.

Next we outline the main properties of POVMs, then we show how they emerge in the quantum context and briefly outline the main objectives and results of the present work.

1.1 Main properties of POVMs

In the present section we recall the main properties of POVMs. We restrict ourselves to the case of POVMs defined on the Borel σ-algebra of a topological set X. For a more general exposition we refer to the book by Berberian [24]. There are other concepts and properties concerning POVMs that, when necessary, will be introduced in each single chapter.

Definition 1.1.1. *Let X be a topological space and $\mathcal{B}(X)$ the Borel σ-algebra on X. A POVM is a map $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ such that:*

$$F\Big(\bigcup_{n=1}^{\infty} \Delta_n\Big) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(X)$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(X) = \mathbf{1}.$$

Definition 1.1.2. *A POVM is said to be commutative if*

$$\big[F(\Delta_1), F(\Delta_2)\big] = \mathbf{0} \;\; \forall\, \Delta_1\,, \Delta_2 \in \mathcal{B}(X). \tag{1.1}$$

Definition 1.1.3. *A POVM is said to be orthogonal if*

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \;\; \textit{if}\; \Delta_1 \cap \Delta_2 = \emptyset. \tag{1.2}$$

Definition 1.1.4. *A PVM is an orthogonal, normalized POVM.*

In the case of a PVM E we have $\mathbf{0} = E(\Delta)[1 - E(\Delta)] = E(\Delta) - E^2(\Delta)$. Therefore, $E(\Delta)$ is a projection operator for every $\Delta \in \mathcal{B}(X)$. We have proved the following proposition.

Proposition 1.1.5. *A PVM E on X is a map $E : \mathcal{B}(X) \to \mathcal{E}(\mathcal{H})$ from the Borel σ-algebra of $\mathcal{B}(X)$ to the space of projection operators on $\mathcal{H}$.*

Definition 1.1.6. *A real PVM $E : \mathcal{B}(R) \to \mathcal{F}(\mathcal{H})$ is said to be a spectral measure.*

Definition 1.1.7. *The spectrum $\sigma(F)$ of a POVM F is the set of points $x \in X$ such that $F(\Delta) \neq \mathbf{0}$, for any open set Δ containing x.*

The spectrum $\sigma(F)$ of a POVM F is a closed set since its complement $X - \sigma(F)$ is the union of all the open sets $\Delta \subset X$ such that $F(\Delta) = \mathbf{0}$.

Definition 1.1.8. *The von Neumann algebra $\mathcal{A}^W(F)$ generated by the POVM F is the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta \in \mathcal{B}(X)}$.*

In the following we use the symbols $w - \lim$ and $u - \lim$ to denote the limit in the weak operator topology and the limit in the uniform operator topology respectively.

Definition 1.1.9. *A POVM is regular if for any Borel set Δ,*

$$w - \lim_{i\to\infty} F(G_j) = F(\Delta) = w - \lim_{i\to\infty} F(O_j)$$

where, $\{G_j\}_{j\in\mathbb{N}}$, $\Delta \subset G_j$, is a decreasing family of open sets and $\{O_j\}_{j\in\mathbb{N}}$, $O_j \subset \Delta$, is a increasing family of compact sets and the convergence is in the weak operator topology.

We recall [64] that a topological space (X, τ) is second countable if it has a countable basis for its topology τ; i.e., if there is a countable subset $\mathcal{B}$ of τ such that each member of τ is the union of members of $\mathcal{B}$.

Proposition 1.1.10. *A POVM defined on a Hausdorff locally compact, second countable space X is regular.*

Proof. A locally compact Hausdorff space is regular. (See Ref. [69], page 205). By the Urysohn's theorem, a second countable regular space is metrizable (see [69], page 215). Moreover, the second countability implies the σ-compactness ([69], page 289). In a metrizable σ-compact space the ring of Borel sets coincides with the ring of Baire sets (see page 25 in [24]) and the thesis comes from the fact that each Baire POVM is regular (see Theorem 18 in [24]). □

We can introduce integration with respect to a POVM. Indeed, for any $\psi \in \mathcal{H}$, the expression $\langle F(\cdot)\psi, \psi\rangle$ defines a probability measure and we will use the symbol $d\langle F_\lambda \psi, \psi\rangle$ to mean integration with respect to the measure $\langle F(\cdot)\psi, \psi\rangle$. We shall say that a measurable function $f : N \subset X \to f(N) \subset \mathbb{R}$ is almost everywhere (a.e.) one-to-one with respect to a POVM F if it is one-to-one on a subset $N' \subset N$ such that $N - N'$ is a null set with respect to F. We shall say that a function $f : X \to \mathbb{R}$ is bounded with respect to a POVM F, if it is equal almost everywhere to a bounded function g with respect to F, that is, if $f = g$ a.e. with respect to the measure $\langle F(\cdot)\psi, \psi\rangle$, $\forall \psi \in \mathcal{H}$. For any real, bounded and measurable function f and for any POVM F, there is a unique [24] bounded self-adjoint operator $B \in \mathcal{L}_s(\mathcal{H})$ such that

$$\langle B\psi, \psi\rangle = \int f(\lambda) d\langle F_\lambda \psi, \psi\rangle, \quad \text{for each} \quad \psi \in \mathcal{H}. \tag{1.3}$$

If equation (1.3) is satisfied, we write $B = \int f(\lambda) dF_\lambda$ or $B = \int f(\lambda) F(d\lambda)$ equivalently.

By the spectral theorem [35, 79], real PVMs E (spectral measures) are in a one-to-one correspondence with self-adjoint operators A, the correspondence being given by

$$A = \int \lambda dE_\lambda.$$

Moreover in this case, a functional calculus can be developed. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable real-valued function, we can define the self-adjoint operator [79]

$$f(A) = \int f(\lambda) dE_\lambda, \tag{1.4}$$

where E is the PVM corresponding to A. If f is bounded, then $f(A)$ is bounded [79]. In particular,

$$E[i] := \int t^i \, dE_t = \Big(\int t \, dE_t \Big)^i = A^i \tag{1.5}$$

and $A = \int t \, dE_t$ is the generator of the von Neumann algebra generated by E.

We point out that if F is not projection valued, equations (1.5) and (1.4) do not hold [59] and, in order to recover the generator of the von Neumann algebra generated by F, we need all the moments of F. In particular, in the case of a real commutative POVM F with bounded spectrum and such that $F(\Delta)$ is discrete for any Δ (see chapter 3 for the details), we have

$$A = \sum_{i=0}^{\infty} \alpha_i \, F[i], \quad \alpha_i \geq 0, \quad \sum_{i=0}^{\infty} \alpha_i < \infty$$

where, A is a generator of the von Neumann algebra $\mathcal{A}^W(F)$.

The following result due to Naimark shows that a POVM F in a Hilbert space $\mathcal{H}$ can always be interpreted as the restriction to $\mathcal{H}$ of a PVM E defined in an extended Hilbert space $\mathcal{H}^+$.

Theorem 1.1.11. (Naimark [72, 1, 50, 71]) *Let F be a POVM of the Hilbert space $\mathcal{H}$. Then, there exist a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a PVM E^+ of the space $\mathcal{H}^+$ such that*

$$F(\Delta) = P^+ E^+(\Delta)_{|\mathcal{H}}$$

where P^+ is the operator of projection onto $\mathcal{H}$.

Naimark's theorem is a powerful result on the relationships between PVMs and POVMs but from the physical viewpoint its interpretation is not clear. That is due to the difficulties in interpreting the Hilbert space $\mathcal{H}^+$. As we shall see in chapter 3, a relationships between the Naimark's extension of F and the sharp version A of F can be established in the commutative case. That could provide new insights in the problem of the interpretation of the Naimark's extension.

1.2 Positive operator valued measures in the quantum mechanical framework

As we have already seen the set of PVMs is a subset of the set of POVMs. Moreover, real PVMs (spectral measures) are in one-to-one correspondence with self-adjoint operators (spectral theorem) [79] and are used in standard quantum mechanics to represent quantum observables. It was pointed out [2, 27, 32, 50, 78, 82] that POVMs are more suitable than spectral measures in representing quantum observables.

From a general theoretical viewpoint, the introduction of POVMs can be justified by analyzing the statistical description of a measurement. Indeed, a measurement procedure can be described as an affine map from the set of states

$\mathcal{S}$ into the set of probability measures on $\mathcal{B}(X)$. The set of states represents the set of possible preparation procedures of the system while the set of probability measures represents the statistical distribution of the results of the possible measurements. It was shown [50, 51] that there exists a one-to-one correspondence between POVMs $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ and affine maps $S \mapsto \mu_S^F(\cdot)$ from the set of states $\mathcal{S}$ into the set of probability measures on $\mathcal{B}(X)$. Moreover, this correspondence is determined by the relation $\mu_S^F(\Delta) = Tr[SF(\Delta)]$. That allows one to interpret the number

$$\mu_S^F(\Delta) = Tr[SF(\Delta)]$$

as the probability that the outcomes of a measurement of the observable $\mathcal{F}$ (corresponding to a POVM F) is in Δ when the physical system is prepared in the state $S \in \mathcal{S}$. We recall that an analogous relation holds for standard observables which are represented by real PVMs $E : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$, that is:

$$\mu_S^E(\Delta) = Tr[SE(\Delta)].$$

That shows again why the quantum observables described by POVMs are a generalization of the standard quantum observables. They are called generalized observables or unsharp observables and, as we already pointed out, play a key role in quantum information theory, quantum optics, quantum estimation theory [45, 50, 66, 27, 83, 90], quantum measurement theory, and in the phase space formulation of quantum mechanics [83, 19, 20].

1.3 Objectives and results of the present work

The brief outline we made above raises the problem of giving a clear physical meaning to the POVMs as well as to study the relationships between POVMs and PVMs. As we shall see the answer to the second problem shed light also on the first problem.

The main aim of the present thesis is to answer the following questions:

1) What are the relationships between POVMs and spectral measures? As we shall see a commutative POVM F is the smearing of a spectral measure E (called the sharp version of F). Then, the following question raises. What are the physical meaning and the mathematical structure of the smearing which connects F to E? (Chapter 2).

2) In chapter 2 we characterize a commutative POVM as the smearing of its sharp version. What are the relationships between this characterization and Naimark's dilation theorem? Is there a universal one-to-one function f such that the sharp version A^F of any commutative POVM F can be written in the form $A^F = \int f(t)\, dF_t$? (Chapter 3).

3) Is there any loss of "information" during the smearing from the sharp version A of F to F? (Chapter 4).

4) Can we give conditions for a POVM to have the norm-1 property? Is the norm-1 property relevant to localization observables? (Chapter 5).

Here is a brief list of the results we get. A more detailed overview can be found in the summary at the end of the thesis.

1. **Characterization of commutative POVMs [22, 11]**: We generalize the results I got in Ref.s [22, 11]. In particular, we prove that each commutative POVM F is the smearing of a spectral measure E realized by a Feller Markov Kernel. That suggests an interpretation of commutative POVMs as the randomization of real PVMs. Moreover, we characterize the POVMs whose smearing can be realized by strong Feller Markov kernels.

2. **Analysis of the relationships between the results in item1 and Naimark's dilation theorem [16, 17, 18]**. We prove that the self-adjoint operator corresponding to the spectral measure E, of which F is the smearing, is the projection of a Naimark's operator [12, 13, 14, 15, 16, 17, 18]. That suggests an interpretation of the Naimark's dilation of a commutative POVM.

3. **Analysis of the informational content of a POVM [15]**. We introduce an equivalence relation on the set of observables which we compare with other well known equivalence relations and prove that it is the only one for which E is always equivalent to F [15].

4. **The study of the uniform continuity and norm-1 property of a POVM [21, 23]**. We characterize the uniform continuity of a POVM, give a necessary condition for the norm-1 property ($\|F(\Delta)\| = 1$, whenever $F(\Delta) \neq \mathbf{0}$) and explain its physical meaning and its relevance to the localization problem [21]. Then, we prove that several relevant localization observables cannot have such a property [23].

Chapter 2

Characterization of commutative POVMs

The present chapter focuses on the analysis of commutative POVMs. We recall that the set of PVMs is a subset of the set of commutative POVMs. That suggests to start our analysis of the mathematical structure of POVMs by studying the relationships between commutative POVMs and PVMs.

We prove a strong connection between POVMs and spectral measures, i.e., each commutative POVM F is the smearing (realized by a Feller Markov kernel) of a spectral measure E. In particular we generalize the results I proved in Ref.s [22, 11] to the case of a POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ defined on a Hausdorff, locally compact, second countable space X. Some previous results on commutative POVMs of which the ones we present here are a strengthening are contained in Ref.s [10, 11, 12, 13, 14, 15, 16, 17, 18, 53, 54]. Other important characterizations of POVMs and some analysis of their relationships can be found in Ref.s [2, 48, 6, 55].

First we introduce the concepts of weak Markov kernel, Markov kernel and smearing. In the following the symbols $(\Lambda, \mathcal{B}(\Lambda))$ and $(X, \mathcal{B}(X))$ denote measurable spaces where Λ and X are topological spaces and $\mathcal{B}(\Lambda)$ and $\mathcal{B}(X)$ the corresponding Borel σ-algebras.

Definition 2.0.1. *A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that,*

1. *$\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,*
2. *$\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.*

Definition 2.0.2. *Let ν be a measure on Λ. A map $\mu : \Lambda \times \mathcal{B}(X) \to \mathbb{R}$ is a weak Markov kernel with respect to ν if:*

1. $\mu_\Delta(\cdot)$ *is a measurable function for each* $\Delta \in \mathcal{B}(X)$*,*
2. *for each* $\Delta \in \mathcal{B}(X)$*,* $0 \leq \mu_\Delta(\lambda) \leq 1, \quad \nu - a.e.$*,*
3. $\mu_X(\lambda) = 1$*,* $\mu_\emptyset(\lambda) = 0, \quad \nu - a.e.$*,*
4. *for any sequence* $\{\Delta_i\}_{i\in\mathbb{N}}$*,* $\Delta_i \cap \Delta_j = \emptyset$*,*

$$\sum_i \mu_{(\Delta_i)}(\lambda) = \mu_{(\cup_i \Delta_i)}(\lambda), \quad \nu - a.e.$$

Definition 2.0.3. *The map* $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ *is a weak Markov kernel with respect to a PVM* $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ *if it is a weak Markov kernel with respect to each measure* $\nu_\psi(\cdot) := \langle E(\cdot)\,\psi, \psi\rangle$*,* $\psi \in \mathcal{H}$*.*

In the following, by a weak Markov kernel μ we mean a weak Markov kernel with respect to a PVM E. Moreover the function $\lambda \mapsto \mu_\Delta(\lambda)$ will be denoted indifferently by μ_Δ or $\mu_\Delta(\cdot)$.

Definition 2.0.4. *A POVM* $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ *is said to be a smearing of a POVM* $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ *if there exists a weak Markov kernel* $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ *such that,*

$$F(\Delta) = \int_\Lambda \mu_\Delta(\lambda) dE_\lambda, \quad \Delta \in \mathcal{B}(X).$$

Example 2.0.5. *In the standard formulation of quantum mechanics, the operator*

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$\psi(x) \in L^2(\mathbb{R}) \mapsto Q\psi := x\psi(x)$$

is used to represent the position observable. A more realistic description of the position observable of a quantum particle is given by a smearing of Q *as, for example, the optimal position POVM*

$$F^Q(\Delta) = \frac{1}{l\,\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \Big(\int_\Delta e^{-\frac{(x-y)^2}{2\,l^2}}\, dy\Big)\, dE_x^Q = \int_{-\infty}^{\infty} \mu_\Delta(x)\, dE_x^Q$$

where,

$$\mu_\Delta(x) = \frac{1}{l\,\sqrt{2\,\pi}} \int_\Delta e^{-\frac{(x-y)^2}{2\,l^2}}\, dy, \quad \Delta \in \mathcal{B}(\mathbb{R})$$

defines a Markov kernel and E^Q *is the spectral measure corresponding to the position operator* Q*.*

In the following, the symbol μ is used to denote both Markov kernels and weak Markov kernels. The symbols A and B are used to denote self-adjoint operators.

Definition 2.0.6. *Whenever F, A, and μ are such that $F(\Delta) = \mu_\Delta(A)$, $\Delta \in \mathcal{B}(X)$, we say that (F, A, μ) is a von Neumann triplet.*

Definition 2.0.7. *The von Neumann algebra $\mathcal{A}^W(F)$ generated by the POVM F is the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$.*

Definition 2.0.8. *If (F, A, μ) is a von Neumann triplet and A and F generate the same von Neumann algebra then A is named the sharp version of F.*

2.1 On the separation properties of μ

In the following, we assume X to be a Hausdorff, locally compact, second countable topological space. The symbol $\mathcal{S}$ denotes a countable basis for the topology of X. The symbol $\mathcal{R}(\mathcal{S})$ denotes the ring generated by $\mathcal{S}$. Notice that by theorem c, page 24, in Ref. [44], $\mathcal{R}(\mathcal{S})$ is countable too. Moreover, $\mathcal{R}(\mathcal{S})$ generates the Borel σ-algebra $\mathcal{B}(X)$.

A weak Markov kernel μ such that (F, A, μ) is a von Neumann triplet, separates the point of $\Gamma \subset \sigma(A)$ if the family of functions $\{\mu_\Delta\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ separates the points of Γ or, in other words, if the set functions $\{\mu_{(\cdot)}(\lambda)\}_{\lambda\in\Gamma}$ are distinct. It is then natural to ask if in general μ has that property. The following theorem answers in the positive.

Theorem 2.1.1. *Let (F, A, μ) be a von Neumann triplet and suppose that A is a sharp version of F. Then, there exists a set $\Gamma \subseteq \sigma(A)$, $E^A(\Gamma) = \mathbf{1}$, such that the family of functions $\{\mu_\Delta(\cdot)\}_{\Delta\in\mathcal{B}(X)}$ separates the points of Γ.*

Proof. In the following, $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by $\{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$, $O_2 := \{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$ and $\mathcal{A}^C(O_2)$ is the C^*-algebra generated by O_2. The von Neumann algebra generated by $\mathcal{A}^C(O_2)$ coincides with $\mathcal{A}^W(F)$ (see appendix A). Moreover, $\mathcal{A}^W(F) = \mathcal{A}^W(A)$ since A is the sharp version of F and generates $\mathcal{A}^W(F)$. By the Gelfand-Naimark theorem [35, 72], there is a * isomorphism ϕ between $\mathcal{A}^C(O_2)$ and the algebra of continuous functions $\mathcal{C}(\Lambda_2)$ where Λ_2 is the spectrum of $\mathcal{A}^C(O_2)$. Moreover,

$$f \in \mathcal{C}(\Lambda_2) \mapsto \phi(f) = \int_{\Lambda_2} f(\lambda)\, d\widetilde{E}_\lambda$$

where, $\widetilde{E}$ is the spectral measure from the Borel σ algebra $\mathcal{B}(\Lambda_2)$ to $\mathcal{E}(\mathcal{H})$ whose existence is assured by theorem 1, page 895, in Ref. [35]. The Gelfand-Naimark isomorphism ϕ can be extended to a homomorphism between the algebra of the Borel functions on Λ_2 and the von Neumann algebra $\mathcal{A}^W(F) = \mathcal{A}^W(A)$ generated by $\mathcal{A}^C(O_2)$ (see Ref. [34], page 360, section 3). Therefore, there is a Borel function h such that

$$A = \int_{\Lambda_2} h(\lambda)\, d\widetilde{E}_\lambda \tag{2.1}$$

Let $\mathcal{S}$ be a countable basis for the topology of X. Let $\{\Delta_i\}_{i\in\mathbb{N}}$ denote an enumeration of the set $\mathcal{R}(\mathcal{S})$. Since $\mathcal{A}^C(O_2)$ is the smallest uniform closed algebra containing $\{F(\Delta_i)\}_{i\in\mathbb{N}}$, $\mathcal{C}(\Lambda_2)$ is the smallest uniform closed algebra of functions containing $\{\nu_{\Delta_i} := \phi^{-1}(F(\Delta_i))\}_{i\in\mathbb{N}}$. In other words $\{\nu_{\Delta_i}\}_{i\in\mathbb{N}}$ generates $\mathcal{C}(\Lambda_2)$. The Stone-Weierstrass theorem [35] assures that $\{\nu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points in Λ_2.

On the other hand, the fact that (F, A, μ) is a von Neumann triplet, implies that, for each $\Delta_i \in \mathcal{R}(\mathcal{S})$, there is a Borel function μ_{Δ_i} such that

$$\int_{\Lambda_2} \nu_{\Delta_i}(\lambda)\, d\widetilde{E}_\lambda = F(\Delta_i) = \mu_{\Delta_i}(A) = \int_{\Lambda_2} \mu_{\Delta_i}(h(\lambda))\, d\widetilde{E}_\lambda.$$

Then, for each $\Delta_i \in \mathcal{R}(\mathcal{S})$, there is a set $M_i \subset \Lambda_2$, $\widetilde{E}(M_i) = \mathbf{1}$, such that

$$\mu_{\Delta_i}(h(\lambda)) = \nu_{\Delta_i}(\lambda), \quad \lambda \in M_i. \tag{2.2}$$

Let $M := \cap_{i=1}^{\infty} M_i$. Then,

$$\widetilde{E}(M) = \lim_{n\to\infty} \widetilde{E}(\cap_{i=1}^{n} M_i) = \lim_{n\to\infty} \prod_{i=1}^{n} \widetilde{E}(M_i) = \mathbf{1}$$

and, for each $i \in \mathbb{N}$,

$$(\mu_{\Delta_i} \circ h)(\lambda) = \nu_{\Delta_i}(\lambda), \quad \lambda \in M \subseteq \Lambda_2. \tag{2.3}$$

Since $\{\nu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points in Λ_2, it separates the points in M. Then, equation (2.3) implies that $\{\mu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points in $\Gamma := h(M)$. Moreover[1],

$$E^A(\Gamma) = E^A(h(M)) = \widetilde{E}[h^{-1}(h(M))] = \mathbf{1}$$

---

[1] Notice that $h(M)$ is a Borel set. In order to prove that, we first recall that Λ_2 is a Polish space (that is, a complete, separable, space [63]). Indeed, by theorem 11, page 871, in Ref. [35], it is homeomorphic to a closed subspace of the Cartesian product $\prod_{i=1}^{\infty} \sigma(F(\Delta_i))$, where $\sigma(F(\Delta_i))$ is a complete separable metric space, and by theorem 2, page 406, and theorem 6, page 156, in Ref. [64], it is complete and separable. Moreover, h is measurable and injective on M. Therefore, Soulsin's theorem (see theorem 9 page 440 and Corollary 1 page 442 in Ref. [63]) assures that $h(M)$ is a Borel set.

where, E^A is the spectral measure defined by the relation

$$E^A(\Delta) = \widetilde{E}(h^{-1}(\Delta))$$

and such that,

$$A = \int x \, dE^A_x$$

while, $h^{-1}(h(M))$ is a Borel set containing M.
We have proved that the set of functions $\{\mu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points of Γ and that $E^A(\Gamma) = \mathbf{1}$. In other words,

$$\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda'), \quad \lambda \neq \lambda', \quad \lambda, \lambda' \in \Gamma.$$

□

2.2 Characterization by means of Feller Markov kernels

In the present section we give a characterization of the commutative POVMs. First we introduce the concept of strong Markov kernel, i.e., a weak Markov kernel $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(X) \to [0,1]$ with respect to a PVM $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ such that $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Gamma \subset \Lambda$, $E(\Gamma) = \mathbf{1}$. Then, we prove (theorem 2.2.3) that a POVM F is commutative if and only if there are a self-adjoint operator A and a strong Markov kernel μ such that (F, A, μ) is a von Neumann triplet, A is the sharp version of F, and μ_Δ is continuous for each $\Delta \in R$, where R is a ring which generates $\mathcal{B}(X)$. It is worth remarking that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(X) \to [0,1]$ is a Feller Markov kernel. Therefore, F is commutative if and only if there exists a bounded self-adjoint operator A and a Feller Markov kernel μ such that

$$F(\Delta) = \int_\Gamma \mu_\Delta(\lambda) \, dE_\lambda.$$

Moreover, the family of functions $\{\mu_\Delta\}_{\Delta\in R}$ separates the points in Γ (see theorems 2.1.1 and 2.2.3).

In order to prove the main theorem we need the following definitions.

Definition 2.2.1. *Let $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ be a PVM. The map $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(X) \to [0,1]$ is a strong Markov kernel with respect to E if it is a weak Markov kernel and there exists a set $\Gamma \subset \Lambda$, $E(\Gamma) = \mathbf{1}$, such that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(X) \to [0,1]$ is a Markov kernel with respect to E. A strong Markov kernel is denoted by the symbol $(\mu, E, \Gamma \subset \Lambda)$.*

Definition 2.2.2. *A Feller Markov kernel is a Markov kernel* $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(X) \to [0,1]$ *such that the function*

$$G(\lambda) = \int_X f(x)\, d\mu_x(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever f *is continuous and bounded.*

Theorem 2.2.3. *A POVM* $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ *is commutative if and only if, there exists a bounded self-adjoint operator* $A = \int \lambda\, dE_\lambda$ *with spectrum* $\sigma(A) \subset [0,1]$ *and a strong Markov Kernel* $(\mu, E, \Gamma \subset \sigma(A))$ *such that:*

1) $\mu_\Delta(\cdot) : \sigma(A) \to [0,1]$ *is continuous for each* $\Delta \in \mathcal{R}(\mathcal{S})$*,*

2) $F(\Delta) = \int_\Gamma \mu_\Delta(\lambda)\, dE_\lambda, \quad \Delta \in \mathcal{B}(X)$*.*

3) $\mathcal{A}^W(F) = \mathcal{A}^W(A)$*.*

4) μ *separates the points in* Γ*.*

Moreover, $\mu : \Gamma \times \mathcal{B}(X) \to [0,1]$ *is a Feller Markov kernel.*

Proof. Let $\mathcal{A}^W(F)$ be the von Neumann algebra generated by F. $\mathcal{A}^W(F)$ coincides with the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta \in \mathcal{R}(\mathcal{S})}$ where, $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(X)$ is the ring generated by $\mathcal{S}$, the countable sub-basis for the topology of X (see appendix A for the proof). We recall that both $\mathcal{S}$ and $\mathcal{R}(\mathcal{S})$ are countable (see theorem c, page 24, in Ref. [44]).

Now, we proceed to the proof of the existence of A. Let $\{\Delta_i\}_{i \in \mathbb{N}}$ be an enumeration of the set $\mathcal{R}(\mathcal{S})$ and $O_2 := \{F(\Delta)\}_{\Delta \in \mathcal{R}(\mathcal{S})}$. Let $E^{(i)}$ denote the spectral measure corresponding to $F(\Delta_i) \in O_2$. We have $F(\Delta_i) = \int x\, dE^{(i)}_x$. Therefore, for each $i, k \in \mathbb{N}$ there exists a division $\{\Delta^{(i,k)}_j\}_{j=1,\dots,m_{i,k}}$ of $[0,1]$ such that

$$\| \sum_{j=1}^{m_{i,k}} x^{(i,k)}_j\, E^{(i)}(\Delta^{(i,k)}_j) - F(\Delta_i) \| \leq \frac{1}{k}. \tag{2.4}$$

By the spectral theorem [35] the von Neumann algebra $\mathcal{A}^W(F)$ contains all the projection operators in the spectral resolution of $F(\Delta)$, $\Delta \in \mathcal{B}(X)$. Therefore, the von Neumann algebra $\mathcal{A}^W(D)$ generated by the set $D := \{E^{(i)}(\Delta^{i,k}_j),\, j \leq m_{i,k},\, i, k \in \mathbb{N}\}$ is contained in $\mathcal{A}^W(F)$ and then

$$\mathcal{A}^W(D) \subset \mathcal{A}^W(F) = \mathcal{A}^W(O_2). \tag{2.5}$$

Moreover, the C^*-algebra $\mathcal{A}^C(D)$ generated by D contains the C^*-algebra $\mathcal{A}^C(O_2)$ generated by O_2 (see equation (2.4)). Summing up the preceding observations, we have

$$\mathcal{A}^C(O_2) \subset \mathcal{A}^C(D) \subset \mathcal{A}^W(F).$$

By the double commutant theorem [56],

$$\mathcal{A}^W(F) = [\mathcal{A}^C(O_2)]'' \subset [\mathcal{A}^C(D)]'' = \mathcal{A}^W(D)$$

so that (see equation 2.5),

$$\mathcal{A}^W(D) = \mathcal{A}^W(F). \tag{2.6}$$

By theorem 11, page 871 in Ref. [35], the spectrum Λ of $\mathcal{A}^C(D)$ is homeomorphic to a closed subset of $\prod_{i=1}^{\infty}\{0,1\}$. Let $\pi : \Lambda \to \prod_{i=1}^{\infty}\{0,1\}$ denote the homeomorphism between the two spaces.
Now, if we identify Λ with a closed subset of $\prod_{i=1}^{\infty}\{0,1\}$, we can prove the existence of a continuous function distinguishing the points of Λ. Indeed, let $\pi(\lambda) = \bar{x} := (x_1, \dots, x_n, \dots) \in \prod_{i=1}^{\infty}\{0,1\}$. The function

$$f(\lambda) = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$$

is continuous and injective and then it distinguishes the points of Λ. Moreover, since Λ and $[0,1]$ are Hausdorff, the map $f : \Lambda \to f(\Lambda)$ is a homeomorphism.
By theorem 1, page 895, in Ref. [35], there exists a spectral measure $\widetilde{E} : \mathcal{B}(\Lambda) \to \mathcal{F}(\mathcal{H})$ such that the map

$$\begin{aligned} T : \mathcal{C}(\Lambda) &\to B(\mathcal{H}) \\ g \mapsto T(g) &= \int_{\Lambda} g(\lambda) d\widetilde{E}_{\lambda} \end{aligned} \tag{2.7}$$

defines an isometric *-isomorphism between $\mathcal{A}^C(D)$ and $\mathcal{C}(\Lambda)$.
The fact that f distinguishes the points of Λ, implies that the self-adjoint operator

$$A = \int_{\Lambda} f(\lambda)\, d\widetilde{E}_{\lambda}$$

is a generator of the von Neumann algebra $\mathcal{A}^W(D) = \mathcal{A}^W(F)$. Indeed, by the Stone-Weierstrass theorem, $\mathcal{C}(\Lambda)$ is singly generated, in particular f is a generator. Then, the isomorphism between $\mathcal{A}^C(D)$ and $\mathcal{C}(\Lambda)$ assures that $\mathcal{A}^C(D)$ is singly generated and that A is a generator. Hence, $\mathcal{A}^W(F) = \mathcal{A}^W(D) = [\mathcal{A}^C(D)]''$ is singly generated. In particular, A generates $\mathcal{A}^W(F)$, i.e., $\mathcal{A}^W(F) = \mathcal{A}^W(A)$.

Now, we proceed to the proof of the existence of the weak Markov kernel $\widetilde{\nu}$ such that $(F, A, \widetilde{\nu})$ is a von Neumann triplet.

By (2.7), for each $\Delta \in \mathcal{R}(\mathcal{S})$, there exists a continuous function $\gamma_\Delta \in \mathcal{C}(\Lambda)$ such that

$$F(\Delta) = \int_\Lambda \gamma_\Delta(\lambda)\, d\widetilde{E}_\lambda.$$

Now, we show that, for each $\Delta \in \mathcal{R}(\mathcal{S})$, there is a continuous function $\nu_\Delta : \sigma(A) \to [0,1]$ from the spectrum of A to the interval $[0,1]$ such that $\nu_\Delta(f(\lambda)) = \gamma_\Delta(\lambda)$, $\lambda \in \Lambda$, and $F(\Delta) = \nu_\Delta(A)$.
To prove this, let us consider the function

$$\nu_\Delta(t) := (\gamma_\Delta \circ f^{-1})(t), \quad \Delta \in \mathcal{R}(\mathcal{S}).$$

It is continuous since it is the composition of continuous functions. Moreover, since

$$\nu_\Delta(f(\lambda)) = \gamma_\Delta(f^{-1}(f(\lambda))) = \gamma_\Delta(\lambda).$$

we have,

$$\nu_\Delta(A) = F(\Delta), \quad \forall \Delta \in \mathcal{R}(\mathcal{S}).$$

Indeed, by the change of measure principle (page 894, ref. [35]),

$$\begin{aligned} F(\Delta) &= \int_\Lambda \gamma_\Delta(\lambda)\, d\widetilde{E}_\lambda = \int_\Lambda \gamma_\Delta(f^{-1}(f(\lambda)))\, d\widetilde{E}_\lambda \\ &= \int_{\sigma(A)} \gamma_\Delta(f^{-1}(t))\, dE_t = \int_{\sigma(A)} \nu_\Delta(t)\, dE_t = \nu_\Delta(A) \end{aligned}$$

where $\sigma(A) = f(\Lambda)$ is the spectrum of A and E is the spectral measure corresponding to A defined by the relation $E(\Delta) = \widetilde{E}(f^{-1}(\Delta))$, $\Delta \in \mathcal{B}(\sigma(A))$ (see corollary 10, page 902, in Ref. [35]).
For each $\lambda \in \sigma(A)$, the map $\nu_{(\cdot)}(\lambda) : \mathcal{R}(\mathcal{S}) \to [0,1]$ defines an additive set function. Indeed, let $\Delta \in \mathcal{R}(\mathcal{S})$ be the disjoint union of the sets $\Delta_1, \Delta_2 \in \mathcal{R}(\mathcal{S})$. Then,

$$\begin{aligned} \int \nu_{(\Delta_1 \cup \Delta_2)}(\lambda)\, dE_\lambda = F(\Delta_1 \cup \Delta_2) &= F(\Delta_1) + F(\Delta_1) \\ &= \int \nu_{\Delta_1}(\lambda)\, dE_\lambda + \int \nu_{\Delta_2}(\lambda)\, dE_\lambda \\ &= \int \left[\nu_{\Delta_1}(\lambda) + \nu_{\Delta_2}(\lambda)\right] dE_\lambda \end{aligned}$$

so that, by the continuity of the functions $\nu_{(\Delta_1)}(\lambda)$ and $\nu_{(\Delta_2)}(\lambda)$, we get (see theorem 1, page 895, in Ref. [35])

$$\nu_{(\Delta_1)}(\lambda) + \nu_{(\Delta_2)}(\lambda) = \nu_{(\Delta_1 \cup \Delta_2)}(\lambda), \quad \forall \lambda \in \sigma(A).$$

Now, we extend ν to all $\mathcal{B}(X)$.
Since A is the generator of $\mathcal{A}^W(F)$, for each $\Delta \in \mathcal{B}(X)$, there exists a Borel function ω_Δ such that.

$$F(\Delta) = \int_{\sigma(A)} \omega_\Delta(t)\, dE_t = \int_\Lambda (\omega_\Delta \circ f)(\lambda)\, d\widetilde{E}_\lambda$$

Then, we can consider the map $\widetilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ defined as follows

$$\widetilde{\nu}_\Delta(\lambda) = \begin{cases} \nu_\Delta(\lambda) & if \quad \Delta \in \mathcal{R}(\mathcal{S}) \\ \omega_\Delta(\lambda) & if \quad \Delta \notin \mathcal{R}(\mathcal{S}). \end{cases} \tag{2.8}$$

Since $\widetilde{\nu}$ coincides with ν on $\mathcal{R}(\mathcal{S})$ it is additive on $\mathcal{R}(\mathcal{S})$.
In order to prove that $\widetilde{\nu}$ is a weak Markov kernel, let us consider a set $\Delta \in \mathcal{B}(X)$ which is the disjoint union of the sets $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \in \mathcal{B}(X)$. Then,

$$\begin{aligned} \int \widetilde{\nu}_{(\cup_{i=1}^\infty \Delta_i)}(x)\, dE_x = \int \widetilde{\nu}_\Delta(x) dE_x = F(\Delta) = \sum_{i=1}^\infty F(\Delta_i) \\ = \sum_{i=1}^\infty \int \widetilde{\nu}_{\Delta_i}(x)\, dE_x = \int \sum_{i=1}^\infty \widetilde{\nu}_{\Delta_i}(x)\, dE_x \end{aligned}$$

so that, by Corollary 9, page 900, in Ref. [35],

$$\sum_{i=1}^\infty \widetilde{\nu}_{\Delta_i}(x) = \widetilde{\nu}_\Delta(x), \quad E - a.e,$$

which implies that $\widetilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ is a weak Markov kernel. In particular $(F, A, \widetilde{\nu})$ is a von Neumann triplet.

Now, we proceed to prove the existence of the Markov kernel $\mu : \Gamma \times \mathcal{B}(X) \to [0,1]$ such that items 1, 2, and 3 of the theorem are satisfied.

Since X is Hausdorff locally compact second countable, it is a Polish space (theorem 5.3 in [62]). Then, to each weak Markov kernel $\widetilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ such that $(F, A, \widetilde{\nu})$ is a von Neumann triplet, there corresponds a Markov kernel $\phi : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ such that (F, A, ϕ) is a von Neumann triplet [53, 54, 11]. Then, for each $\Delta \in \mathcal{B}(X)$,

$$\int \widetilde{\nu}_\Delta(\lambda)\, dE_\lambda = F(\Delta) = \int \phi_\Delta(\lambda)\, dE_\lambda$$

hence,

$$\phi_\Delta(\lambda) = \widetilde{\nu}_\Delta(\lambda), \quad E - a.e. \tag{2.9}$$

Now, let $\{\Delta_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathcal{R}(\mathcal{S})$. By equation (2.9), for each $i \in \mathbb{N}$, there is a set $N_i \subset \sigma(A)$, $E(N_i) = \mathbf{0}$, such that

$$\phi_{\Delta_i}(\lambda) = \widetilde{\nu}_{\Delta_i}(\lambda), \quad \lambda \in \sigma(A) - N_i. \tag{2.10}$$

Then, for each $i \in \mathbb{N}$,

$$\phi_{\Delta_i}(\lambda) = \widetilde{\nu}_{\Delta_i}(\lambda), \quad \lambda \in \sigma(A) - N \tag{2.11}$$

where,

$$N := \cup_{i=1}^{\infty} N_i, \quad E(N) = \mathbf{0}.$$

Therefore, for almost all $\lambda \in \sigma(A)$, $\widetilde{\nu}_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$.
Now, we can define the map

$$\mu_{(\cdot)}(\lambda) = \begin{cases} \widetilde{\nu}_{(\cdot)}(\lambda) & \lambda \in N \\ \phi_{(\cdot)}(\lambda) & \lambda \in \sigma(A) - N \end{cases}$$

If we put $\Gamma = \sigma(A) - N$, we have that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(X) \to [0,1]$ is a Markov kernel. Therefore, $\mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ is a strong Markov kernel.
Notice that, for each $\Delta \in \mathcal{R}(\mathcal{S})$ and $\lambda \in \sigma(A)$,

$$\mu_\Delta(\lambda) = \widetilde{\nu}_\Delta(\lambda)$$

so that, μ_Δ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$ and additive on $\mathcal{R}(\mathcal{S})$. We also have,

$$\mu_\Delta(A) = \phi_\Delta(A) = F(\Delta), \quad \Delta \in \mathcal{R}(\mathcal{S}).$$

We have proved items 1, 2, and 3. Item 4 comes from theorem 2.1.1.

It remains to prove that μ is a Feller Markov kernel. By item 1, μ_Δ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$. Notice that for each open set $O \in \mathcal{B}(X)$, there is a countable family of sets $\Delta_i \in \mathcal{R}(\mathcal{S})$ such that $O = \cup_{i=1}^{\infty} \Delta_i$. Therefore, by theorem 2.2 in Ref. [25], and the continuity of μ_Δ for each $\Delta \in \mathcal{R}(S)$, $\lim_{n\to\infty} \lambda_n = \lambda$ implies,

$$\lim_{n\to\infty} \int f(t)\, \mu_t(\lambda_n) = \int f(t)\, \mu_t(\lambda), \quad f \in \mathcal{C}_b(X)$$

where, $\mathcal{C}_b(X)$ is the space of bounded, continuous real functions. Than, $G(\lambda) := \int f(t)\, \mu_t(\lambda)$ is continuous whenever f is continuous and μ is a Feller Markov kernel.

Finally, we note that $F(\Delta) = \mu_\Delta(A)$ implies the commutativity of F and that ends the proof. □

In the proof of theorem 2.2.3 we have also proved the following theorem.

Theorem 2.2.4. *A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if, there exists a bounded self-adjoint operator $A = \int \lambda \, dE_\lambda$ with spectrum $\sigma(A) \subset [0,1]$ and a Markov Kernel $\mu : \mathcal{B}(X) \times \sigma(A) \to [0,1]$ such that*

$$F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda, \quad \Delta \in \mathcal{B}(X).$$

Proof. We have already shown the existence of a weak Markov kernel $\widetilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ which is additive on the the ring $\mathcal{R}(\mathcal{S})$ and such that $F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda) \, dE_\lambda$, for each $\Delta \in \mathcal{B}(X)$ (see equation 2.8). Moreover, since X is a Polish space, to each weak Markov kernel $\widetilde{\nu} : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ there corresponds a Markov kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ such that (F, A, μ) is a von Neumann triplet [53, 54, 11]. □

Finally, we note that the sharp version A of F is unique up to almost everywhere bijections.

Theorem 2.2.5. *[14] Let $(F, A; \mu)$ be a von Neumann triplet such that A is the sharp version of F. Than, i) for any for von Neumann triplet $\big(F, B, \, \mu\big)$, there exists a real function g such that $A = g(B)$, ii) for any von Neumann triplet (F, A', ν) satisfying item i) there exists an almost everywhere one-to-one function h such that $A' = h(A)$.*

2.3 Characterization of POVMs admitting strong Feller Markov Kernels

In the last section we proved that each commutative POVM admits a strong Markov kernel μ such that μ_Δ is a continuous function for each $\Delta \in \mathcal{R}(\mathcal{S})$ where, $\mathcal{R}(\mathcal{S})$ is a ring which generates the Borel σ-algebra $\mathcal{B}(X)$.
In the present section we characterize the commutative POVMs for which the Markov kernel μ, whose existence was proved in theorem 2.2.4, is such that μ_Δ is continuous for each $\Delta \in \mathcal{B}(X)$. Whenever such a Markov kernel exists, we say that the POVM admits a strong Feller Markov kernel. In particular, we prove that a commutative POVM F admits a strong Feller Markov kernel if and only if F is uniformly continuous.

Definition 2.3.1. *Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$. Let $\Delta = \cup_{i=1}^\infty \Delta_i$, $\Delta_i \cap \Delta_j = \emptyset$. If*

$$\lim_{n\to\infty} \sum_{i=1}^n F(\Delta_i) = F(\Delta)$$

in the uniform operator topology then we say that F is uniformly continuous.

Notice that the term uniformly continuous derives from the fact that the σ-additivity of F in the uniform operator topology is equivalent to the continuity in the uniform operator topology. Analogously, the σ-additivity of F in the weak operator topology is equivalent to the continuity of F in the weak operator topology [24].

Definition 2.3.2. *A Markov kernel $\mu_{(\cdot)}(\cdot) : [0,1] \times \mathcal{B}(X) \to [0,1]$ is said to be strong Feller if μ_Δ is a continuous function for each $\Delta \in \mathcal{B}(X)$.*

Definition 2.3.3. *We say that a commutative POVM admits a strong Feller Markov kernel if there exists a strong Feller Markov kernel μ such that $F(\Delta) = \int \mu_\Delta(\lambda)\, dE_\lambda$, where E is the sharp version of F.*

In order to prove the main theorem of the section we need the following lemma.

Lemma 2.3.4. *Let F be uniformly continuous. Let μ be a weak Markov kernel and (F, A, μ) a von Neumann triplet. Suppose that μ_Δ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$. Then, for each $\lambda \in \sigma(A)$, $\mu_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$.*

Proof. Let $\Delta, \Delta_i \in \mathcal{R}(\mathcal{S})$, $\Delta_i \cap \Delta_j = \emptyset$, $\cup_{i=1}^{\infty} \Delta_i = \Delta$. Then,

$$\begin{aligned}\mathbf{0} &= u - \lim_{n\to\infty} \big(F(\Delta) - F(\cup_{i=1}^{n} \Delta_i)\big)\\ &= u - \lim_{n\to\infty} \int \big(\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\big)\, dE_\lambda.\end{aligned}$$

By the uniform continuity of F and theorem 1, page 895, in Ref. [35], it follows that, $\forall \epsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n > \bar{n}$ implies,

$$\begin{aligned}\|\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\|_\infty &= \| \int \big(\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\big)\, dE_\lambda \|\\ &= \|F(\Delta) - F(\cup_{i=1}^{n} \Delta_i)\| \le \epsilon.\end{aligned} \tag{2.12}$$

By equation (2.12),

$$|\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)| \le \epsilon, \quad \forall \lambda \in \sigma(A).$$

□

Theorem 2.3.5. *A commutative POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ admits a strong Feller Markov kernel if and only if it is uniformly continuous.*

Proof. Suppose F is uniformly continuous. By theorem 2.2.3, there is a weak Markov kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ such that $\mu_\Delta(\cdot)$ is continuous for every $\Delta \in \mathcal{R}(\mathcal{S})$ and a self-adjoint operator A such that (F, A, μ) is a von Neumann triplet. By lemma 2.3.4, μ is σ-additive on $\mathcal{R}(\mathcal{S})$. Therefore Charateodory theorem [65] assures that the map $\mu : \sigma(A) \times \mathcal{R}(\mathcal{S}) \to [0, 1]$ can be extended to a map $\widetilde{\mu} : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ whose restriction to $\mathcal{R}(\mathcal{S})$ coincides with μ. Now we prove that $\widetilde{\mu}$ is a Markov kernel such that $F(\Delta) = \widetilde{\mu}_\Delta(A)$ and that $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in \mathcal{B}(X)$. We proceed by steps.

1) Open sets. Each open set G is the union of a countable family of sets in S, i.e., $G = \cup_{i=1}^\infty \Delta_i$, $\Delta_i \in S$. Let us define the set $G_n := \cup_{i=1}^n \Delta_i$. Therefore, $G_n \uparrow G$. Moreover, μ_{G_n} is continuous for each $n \in \mathbb{N}$, and

$$u - \lim_{n\to\infty} F(G_n) = F(G).$$

Then,

$$F(G) = u - \lim_{i\to\infty} F(G_i) = u - \lim_{i\to\infty} \int \widetilde{\mu}_{G_i}(\lambda)\, dE_\lambda.$$

By the uniform continuity of F, it follows that, $\forall\epsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n, m > \bar{n}$ implies,

$$\|\widetilde{\mu}_{G_n}(\lambda) - \widetilde{\mu}_{G_m}(\lambda)\|_\infty = \| \int [\widetilde{\mu}_{G_n}(\lambda) - \widetilde{\mu}_{G_m}(\lambda)]\, dE_\lambda \| = \|F(G_n) - F(G_m)\| \le \epsilon. \tag{2.13}$$

By equation (2.13),

$$|\widetilde{\mu}_{G_n}(\lambda) - \widetilde{\mu}_{G_m}(\lambda)| \le \epsilon, \quad \forall\lambda \in \sigma(A). \tag{2.14}$$

Since $\widetilde{\mu}_{(\cdot)}(\lambda)$ is a probability measure,

$$\lim_{i\to\infty} \widetilde{\mu}_{G_i}(\lambda) = \widetilde{\mu}_G(\lambda), \quad \forall\lambda \in \sigma(A).$$

By equation (2.14), the convergence is uniform and this proves the continuity of $\widetilde{\mu}_G$. Moreover,

$$F(G) = \lim_{i\to\infty} F(G_i) = \lim_{i\to\infty} \int \widetilde{\mu}_{G_i}(\lambda)\, dE_\lambda = \int \widetilde{\mu}_G(\lambda)\, dE_\lambda = \widetilde{\mu}_G(A).$$

2) $\mathbf{G}_\delta$ sets. For each G_δ set there exists [24] a family of open sets $\{G_i\}_{i\in\mathbb{N}}$, $G_\delta \subset G_i$, such that $\cap_{i=1}^\infty G_i = G_\delta$. Then, by the uniform continuity of F,

$$F(G_\delta) = F(\cap_{i=1}^\infty G_i) = u - \lim_{n\to\infty} F(\cap_{i=1}^n G_i) = u - \lim_{n\to\infty} F(\widetilde{G}_n)$$

where, $\widetilde{G}_n := \cap_{i=1}^n G_i$ and $\widetilde{G}_n \downarrow G_\delta$.
By theorem 1, page 895, in Ref. [35], it follows that, $\forall \epsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n, m > \bar{n}$ implies,

$$\|\mu_{\widetilde{G}_n}(\lambda) - \mu_{\widetilde{G}_m}(\lambda)\|_\infty = \| \int (\mu_{\widetilde{G}_n}(\lambda) - \mu_{\widetilde{G}_m}(\lambda))\, dE_\lambda\| \leq \epsilon. \tag{2.15}$$

Since $\widetilde{\mu}_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \sigma(A)$,

$$\lim_{i\to\infty} \widetilde{\mu}_{\widetilde{G}_i}(\lambda) = \widetilde{\mu}_{G_\delta}(\lambda).$$

By equation (2.15) the convergence is uniform and then $\widetilde{\mu}_{G_\delta}$ is continuous. Moreover,

$$F(G_\delta) = \lim_{i\to\infty} F(\widetilde{G}_i) = \lim_{i\to\infty} \int \widetilde{\mu}_{\widetilde{G}_i}(\lambda)\, dE_\lambda = \int \widetilde{\mu}_{G_\delta}(\lambda)\, dE_\lambda = \widetilde{\mu}_{G_\delta}(A).$$

3) Borel sets. We use transfinite induction [63, 33]. Let G_0 be the family of open sets in X, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy (see page 236 in Ref. [63]). In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \dots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. By means of the same reasoning that we used in items 1 and 2, one can prove the continuity of $\widetilde{\mu}_\Delta$ as well as that $\widetilde{\mu}_\Delta(A) = F(\Delta)$ whenever Δ is of the kind $G_{\delta,\sigma}, G_{\delta\sigma\delta} \dots$. Analogously, if $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in G_\alpha$ then, $\widetilde{\mu}_\Delta$ is continuous for each Δ in $G_{\alpha+1}$ and $\widetilde{\mu}_\Delta(A) = F(\Delta)$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the reasoning in items 1 and 2 can be used. If α is a limit ordinal and $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in G_\beta$, $\beta < \alpha$, then, $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ and $\widetilde{\mu}_\Delta(A) = F(\Delta)$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the reasoning used in item 1 can be used. Therefore, by transfinite induction, $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}(X)$ [63] and $\widetilde{\mu}_\Delta(A) = F(\Delta)$.

In order to prove the second part of the theorem we show that the existence of a strong Feller Markov kernel implies the uniform continuity of F. Suppose that there exists a strong Feller Markov kernel μ such that $F(\Delta) = \mu_\Delta(\lambda)$. Since μ is a Markov kernel it is σ-additive. Then,

$$\lim_{n\to\infty} \big(\mu_\Delta(\lambda) - \sum_{i=1}^n \mu_{\Delta_i}(\lambda)\big) = 0, \quad \lambda \in \sigma(A).$$

where, $\Delta, \Delta_i \in \mathcal{B}(X)$, $\cup_{i=1}^\infty \Delta_i = \Delta$.
By hypothesis,

$$\mu_\Delta(\lambda) - \sum_{i=1}^n \mu_{\Delta_i}(\lambda) \in \mathcal{C}(\sigma(A)), \quad \forall n \in \mathbb{N}.$$

Then, by theorem 2.6.1 in appendix B,

$$u-\lim_{n\to\infty}\big(\mu_\Delta(\lambda)-\sum_{i=1}^{n}\mu_{\Delta_i}(\lambda)\big)=0.$$

By theorem 1, page 895, in Ref. [35], $\|F(\Delta)\|=\|\mu_\Delta\|_\infty$, hence

$$\lim_{n\to\infty}\|F(\Delta)-F(\cup_{i=1}^{n}\Delta_i)\|=\lim_{n\to\infty}\|\mu_\Delta-\sum_{i=1}^{n}\mu_{\Delta_i}\|_\infty=0.$$

which proves that F is uniformly continuous. □

Example 2.3.6. *Let us consider the following unsharp position observable*

$$Q^f(\Delta):=\int_{[0,1]}\mu_\Delta(x)\,dQ_x,\quad \Delta\in\mathcal{B}(\mathbb{R}), \tag{2.16}$$
$$\mu_\Delta(x):=\int_{\mathbb{R}}\chi_\Delta(x-y)\,f(y)\,dy,\quad x\in[0,1]$$

where, f is a bounded, continuous function such that $f(y)=0$, $y\notin[0,1]$ and

$$\int_{[0,1]}f(y)\,dy=1,$$

and Q_x is the spectral measure corresponding to the position operator

$$Q:L^2([0,1])\to L^2([0,1])$$
$$\psi(x)\mapsto(Q\psi)(x):=x\psi(x)$$

Notice that, for each $\Delta\in\mathcal{B}(\mathbb{R})$, $\mu_\Delta:[0,1]\to[0,1]$ is continuous. Indeed, by the uniform continuity of f, for each $\epsilon>0$, there is a $\delta>0$ such that $|x-x'|\leq\delta$ implies $|f(x-y)-f(x'-y)|\leq\epsilon$, for each y. Therefore,

$$\begin{aligned}|\mu_\Delta(x)-\mu_\Delta(x')|&=\Big|\int_{\mathbb{R}}\chi_\Delta(x-y)\,f(y)\,dy-\int_{\mathbb{R}}\chi_\Delta(x'-y)\,f(y)\,dy\Big|\\&=\Big|\int_\Delta[f(x-y)-f(x'-y)]\,dy\Big|\\&\leq\epsilon\int_{\Delta\cap[-1,1]}dy\leq 2\epsilon\end{aligned}$$

By theorem 2.3.5 and the continuity of μ_Δ, $\Delta\in\mathcal{B}(\mathbb{R})$, Q^f is uniformly continuous. That can be proved as follows. Suppose $\Delta_i\downarrow\Delta$ and $f(y)\leq M$, $y\in\mathbb{R}$. Since, for each $x\in[0,1]$,

$$\mu_{\Delta_i-\Delta}(x)=\int_{\Delta_i-\Delta}f(x-y)\,dy\leq M\int_{(\Delta_i-\Delta)\cap[-1,1]}dx$$

we have that, for each $\psi \in \mathcal{H}$, $|\psi|^2 = 1$,

$$\langle \psi, Q^f(\Delta_i - \Delta)\psi \rangle = \int_{[0,1]} \mu_{\Delta_i - \Delta}(x)\, |\psi|^2(x)\, dx \leq M \int_{(\Delta_i - \Delta)\cap[-1,1]} dx$$

which proves the uniform continuity of Q^f.

2.4 Absolutely continuous POVMs

In the present section, we prove that absolutely continuous commutative POVMs admit a strong Feller Markov kernel. Then, we apply the result to the case of the unsharp position observable.

Definition 2.4.1. *[82, 83] A POVM* $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ *is absolutely continuous with respect to a measure* $\nu : \mathcal{B}(X) \to [0,1]$ *if there exists a positive number* c *such that* $\|F(\Delta)\| \leq c\,\nu(\Delta)$, *for each* $\Delta \in \mathcal{B}(X)$.

Theorem 2.4.2. *Let* F *be absolutely continuous with respect to a finite measure* ν. *Then,* F *is uniformly continuous.*

Proof. Suppose $\Delta_i \uparrow \Delta$. We have

$$\begin{aligned}\lim_{i\to\infty} \|F(\Delta) - F(\Delta_i)\| &= \lim_{i\to\infty} \|F(\Delta - \Delta_i)\| \\ &\leq c \lim_{i\to\infty} \nu(\Delta - \Delta_i) = 0.\end{aligned}$$

which proves that F is uniformly continuous. □

Corollary 2.4.3. *Let* $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ *be absolutely continuous with respect to a finite measure* ν. *Then,* F *is commutative if and only if there exist a self-adjoint operator* A *and a strong Feller Markov kernel* $\mu : \sigma(A) \times \mathcal{B}(X) \to [0,1]$ *such that:*

$$F(\Delta) = \mu_\Delta(A), \quad \Delta \in \mathcal{B}(X). \tag{2.17}$$

Proof. By theorem 2.4.2, F is uniformly continuous. Then, theorem 2.3.5 implies the thesis. □

Example 2.4.4. *Let us consider the unsharp position operator defined as follows.*

$$\begin{aligned} Q^f(\Delta) &:= \int_{[0,1]} \mu_\Delta(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \\ \mu_\Delta(x) &:= \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1] \end{aligned} \tag{2.18}$$

where, f is a positive, bounded, Borel function such that $f(x) = 0$, $x \notin [0,1]$,

$$\int_{[0,1]} f(x)dx = 1,$$

and Q_x is the spectral measure corresponding to the position operator

$$Q : L^2([0,1]) \to L^2([0,1])$$
$$\psi(x) \mapsto Q\psi := x\psi(x)$$

Q^f is absolutely continuous with respect to the measure

$$\nu(\Delta) = M \int_{\Delta\cap[-1,1]} dx.$$

Indeed, for each $\psi \in \mathcal{H}$, $|\psi|^2 = 1$,

$$\langle\psi, Q^f(\Delta)\psi\rangle = \int_{[0,1]} \mu_\Delta(x)\,\psi^2(x)\,dx \leq M \int_{\Delta\cap[-1,1]} dx$$

where, the inequality

$$\mu_\Delta(x) = \int_\Delta f(x-y)\,dy \leq M \int_{\Delta\cap[-1,1]} dx$$

has been used.
Therefore, by theorem 2.4.2, $Q^f(\Delta)$ is uniformly continuous. Moreover, the continuity of f assures the continuity of μ_Δ for each $\Delta \in \mathcal{B}(\mathbb{R})$ so that μ is a Feller Markov kernel.

2.4.1 Unsharp Position Observable

In the present subsection, we study an important kind of absolutely continuous POVMs, the unsharp position observables obtained as the marginals of a covariant phase space observable.
In the following $\mathcal{H} = L^2(\mathbb{R})$, Q and P denote position and momentum observables respectively and $*$ denotes convolution, i.e. $(f * g)(x) = \int f(y)g(x-y)dy$.
Let us consider the joint position-momentum POVM [2, 27, 32, 41, 50, 78, 83, 85]

$$F(\Delta \times \Delta') = \int_{\Delta\times\Delta'} U_{q,p}\,\gamma\,U^*_{q,p}\,dq\,dp$$

where, $U_{q,p} = e^{-iqP}e^{ipQ}$ and $\gamma = |f\rangle\langle f|$, $f \in L^2(\mathbb{R})$, $\|f\|_2 = 1$. The marginal

$$Q^f(\Delta) := F(\Delta \times \mathbb{R}) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |f|^2)(x)\,dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{2.19}$$

is an unsharp position observable. Notice that the map $\mu_\Delta(x) := \mathbf{1}_\Delta * |f(x)|^2$ defines a Markov kernel.
Moreover, Q^f is absolutely continuous with respect to the Lebesgue measure. Indeed,

$$\begin{aligned} Q^f(\Delta) = F(\Delta\times\mathbb{R}) &= \int_{\Delta\times\mathbb{R}} U_{q,p}\,\gamma\,U^*_{q,p}\,dq\,dp \\ &= \int_\Delta dq \int_{\mathbb{R}} U_{q,p}\,\gamma\,U^*_{q,p}\,dp \\ &= \int_\Delta \widehat{Q}(q)\,dq \leq \int_\Delta \mathbf{1}\,dq \end{aligned}$$

where,

$$\widehat{Q}(q) = \int_{\mathbb{R}} U_{q,p}\,\gamma\,U^*_{q,p}\,dp.$$

Although Q^f is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, it is not uniformly continuous. That does not contradict theorem 2.4.2 since the Lebesgue measure on $\mathbb{R}$ is not finite. Anyway, Q^f is uniformly continuous on each Borel set Δ with finite Lebesgue measure.
Now, we show that Q^f is not in general uniformly continuous. We give the details of the following particular case.

Example 2.4.5 (Optimal Phase Space Representation)**.** *If we choose*

$$f^2(x) = \frac{1}{l\sqrt{2\pi}}\,e^{(-\frac{x^2}{2l^2})}, \quad l\in\mathbb{R}-\{0\}.$$

in (2.19), we get an optimal phase space representation of quantum mechanics [78]. In this case,

$$\begin{aligned} Q^f(\Delta) &= \int_{-\infty}^{\infty}\Big(\int_\Delta |f(x-y)|^2)\,dy\Big)\,dQ_x \\ &= \frac{1}{l\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big(\int_\Delta e^{-\frac{(x-y)^2}{2l^2}}\,dy\Big)\,dQ_x = \int_{-\infty}^{\infty}\mu_\Delta(x)\,dQ_x \end{aligned}$$

where,

$$\mu_\Delta(x) = \frac{1}{l\sqrt{2\pi}}\int_\Delta e^{-\frac{(x-y)^2}{2l^2}}\,dy \tag{2.20}$$

defines a Markov kernel.

In order to prove that Q^f is not uniformly continuous we consider the family of sets $\Delta_i = (-\infty, a_i)$, $\lim_{i\to\infty} a_i = -\infty$ such that $\Delta_i \downarrow \emptyset$, and prove that $\lim_{i\to\infty} \|Q^f(\Delta_i)\| = 1$. For each $i \in \mathbb{N}$,

$$\begin{aligned}\lim_{x\to-\infty} \mu_{\Delta_i}(x) &= \lim_{x\to-\infty} \frac{1}{l\sqrt{2\pi}} \int_{\Delta_i} e^{-\frac{(x-y)^2}{2l^2}}\, dy \\ = \lim_{x\to-\infty} \frac{1}{l\sqrt{2\pi}} \int_{(-\infty,\, a_i-x)} e^{-\frac{y^2}{2l^2}}\, dy &= \frac{1}{l\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2l^2}}\, dy = 1.\end{aligned}$$

Now, we prove that $\|Q^f(\Delta_i)\| = 1$, $i \in \mathbb{N}$. Indeed, if

$$\psi_n = \chi_{[-n,-n+1]}(x),$$

$$\lim_{n\to\infty} \langle \psi_n, Q^f(\Delta_i)\psi_n\rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \mu_{\Delta_i}(x) |\psi_n(x)|^2\, dx \tag{2.21}$$

$$= \lim_{n\to\infty} \int_{[-n,-n+1]} \mu_{\Delta_i}(x)\, dx = 1. \tag{2.22}$$

Since, for each $\Delta \in \mathcal{B}(\mathbb{R})$, $\|Q^f(\Delta)\| \le 1$, equation (2.21) implies that $\|Q^f(\Delta_i)\| = 1$, for each $i \in \mathbb{N}$. Hence, $\lim_{i\to\infty} \|Q^f(\Delta_i)\| = 1$ and Q^f cannot be uniformly continuous.

It is worth noticing that although Q^f is not uniformly continuous, μ_Δ is continuous for each interval $\Delta \in \mathcal{B}(\mathbb{R})$. Indeed,

$$\begin{aligned}|\mu_\Delta(x) - \mu_\Delta(x')| &= \frac{1}{l\sqrt{2\pi}} \Big| \int_\Delta e^{-\frac{(x-y)^2}{2l^2}}\, dy - \int_\Delta e^{-\frac{(x'-y)^2}{2l^2}}\, dy \Big| \\ &= \frac{1}{l\sqrt{2\pi}} \Big| \int_{\Delta_x} e^{-\frac{(y)^2}{2l^2}} - \int_{\Delta_{x'}} e^{-\frac{(y)^2}{2l^2}}\, dy \Big| \\ &\le \frac{1}{l\sqrt{2\pi}} \Big| \int_{\overline{\Delta}} e^{-\frac{(y)^2}{2l^2}}\, dy \Big|\end{aligned}$$

where,

$$\Delta_x = \{z \in \mathbb{R} \,|\, z = y - x,\, y \in \Delta\}, \quad \Delta_{x'} = \{z \in \mathbb{R} \,|\, z = y - x',\, y \in \Delta\}$$

and,

$$\overline{\Delta} = (\Delta_x - \Delta_{x'}) \cup (\Delta_{x'} - \Delta_x).$$

Therefore, $|x - x'| \le \epsilon$ implies,

$$|\mu_\Delta(x) - \mu_\Delta(x')| \le \frac{1}{l\sqrt{2\pi}} \Big| \int_{\overline{\Delta}} e^{-\frac{(y)^2}{2l^2}}\, dy \Big| \le \frac{1}{l\sqrt{2\pi}} \int_{\overline{\Delta}} dy = \frac{\sqrt{2}}{l\sqrt{\pi}}\, \epsilon.$$

2.5 Interpretation of the smearing

In order to discuss a possible interpretations of the results of the chapter we go back to the example of the unsharp position observable. Let $\Delta \in \mathcal{B}(\mathbb{R})$, $\psi \in L^2([0,1])$ and

$$\langle \psi, Q^f(\Delta)\psi\rangle := \int_{[0,1]} \mu_\Delta(x)\, d\langle \psi, Q_x\psi\rangle, \tag{2.23}$$

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a positive, bounded, Borel function such that $f(y) = 0$, $y \notin [0,1]$, and $\int_{[0,1]} f(y)dy = 1$, while Q_x is the spectral measure corresponding to the position operator

$$Q : L^2([0,1]) \to L^2([0,1])$$
$$\psi(x) \mapsto Q\psi := x\psi(x)$$

We recall that $\langle \psi, Q(\Delta)\psi\rangle$ is interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in Δ. Then, a possible interpretation of equation (2.23) is that Q^f is a randomization of Q. Indeed [78], the outcomes of the measurement of the position of a particle depend on the measurement imprecision[2] so that, if the sharp value of the outcome of the measurement of Q is x then the apparatus produces with probability $\mu_\Delta(\lambda)$ a reading in Δ.

It is worth noting that (see example 2.3.6) the Markov kernel

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

in equation (2.23) above is such that the function $x \mapsto \mu_\Delta(x)$ is continuous for each $\Delta \in \mathcal{B}(\mathbb{R})$. The continuity of μ_Δ means that if two sharp values x and x' are very close to each other then, the corresponding random diffusions are very similar, i.e., the probability to get a result in Δ if the sharp value is x is very close to the probability to get a result in Δ if the sharp value is x'. That is quite common in important physical applications and seems to be reasonable from the physical viewpoint. It is then natural to look for general conditions which ensure the continuity of $\lambda \mapsto \mu_\Delta$. We have proved that it is always possible to choose the Markov kernel μ to be continuous on a ring which generates the Borel σ-algebra $\mathcal{B}(X)$. Anyway, that is the most we can do in the general case. Indeed,

[2] There are other possible interpretations of the randomization. For example, it could be due to the existence of a no-detection probability depending on hidden variables [39].

we proved (see theorem 2.3.5) that the continuity for each Borel set is equivalent to the uniform continuity of F which in its turn is equivalent to require that the smearing $F(\Delta) = \int \mu_\Delta(\lambda)\, dE_\lambda$ can be realized by a strong Feller Markov kernel. It is worth remarking that although in the general case the continuity holds only for a ring of subsets which generates $\mathcal{B}(X)$, that is sufficient to prove the weak convergence of $\mu_{(\cdot)}(x)$ to $\mu_{(\cdot)}(x')$.

2.6 Appendix

2.6.1 On the von Neumann algebra generated by F

We recall that $\mathcal{S} \subset \mathcal{B}(X)$ is a countable basis for the topology of X and $\mathcal{R}(\mathcal{S})$ is the ring generated by $\mathcal{S}$. Theorem c, page 24, in Ref. [65] ensures the countability of $\mathcal{R}(\mathcal{S})$.

Proof. Let $M := \{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$, and $\mathcal{A}^W(F) = \mathcal{A}^W(M)$ the von Neumann algebra generated by F. Let G denote the family of open subsets of X and $O := \{F(\Delta),\ \Delta \in G\}$. Since the POVM F is regular, for each Borel set Δ, there exists a decreasing family of open sets G_i such that $F(G_i) \to F(\Delta)$ strongly. Then, O is dense in M and the von Neumann algebra generated by M coincides with the von Neumann algebra generated by O. Hence,

$$\mathcal{A}^W(F) = \mathcal{A}^W(M) = \mathcal{A}^W(O). \tag{2.24}$$

Now, we prove that the von Neumann algebra $\mathcal{A}^W(O_1)$ generated by $O_1 = \{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$ coincides with $\mathcal{A}^W(O)$.
For each open set G, there exists a family of sets $\{\Delta_i\}_{i\in\mathbb{N}} \subset \mathcal{S}$, such that $G = \cup_{i=1}^{\infty}\Delta_i$. Let $G_n = \cup_{i=1}^{n}\Delta_i$. Then, $G_n \uparrow G$ and

$$F(G) = \lim_{n\to\infty} F(G_n) = \lim_{n\to\infty} F(\cup_{i=1}^{n}\Delta_i).$$

Since the von Neumann algebra generated by O_1 contains $F(\cup_{i=1}^{n}\Delta_i)$ for each $n \in \mathbb{N}$, it must contain $F(G) = \lim_{n\to\infty} F(\cup_{i=1}^{n}\Delta_i)$. Therefore, $\mathcal{A}^W(O) = \mathcal{A}^W(O_1)$ and, by equations (2.24),

$$\mathcal{A}^W(O_1) = \mathcal{A}^W(O) = \mathcal{A}^W(F) \tag{2.25}$$

which proves that $\mathcal{A}^W(F)$ coincides with the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$.

□

2.6.2 Sequences of continuous functions

The following theorem is due to Dini. We give a proof based on the use of sequences.

Theorem 2.6.1. *Let* $\{f_n(\lambda)\}_{n\in\mathbb{N}}$ *be a non increasing sequence of continuous functions defined on a compact set* $B \subset [0,1]$ *with values in* $[0,1]$ *and such that* $f_n(\lambda) \to 0$ *point-wise. Then,* $f_n(\lambda) \to 0$ *uniformly.*

Proof. Since $f_{n+1}(\lambda) \leq f_n(\lambda)$ for each $\lambda \in B$, we have $\|f_{n+1}\|_\infty \leq \|f_n\|_\infty$. If $\|f_n\|_\infty \to 0$ clearly $f_n(\lambda) \to 0$ uniformly.
Then, suppose $\|f_n\|_\infty \to a > 0$. Since $\|f_{n+1}\|_\infty \leq \|f_n\|_\infty$, we have $\|f_n\|_\infty \geq a$, for each $n \in \mathbb{N}$.
Let λ_n be such that $f_n(\lambda_n) = \|f_n\|_\infty$. Since $\{\lambda_n\}$ is a bounded sequence of real numbers, there exists a convergent subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Let β be its limit, i.e., $\beta := \lim_{k\to\infty} \lambda_{n_k}$. The compactness of B assures that $\beta \in B$. Moreover, $\lim_{k\to\infty} f_{n_k}(\lambda_{n_k}) = a$.
Let us consider the sequence of numbers $f_{n_k}(\beta)$. We prove that $f_{n_k}(\beta) \geq a$ for each $k \in \mathbb{N}$. We proceed by contradiction. Suppose that there exists $\bar{k} \in \mathbb{N}$ such that $f_{n_{\bar{k}}}(\beta) < a$. Then, there exists a neighborhood $I(\beta)$ of β such that $f_{n_{\bar{k}}}(\lambda) < a$ for each $\lambda \in I(\beta)$. Moreover, since $\lambda_{n_k} \to \beta$, there exists $l \in \mathbb{N}$ such that $k > l$ implies $\lambda_{n_k} \in I(\beta)$. Take $k > \max\{\bar{k}, l\}$. Then, $\lambda_{n_k} \in I(\beta)$ and $f_{n_k}(\lambda) \leq f_{n_{\bar{k}}}(\lambda)$, for each $\lambda \in B$. Therefore,

$$f_{n_k}(\lambda_{n_k}) \leq f_{n_{\bar{k}}}(\lambda_{n_k}) < a$$

which contradicts the fact that $f_{n_k}(\lambda_{n_k}) = \|f_{n_k}\|_\infty \geq a$, for each $k \in \mathbb{N}$.
We have proved that $f_{n_k}(\beta) \geq a$, for each $k \in \mathbb{N}$. This implies that $\lim_{k\to\infty} f_{n_k}(\beta) \geq a$ and contradicts one of the hypothesis of the lemma, i.e., $\lim_{n\to\infty} f_n(\lambda) = 0$ for each $\lambda \in B$. □

Chapter 3

POVMs and Naimark's dilation

As we have seen in the introduction, an important characterization of general POVMs, not necessarily commutative, has been given by Naimark.

Theorem 3.0.2. (Naimark [72, 1, 50, 71]) *Let F be a POVM of the Hilbert space $\mathcal{H}$. Then, there exist a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a PVM E^+ of the space $\mathcal{H}^+$ such that*

$$F(\Delta) = P^+ E^+(\Delta)_{|\mathcal{H}}$$

where P^+ is the operator of projection onto $\mathcal{H}$.

Naimark's theorem assures that to each POVM F there corresponds a PVM E^+ acting on an extended Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ while, Theorem 2.2.3 in section 2 assures that for any commutative POVM F there exists a PVM E (the sharp version of F) acting on $\mathcal{H}$. It is worth remarking that both theorems 3.2 and 2.2.3 establish a relationship between a POVM and a PVM. In theorem 3.2, the PVM corresponding to the POVM F acts on an extended Hilbert space while, in theorem 2.2.3, the PVM corresponding to F acts on the same Hilbert space on which F acts. Moreover, Theorem 2.2.3 allows us to interpret a commutative POVM as a randomization of a spectral measure. All that raises the question of what are the relationships between the PVM introduced by the Naimark's theorem and the one introduced by theorem 2.2.3. An answer to that question can be given by using the main results of the present chapter. In particular, we establish a connection between the sharp version A of a commutative POVM F and the Naimark's operator $A^+ = \int \lambda \, dE^+_\lambda$ corresponding to E^+. As a consequence, we show that the Naimark's extension theorem can be used in order to characterize the generator of the von Neumann algebra generated by F. The results we present here are published in Ref.s [16, 17, 18].

First, we need to introduce some definitions and to prove some preliminary results. In the following, we restrict ourselves to the case $X = \mathbb{R}$.

Definition 3.0.3. *Each operator $\int f(\lambda) dE^+_\lambda$, where f is a real, one-to-one, measurable, function, is said to be a Naimark operator corresponding to F. The Naimark operator $\int \lambda dE^+_\lambda$ is denoted by A^+.*

Theorem 3.0.4. *Let E^+ be a Naimark's extension of F and $A^+ = \int \lambda dE^+_\lambda$. Let f be a measurable function which is bounded with respect to E^+. Then*

$$P^+ f(A^+)_{|\mathcal{H}} = \int f(\lambda) dF_\lambda$$

and $P^+ f(A^+)_{|\mathcal{H}}$ is a bounded self-adjoint operator.

Proof.

$$\langle P^+ f(A^+)x, y\rangle = \int_{-\infty}^{\infty} f(\lambda) d\langle E^+_\lambda x, P^+ y\rangle = \int_{-\infty}^{\infty} f(\lambda) d\langle F_\lambda x, y\rangle$$

for every $x, y \in \mathcal{H}$.
The boundedness and the self-adjointness of $P^+ f(A^+)_{|\mathcal{H}}$ come, respectively, from the boundedness and the real-valuedness of f with respect to E^+ (see Theorem 10 in Ref. [24]). □

In case f is unbounded, the domain of definitions of the operators must be taken into account [59].

Definition 3.0.5. *Let A be the sharp version of F. Suppose there exist f and G one-to-one and bounded such that*

$$G(A) = \int f(t)\, dF_t = P^+ f(A^+)_{|\mathcal{H}}.$$

Then we say that the sharp version A is equivalent to the projection of a Naimark operator and write

$$A \leftrightarrow \mathrm{Pr}\, A^+.$$

The following theorem is a consequence of definition 4.2.3 and theorem 4.2.5 in section 4.2, chapter 4.

Theorem 3.0.6. *([12, 13, 15]) Let F be commutative and (F, A, μ) be the von Neumann triplet whose existence is proved in Theorem 2.2.3. Let A^+ be as in theorem 3.0.4. Then, for every real Borel function f,*

$$P^+ f(A^+)_{|\mathcal{H}} = F(f) = \int f(t)\, dF_t = \int G_f(\lambda)\, dE_\lambda = G_f(A),$$

where,

$$G_f(\lambda) = \int f(t)\, d\mu_t(\lambda).$$

Theorem 3.0.6 assures that for each operator $F(f) = \int f(t)\, dF_t$ there is a function G_f such that $F(f) = G_f(A)$. We want to show that there is a one-to-one function f such that $F(f)$ is equivalent to the sharp version of F. In other words, we want to show that there is a one-to-one function f such that G_f is one-to-one. That would be an important strengthening of theorem 3.0.6 since it would assure that the sharp version of F can be obtained from a Naimark's extension of F. Such a strengthening can be proved in the case of a POVM $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ such that the operators in the range $\mathcal{R}(F) := \{F(\Delta) \mid \Delta \in \mathcal{B}([0,1])\}$ are discrete (i.e., they have a complete set of eigenvectors). In this case, the sharp version of F is discrete (see theorem 3.2.1).
Before we can proceed to the proof of the main theorem of the chapter we need some preliminary results about sequences of probability measures.

3.1 Sequences of probability measures

In what follows $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ denotes a sequence of probability measures on the Borel σ-algebra $\mathcal{B}([0,1])$. In particular, we focus on sequences of probability measures such that for every non ordered couple of indexes $(i,j) = (j,i)$, $i,j \in \mathbb{N}$, $i \neq j$, there exists a Borel set $\Delta_{i,j}$ such that $\mu_{\Delta_{i,j}}(j) \neq \mu_{\Delta_{i,j}}(i)$. The symbol $\mathcal{C}[0,1]$ denotes the space of bounded continuous functions on $[0,1]$.

Definition 3.1.1. *A sequence of probability measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ on $\mathcal{B}([0,1])$ such that, for every non ordered couple of indexes (i,j), $i,j \in \mathbb{N}$, $i \neq j$, there exists a Borel set $\Delta_{i,j}$ such that $\mu_{\Delta_{i,j}}(j) \neq \mu_{\Delta_{i,j}}(i)$ is named a sequence of distinct probability measures.*

The following Theorem is the main result of the present section. It will be applied to quantum mechanics in subsection 3.2 (see theorem 6). Moreover, it could conceivably be of interest in other areas of mathematics.

Theorem 3.1.2. *[16] Let $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ be a sequence of distinct probability measures on $\mathcal{B}([0,1])$. Let us consider the infinite sequence of linear functionals*

$\{T_i\}_{i\in\mathbb{N}}$ defined as follows

$$T_i f := \int f(t)\, d\mu_t(i) =: G_f(i), \quad i \in \mathbb{N} \tag{3.1}$$

where, $f : [0,1] \to \mathbb{R}$, is a bounded Borel function and the integration is in the sense of Lebesgue-Stieltjes.
There exists a continuous one-to-one function $f(t)$ such that G_f is one-to-one

$$G_f(i) = \int f(t)\, d\mu_t(i) \neq \int f(t)\, d\mu_t(j) = G_f(j), \quad i, j \in \mathbb{N},\, i \neq j.$$

Moreover,

$$f(t) = \sum_{i=1}^{\infty} \alpha_i f_i \tag{3.2}$$

where, $\alpha_i \geq 0$, $\sum_i \alpha_i < \infty$ and $f_i = t^i$ for $i \geq 0$.

Proof. The problem of finding a function f such that $\int f(t) d\mu_t(i) \neq \int f(t) d\mu_t(j)$, for any $i, j \in \mathbb{N}$, is equivalent to the problem of finding a function f such that $\int f(t) d\mu_t(i,j) \neq 0$, for any $i, j \in \mathbb{N}$, where $d\mu_t(i,j)$ denotes integration with respect to the signed measure $\mu_{(\cdot)}(i) - \mu_{(\cdot)}(j)$. Therefore, our problem can be restated as follows. We have a family of signed measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ such that $\mu_{(\cdot)}(i) \neq 0$ for any $i \in \mathbb{N}$ and we want find a continuous one-to-one function f such that $\int f(t) d\mu_t(i) \neq 0$ for any $i \in \mathbb{N}$. We proceed as follows. Let us consider the space $\mathcal{I} = \{f \in \mathcal{C}[0,1] \,|\, f = \sum_{i=0}^{\infty} \alpha_i f_i,\, \alpha_i \geq 0,\, \sum_{i=0} \alpha_i \geq 1\}$. $\mathcal{I}$ with the norm $\|f(t)\| = \sup_{t\in[0,1]} |f(t)|$ is isometric to the metric space $C = \{(\alpha_0, \ldots, \alpha_n, \ldots) \in l_1,\, \alpha_i \geq 0,\, \sum_i \alpha_i \geq 1\}$ which is a closed subset of the complete metric space l_1. Therefore, $\mathcal{I}$ is a complete metric space. Now, for any functional T_i, let us consider the set $\mathcal{A}_i = \{f \in \mathcal{I} \,|\, T_i f \neq 0\}$. Since the family of functions $\{f_n\}$ generates $\mathcal{C}[0,1]$, for any T_i there is a function $f_{n_i} = t^{n_i}$ such that $T_i f_{n_i} \neq 0$. Therefore, $\mathcal{A}_i \neq \emptyset$ for any $i \in \mathbb{N}$. The sets $\mathcal{A}_i$ are also open since $\mathcal{I} - \mathcal{A}_i = \{f \in \mathcal{I} \,|\, T_i f = 0\}$ is closed. Moreover, $\mathcal{A}_i$ is dense in $\mathcal{I}$ for any $i \in \mathbb{N}$. Indeed, for any function $f \in \mathcal{I} - \mathcal{A}_i$, the sequence of functions $f_n = \frac{1}{n} f_0 + f$ with $f_0 \in \mathcal{A}_i$ is such that $f_n \in \mathcal{A}_i$ and $\lim_{n\to\infty} f_n = f$. Now, by Baire category theorem [38], $A = \cap_{i=1}^{\infty} \mathcal{A}_i \neq \emptyset$. Therefore, there exists a function $f(t) = \sum_{i=0}^{\infty} \alpha_n f_n \in A$ such that $T_i f \neq 0$, for any $i \in \mathbb{N}$. Moreover, also the function $f(t) = \sum_{i=1}^{\infty} \alpha_n f_n$ satisfies the thesis of the theorem since $T_i t^0 = T_i 1 = 0$, for any $i \in \mathbb{N}$) and this ends the proof. $\square$

3.2 Sharp versions as projections of Naimark operators

Theorem 3.2.1 ([16]). *Let $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ be a commutative POVM such that $F(\Delta)$ is a discrete operator for every $\Delta \in \mathcal{B}([0,1])$. Then, the sharp version A of F is discrete.*

Proof. Let us consider the set $\mathcal{R}(F)$. By the Theorem at page 358 in [71], there exists a countable subset $\mathcal{B} := \{F_i\}_{i\in\mathbb{N}} \subset \mathcal{R}(F)$ such that the closure of $\mathcal{B}$ in the strong operator topology contains $\mathcal{R}(F)$. This implies that the von Neumann algebra $\mathcal{A}^W(F)$ generated by $\mathcal{R}(F)$ coincides with the von Neumann algebra $\mathcal{A}^W(\mathcal{B})$ generated by $\mathcal{B} := \{F_i\}_{i\in\mathbb{N}}$, $\mathcal{A}^W(F) = \mathcal{A}^W(\mathcal{B})$. It can be proved (see theorem 3.5.1 in the appendix for the details) that the generator of the von Neumann algebra $\mathcal{A}^W(\mathcal{B}) = \mathcal{A}^W(F)$ is a discrete operator. Moreover, by theorem 2.2.3, the sharp version A of F coincides with the generator of $\mathcal{A}^W(F)$. Therefore, A is a discrete operator. □

In the following, by $A = \sum_{i=1}^{\infty} \lambda_i E_i^A$ and $\{\mu_{(\cdot)}(\lambda_i)\}_{i\in\mathbb{N}}$ we denote respectively the sharp version of F and the sequence of probability measures such that $F(\Delta) = \mu_\Delta(A)$, $\Delta \in \mathcal{B}([0,1])$. Notice that $\lambda_i \neq \lambda_j,\ i \neq j$.
The following theorems are the main results of the section and are a direct consequence of Theorem 2.1.1 and Theorem 3.1.2.

Theorem 3.2.2 ([16]). *Let $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ be a commutative POVM such that the operators in the range of F are discrete. Then, the sharp version of F is a linear combination of the moments of F.*

Proof. By Theorem 3.1.2, there is a function f of the form $f(t) = \sum_{i=1}^{\infty} \alpha_i t^i$ such that the function G_f in equation (3.1) is one-to-one. Therefore, (see item d in Theorem VII.2, Ref. [24])

$$G_f(A) = \int f(t)\, dF_t = \sum_{i=1}^{\infty} \alpha_i \int t^i dF_t = \sum_{i=1}^{\infty} \alpha_i\, F[i]. \tag{3.3}$$

Moreover, $G_f(A) \leftrightarrow A$. □

The equivalence between sharp versions and projections of Naimark operators can be generalized to the case of POVM with spectrum in $\mathbb{R}$.

Theorem 3.2.3 ([16]). *Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ be a commutative POVM such that the operators in the range of F are discrete. Let A be the sharp version of F, E^+ an extension of F whose existence is asserted by Naimark's theorem and A^+ the Naimark operator $\int \lambda\, dE^+_\lambda$. Then, $A \leftrightarrow \Pr A^+$.*

Proof. By theorem 10 in [13], we can restrict ourselves, without loss of generality, to the case of POVMs with spectrum in $[0, 1]$. Therefore, let F be a POVM with spectrum in $[0, 1]$ and such that $F(\Delta)$ is discrete for every $\Delta \in \mathcal{B}([0, 1])$. By Theorem 3.2.1, A is discrete so that we can write $A = \sum_{i=1}^{\infty} \lambda_i E_i^A$. Let $\{\mu_{(\cdot)}(\lambda_i)\}_{i\in\mathbb{N}}$ be the sequence of probability measures such that $F(\Delta) = \mu_\Delta(A)$. By Theorem 2.1.1, $\{\mu_{(\cdot)}(\lambda_i)\}_{i\in\mathbb{N}}$ is a sequence of distinct probability measures. Theorem 3.1.2 ensures the existence of a measurable, one-to-one function $f(t)$ such that the function

$$G_f(\lambda_i) = \int f(t) d\mu_t(\lambda_i)$$

is one-to-one. Theorem 3.0.6 ends the proof. □

In the language of the theory of operator algebras, we can state the following corollary of Theorem 3.2.3:

Corollary 3.2.4 ([16])**.** *Let F, A and E^+ be as in Theorem 3.2.3. Let $\mathcal{A}^W(E^+)$ be the von Neumann-algebra of bounded self-adjoint operators generated by E^+ and $\mathcal{A}^W(A)$ the von Neumann algebra generated by A. Then, there exist a generator B^+ of $\mathcal{A}^W(E^+)$ and a generator B of $\mathcal{A}^W(A)$ such that $B = P^+ B^+_{|\mathcal{H}}$.*

Remark 3.2.5. *Notice that P^+ does not commute with the operators in $\mathcal{A}^W(E^+)$. In fact, if these operators commuted F would be orthogonal.*

3.3 Open problems

In the previous section we characterized the sharp version A of a POVM F as the projection of a Naimark operator A^+. In order for such characterization to be complete, it is necessary to extend theorem 3.2.3 to the general case. That seems not to be straitforward. In the present section we introduce [18] two conjectures (conjecture 3.3.1) which suggest a possible path to such extension and generalize theorem 3.2.3.

Conjecture 3.3.1 ([18])**.** *For each von Neumann triplet (F, A, μ) there exists a one-to-one, measurable function $f : \mathbb{R} \to [0, 1]$ and a real, E-a.e. one-to-one, measurable function $G(\lambda)$ such that*

$$G(A) = \int f(t)\, dF_t \tag{3.4}$$

where, E is the spectral measure corresponding to A.

Notice that, $G(A)$ is a sharp version for F. In order to see that, let us consider the measurable function

$$\widetilde{G}(\lambda) = \begin{cases} G(\lambda), & \lambda \in M \\ x_0, & \lambda \notin M \end{cases}$$

where, x_0 is a number such that $x_0 \notin G(M)$, $E(M) = \mathbf{1}$ and G is one-to-one on M.
We have,

$$\int \widetilde{G}(\lambda)\, dE_\lambda = \int G(\lambda)\, dE_\lambda = G(A).$$

Now, we define the measurable function

$$H(x) = \begin{cases} \widetilde{G}^{-1}(x), & x \in G(M) \\ 0, & x \in \overline{G(\sigma(A))} - G(M) \end{cases}$$

We have,

$$(H \circ \widetilde{G})(\lambda) = \begin{cases} \lambda, & \lambda \in M \\ 0, & \lambda \in \sigma(A) - M \end{cases}$$

so that,

$$A = (H \circ \widetilde{G})(A) = H(\widetilde{G}(A)) = H(G(A)).$$

Moreover, $F(\Delta) = (\mu_\Delta \circ H)(G(A))$ which proves that $(F, G(A), (\mu \circ H))$ is a von Neumann triplet.
Now, we can prove that the von Neumann algebra $\mathcal{A}^W(G(A))$ generated by $G(A)$ coincides with the von Neumann algebra $\mathcal{A}^W(A)$ generated by A. Indeed, $G(A) \in \mathcal{A}^W(A)$ implies $\mathcal{A}^W(G(A)) \subset \mathcal{A}^W(A)$. On the other hand, for each Δ, $F(\Delta) \in \mathcal{A}^W(G(A))$. Then, $\mathcal{A}^W(A) = \mathcal{A}^W(F) \subset \mathcal{A}^W(G(A))$.

Now, we give a necessary and sufficient condition for the conjecture to be true.

Theorem 3.3.2 ([18])**.** *Conjecture 3.4 is true if and only if, for any von Neumann triplet (F, A, μ), there exists a measurable, one-to-one function $f : \mathbb{R} \to [0, 1]$ such that the function*

$$G(\lambda) := \int f(t)\, d\mu_t(\lambda)$$

is E-a.e. one-to-one on $\sigma(A)$.

Proof. Let (F, A, μ) be a von Neumann triplet. Suppose that the conjecture is true. Then, there exists a real, measurable, one-to-one function f and a real, E-a.e. one-to-one, measurable function G such that

$$G(A) = \int f(t)\, dF_t.$$

Then (see theorem 9 in Ref. [15]),

$$\int G(\lambda)\, dE_\lambda = G(A) = \int f(t)\, dF_t = \int \Big(\int f(t)\, d\mu_t(\lambda) \Big)\, dE_\lambda$$

so that,

$$G(\lambda) = \int f(t)\, d\mu_t(\lambda), \quad E - a.e.$$

Conversely, suppose that, for any von Neumann triplet (F, A, μ), there exists a real, measurable, one-to-one function f such that the function

$$G(\lambda) = \int f(t)\, d\mu_t(\lambda)$$

is E-a.e. one-to-one on $\sigma(A)$. Then,

$$G(A) = \int \Big(\int f(t)\, d\mu_t(\lambda) \Big)\, dE_\lambda = \int f(t)\, dF_t.$$

□

Theorem 3.3.2 shows that conjecture 1 is true if and only theorem 3.1.2 can be extended to the continuous case (notice that we only require that G is E-a.e. one-to-one).
It is easy to see that if conjecture 3.3.1 is true, theorem 3.2.3 can be extended to any commutative POVM.
Now, we introduce a second conjecture which is stronger than conjecture 3.3.1. It corresponds to an extension of theorem 3.1.2 to the continuous case but, at variance with conjecture 1, we require that G is one-to-one. Some of its consequences are analyzed below.

Conjecture 3.3.3 ([18])**.** *For each family of probability measures* $\{\mu_{(\cdot)}(\lambda)\}_{\lambda \in B}$*,* $B \subseteq [0, 1]$*,* $\mu_{(\cdot)}(\lambda) : \mathcal{B}(\mathbb{R}) \to [0, 1]$*, there exists a real, measurable, one-to-one function* $f : \mathbb{R} \to [0, 1]$ *such that,*

$$G(\lambda) := \int f(t)\, d\mu_t(\lambda) \neq \int f(t)\, d\mu_t(\lambda') =: G(\lambda')$$

whenever $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$*.*

Notice that, both conjecture 1 and conjecture 2 correspond to an extension of theorem 3.1.2 to the continuous case but, while the former requires that G be E-a.e one-to-one, the latter requires that G be one-to-one on every set $B \subset [0,1]$ such that $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$, $\lambda, \lambda' \in B$. Indeed, conjecture 3.3.3 is stronger than conjecture 3.3.1.

Theorem 3.3.4 ([18]). *Conjecture 3.3.3 implies conjecture 3.3.1*

Proof. Let (F, A, μ) be a von Neumann triplet. By theorem 2.1.1, there exists a set $B \in \sigma(A)$ such that, $E(B) = \mathbf{1}$ and $\{\mu_{(\cdot)}(\lambda)\}_{\lambda \in B}$ is a family of distinct probability measures. Suppose that conjecture 3.3.3 is true. Then, there exists a real, measurable, one-to-one function f such that the function

$$G(\lambda) := \int f(t)\, d\mu_t(\lambda)$$

is one-to-one on B. Then, theorem 3.3.2 ends the proof. □

The following theorem shows an important consequence of conjecture 3.3.3. Its proof requires some results obtained in Ref.s [31, 10] which we briefly recall. There exists an algorithmic procedure [31] for the construction of a family of set functions $\omega_{(\cdot)}(\lambda)$, $\lambda \in [0,1]$, such that, for any commutative POVM F, there exists a self-adjoint operator A^F such that,

$$F(\Delta) = \int \omega_\Delta(\lambda)\, dE^F_\lambda = \omega_\Delta(A^F), \quad \Delta \in \mathcal{B}(\mathbb{R}) \tag{3.5}$$

where, E^F is the spectral resolution corresponding to A^F.
Another property of the family of set functions $\{\omega_{(\cdot)}(\lambda)\}_{\lambda \in [0,1]}$ is that there exists a countable semi-ring $\mathcal{S}$ which generates the σ-algebra of the reals $\mathcal{B}(\mathbb{R})$, such that, for each $\lambda \in \sigma(A^F)$, $\omega_{(\cdot)}(\lambda)$ is additive on the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$.

Theorem 3.3.5 ([18]). *If the conjecture 3.3.3 is true, the functions f and G in the equation*

$$G(A) = \int f(t)\, dF_t$$

do not depend on the POVM F.

Proof. Let $\mathcal{D}$ be the set of the commutative POVMs with spectrum in $[0,1]$, $\{\omega_\Delta\}_{\Delta \in \mathcal{B}(\mathbb{R})}$ the family of functions whose existence is proved in Ref. [31] and $I := \cup_{F \in \mathcal{D}}\, \sigma(A^F) \subset [0,1]$. What we said above implies that, for each $\lambda \in I$, the set function $\omega_{(\cdot)}(\lambda)$ is additive on $\mathcal{R}(\mathcal{S})$.

Theorem 2 in Ref. [11] assures that, starting from the family of additive (on $\mathcal{R}(\mathcal{S})$) set functions $\{\omega_{(\cdot)}(\lambda)\}_{\lambda\in I}$, it is possible to build a family of probability measures $\{\mu_{(\cdot)}(\lambda)\}_{\lambda\in I}$ such that, for any pair (F, A^F), we have $\mu_\Delta(A^F) = F(\Delta)$. In particular, for each pair (F, A^F), the triplet (F, A^F, μ) is a von Neumann triplet and $\mu_{(\cdot)}(\cdot)$ does not depend on F.
Moreover, by theorem 2.1.1, for any triplet (F, A^F, μ), there is a set $B_F \subset \sigma(A^F)$ such that $E^F(B_F) = \mathbf{1}$ and the probability measures in the family $\{\mu_{(\cdot)}(\lambda)\}_{\lambda\in B_F}$ are distinct.
In other words, we can define a family of probability measures $\{\mu_{(\cdot)}(\lambda)\}_{\lambda\in I}$ such that, for every commutative POVM F, there exists a self-adjoint operator A^F such that (F, A^F, μ) is a von Neumann triplet. Moreover, for any F, the probability measures in the subfamily $\{\mu_{(\cdot)}(\lambda)\}_{\lambda\in B_F}$ are distinct.
Now, if conjecture 3.3.3 is true, there exists a one-to-one function $f : \mathbb{R} \to [0, 1]$ such that the function

$$G(\lambda) = \int f(t)\, d\mu_t(\lambda) \tag{3.6}$$

is one-to-one on $B_F \subset \sigma(A^F)$, for each $F \in \mathcal{D}$. Therefore, by theorem 2.2.3,

$$A^F \equiv G(A^F) = \int G(\lambda)\, dE^F_\lambda \tag{3.7}$$

$$= \int \Big(\int f(t)\, d\mu_t(\lambda)\Big)\, dE^F_\lambda = \int f(t)\, dF_t. \tag{3.8}$$

Since μ in equation (3.6) does not depend on F the same will be true for f and for G in (3.7), i.e., f and G do not depend on F (notice that the spectrum of A^F is contained in $I \subseteq [0, 1]$). □

In other words, f, G and μ are fixed and, for any F, there is a self-adjoint operator A_F such that (F, A^F, μ) is a von Neumann triplet and $G(A^F) = \int f(t)\, dF_t$. Therefore, A^F is the only object which changes when F changes.

3.4 Construction of the universal antismearing function

Theorem 3.1.2 can be proved by construction. In particular, one can prove the following theorem.

Theorem 3.4.1 ([17])**.** *Let $F : \mathcal{B}([0, 1]) \to \mathcal{F}(\mathcal{H})$ be a commutative POVM such that the operators in the range of F are discrete. Then, one can build a real,* *one-to-one, continuous from the left function f such that the function*

$$G(i) := \int_{[0,1]} f(t)\, d\mu_t(i).$$

is injective, i.e., $G(i) \neq G(j)$, $i \neq j$, and

$$G(A) = \int_{[0,1]} f(t)\, dF_t. \tag{3.9}$$

Proof. Let F be a POVM with spectrum in $[0,1]$ such that $F(\Delta)$ is discrete for every $\Delta \in \mathcal{B}([0,1])$. By Theorem 3.2.1, the sharp version A is discrete so that we can write $A = \sum_{i=1}^{\infty} \lambda_i\, E_i^A$. Let $\{\mu_{(\cdot)}(\lambda_i),\, i \in N\}$ be the sequence of probability measures such that $F(\Delta) = \mu_\Delta(A)$. By Theorem 2.1.1, $\{\mu_{(\cdot)}(\lambda_i),\, i \in N\}$ is a sequence of distinct probability measures. Theorem 3.4.5 below ensures the existence of a measurable, one-to-one, continuous from the left function $f(t)$ such that the function $G_f(\lambda_i) = \int f(t)\, d\mu_t(\lambda_i)$ is one-to-one. Theorem 2.2.3 ends the proof. □

Theorem 3.4.1 suggests a procedure for extending theorem 3.1.2 to the general case as well as to prove conjecture 3.3.3 constructively. It is worth remarking that it would be very helpful to have a procedure for the construction of f in the general case. Indeed, as we explained in the preceding section, if conjecture 3.3.3 is true, the function f is universal, i.e., it does not depend on F. Therefore, once f is constructed, it can be used to recover the sharp version of any commutative POVM. That is why f was called the universal antismearing function.

On a property of stochastic matrices

The following theorem on stochastic matrices is the starting point in the constructive proof of Theorem 3.4.1.

Theorem 3.4.2 ([17]). *A matrix of real numbers:*

$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} \\ \dots & \dots & \dots & \dots \\ \lambda_N^{(1)} & \lambda_N^{(2)} & \dots & \lambda_N^{(m)} \end{pmatrix} \tag{3.10}$$

such that:

i) for every couple of indexes (i,j), $i,j = 1,\dots,N$, there exists an index $l \in \{1,\dots,m\}$ such that $\lambda_i^{(l)} \neq \lambda_j^{(l)}$;

ii) the matrix is stochastic, i.e., $\sum_{i=1}^{m} \lambda_j^{(i)} = 1,\ \forall j \in \{1, \ldots, N\}$,

defines an operator $T : \mathbb{C}^m \to \mathbb{C}^N$

$$T\mathbf{k} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} := \begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \ldots & \lambda_1^{(m)} \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \ldots & \lambda_2^{(m)} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_N^{(1)} & \lambda_N^{(2)} & \ldots & \lambda_N^{(m)} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix}, \tag{3.11}$$

with the property that there exists a real vector $\mathbf{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m$*;* $k_i \neq k_j$*,* $i \neq j$*;* $0 < k_i \leq 1$*, such that,* $(T\,\mathbf{k})_i \neq (T\,\mathbf{k})_j$ *if* $i \neq j$*.*

Proof. We proceed by steps.
Step 1: An arbitrary vector $\mathbf{k}^{(1)} = (k_1^{(1)}, \ldots, k_m^{(1)})$, $0 < k_i^{(1)} \leq 1$, $k_i^{(1)} \neq k_j^{(1)}$, is chosen as the first vector of the sequence.
Step 2: if $(T\,\mathbf{k}^{(1)})_2 \neq (T\,\mathbf{k}^{(1)})_1$ we set $\mathbf{k}^{(2)} = \mathbf{k}^{(1)}$ and proceed to the next step. If instead, $(T\,\mathbf{k}^{(1)})_2 = (T\,\mathbf{k}^{(1)})_1$ then, by item i), there exists an index q_2 such that $\lambda_2^{(q_2)} \neq \lambda_1^{(q_2)}$. We define $\mathbf{k}^{(2)} = (k_1^{(2)} = k_1^{(1)}, \ldots, k_{q_2}^{(2)}, k_{q_2+1}^{(2)} = k_{q_2+1}^{(1)}, \ldots, k_m^{(2)} = k_m^{(1)})$, where $k_{q_2}^{(2)} \in \mathbb{R}$ is such that

$$\begin{cases} k_{q_2}^{(2)} \neq k_j^{(1)}, & 1 \leq j \leq m \\ 0 < k_{q_2}^{(2)} \leq 1 \end{cases} \tag{3.12}$$

We have $(T\,\mathbf{k}^{(2)})_2 \neq (T\,\mathbf{k}^{(2)})_1$. Indeed,

$$(T\,\mathbf{k}^{(2)})_2 - (T\,\mathbf{k}^{(2)})_1 = (T\,\mathbf{k}^{(1)})_2 - (T\,\mathbf{k}^{(1)})_1 + (k_{q_2}^{(2)} - k_{q_2}^{(1)})(\lambda_2^{(q_2)} - \lambda_1^{(q_2)}) =$$

$$= (k_{q_2}^{(2)} - k_{q_2}^{(1)})(\lambda_2^{(q_2)} - \lambda_1^{(q_2)}) \neq 0.$$

Step n ($n < N$): if $(T\,\mathbf{k}^{(n-1)})_n \neq (T\,\mathbf{k}^{(n-1)})_l$ for every $l < n$, we set $\mathbf{k}^{(n)} = \mathbf{k}^{(n-1)}$ and proceed to the next step. If instead, there exists an index $l < n$ such that $(T\,\mathbf{k}^{(n-1)})_n = (T\,\mathbf{k}^{(n-1)})_l$ then, by item i), there exists an index q_n such that, $\lambda_n^{(q_n)} \neq \lambda_l^{(q_n)}$. Therefore, we define $\mathbf{k}^{(n)} = (k_1^{(n)} = k_1^{(n-1)}, \ldots, k_{q_n}^{(n)}, k_{q_n+1}^{(n)} = k_{q_n+1}^{(n-1)}, \ldots, k_m^{(n)} = k_m^{(n-1)})$, where $k_{q_n}^{(n)} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \ldots, m\}$,

$$\begin{cases} 1)\, 0 < k_{q_n}^{(n)} \leq 1 \\ 2)\, k_{q_n}^{(n)} \neq k_j^{(n-1)} \\ 3)\, k_{q_n}^{(n)} \neq k_{q_n}^{(n-1)} - \frac{(T\,\mathbf{k}^{(n-1)})_j - (T\,\mathbf{k}^{(n-1)})_n}{(\lambda_j^{(q_n)} - \lambda_n^{(q_n)})}, \quad if\ \lambda_j^{(q_n)} \neq \lambda_n^{(q_n)},\ j \neq l,\ j < n \\ 4)\, |k_{q_n}^{(n)} - k_{q_n}^{(n-1)}| \leq \frac{\min\limits_{p=j,\ldots,n-1} \{|(T\mathbf{k}^{(p)})_j - (T\mathbf{k}^{(p)})_i|\}}{8 \cdot 2^n},\ i < j < n \end{cases}$$

Notice that, by items n.2, n.3 and n.4,

$$(k_{q_n}^{(n)} - k_{q_n}^{(n-1)}) \neq -\frac{(T\,\mathbf{k}^{(n-1)})_j - (T\,\mathbf{k}^{(n-1)})_i}{(\lambda_j^{(q_n)} - \lambda_i^{(q_n)})}, \tag{3.13}$$

for every $i, j = 1, \ldots, n$, such that $(\lambda_j^{(q_n)} - \lambda_i^{(q_n)}) \neq 0$.
Indeed, by items n.2 and n.3, equation (3.13) holds for every $j = 1, \ldots, n-1$, $i = n$ and, by items n.4, we have:

$$|k_{q_n}^{(n)} - k_{q_n}^{(n-1)}| \leq \frac{|(T\mathbf{k}^{(n-1)})_j - (T\mathbf{k}^{(n-1)})_i|)}{8 \cdot 2^n} < \frac{|(T\mathbf{k}^{(n-1)})_j - (T\mathbf{k}^{(n-1)})_i|}{|\lambda_j^{(q_n)} - \lambda_i^{(q_n)}|}$$

for all $i, j < n$, such that $(\lambda_j^{(q_n)} - \lambda_i^{(q_n)}) \neq 0$.
By equation (3.13),

$$(T\,\mathbf{k}^{(n)})_j \neq (T\,\mathbf{k}^{(n)})_i, \quad i, j = 1, \ldots, n.$$

Indeed,

$$(T\,\mathbf{k}^{(n)})_i - (T\,\mathbf{k}^{(n-1)})_i = (k_{q_n}^{(n)} - k_{q_n}^{(n-1)})\lambda_i^{(q_n)} \tag{3.14}$$

and, by subtracting equation (3.14) from

$$(T\,\mathbf{k}^{(n)})_j - (T\,\mathbf{k}^{(n-1)})_j = (k_{q_n}^{(n)} - k_{q_n}^{(n-1)})\lambda_j^{(q_n)},$$

we get

$$(T\,\mathbf{k}^{(n)})_j - (T\,\mathbf{k}^{(n)})_i =$$
$$= (k_{q_n}^{(n)} - k_{q_n}^{(n-1)})(\lambda_j^{(q_n)} - \lambda_i^{(q_n)}) + (T\,\mathbf{k}^{(n-1)})_j - (T\,\mathbf{k}^{(n-1)})_i \neq 0$$

for every $i, j = 1, \ldots, n$. Therefore, the vector $\mathbf{k}^{(n)} = (k_1^{(n)}, \ldots, k_m^{(n)})$ is such that $(T\,\mathbf{k}^{(n)})_j - (T\,\mathbf{k}^{(n)})_i \neq 0,\ i, j \in \{1, \ldots, n\},\ i \neq j$.
At step $\mathbf{n = N}$, we get a vector $\mathbf{k}^{(N)} = (k_1^{(N)}, \ldots, k_m^{(N)})$ such that $0 < k_i^{(N)} \leq 1$, $k_i^{(N)} \neq k_j^{(N)}$, $i, j = 1 \ldots, m$, and $(T\,\mathbf{k}^{(N)})_j - (T\,\mathbf{k}^{(N)})_i \neq 0,\ i, j \in \{1, \ldots, N\}$, $i \neq j$. □

Stochastic Matrices and Infinite Sequences of Probability Measures

Before we can proceed to the construction of f we need to recall some results in the theory of family of sets.

Definition 3.4.3. *A nonempty family $\mathcal{D}$ of subsets of a set X is said to be a Dynkin system or a σ-class if $\mathcal{D}$ is closed under complements and countable disjoint unions.*

It is worth remarking that σ-class of sets were introduced by Suppes [86] who showed that quantum mechanical phenomena are suitably described by them. In the context of quantum mechanics they are indeed known as quantum probability spaces. They are an interesting example of a non-classical logic. Later, Gudder [40] began the study of the mathematical properties of these spaces.

Theorem 3.4.4 ([58, 75, 77, 91]). *Let $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$ be, respectively, the Dynkin system and the Borel σ-algebra generated by the open balls in $\mathbb{R}^n$. Then $\mathcal{D}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)$. Let $\mathcal{D}(\mathbb{P}^1)$ and $\mathcal{B}(\mathbb{P}^1)$ be, respectively, the Dynkin system and the Borel σ-algebra generated by the half-open intervals $(a, b]$ in $[0, 1]$. Then, $\mathcal{D}(\mathbb{P}^1) = \mathcal{B}(\mathbb{P}^1)$.*

The importance of theorem 3.4.4 derives from the fact that two probability measures $\mu_{(\cdot)}(1)$ and $\mu_{(\cdot)}(2)$ which agree on each open ball must agree on $\mathcal{D}(\mathbb{R}^n)$, so that if we know that $\mathcal{D}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)$ then, we can conclude that the two probability measures are the same. In other words, if $\mu_{(\cdot)}(1) \neq \mu_{(\cdot)}(2)$ then, there must exist an open ball Δ such that $\mu_{(\Delta)}(1) \neq \mu_{(\Delta)}(2)$. In the case of probability measures defined on $\mathcal{B}([0, 1])$, if $\mu_{(\cdot)}(1) \neq \mu_{(\cdot)}(2)$ then, there must exist a half-open interval Δ such that $\mu_{(\Delta)}(1) \neq \mu_{(\Delta)}(2)$.

Now, let us consider a sequence of distinct probability measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ on $\mathcal{B}([0, 1])$. By Theorem 3.4.4, for every non ordered couple of indexes (i, j), there exists a half-open interval $\Delta_n = (a_n, b_n]$ such that $\mu_{\Delta_n}(i) \neq \mu_{\Delta_n}(j)$ where, $n = n(i, j)$. Let us denote by $\mathcal{N}$ the family $\{\Delta_n\}_{n\in\mathbb{N}}$. Notice that such a family is not generally unique. In the following we assume $\mathcal{N}$ to be chosen once and for all. Moreover, we assume that to a partition $\sigma = \{\gamma_1, \ldots, \gamma_n\}$ of $[0, 1]$ there corresponds the family of intervals $\{[0, \gamma_1], (\gamma_1, \gamma_2], \ldots, (\gamma_{n-1}, \gamma_n]\}$.

Theorem 3.4.1 is an immediate consequence of the following theorem.

Theorem 3.4.5 ([17]). *Let $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ be a sequence of distinct probability measures on $\mathcal{B}([0, 1])$. Let us consider the infinite system of linear functionals $\{T_i\}_{i\in\mathbb{N}}$ defined as follows*

$$T_i f := \int f(t)\, d\mu_t(i) =: G_f(i), \quad i \in \mathbb{N}$$

where, $f : [0, 1] \to \mathbb{R}$, is a bounded Borel function and the integration is in the sense of Lebesgue-Stieltjes.
There exists a one-to-one function $f(t)$ such that G_f is one-to-one

$$G_f(i) = \int f(t)\, d\mu_t(i) \neq \int f(t)\, d\mu_t(j) = G_f(j), \quad i, j \in \mathbb{N},\, i \neq j.$$

Moreover, f is continuous from the left.

Proof. In order to construct the one-to-one function f we proceed as follows.
Step 1. Let us consider the first h (with $h > 1$) probability measures, $\{\mu_{(\cdot)}(i)\}_{i=1,\dots,h}$ and the subfamily $\mathcal{D}_1 := \{\Delta_n = \Delta_{n(i,j)} =: (\alpha_{i,j}, \beta_{i,j}]\}_{i,j\le h} \subset \mathcal{N}$. The family $\mathcal{D}_1$ is such that, for every non ordered couple (i,j), $i,j \le h$, there exists an interval $\Delta_n = \Delta_{n(i,j)} \in \mathcal{D}_1$ such that $\mu_{(\Delta_n)}(i) \ne \mu_{(\Delta_n)}(j)$. Moreover, $\mathcal{D}_1$ defines a partition $\sigma^{(1)}$ of $[0,1]$. Indeed, if we arrange the numbers $\alpha_{i,j}$ and $\beta_{i,j}$ in increasing order we get a sequence $\gamma_1^{(1)} < \gamma_2^{(1)} < \cdots < \gamma_{s_1-1}^{(1)}$ which decomposes the interval $[0,1]$ into the family of sets $\mathcal{A}_1 = \{\Delta_1^{(1)} := [0,\gamma_1^{(1)}], \Delta_2^{(1)} := (\gamma_1^{(1)},\gamma_2^{(1)}],\dots,\Delta_{s_1}^{(1)} := (\gamma_{s_1-1}^{(1)},1]\}$ where, $s_1 - 1$ denotes the number of distinct elements in the set $\{\alpha_{i,j},\beta_{i,j}\}_{i<j\le h} = \{\alpha_{i,j},\beta_{i,j}\}_{i,j\le h}$. Notice that, each interval $(\alpha_{i,j},\beta_{i,j}] \in \mathcal{D}_1$ is the union of a finite number of half-open intervals in $\mathcal{A}_1$, so that, we write $\mathcal{D}_1 \prec \mathcal{A}_1$. Now, let us consider the stochastic matrix

$$T^{(1)} := \begin{pmatrix} \mu_{\Delta_1^{(1)}}(1) & \mu_{\Delta_2^{(1)}}(1) & \dots & \mu_{\Delta_{s_1}^{(1)}}(1) \\ \mu_{\Delta_1^{(1)}}(2) & \mu_{\Delta_2^{(1)}}(2) & \dots & \mu_{\Delta_{s_1}^{(1)}}(2) \\ \dots & \dots & \dots & \dots \\ \mu_{\Delta_1^{(1)}}(h) & \mu_{\Delta_2^{(1)}}(h) & \dots & \mu_{\Delta_{s_1}^{(1)}}(h) \end{pmatrix} \tag{3.15}$$

Since $\mathcal{D}_1 \prec \mathcal{A}_1$, $T^{(1)}$ satisfies item i) in Lemma 3.4.2. Therefore, there exists a vector $\mathbf{k}^{(1)} \in \mathbb{R}^{s_1}$ such that $[T^{(1)}\mathbf{k}^{(1)}]_i \ne [T^{(1)}\mathbf{k}^{(1)}]_j$, if $i \ne j$. Moreover $\mathbf{k}^{(1)}$ can be chosen such that $0 < k_i^{(1)} \le 1$, $k_i^{(1)} \ne k_j^{(1)}$, $i = 1,\dots,s_1$, $i \ne j$.
Step 2. Let us set $2_h := h+2-1$, $s_2 := s_1[2(2_h)+1]$ and consider the probability measure $\mu_{(\cdot)}(h+1)$, and the h half-open intervals $\{(\alpha_j^{(2)},\beta_j^{(2)}] := \Delta_{h+1,j}\}_{j=1,\dots,h}$ such that $\mu_{(\Delta_{h+1,j})}(h+1) \ne \mu_{(\Delta_{h+1,j})}(j)$, $j = 1,\dots,h$. Now, let us define an arbitrary partition $\sigma^{(2)} \supset \sigma^{(1)}$ of $[0,1]$ which is obtained from $\sigma^{(1)}$ by dividing each interval $\Delta_i^{(1)}$ into $2(2_h)+1$ intervals in such a way that $\{\alpha_j^{(2)},\beta_j^{(2)}\}_{j=1,\dots,h} \subset \sigma^{(2)}$. Let $\sigma^{(2)} = \{\gamma_1^{(2)},\gamma_2^{(2)},\dots,\gamma_{s_2-1}^{(2)}\}$ be such a partition. Then, the family of intervals corresponding to $\sigma^{(2)}$ is $\mathcal{A}_2 = \{\Delta_1^{(2)} := [0,\gamma_1^{(2)}],\dots,\Delta_{j+1}^{(2)} := (\gamma_j^{(2)},\gamma_{j+1}^{(2)}],\dots,\Delta_{s_2} := (\gamma_{s_2-1}^{(2)},1]\}$. Notice that $\mathcal{A}_2$ decomposes $[0,1]$ in such a way that each half-open interval in $\mathcal{A}_1$ is decomposed into $2(2_h)+1$ half-open intervals in $\mathcal{A}_2$, so that, we write $\mathcal{A}_1 \prec \mathcal{A}_2$.

Now, let us consider the stochastic matrix

$$T^{(2)} := \begin{pmatrix} \mu_{\Delta_1^{(2)}}(1) & \mu_{\Delta_2^{(2)}}(1) & \dots & \mu_{\Delta_{s_2}^{(2)}}(1) \\ \mu_{\Delta_1^{(2)}}(2) & \mu_{\Delta_2^{(2)}}(2) & \dots & \mu_{\Delta_{s_2}^{(2)}}(2) \\ \dots & \dots & \dots & \dots \\ \mu_{\Delta_1^{(2)}}(h+1) & \mu_{\Delta_2^{(2)}}(h+1) & \dots & \mu_{\Delta_{s_2}^{(2)}}(h+1) \end{pmatrix} \tag{3.16}$$

Since $\mathcal{A}_1 \prec \mathcal{A}_2$, $T^{(2)}$ satisfies item i) in Lemma 3.4.2. Therefore, by Lemma 3.4.2, there is a vector $\mathbf{k}^{(2)}$ such that $[T^{(2)}\mathbf{k}^{(2)}]_i \neq [T^{(2)}\mathbf{k}^{(2)}]_j$, $i, j \in \{1, \dots, h+1\}$, $i \neq j$. Now, we show a particular construction of $\mathbf{k}^{(2)}$:

Step 2.1 We start from the vector $\mathbf{k}^{(1,1)} = (k_1^{(1,1)}, k_2^{(1,1)}, \dots, k_{s_2}^{(1,1)})$, where $k_i^{(1,1)} = k_l^{(1)} + a_i^{(1)}$ if $(l-1)[2(2_h)+1] < i \leq l[2(2_h)+1]$, $l = 1, \dots, s_1$, and $a_i^{(1)}$ are real numbers such that (see Lemma 3.5.2 in appendix B), for any $l, q \in \{1, \dots, s_1\}$,

$$\begin{cases} 1)\, a_r^{(1)} = 0 & r = d_l^{(2)} \\ 2)\, a_r^{(1)}(k_{l+1}^{(1)} - k_l^{(1)}) > 0, & r \in (d_l^{(2)}, D_l^{(2)}],\ l < s_1 \\ 3)\, |a_r^{(1)}| \leq b_{(i,j)}^{(2,r)}, & 1 \leq i < j \leq 2 \\ 4)\, |a_r^{(1)}| \leq \delta^{(2,r)} & \\ 5)\, a_r^{(1)} \neq -k_l^{(1)}, & r \in (d_l^{(2)}, D_l^{(2)}] \\ 6)\, a_j^{(1)} - a_i^{(1)} \neq -(k_q^{(1)} - k_l^{(1)}), & i \in (d_l^{(2)}, D_l^{(2)}] \\ & j \in (d_q^{(2)}, D_q^{(2)}] \end{cases}$$

where,

$$\begin{cases} d_l^{(2)} := (l-1)[2(2_h)+1]+1 \\ D_l^{(2)} := l[2(2_h)+1] \\ b_{(i,j)}^{(2,r)} = \frac{|\sum_{l=1}^{s_1} k_l^{(1)}(\mu_{\Delta_l^{(1)}}(j) - \mu_{\Delta_l^{(1)}}(i))|}{32 \cdot 2^2 \cdot 2^r} \\ \delta^{(2,r)} = \frac{\min\left\{|k_j^{(1)} - k_i^{(1)}|,\ |1 - k_j^{(1)}|;\ i < j \leq s_1\right\}}{32 \cdot 2^2 \cdot 2^r} \end{cases}$$

Notice that (see item 2.1.5) $k_i^{(1,1)} \neq 0$ and (see item 2.1.6) $k_i^{(1,1)} \neq k_j^{(1,1)}$ for every $i, j = 1, \dots, s_2$. Moreover (see items 2.1.2 and 2.1.4), $0 < k_j^{(1,1)} \leq 1$, $j = 1, \dots, s_2$.

Step 2.2 if $(T^{(2)}\,\mathbf{k}^{(1,1)})_2 \neq (T^{(2)}\,\mathbf{k}^{(1,1)})_1$, we set $\mathbf{k}^{(1,2)} = \mathbf{k}^{(1,1)}$ and proceed to the next step. If instead, $(T^{(2)}\,\mathbf{k}^{(1,1)})_2 = (T^{(2)}\,\mathbf{k}^{(1,1)})_1$ then, by item i) in Lemma 3.4.2, there exists an index $q_{2,2}$ such that, $\mu_{\Delta_{q_{2,2}}^{(2)}}(1) \neq \mu_{\Delta_{q_{2,2}}^{(2)}}(2)$. Therefore, we define

$$\mathbf{k}^{(1,2)} = (k_1^{(1,2)} = k_1^{(1,1)}, \dots, k_{q_{2,2}}^{(1,2)}, k_{q_{2,2}+1}^{(1,2)} = k_{q_{2,2}+1}^{(1,1)}, \dots, k_{s_2}^{(1,2)} = k_{s_2}^{(1,1)}),$$

where, $k^{(1,2)}_{q_{2,2}} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \dots, s_2\}$,

$$\begin{cases} 1)\, 0 < k^{(1,2)}_{q_{2,2}} \leq 1 & \\ 2)\, k^{(1,2)}_{q_{2,2}} \neq k^{(1,1)}_j & \\ 3)\, |(k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})| \leq \beta^{(2,2)}_{i,j}, & 1 \leq i < j \leq s_2 \\ 4)\, (k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})(k^{(1,1)}_{q_{2,2}+1} - k^{(1,1)}_{q_{2,2}}) > 0, & if \quad q_{2,2} < s_2 \\ 5)\, (k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})(k^{(1,1)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}-1}) < 0, & if \quad q_{2,2} = s_2 \\ 6)\, |(k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})| \leq \gamma^{(2,2)}_{i,j}, & 1 \leq i < j \leq 2 \\ 7)\, |(k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})| \leq \frac{|k^{(1)}_j - k^{(1)}_i|}{32 \cdot 2^2 \cdot 2^2}, & 1 \leq i < j \leq s_1 \end{cases}$$

where,

$$\begin{cases} \beta^{(2,2)}_{i,j} = \frac{|k^{(1,1)}_j - k^{(1,1)}_i|}{8 \cdot 2^2}, \\ \gamma^{(2,2)}_{i,j} = \frac{\left| \sum_{l=1}^{s_1} k^{(1)}_l \left(\mu_{\Delta^{(1)}_l}(j) - \mu_{\Delta^{(1)}_l}(i) \right) \right|}{32 \cdot 2^2 \cdot 2^2 \cdot 2^{q_{2,2}}}. \end{cases}$$

By proceeding as in step 1 of the proof of Lemma 3.4.2, one can prove that

$$[T^{(2)} \mathbf{k}^{(1,2)}]_2 \neq [T^{(2)} \mathbf{k}^{(1,2)}]_1.$$

Step 2.n ($n < 2_h$). If $(T^{(2)}\, \mathbf{k}^{(1,n-1)})_n \neq (T^{(2)}\, \mathbf{k}^{(1,n-1)})_l$ for every $l < n$, we set $\mathbf{k}^{(1,n)} = \mathbf{k}^{(1,n-1)}$ and proceed to the next step. If instead, there exists an index $l < n$ such that $(T^{(2)}\, \mathbf{k}^{(1,n-1)})_n = (T^{(2)}\, \mathbf{k}^{(1,n-1)})_l$ then, by item i) in Lemma 3.4.2, there exists an index $q_{2,n}$ such that, $\mu_{\Delta^{(2)}_{q_{2,n}}}(l) \neq \mu_{\Delta^{(2)}_{q_{2,n}}}(n)$. Therefore, we define

$$\mathbf{k}^{(1,n)} = (k^{(1,n)}_1 = k^{(1,n-1)}_1, \dots, k^{(1,n)}_{q_{2,n}}, k^{(1,n)}_{q_{2,n}+1} = k^{(1,n-1)}_{q_{2,n}+1}, \dots, k^{(1,n)}_{s_2} = k^{(1,n-1)}_{s_2}),$$

where, $k^{(1,n)}_{q_{2,n}} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \dots, s_2\}$,

$$\begin{cases} 1)\, 0 < k^{(1,n)}_{q_{2,n}} \leq 1 & \\ 2)\, k^{(1,n)}_{q_{2,n}} \neq k^{(1,n-1)}_j & \\ 3)\, k^{(1,n)}_{q_{2,n}} \neq k^{(1,n-1)}_{q_{2,n}} - \frac{(T^{(2)}\, \mathbf{k}^{(1,n-1)})_j - (T^{(2)}\, \mathbf{k}^{(1,n-1)})_n}{\mu_{\Delta^{(2)}_{q_{2,n}}}(j) - \mu_{\Delta^{(2)}_{q_{2,n}}}(n)}, & \mu_{\Delta^{(2)}_{q_{2,n}}}(j) \neq \mu_{\Delta^{(2)}_{q_{2,n}}}(n), \\ & j \neq l,\, j < n \\ 4)\, |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leq \alpha^{(2,n)}_{i,j}, & 1 \leq i < j < n \\ 5)\, |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leq \beta^{(2,n)}_{i,j}, & 1 \leq i < j \leq s_2 \\ 6)\, (k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})(k^{(1,n-1)}_{q_{2,n}+1} - k^{(1,n-1)}_{q_{2,n}}) > 0, & if \quad q_{2,n} < s_2 \\ 7)\, (k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})(k^{(1,n-1)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}-1}) < 0, & if \quad q_{2,n} = s_2 \\ 8)\, |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leq \gamma^{(2,n)}_{i,j}, & 1 \leq i < j \leq 2 \\ 9)\, |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leq \frac{|k^{(1)}_j - k^{(1)}_i|}{32 \cdot 2^n \cdot 2^2}, & 1 \leq i < j \leq s_1 \end{cases}$$

where,

$$\begin{cases} \alpha_{i,j}^{(2,n)} = \dfrac{\min\limits_{p=j,\dots,n-1}\{|(T^{(2)}\mathbf{k}^{(1,p)})_j - (T^{(2)}\mathbf{k}^{(1,p)})_i|\}}{8\cdot 2^n}, \\ \beta_{i,j}^{(2,n)} = \dfrac{\min\limits_{p=1,\dots,n-1}\{|k_j^{(1,p)} - k_i^{(1,p)}|\}}{8\cdot 2^n}, \\ \gamma_{i,j}^{(2,n)} = \dfrac{\left|\sum_{l=1}^{s_1} k_l^{(1)}\left(\mu_{\Delta_l^{(1)}}(j) - \mu_{\Delta_l^{(1)}}(i)\right)\right|}{32\cdot 2^n\cdot 2^2\cdot 2^{q_{2,n}}}. \end{cases}$$

By items 2.n.1 and 2.n.2, it follows that the vector $\mathbf{k}^{(1,n)}$ is such that $0 < k_i^{(1,n)} \leq 1$, $k_i^{(1,n)} \neq k_j^{(1,n)}$, $i, j = 1, \dots, s_2$, $i \neq j$. Moreover, by proceeding as in step n of the proof of Lemma 3.4.2 (see items 2.n.3 and 2.n.4 above), one can prove that $[T^{(2)}\mathbf{k}^{(1,n)}]_j \neq [T^{(2)}\mathbf{k}^{(1,n)}]_i$, $i, j \in \{1, \dots, n\}$, $i \neq j$.
Step 2.2$_\mathbf{h}$. For $n = 2_h = h + 1$, we get a vector $\mathbf{k}^{(2)} := \mathbf{k}^{(1,h+1)}$ such that $0 < k_i^{(2)} \leq 1$, $k_i^{(2)} \neq k_j^{(2)}$, $i, j = 1, \dots, s_2$, $i \neq j$. Moreover, $[T^{(2)}\mathbf{k}^{(2)}]_j \neq [T^{(2)}\mathbf{k}^{(2)}]_i$, $i, j \in \{1, \dots, h + 1\}$, $i \neq j$.

Step n, $(n > 1)$. Let us set $n_h := h + n - 1$, $s_n := s_{n-1}(2n_h + 1)$, and consider the probability measure $\mu_{(\cdot)}(n_h)$, and the $n_h - 1$ open intervals $\{(\alpha_j^{(n)}, \beta_j^{(n)}) := \Delta_{n_h,j}\}_{j=1,\dots,n_h-1}$ such that $\mu_{(\Delta_{n_h,j})}(n_h) \neq \mu_{(\Delta_{n_h,j}))}(j)$, $j = 1, \dots, n_h - 1$. Now, let us define an arbitrary partition $\sigma^{(n)} \supset \sigma^{(n-1)}$ of $[0, 1]$ which is obtained from $\sigma^{(n-1)}$ by dividing each interval $\Delta_i^{(n-1)}$ into $2n_h + 1$ intervals in such a way that $\{\alpha_j^{(n)}, \beta_j^{(n)}\}_{j=1,\dots,n_h-1} \subset \sigma^{(n)}$. Let $\sigma^{(n)} = \{\gamma_1^{(n)}, \gamma_2^{(n)}, \dots, \gamma_{s_n-1}^{(n)}\}$ be such a partition. Then, the family of intervals corresponding to $\sigma^{(n)}$ is

$$\mathcal{A}_n = \{\Delta_1^{(n)} := [0, \gamma_1^{(n)}], \dots, \Delta_{j+1}^{(n)} := (\gamma_j^{(n)}, \gamma_{j+1}^{(n)}], \dots, \Delta_{s_n} := (\gamma_{s_n-1}, 1]\}.$$

Notice that $\mathcal{A}_n$ decomposes $[0, 1]$ in such a way that each half-open interval in $\mathcal{A}_{n-1}$ is decomposed into $2n_h + 1$ half-open intervals.

Now, let us consider the stochastic matrix

$$T^{(n)} := \begin{pmatrix} \mu_{\Delta_1^{(n)}}(1) & \mu_{\Delta_2^{(n)}}(1) & \dots & \mu_{\Delta_{s_n}^{(n)}}(1) \\ \mu_{\Delta_1^{(n)}}(2) & \mu_{\Delta_2^{(n)}}(2) & \dots & \mu_{\Delta_{s_n}^{(n)}}(2) \\ \dots & \dots & \dots & \dots \\ \mu_{\Delta_1^{(n)}}(n_h) & \mu_{\Delta_2^{(n)}}(n_h) & \dots & \mu_{\Delta_{s_n}^{(n)}}(n_h) \end{pmatrix}$$

Since $\mathcal{A}_{n-1} \prec \mathcal{A}_n$, $T^{(n)}$ satisfies item i) in Lemma 3.4.2. Therefore, by Lemma 3.4.2, there is a vector $\mathbf{k}^{(n)}$ such that $[T^{(n)}\mathbf{k}^{(n)}]_i \neq [T^{(n)}\mathbf{k}^{(n)}]_j$, $i, j \in \{1, \dots, n_h\}$, $i \neq j$.

Now, we show a particular construction of $\mathbf{k}^{(n)}$:

Step n.1. We start from the vector $\mathbf{k}^{(n-1,1)} = (k_1^{(n-1,1)}, k_2^{(n-1,1)}, \ldots, k_{s_n}^{(n-1,1)})$ where,

$$k_i^{(n-1,1)} = k_l^{(n-1)} + a_i^{(n-1)} \quad \text{if} \quad (l-1)(2n_h+1) < i \leq l(2n_h+1),\ l = 1, \ldots, s_{n-1},$$

and $a_i^{(n-1)}$ are real numbers such that (see lemma 3.5.2 in appendix B) for any $q, l \in \{1, \ldots, s_{n-1}\}$,

$$\begin{cases} 1)\ a_r^{(n-1)} = 0 & r = d_l^{(n)} \\ 2)\ a_r^{(n-1)}(k_{l+1}^{(n-1)} - k_l^{(n-1)}) > 0, & r \in (d_l^{(n)}, D_l^{(n)}],\ l < s_{n-1} \\ 3)\ |a_r^{(n-1)}| \leq b_{(i,j)}^{(n,r)}, & 1 \leq i < j \leq n \\ 4)\ |a_r^{(n-1)}| \leq \delta^{(n,r)} & \\ 5)\ a_r^{(n-1)} \neq -k_l^{(n-1)}, & r \in (d_l^{(n)}, D_l^{(n)}] \\ 6)\ a_j^{(n-1)} - a_i^{(n-1)} \neq -(k_q^{(n-1)} - k_l^{(n-1)}), & i \in (d_l^{(n)}, D_l^{(n)}] \\ & j \in (d_q^{(n)}, D_q^{(n)}] \end{cases}$$

where,

$$\begin{cases} d_l^{(n)} := (l-1)(2n_h+1)+1 \\ D_l^{(n)} := l(2n_h+1) \\ b_{i,j}^{(n,r)} = \dfrac{\min\limits_{p=j-1,\ldots,n-1} \left\{ \left| \sum\limits_{l=1}^{s_p} k_l^{(p)} (\mu_{\Delta_l^{(p)}}(j) - \mu_{\Delta_l^{(p)}}(i)) \right| \right\}}{32 \cdot 2^n \cdot 2^r} \\ \delta^{(n,r)} = \dfrac{\min\limits_{p=1,\ldots,n-1} \left\{ |k_j^{(p)} - k_i^{(p)}|,\ |1 - k_j^{(p)}|;\ i < j \leq s_p \right\}}{32 \cdot 2^n \cdot 2^r} \end{cases}$$

Notice that (item n.1.5) $k_i^{(n-1,1)} \neq 0$, $i = 1, \ldots, s_n$, and (item n.1.6) $k_i^{(n-1,1)} \neq k_j^{(n-1,1)}$, $i \neq j$. Moreover, (items n.1.2 and n.1.4) $0 < k_j^{(n-1,1)} \leq 1$, $j = 1, \ldots, s_n$.

Step n.2 if $(T^{(n)}\,\mathbf{k}^{(n,1)})_2 \neq (T^{(n)}\,\mathbf{k}^{(n,1)})_1$, we set $\mathbf{k}^{(n,2)} = \mathbf{k}^{(n,1)}$ and proceed to the next step. If instead, $(T^{(n)}\,\mathbf{k}^{(n,1)})_2 = (T^{(n)}\,\mathbf{k}^{(n,1)})_1$ then, by item i) in Lemma 3.4.2, there exists an index $q_{n,2}$ such that, $\mu_{\Delta_{q_{n,2}}^{(n)}}(1) \neq \mu_{\Delta_{q_{n,2}}^{(n)}}(2)$. Therefore, we define,

$$\mathbf{k}^{(n,2)} = (k_1^{(n,2)} = k_1^{(n,1)}, \ldots, k_{q_{n,2}}^{(n,2)}, k_{q_{n,2}+1}^{(n,2)} = k_{q_{n,2}+1}^{(n,1)}, \ldots, k_{s_n}^{(n,2)} = k_{s_n}^{(n,1)}),$$

where, $k^{(n,2)}_{q_{n,2}} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \dots, s_n\}$,

$$\begin{cases} 1)\, 0 < k^{(n-1,2)}_{q_{n,2}} \leq 1 & \\ 2)\, k^{(n-1,2)}_{q_{n,2}} \neq k^{(n-1,1)}_{j} & \\ 3)\, |(k^{(n-1,2)}_{q_{n,2}} - k^{(n-1,1)}_{q_{n,2}})| \leq \beta^{(n,2)}_{i,j}, & 1 \leq i < j \leq s_n \\ 4)\, (k^{(n-1,2)}_{q_{n,2}} - k^{(n-1,1)}_{q_{n,2}})(k^{(n-1,1)}_{q_{n,2}+1} - k^{(n-1,1)}_{q_{n,2}}) > 0, & if \quad q_{n,2} < s_n \\ 5)\, (k^{(n-1,2)}_{q_{n,2}} - k^{(n-1,1)}_{q_{n,2}})(k^{(n-1,1)}_{q_{n,2}} - k^{(n-1,1)}_{q_{n,2}-1}) < 0, & if \quad q_{n,2} = s_n \\ 6)\, |(k^{(n-1,2)}_{q_{n,2}} - k^{(n-1,1)}_{q_{n,2}})| \leq \gamma^{(n,2)}_{i,j}, & 1 \leq i < j \leq n \\ 7)\, |(k^{(n-1,2)}_{q_{n,2}} - k^{(n-1,1)}_{q_{n,2}})| \leq \bar{\delta}^{(n,2)} & \end{cases}$$

where,

$$\begin{cases} \beta^{(n,2)}_{i,j} = \frac{|k^{(n-1,1)}_j - k^{(n-1,1)}_i|}{8 \cdot 2^2}, \\ \gamma^{(n,2)}_{i,j} = \frac{\min\limits_{p=j-1,\dots,n-1} \left\{ \left| \sum\limits_{l=1}^{s_p} k^{(p)}_l (\mu_{\Delta^{(p)}_l}(j) - \mu_{\Delta^{(p)}_l}(i)) \right| \right\}}{32 \cdot 2^n \cdot 2^2 \cdot 2^{q_{n,2}}}, \\ \bar{\delta}^{(n,2)} = \frac{\min\limits_{p=1,\dots,n-1} \{ |k^{(p)}_j - k^{(p)}_i|,\ i < j \leq s_p \}}{32 \cdot 2^n \cdot 2^2}. \end{cases}$$

By proceeding as in step 1 of the proof of Lemma 3.4.2, one can prove that

$$[T^{(n)} \mathbf{k}^{(n-1,2)}]_2 \neq [T^{(n)} \mathbf{k}^{(n-1,2)}]_1$$

Step n.m ($m < n_h$). If $(T^{(n)}\, \mathbf{k}^{(n-1,m-1)})_m \neq (T^{(n)}\, \mathbf{k}^{(n-1,m-1)})_l$, for every $l < m$, we set $\mathbf{k}^{(n-1,m)} = \mathbf{k}^{(n-1,m-1)}$ and proceed to the next step. If instead, there exists an index $l < m$ such that $(T^{(n)}\, \mathbf{k}^{(n-1,m-1)})_m = (T^{(n)}\, \mathbf{k}^{(n-1,m-1)})_l$ then, by item i), there exists an index $q_{n,m}$ such that, $\mu_{\Delta^{(n)}_{q_{n,m}}}(l) \neq \mu_{\Delta^{(n)}_{q_{n,m}}}(m)$.

Hence, we define $\mathbf{k}^{(n-1,m)} = (k^{(n-1,m)}_1 = k^{(n-1,m-1)}_1, \dots, k^{(n-1,m)}_{q_{n,m}}, k^{(n-1,m)}_{q_{n,m}+1} = k^{(n-1,m-1)}_{q_{n,m}+1}, \dots, k^{(n-1,m)}_{s_n} = k^{(n-1,m-1)}_{s_n})$, with $k^{(n-1,m)}_{q_{n,m}} \in \mathbb{R}$ such that, for any $i, j \in \{1, \dots, s_n\}$,

$$
\begin{cases}
1)\, 0 < k^{(n-1,m)}_{q_{n,m}} \leq 1 & \\
2)\, k^{(n-1,m)}_{q_{n,m}} \neq k^{(n-1,m-1)}_{j} & \\
3)\, k^{(n-1,m)}_{q_{n,m}} \neq a^{(n,m)}_{q_{n,m}}, & \mu_{\Delta^{(n)}_{q_{n,m}}}(j) \neq \mu_{\Delta^{(n)}_{q_{n,m}}}(m) \\
 & j \neq l,\ j < m \\
4)\, |(k^{(n-1,m)}_{q_{n,m}} - k^{(n-1,m-1)}_{q_{n,m}})| \leq \alpha^{(n,m)}_{i,j}, & 1 \leq i < j < m \\
5)\, |(k^{(n-1,m)}_{q_{n,m}} - k^{(n-1,m-1)}_{q_{n,m}})| \leq \beta^{(n,m)}_{i,j}, & 1 \leq i < j \leq s_n \\
6)\, (k^{(n-1,m)}_{q_{n,m}} - k^{(n-1,m-1)}_{q_{n,m}}) \cdot b^{(n,m)}_{q_{n,m}} > 0, & if \quad q_{n,m} < s_n \\
7)\, (k^{(n-1,m)}_{q_{n,m}} - k^{(n-1,m-1)}_{q_{n,m}}) \cdot b^{(n,m)}_{q_{n,m}-1} < 0, & if \quad q_{n,m} = s_n \\
8)\, |(k^{(n-1,m)}_{q_{n,m}} - k^{(n-1,m-1)}_{q_{n,m}})| \leq \gamma^{(n,m)}_{i,j}, & 1 \leq i < j \leq n \\
9)\, |(k^{(n-1,m)}_{q_{n,m}} - k^{(n-1,m-1)}_{q_{n,m}})| \leq \bar{\delta}^{(n,m)} &
\end{cases}
$$

where,

$$
\begin{cases}
a^{(n,m)}_{q_{n,m}} = k^{(n-1,m-1)}_{q_{n,m}} - \frac{(T^{(n)}\mathbf{k}^{(n-1,m-1)})_j - (T^{(n)}\mathbf{k}^{(n-1,m-1)})_m}{\mu_{\Delta^{(n)}_{q_{n,m}}}(j) - \mu_{\Delta^{(n)}_{q_{n,m}}}(m)} \\
\alpha^{(n,m)}_{i,j} = \frac{\min\limits_{p=j,\ldots,m-1}\{|(T^{(n)}\mathbf{k}^{(n-1,p)})_j - (T^{(n)}\mathbf{k}^{(n-1,p)})_i|\}}{8\cdot 2^m} \\
\beta^{(n,m)}_{i,j} = \frac{\min\limits_{p=1,\ldots,m-1}\{|k^{(n-1,p)}_j - k^{(n-1,p)}_i|\}}{8\cdot 2^m} \\
b^{(n,m)}_{q_{n,m}} = (k^{(n-1,m-1)}_{q_{n,m}+1} - k^{(n-1,m-1)}_{q_{n,m}}) \\
\gamma^{(n,m)}_{i,j} = \frac{\min\limits_{p=j-1,\ldots,n-1}\left\{\left|\sum\limits_{l=1}^{s_p} k^{(p)}_l (\mu_{\Delta^{(p)}_l}(j) - \mu_{\Delta^{(p)}_l}(i))\right|\right\}}{32\cdot 2^n\cdot 2^m\cdot 2^{q_{n,m}}} \\
\bar{\delta}^{(n,m)} = \frac{\min\limits_{p=1,\ldots,n-1}\{|k^{(p)}_j - k^{(p)}_i|;\ i < j \leq s_p\}}{32\cdot 2^n\cdot 2^m}
\end{cases}
$$

By items n.m.1 and n.m.2, it follows that the vector $\mathbf{k}^{(n-1,m)}$ is such that $0 < k^{(n-1,m)}_i \leq 1$, $k^{(n-1,m)}_i \neq k^{(n-1,m)}_j$, $i,j = 1,\ldots,s_n$, $i \neq j$. Moreover, by proceeding as in step n of the proof of Lemma 3.4.2 (see items n.m.3 and n.m.4 above), one can prove that $[T^{(n)}\mathbf{k}^{(n-1,m)}]_j \neq [T^{(n)}\mathbf{k}^{(n-1,m)}]_i$, $i,j \in \{1,\ldots,m\}$, $i \neq j$.

Step n.$\mathbf{n_h}$. For $m = n_h$, we get a vector $\mathbf{k}^{(n)} := \mathbf{k}^{(n-1,n_h)}$ such that $0 \neq k^{(n)}_i \leq 1$, $k^{(n)}_i \neq k^{(n)}_j$, $i,j = 1,\ldots,s_n$, $i \neq j$. Moreover, $[T^{(n)}\mathbf{k}^{(n)}]_j \neq [T^{(n)}\mathbf{k}^{(n)}]_i$, $i,j \in \{1,\ldots,n_h\}$, $i \neq j$.

The procedure outlined above defines inductively a sequence of real vectors

$\{\mathbf{k}^{(n)}\}_{n\in\mathbb{N}}$. Now, let us consider the sequence of uniformly bounded functions $\{f_n(t)\}_{n\in\mathbb{N}}$ defined as follows

$$f_n(t) := \sum_{i=1}^{s_n} k_i^{(n)} \chi_{\Delta_i^{(n)}}(t) \tag{3.17}$$

where, $\chi_\Delta(t)$ denotes the characteristic function of the Borel set Δ.
Clearly, $\|f_n\|_\infty \leq 1, \ \forall n \in \mathbb{N}$. Now, we prove that

a) $\{f_n(t)\}_{n\in\mathbb{N}}$ `is point-wise convergent`

In order to prove item **a)**, we prove that, for any $t \in [0,1]$, the sequence $f_n(t)$ is Cauchy. We proceed as follows. For every $t \in [0,1]$ and $i \in \mathbb{N}$, let us denote by $\Delta_{i(t)}^{(i)}$ the set in $\mathcal{A}_i$ such that $t \in \Delta_{i(t)}^{(i)}$. We have (see items n.1.4 and n.m.9),

$$\begin{aligned}
|f_l(t) - f_{l-1}(t)| &= \Big| \sum_{i=1}^{s_l} k_i^{(l)} \chi_{\Delta_i^{(l)}}(t) - \sum_{i=1}^{s_{l-1}} k_i^{(l-1)} \chi_{\Delta_i^{(l-1)}}(t) \Big| \\
&= \Big| \sum_{i=1}^{s_l} (k_i^{(l)} - \widetilde{k}_i^{(l-1)}) \chi_{\Delta_i^{(l)}}(t) \Big| = |k_{l(t)}^{(l)} - \widetilde{k}_{l(t)}^{(l-1)}| \\
&\leq |k_{l(t)}^{(l-1,l_h)} - k_{l(t)}^{(l-1,1)} + k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)}| \leq \\
&\leq |k_{l(t)}^{(l-1,l_h)} - k_{l(t)}^{(l-1,1)}| + |k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)}| = \\
&= \Big| \sum_{r=2}^{l_h} (k_{l(t)}^{(l-1,r)} - k_{l(t)}^{(l-1,r-1)}) \Big| + |k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)}| \leq \\
&\leq \sum_{r=2}^{l_h} |(k_{l(t)}^{(l-1,r)} - k_{l(t)}^{(l-1,r-1)})| + |a_{l(t)}^{(l-1)}| < \frac{1}{8 \cdot 2^l}
\end{aligned} \tag{3.18}$$

where, for every, $i \in [d_j^{(l)}, D_j^{(l)}]$, $j = 1, \ldots, l_h$, we have defined

$$\widetilde{k}_i^{(l-1)} = k_j^{(l-1)}$$

so that,

$$f_{l-1}(t) = \sum_{i}^{s_{l-1}} k_i^{(l-1)} \chi_{\Delta_i^{(l-1)}}(t) = \sum_{i=1}^{s_l} \tilde{k}_i^{(l-1)} \chi_{\Delta_i^{(l)}}(t).$$

By equation (3.18) the sequence $f_n(t)$ is Cauchy and then convergent for any $t \in [0,1]$. Indeed, for any $\epsilon > 0$ there exists an index $\bar{n}$ such that $\sum_{i=\bar{n}}^{\infty} \frac{1}{2^i} \leq \epsilon$ so that, for any couple of indexes n, m with, $n > m > \bar{n}$, one has

$$
(3.19) \qquad |f_n(t) - f_m(t)| = \Big| \sum_{i=m+1}^{n} f_i(t) - f_{i-1}(t) \Big| \\
\leq \sum_{i=m+1}^{n} |f_i(t) - f_{i-1}(t)| \leq \sum_{i=\bar{n}}^{\infty} \frac{1}{2^i} \leq \epsilon.
$$

Therefore, there exists a function $f(t)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$. Notice that f is Borel measurable because it is the limit of a sequence of Borel measurable functions [79]. We can say more. Indeed, since the inequality in (3.19) does not depend on t, $f_n(t)$ converges uniformly to $f(t)$. This implies that f is continuous from the left since the space of left continuous step functions with the uniform norm is dense in the space of piecewise continuous functions which are continuous from the left [79]. It remains to prove that

b) f `is one-to-one`

c) G_f `is one-to-one`

In order to prove item **b)** we proceed as follows. For every $t, \bar{t} \in [0,1]$, there exists an index s such that $t \in \Delta^{(s)}_{s(t)}$, $\bar{t} \in \Delta^{(s)}_{s(\bar{t})}$, $\Delta^{(s)}_{s(t)} \cap \Delta^{(s)}_{s(\bar{t})} = \emptyset$. Let j be the smallest index such that $t \in \Delta^{(j)}_{j(t)}$, $\bar{t} \in \Delta^{(j)}_{j(\bar{t})}$, $\Delta^{(j)}_{j(t)} \cap \Delta^{(j)}_{j(\bar{t})} = \emptyset$. Moreover, let us suppose, without loss of generality, $j(t) > j(\bar{t})$ (notice that, for every $s > j$, $s(t) > s(\bar{t})$, $s(t) > j(t)$, $s(\bar{t}) > j(\bar{t})$).

For every $n > j, j(t), j(\bar{t})$,

$$
\begin{aligned}
|f_n(t) - f_n(\bar{t})| &= \Big| f_j(t) - f_j(\bar{t}) + \sum_{l=j+1}^{n} [f_l(t) - f_l(\bar{t})] - [f_{l-1}(t) - f_{l-1}(\bar{t})] \Big| \\
&= \Big| f_j(t) - f_j(\bar{t}) + \sum_{l=j+1}^{n} [f_l(t) - f_{l-1}(t)] - [f_l(\bar{t}) - f_{l-1}(\bar{t})] \Big| = \\
&= \Big| (k^{(j)}_{j(t)} - k^{(j)}_{j(\bar{t})}) + \sum_{l=j+1}^{n} [k^{(l)}_{l(t)} - \widetilde{k}^{(l-1)}_{l(t)}] - [k^{(l)}_{l(\bar{t})} - \widetilde{k}^{(l-1)}_{l(\bar{t})}] \Big|
\end{aligned}
$$

Moreover (see items n.1.4 and n.m.9),

$$\begin{aligned}|k_{l(t)}^{(l)} - \widetilde{k}_{l(t)}^{(l-1)}| &= |k_{l(t)}^{(l-1,l_h)} - k_{l(t)}^{(l-1,1)} + k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)}| \leq \\ &\leq |k_{l(t)}^{(l-1,l_h)} - k_{l(t)}^{(l-1,1)}| + |k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)}| = \\ &= \Big|\sum_{r=2}^{l_h}(k_{l(t)}^{(l-1,r)} - k_{l(t)}^{(l-1,r-1)})\Big| + |k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)}| \leq \\ &\leq \sum_{r=2}^{l_h}|(k_{l(t)}^{(l-1,r)} - k_{l(t)}^{(l-1,r-1)})| + |a_{l(t)}^{(l-1)}| < \frac{|(k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)})|}{8 \cdot 2^l}\end{aligned}$$

By the same reasoning applied to the case $\bar{t}$ we get

$$|k_{l(\bar{t})}^{(l)} - \widetilde{k}_{l(\bar{t})}^{(l-1)}| < \frac{|(k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)})|}{8 \cdot 2^l}$$

Therefore,

$$\begin{aligned}&\Big|\lim_{n\to\infty}\sum_{l=j+1}^{n}[k_{l(t)}^{(l)} - \widetilde{k}_{l(t)}^{(l-1)}] - [k_{l(\bar{t})}^{(l)} - \widetilde{k}_{l(\bar{t})}^{(l-1)}]\Big| = \\ &= \lim_{n\to\infty}\Big|\sum_{l=j+1}^{n}[k_{l(t)}^{(l)} - \widetilde{k}_{l(t)}^{(l-1)}] - [k_{l(\bar{t})}^{(l)} - \widetilde{k}_{l(\bar{t})}^{(l-1)}]\Big| < \\ &< \lim_{n\to\infty}\sum_{l=1}^{n}\frac{|(k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)})|}{4 \cdot 2^l} < \frac{|(k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)})|}{2}\end{aligned} \tag{3.20}$$

Then,

$$\lim_{n\to\infty}|f_n(t) - f_n(\bar{t})| \neq 0$$

which proves that f is one-to-one.
Now, we proceed to prove **item c)**.
First we show that $\lim_{n\to\infty}\left(T^{(n)}\mathbf{k}^{(n)}\right)_j \neq \lim_{n\to\infty}\left(T^{(n)}\mathbf{k}^{(n)}\right)_i$.
For every $n > j > i$,

$$\Big|\left(T^{(n)}\mathbf{k}^{(n)}\right)_j - \left(T^{(n)}\mathbf{k}^{(n)}\right)_i\Big| = \Big|\left(T^{(j)}\mathbf{k}^{(j)}\right)_j - \left(T^{(j)}\mathbf{k}^{(j)}\right)_i +$$

$$+\sum_{l=j+1}^{n}\left\{\left[\left(T^{(l)}\mathbf{k}^{(l)}\right)_j - \left(T^{(l)}\mathbf{k}^{(l)}\right)_i\right] - \left[\left(T^{(l-1)}\mathbf{k}^{(l-1)}\right)_j - \left(T^{(l-1)}\mathbf{k}^{(l-1)}\right)_i\right]\right\}\Big| =$$

$$= \Big| \sum_{r=1}^{s_j} k_r^{(j)} [\mu_{\Delta_r^{(j)}}(j) - \mu_{\Delta_r^{(j)}}(i)] + \sum_{l=j+1}^{n} \Big(\sum_{q=1}^{s_l} k_q^{(l)} (\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)) +$$

$$- \sum_{q=1}^{s_{l-1}} k_q^{(l-1)} [\mu_{\Delta_q^{(l-1)}}(j) - \mu_{\Delta_q^{(l-1)}}(i)] \Big) \Big|.$$

Notice that,

$$\mu_{\Delta_q^{(l-1)}}(j) = \sum_{p \in [d_q^l, D_q^l]} \mu_{\Delta_p^{(l)}}(j), \quad j = 1, \ldots, l_h - 1$$

hence,

$$\Big| \Big(T^{(n)} \mathbf{k}^{(n)} \Big)_j - \Big(T^{(n)} \mathbf{k}^{(n)} \Big)_i \Big| =$$

$$= \Big| \sum_{r=1}^{s_j} k_r^{(j)} [\mu_{\Delta_r^{(j)}}(j) - \mu_{\Delta_r^{(j)}}(i)] + \sum_{l=j+1}^{n} \Big[\sum_{q=1}^{s_l} k_q^{(l)} [\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)] +$$

$$- \sum_{q=1}^{s_{l-1}} k_q^{(l-1)} \sum_{p \in [d_q^l, D_q^l]} (\mu_{\Delta_p^{(l)}}(j) - \mu_{\Delta_p^{(l)}}(i)) \Big] \Big| =$$

$$= \Big| \sum_{r=1}^{s_j} k_r^{(j)} [\mu_{\Delta_r^{(j)}}(j) - \mu_{\Delta_r^{(j)}}(i)] + \sum_{l=j+1}^{n} \Big(\sum_{q=1}^{s_l} k_q^{(l)} [\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)] +$$

$$- \sum_{q=1}^{s_l} \widetilde{k}_q^{(l-1)} [\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)] \Big) \Big| =$$

$$= \Big| \sum_{r=1}^{s_j} k_r^{(j)} [\mu_{\Delta_r^{(j)}}(j) - \mu_{\Delta_r^{(j)}}(i)] + \sum_{l=j+1}^{n} \Big(\sum_{q=1}^{s_l} (k_q^{(l)} - \widetilde{k}_q^{(l-1)}) [\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)] \Big) \Big|.$$

Moreover (see items n.m.8 and n.1.3),

$$
\begin{aligned}
|k_q^{(l)} - \widetilde{k}_q^{(l-1)}| &= |k_q^{(l-1,l_h)} - k_q^{(l-1,1)} + k_q^{(l-1,1)} - \widetilde{k}_q^{(l-1)}| \leq \\
&\leq |k_q^{(l-1,l_h)} - k_q^{(l-1,1)}| + |k_q^{(l-1,1)} - \widetilde{k}_q^{(l-1)}| = \\
&= \Big| \sum_{r=2}^{l_h} (k_q^{(l-1,r)} - k_q^{(l-1,r-1)}) \Big| + |k_q^{(l-1,1)} - \widetilde{k}_q^{(l-1)}| \leq \\
&\leq \sum_{r=2}^{l_h} |k_q^{(l-1,r)} - k_q^{(l-1,r-1)}| + |a_q^{(l-1)}| \leq \\
&\leq \frac{|\sum_{s=1}^{s_j} k_s^{(j)} (\mu_{\Delta_s^{(j)}}(j) - \mu_{\Delta_s^{(j)}}(i))|}{8 \cdot 2^l \cdot 2^q}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big| \lim_{n\to\infty} \sum_{l=j+1}^{n} \Big(\sum_{q=1}^{s_l} (k_q^{(l)} - \widetilde{k}_q^{(l-1)}) [\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)] \Big) \Big| \leq \\
&\leq \lim_{n\to\infty} \sum_{l=j+1}^{n} \sum_{q=1}^{s_l} \frac{|\sum_{s=1}^{s_j} k_s^{(j)} (\mu_{\Delta_s^{(j)}}(j) - \mu_{\Delta_s^{(j)}}(i))|}{8 \cdot 2^l \cdot 2^q} = \\
&= \lim_{n\to\infty} \sum_{l=j+1}^{n} \frac{|\sum_{s=1}^{s_j} k_s^{(j)} (\mu_{\Delta_s^{(j)}}(j) - \mu_{\Delta_s^{(j)}}(i))|}{4 \cdot 2^l} < \\
&< \frac{|\sum_{s=1}^{s_j} k_s^{(j)} (\mu_{\Delta_s^{(j)}}(j) - \mu_{\Delta_s^{(j)}}(i))|}{2}
\end{aligned}
$$

which implies,

$$
\sum_{r=1}^{s_j} k_r^{(j)} [\mu_{\Delta_r^{(j)}}(j) - \mu_{\Delta_r^{(j)}}(i)] \neq \lim_{n\to\infty} \sum_{l=j+1}^{n} \Big(\sum_{q=1}^{s_l} (k_q^{(l)} - \widetilde{k}_q^{(l-1)}) [\mu_{\Delta_q^{(l)}}(j) - \mu_{\Delta_q^{(l)}}(i)] \Big)
$$

and then,

$$
\lim_{n\to\infty} \left(T^{(n)} \mathbf{k}^{(n)} \right)_j \neq \lim_{n\to\infty} \left(T^{(n)} \mathbf{k}^{(n)} \right)_i .
$$

By the dominated convergence theorem [65], we get

$$\begin{aligned} G_f(i) &= \int f(t)\, d\mu_t(i) = \lim_{n\to\infty} \int f_n(t)\, d\mu_t(i) \\ &= \lim_{n\to\infty} \left(T^{(n)}\mathbf{k}^{(n)}\right)_i \neq \lim_{n\to\infty} \left(T^{(n)}\mathbf{k}^{(n)}\right)_j \\ &= \lim_{n\to\infty} \int f_n(t)\, d\mu_t(j) = \int f(t)\, d\mu_t(j) = G_f(j) \end{aligned}$$

which proves item c and ends the proof of the theorem.

□

3.5 Appendix

3.5.1 On the discreteness of the sharp version A

Theorem 3.5.1 ([16]). *Let $\mathcal{B} := \{F_i\}_{i\in\mathbb{N}}$ be a commutative family of bounded self-adjoint operators such that F_i is discrete for every $i \in \mathbb{N}$ and, $\mathbf{0} \leq F_i \leq \mathbf{1}$. Let $\mathcal{A}^W(\mathcal{B})$ be the von Neumann algebra generated by $\mathcal{B}$. Let A be a self-adjoint operator which generates $\mathcal{A}^W(\mathcal{B})$. Then, A is discrete.*

Proof. Since the operators F_n are discrete, we can write

$$F_n = \sum_{i=1}^{\infty} \lambda_i^{(n)} E_n(\{\lambda_i^{(n)}\})$$

where, E_n is the spectral measure corresponding to F_n and $\widetilde{\sigma}(F_n) := \{\lambda_i^{(n)}\}_{i\in\mathbb{N}} \subset \sigma(F_n)$ the set of eigenvalues of F_n. Let $\mathcal{A}(\mathcal{B})$ denote the C^*-algebra generated by $\mathcal{B}$. Let $E(\cdot)$ be the spectral measure corresponding to $\mathcal{A}(\mathcal{B})$ whose existence is assured by Theorem 1, page 895, in [35]. By the same theorem, there exists a set of continuous functions $\{f_i\}_{i\in\mathbb{N}}$ such that $\pi(f_i) := \int_\Lambda f_i(\lambda)E(d\lambda) = F_i$ for every $i \in \mathbb{N}$, where Λ is the spectrum of $\mathcal{A}(\mathcal{B})$. Moreover, for every $\mu, \mu' \in \Lambda$ such that $\mu \neq \mu'$, there exists an index l such that $f_l(\mu) \neq f_l(\mu')$. In order to prove this, we proceed by contradiction. Suppose that there exist $\mu, \mu' \in \Lambda$ such that $\mu \neq \mu'$ and $f_i(\mu) = f_i(\mu')$ for every $i \in \mathbb{N}$ and consider the C^*-subalgebra $\mathcal{D} := \{\pi(f) \,|\, f \in C(\sigma(A)),\, f(\mu) = f(\mu')\} \subset \mathcal{A}(\mathcal{B})$. Since $\{\pi(f_i)\}_{i\in\mathbb{N}} \subset \mathcal{D}$, the smallest C^*-algebra containing $\{\pi(f_i)\}_{i\in\mathbb{N}}$ is $\mathcal{D}$. Therefore, $\mathcal{D} = \mathcal{A}(\mathcal{B})$ or, in other words, $\{\pi(f) \,|\, f \in C(\sigma(A))\} = \{\pi(f) \,|\, f \in C(\sigma(A)),\, f(\mu) = f(\mu')\}$. This contradicts Theorem 1, page 895, in [35], which asserts that the map $\pi : f \mapsto \pi(f)$ is an isometric *-isomorphism. Since $\sigma(F_i) = f_i(\Lambda)$, the above reasoning shows that there is a one-to-one correspondence between Λ and a subset X of the

Cartesian product $\sigma(F_1) \times \sigma(F_2) \times \cdots \sigma(F_n) \times \cdots$. One has,

$$\mathbf{1} = \prod_{i=1}^{\infty} E_i(\widetilde{\sigma}(F_i)) = \prod_{i=1}^{\infty}\sum_{j=1}^{\infty} E[f_i^{-1}(\{\lambda_j^{(i)}\})]$$

$$= \sum_{(l_1,\dots,l_n,\dots)\in\mathbb{N}^\infty} E[f_1^{-1}(\lambda_{l_1}^{(1)})]\cdots E[f_n^{-1}(\lambda_{l_n}^{(n)})]\cdots$$

$$= \sum_{(l_1,\dots,l_n,\dots)\in\mathbb{N}^\infty} E[\cap_{i=1}^{\infty} f_i^{-1}(\lambda_{l_i}^{(i)})] = \sum_{(\lambda_{l_1}^{(1)},\dots,\lambda_{l_n}^{(n)},\dots)\in X} E(\{\mu_{l_1,\dots,l_n,\dots}\})$$

where, $\mathbb{N}^\infty = \mathbb{N} \times \cdots \times \mathbb{N} \times \cdots$ is the space of infinite sequences of natural numbers, $\mu_{l_1,\dots,l_n,\dots} \in \Lambda$ is the unique point in Λ such that $f_i(\mu_{l_1,\dots,l_n}) = \lambda_{l_i}^{(i)}$ for every $i \in \mathbb{N}$, and the continuity from above and below of E is used. Since $\mathcal{H}$ is separable, in the last sum, all the terms but a countable number must be $\mathbf{0}$. This implies that E is a discrete spectral measure. Let us denote by μ_i, $i \in \mathbb{N}$, the points in Λ such that $E\{\mu_i\} \neq \mathbf{0}$ and consider the Borel function $g(\mu_i) = \frac{1}{i}$. The operator $A = \sum_{i=1}^{\infty} g(\mu_i)E(\mu_i)$ is a generator of $\mathcal{A}^W(B)$. Indeed, the functions $h_i := f_i \circ g^{-1}$ are such that $F_i = \sum_{i=1}^{\infty} h_i(g(\mu_i))E(\mu_i) = h_i(A)$ so that the von Neumann algebra $\mathcal{A}^W(A)$ generated by A contains $\mathcal{A}^W(\mathcal{B})$, $\mathcal{A}^W(\mathcal{B}) \subset \mathcal{A}^W(A)$. Conversely, the family of elementary functions

$$g_n(\mu_i) = \begin{cases} g(\mu_i) & if \quad i \leq n \\ 0 & if \quad i > n \end{cases}$$

is such that $\lim_{n\to\infty} g_n = g$. Therefore, A is contained in the von Neumann algebra $\mathcal{A}^W(E)$ generated by $\{E(\mu_i)\}_{i\in\mathbb{N}}$. Since $E(\mu_i) \in \mathcal{A}^W(\mathcal{B})$ (see Theorem 5.2.6, page 315 in Ref. [56]), $A \in \mathcal{A}^W(E) \subset \mathcal{A}^W(\mathcal{B})$. Then, $\mathcal{A}^W(A) \subset \mathcal{A}^W(\mathcal{B})$. Moreover, since we have also proved that $\mathcal{A}^W(\mathcal{B}) \subset \mathcal{A}^W(A)$, we must have $\mathcal{A}^W(\mathcal{B}) = \mathcal{A}^W(A)$. The fact that A is discrete ends the proof. □

3.5.2 A Useful Lemma

Lemma 3.5.2 ([17]). *Let us consider step n.1 in the proof of Theorem 3.1.2. There exists a sequence of real numbers* $a_i^{(n-1)}$ *which satisfies the items from n.1.1 to n.1.6.*

Proof. We set

$$\begin{cases} b^{(n,r)} = \dfrac{\min\limits_{p=j-1,\dots,n-1}\{|\sum\limits_{l=1}^{s_p} k_l^{(p)}(\mu_{\Delta_l^{(p)}}(j) - \mu_{\Delta_l^{(p)}}(i))|;\ i < j \leq n\}}{32 \cdot 2^n \cdot 2^r} \\ \delta^{(n,r)} = \dfrac{\min\limits_{p=1,\dots,n-1}\{|k_j^{(p)} - k_i^{(p)}|,\ |1 - k_j^{(p)}|;\ i < j \leq s_p\}}{32 \cdot 2^n \cdot 2^r} \\ B^{(n,r)} := \min\{b^{(n,r)}, \delta^{(n,r)}\} \\ C^{(n,l)} := \frac{(k_{l+1}^{(n-1)} - k_l^{(n-1)})}{|k_{l+1}^{(n-1)} - k_l^{(n-1)}|} \end{cases}$$

In order to prove the Lemma, we set, for every $l \in \{1, \dots, s_{n-1}\}$,

$$a_r^{(n-1)} = \begin{cases} 0, & r = d_l^{(n)} \\ C^{(n,l)} B^{(n,r)} k_l^{(n-1)}, & r \in (d_l^{(n)}, D_l^{(n)}],\ l < s_{n-1} \\ B^{(n,r)} k_{s_{n-1}}^{(n-1)}, & r \in (d_{s_{n-1}}^{(n)}, D_{s_{n-1}}^{(n)}] \end{cases}$$

Then, items n.1.1, n.1.2, n.1.3, n.1.4, n.1.5 are obviously satisfied. It remains to prove item n.1.6. We have, for every $q, l \in \{1, \dots, s_{n-1}\},\ l, q \neq s_{n-1},\ q \neq l$,
(3.21)

$$|a_r^{(n-1)} - a_j^{(n-1)}| = \begin{cases} |C^{(n,l)} k_l^{(n-1)} (B^{(n,r)} - B^{(n,j)})| \neq 0, & j, r \in (d_l^{(n)}, D_l^{(n)}] \\ |C^{(n,l)} B^{(n,r)} k_l^{(n-1)} - C^{(n,q)} B^{(n,j)} k_q^{(n-1)}| < & \\ < |k_l^{(n-1)} - k_q^{(n-1)}|, & r \in (d_l^{(n)}, D_l^{(n)}] \\ & j \in (d_q^{(n)}, D_q^{(n)}] \\ |B^{(n,r)} k_{s_{n-1}}^{(n-1)} - C^{(n,l)} B^{(n,j)} k_l^{(n-1)}| < & \\ < |k_{s_{n-1}}^{(n-1)} - k_l^{(n-1)}|, & r \in (d_{s_{n-1}}^{(n)}, D_{s_{n-1}}^{(n)}] \\ & j \in (d_l^{(n)}, D_l^{(n)}] \\ |B^{(n,r)} k_{s_{n-1}}^{(n-1)} - B^{(n,j)} k_{s_{n-1}}^{(n-1)}| = & \\ = k_{s_{n-1}}^{(n-1)} |B^{(n,r)} - B^{(n,j)}| \neq 0, & r, j \in (d_{s_{n-1}}^{(n)}, D_{s_{n-1}}^{(n)}] \end{cases}$$

In order to explain the second and the third inequalities in (3.21), let us assume $r > j$. Then (see the definition of $B^{(n,r)}$),

$$\begin{aligned} |C^{(n,l)} B^{(n,r)} k_l^{(n-1)} - C^{(n,q)} B^{(n,j)} k_q^{(n-1)}| &\leq \frac{|k_l^{(n-1)} - k_q^{(n-1)}|}{32 \cdot 2^j \cdot 2^n} \Big(\frac{k_l^{(n-1)}}{2^{r-j}} + k_q^{(n-1)} \Big) \\ &< |k_l^{(n-1)} - k_q^{(n-1)}|. \end{aligned}$$

An analogous reasoning can be used to prove the third inequality in 3.21. □

Chapter 4

On the informational content of a POVM

In this chapter we analyze the concept of informational content of an observable. The starting point is the fact that, starting from a commutative POVM F, it is possible to recover the corresponding sharp version E by means of an algorithmic procedure. The procedure was developed in Ref. [31]. In Ref. [10, 11, 14] it was proved that the PVM coming from such a procedure is the sharp version of F (see chapter 2, theorem 2.2.5). We state this results in the form of a theorem.

Theorem 4.0.3. *[31, 10, 11, 14] There exists an algorithmic procedure that starting from a commutative POVM F allows the construction of the corresponding sharp version E.*

As we said above, Theorem 2.2.3 suggests an interpretation of the unsharp observable F as randomization of the sharp observable E. Therefore, we can interpret Theorem 4.0.3 as suggesting that there should be a kind of information contained in E which is not lost during the randomization process. This means that, in some sense, F and E should have the same informational content. Therefore, we look for an equivalence relation between observables which capture this kind of informational content. The necessary condition which such a relation must satisfy is that E and F must be equivalent.

In the next section, we recall some well known partial order relations on the set of observables and show that no one of them satisfies this condition. Therefore, in section 4.2, we propose a partial ordering for which E and F are always equivalent and establish some relationships between this one and the others well known partial order relations. The results we present here have been published in Ref. [15].

4.1 Informational content of a quantum observable

Definition 4.1.1 (Smearing). *Let F_1 and F_2 be two observables. If there exists a Markov kernel $\mu_{(\cdot)}(\lambda)$ such that,*

$$F_1(\Delta) = \int \mu_\Delta(\lambda) F_2(d\lambda),$$

we say that F_1 is a smearing of F_2 and write $F_1 \preceq_f F_2$. If $F_1 \preceq_f F_2 \preceq_f F_1$, we say that F_1 and F_2 are $\sim_f$ equivalent and write $F_1 \sim_f F_2$.

Theorem 4.1.2 ([53]). *A PVM E is a smearing of a POVM F if and only if the range of E is contained in the range of F, $\mathcal{R}(E) \subset \mathcal{R}(F)$.*

Theorem 4.1.3 ([36]). *Let E and F be respectively a PVM and a POVM. Then, $\mathcal{R}(E) \subset \mathcal{R}(F)$ if and only if there exists a measurable function f such that $E(\Delta) = F(f^{-1}(\Delta))$.*

Notice that, if E_1 and E_2 are two PVMs, $E_1 \preceq_f E_2$ if and only if there exists a measurable function f such that $E_1(\Delta) = E_2(f^{-1}(\Delta))$ (see theorems 4.1.2 and 4.1.3). This is equivalent to the fact that $A_1 = f(A_2)$ where, A_1 and A_2 are the self-adjoint operators corresponding to E_1 and E_2 respectively.

In the following, we denote by $\mathcal{T}_1^+(\mathcal{H})$ the space of trace class operators with trace one on the Hilbert space $\mathcal{H}$. The states of a system are represented by operators in $\mathcal{T}_1^+(\mathcal{H})$.

Definition 4.1.4. *Let ρ_1 and ρ_2 be two states. Let F be an observable. If there exists a set $\Delta \in \mathcal{B}(\mathbb{R})$ such that $Tr[F(\Delta)\rho_1] \neq Tr[F(\Delta)\rho_2]$ we say that F can distinguish between the states ρ_1 and ρ_2.*

Definition 4.1.5 (State distinction). *If for all ρ_1, ρ_2*

$$Tr[F_2(\Delta)\rho_1] = Tr[F_2(\Delta)\rho_2], \quad \forall \Delta \in \mathcal{B}(\mathbb{R})$$

$$\Downarrow$$

$$Tr[F_1(\Delta)\rho_1] = Tr[F_1(\Delta)\rho_2], \quad \forall \Delta \in \mathcal{B}(\mathbb{R})$$

we say that the state distinction power of F_2 is greater than or equal to F_1 and write $F_1 \preceq_i F_2$. If $F_1 \preceq_i F_2 \preceq_i F_1$, we say that F_1 and F_2 are $\sim_i$ equivalent or that they have the same informational content and write $F_1 \sim_i F_2$.

Definition 4.1.6. *The set of the states determined by the observable F is $\mathcal{O}_F := \{\rho |\, \forall \rho' \neq \rho,\ \exists \Delta,\ Tr[F(\Delta)(\rho - \rho')] \neq 0\}$*

Definition 4.1.7 (State determination). *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the sets of states determined by F_1 and F_2 respectively. If $\mathcal{O}_1 \subset \mathcal{O}_2$ we say that F_2 has a state determination power greater or equal than F_1 and write $F_1 \preceq_d F_2$. If $F_1 \preceq_d F_2 \preceq_d F_1$, we say that F_1 and F_2 are $\sim_d$ equivalent and write $F_1 \sim_d F_2$.*

Theorem 4.1.8 ([43]). $F_1 \preceq_f F_2 \Rightarrow F_1 \preceq_i F_2 \Rightarrow F_1 \preceq_d F_2$

4.2 On the informational content of E and F

Now we prove that all the equivalence relations between observables introduced above do not ensure that a commutative POVM is equivalent to its sharp version. We begin with the concept of smearing. The following example shows that the sharp version E of a commutative POVM F need not be a smearing of F, while F is always a smearing of E (see theorem 2.2.3).

Example 4.2.1. *Let us consider a physical system with spin $J = 1$. The corresponding Hilbert space is $\mathcal{H} = \mathbb{C}^3$. Let E_{-1}, E_0, E_1 be the projection operators corresponding to the eigenvectors of the spin observable $J_3 = \sum_{m=-1}^{1} mE_m$. Let us consider the POVM $F : \{1, 2, 3\} \to \{F_1, F_2, F_3\}$ where,*

$$F_1 = 1/2E_{-1} + 1/2E_0 + 1/4E_1,\ F_2 = 1/5E_{-1} + 1/5E_0 + 1/4E_1,\ F_3 = 3/10E_{-1} + 3/10E_0 + 1/2E_1.$$

The sharp version of F is the PVM $E : \{1, 2\} = [(E_{-1} + E_0), E_1]$. Since E_1, $(E_{-1} + E_0) \notin \mathcal{R}(F)$, the range of E is not contained in the range of F. Therefore, by theorem 4.1.2, $E \nsim_f F$.

Now, we proceed analogously for the concepts of state distinction and state determination.

Example 4.2.2. *Let $\{|\psi_i\rangle\}_{i\in\mathbb{N}}$ be a basis of $\mathcal{H}$. Let $E_i = |\psi_i\rangle\langle\psi_i|$ be the projector corresponding to $|\psi_i\rangle$. Let us consider the POVM*

$$F(\Delta) = \begin{cases} \Phi & \text{if } 1 \in \Delta \text{ and } 0 \notin \Delta \\ C = I - \Phi & \text{if } 1 \notin \Delta \text{ and } 0 \in \Delta \\ \mathbf{I} & \text{if } 1, 0 \in \Delta \\ \mathbf{0} & \text{if } 1 \notin \Delta \text{ and } 0 \notin \Delta. \end{cases}$$

where,

$$\Phi = \sum_{i=1}^{\infty} \lambda_i E_i = 1/2E_1 + 1/3E_2 + 2/3E_3 + \sum_{i=4}^{\infty} 1/iE_i.$$

The sharp version of F is (see [12]) $E(\Delta) = \sum_{\lambda_i \in \Delta} E_i$*. Let us consider the states* $\rho_1 = E_1$ *and* $\rho_2 = 1/2(E_2 + E_3)$*. We have,*

$$Tr[\rho_1 F(\Delta)] = Tr[\rho_2 F(\Delta)], \quad \forall \Delta \subset \mathbb{R}$$

while,

$$Tr[\rho_1 E((1/3, 1/2])] = 1, \; Tr[\rho_2 E((1/3, 1/2]) = 0$$

so that, $E \preceq_i F$ ***is false****.*
Notice that, $\rho \in \mathcal{O}_E$ *if and only if* ρ *is a one dimensional spectral projection of* E *(see [29]) so that* $\rho_1 \in \mathcal{O}_E$*, while* $\rho_1 \notin \mathcal{O}_F$*. Therefore,* $E \preceq_d F$ ***is false****.*

Now, we introduce an equivalence relation between observables which should capture the meaning of equivalence between a commutative POVM and its sharp version outlined above. Moreover, the relation we are going to introduce is not restricted to commutative POVMs. The analysis of its meaning in the general case will be the aim of a future work.

Definition 4.2.3. *Let* F_1 *and* F_2 *be two POVMs. We say that* $F_1 \preceq_a F_2$ *if, for each real, bounded, measurable function* f *there exists a real, bounded, measurable function* g_f *such that*

$$B_1 := \int f \, dF_1 \preceq_f \int g_f \, dF_2 =: B_2.$$

If $F_1 \preceq_a F_2 \preceq_a F_1$ *we say that* F_1 *and* F_2 *are* $\sim_a$ *equivalent and write* $F_1 \sim_a F_2$*.*

Notice that, B_1 and B_2 are self-adjoint operators, so that $B_1 \preceq_f B_2$ means that there exists a measurable function h such that $B_1 = h(B_2)$ (see the comment to theorems 4.1.2 and 4.1.3). The following theorem shows that, at least in the hypothesis of Theorem 3.2.3 in section 3.2, a commutative POVM is always equivalent to its sharp version in the sense of Definition 3.0.5.

Theorem 4.2.4 ([15])**.** *Let* F *be a commutative POVM with discrete spectrum such that the operators in the range of* F *are discrete. Let* E *be its sharp version. Then,* $E \sim_a F$*.*

Proof. Let $A = \int \lambda \, dE_\lambda$ be the self-adjoint operator corresponding to E. By Theorem 2.2.3, $\mathcal{A}^W(E) = \mathcal{A}^W(A) = \mathcal{A}^W(F)$, where $\mathcal{A}^W(A)$ and $\mathcal{A}^W(F)$ are the von Neumann algebras generated by A and F respectively. Notice that, for each real bounded (with respect to F) measurable function f, the self-adjoint operator $\int h(t)F(dt)$ is contained in $\mathcal{A}^W(F)$. Therefore, for each real, bounded and measurable function h, there exists a function G_h such that $\int h(t)F(dt) = G_h(A)$. This means that $F \preceq_a E$. Moreover, by Theorem 3.2.3, there exist two one-to-one functions f and G_f such that $G_f(A) = \int f \, F(dt)$. Since $G_f(A)$ is a generator of $\mathcal{A}^W(A)$, one has, $\int h dE \preceq_f \int f(t)F(dt)$, for each real bounded, measurable function h. That implies $E \preceq_a F$ and ends the proof. □

Now, we point out some relationships between Definition 4.2.3 and the other partial order relations on the space of observables introduced above.

Theorem 4.2.5 ([15]). *Let F_1 and F_2 be two POVMs. If $F_1 \preceq_f F_2$, then $F_1 \preceq_a F_2$.*

Proof. Since $F_1 \preceq_f F_2$, there exists a Markov kernel $\mu_{(\cdot)}(\lambda)$ such that

$$F_1([t, t+dt)) = \int_{-\infty}^{+\infty} \mu_{[t,t+dt)}(\lambda) F_2(d\lambda).$$

Then, for each $x \in \mathcal{H}$,

$$\int_{-\infty}^{+\infty} f(t)\langle F_1([t,t+dt))x, x\rangle = \int_{-\infty}^{+\infty} f(t) \int_{-\infty}^{+\infty} \mu_{[t,t+dt)}(\lambda)\langle F_2(d\lambda)x, x\rangle$$

(4.1)

$$= \int_{-\infty}^{+\infty} \langle F_2(d\lambda)x, x\rangle \int_{-\infty}^{+\infty} f(t)\, \mu_{[t,t+dt)}(\lambda) = \int_{-\infty}^{+\infty} g_f(\lambda)\langle F_2(d\lambda)x, x\rangle$$

where, f is a real bounded and measurable function whose infimum and supremum are denoted by m and M respectively and $g_f(\lambda) := \int f(t)\, \mu_{[t,t+dt)}(\lambda) \leq M$. Therefore, by the polarization identity, $\int f\, dF_1 = \int g_f\, dF_2$ so that, $F_1 \preceq_a F_2$.
In order to justify the change in the order of integration in equation (4.1) we proceed as follows. First, we notice that

$$\omega(\cdot) = \int_{-\infty}^{+\infty} \mu_{(\cdot)}(\lambda)\langle F_2(d\lambda)x, x\rangle = \langle F_1(\cdot)x, x\rangle$$

is, for every $x \in \mathcal{H}$, a Lebesgue-Stieltjes measure. Therefore, by the definition of Lebesgue-Stieltjes integral [47],

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(t)\omega(dt) &= \lim_{\substack{n\to\infty \\ |\delta_n|\to 0}} \sum_{k=1}^{n} f_{k-1}^{(n)}\, \omega\Big\{t \in \mathbb{R} \,:\, f(t) \in (f_{k-1}^{(n)}, f_k^{(n)}]\Big\} \\
&= \lim_{\substack{n\to\infty \\ |\delta_n|\to 0}} \sum_{k=1}^{n} f_{k-1}^{(n)} \int_{-\infty}^{+\infty} \mu_{(E_{k-1}^{(n)})}(\lambda)\langle F_2(d\lambda)x, x\rangle \\
&= \lim_{\substack{n\to\infty \\ |\delta_n|\to 0}} \int_{-\infty}^{+\infty} \sum_{k=1}^{n} f_{k-1}^{(n)}\, \mu_{(E_{k-1}^{(n)})}(\lambda)\langle F_2(d\lambda)x, x\rangle
\end{aligned}$$

where it was introduced a sequence of subdivisions $\delta_n = \Big\{[f_0, f_1^{(n)}],\, (f_1^{(n)}, f_2^{(n)}],\, \dots, (f_{n-1}^{(n)}, f_n]\Big\}$, $m = f_0 < f_1 < \cdots < f_n = M$, of the interval $[m, M]$, such that

$|\delta_n| = \max_{1\leq k\leq n}\left\{(f_k^{(n)} - f_{k-1}^{(n)})\right\} \to 0$, when $n \to \infty$, and it was set $E_{k-1}^{(n)} = \left\{t \in \mathbb{R} : f(t) \in (f_{k-1}^{(n)}, f_k^{(n)}]\right\}$.
Now let us consider the sequence of functions

$$H_n(\lambda) = \sum_{k=1}^{n} f_{k-1}^{(n)}\, \mu_{(E_{k-1}^{(n)})}(\lambda).$$

One has $H_n(\lambda) \leq \sup\{|f|\}\mu_{(\mathbb{R})}(\lambda) = M < \infty$, for each $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. Moreover, by the integrability of f with respect to the measure $\mu_{(\cdot)}(\lambda)$,

$$\lim_{n\to\infty} H_n(\lambda) = \int_{-\infty}^{+\infty} f(t)\, \mu_{[t,t+dt)}(\lambda) = g_f(\lambda).$$

By theorem 11 in Ref. [24],

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \sum_{k=1}^{n} f_{k-1}^{(n)}\, \mu_{(E_{k-1}^{(n)})}(\lambda)\langle F_2(d\lambda)x, x\rangle = \int_{-\infty}^{+\infty} \lim_{n\to\infty} H_n(\lambda)\langle F_2(d\lambda)x, x\rangle$$
$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(t) d_t\mu_t(\lambda)\right]\langle F_2(d\lambda)x, x\rangle = \int_{-\infty}^{+\infty} g_f(\lambda)\langle F_2(d\lambda)x, x\rangle.$$

□

Up to now, we have proved that $F_1 \preceq_f F_2$ implies $F_1 \preceq_a F_2$ so that, smearing is stronger than definition 4.2.3. We end this section by observing that neither state distinction nor state determination are weaker than definition 4.2.3. This can be seen by resorting to example 4.2.2 where, $E \sim_a F$ while, $E \preceq_i F$ and $E \preceq_d F$ were proved to be false.

Chapter 5

Uniform continuity, norm-1 property and localization

5.1 Characterization of uniform continuity

In section 2.3 we have shown that a commutative POVM is uniformly continuous if and only if it admits a strong Feller Markov kernel. In the present chapter we study the uniform continuity of a general POVM (not necessarily commutative). We also analyze the concept of absolute continuity of POVMs and show the relationships with the concept of uniform continuity. At last we give a necessary condition for the norm-1 property of a uniformly continuous POVM which we use in order to get some new results about localization observables. In particular we analyze the localization of massless relativistic particles in phase space, and the localization of non-relativistic particles both in phase space and in configuration space. The results of the present chapter are contained in the papers [21, 23].

First we recall the definition of uniform continuity.

Definition 5.1.1. *[21, 23] Let $F:\mathcal{B}(X)\to\mathcal{F}(\mathcal{H})$ be a POVM. F is said to be uniformly continuous at Δ if, for any disjoint decomposition $\Delta=\cup_{i=1}^{\infty}\Delta_i$,*

$$\lim_{n\to\infty}\sum_{i=1}^{n}F(\Delta_i)=F(\Delta)$$

in the uniform operator topology. F is said uniformly continuous if it is uniformly continuous at each $\Delta\in\mathcal{B}(X)$.

We recall that a sequence of sets Δ_i is increasing if $\Delta_i\subset\Delta_{i+1}$. In such a case $\Delta=\cup_{i=1}^{\infty}\Delta_i=\lim_{i\to\infty}\Delta_i$ and we write $\Delta_i\uparrow\Delta$. Analogously, Δ_i is a decreasing family of sets if $\Delta_{i+1} \subset \Delta_i$. In such a case $\Delta = \cap_{i=1}^{\infty} \Delta_i = \lim_{i\to\infty} \Delta_i$ and we write $\Delta_i \downarrow \Delta$.

Proposition 5.1.2 ([21, 23]). *A POVM F is uniformly continuous at Δ if and only if, it is uniformly continuous from below at Δ, i.e., for any increasing sequence $\Delta_i \uparrow \Delta$,*

$$\lim_{n\to\infty} \|F(\Delta) - F(\Delta_i)\| = 0.$$

F is uniformly continuous if and only if it is uniformly continuous from below at each Δ.

Proof. Suppose that $\lim_{n\to\infty} \|F(\Delta)-F(\Delta_i)\| = 0$ whenever $\Delta_i \uparrow \Delta$. Let $\{\Delta_i\}_{i\in\mathbb{N}}$ be a sequence of disjoint sets such that $\cup_{i=1}^{\infty} \Delta_i = \Delta$. Then, we can define the family of sets $\overline{\Delta}_n = \cup_{i=1}^{n} \Delta_i$. We have $\overline{\Delta}_i \uparrow \Delta$. Therefore,

$$\lim_{n\to\infty} \|F(\Delta) - \sum_{i=1}^{n} F(\Delta_i)\| = \lim_{n\to\infty} \|F(\Delta) - F(\overline{\Delta}_n)\| = 0$$

Conversely, suppose that F is uniformly continuous at Δ. Let Δ_i be such that $\Delta_i \uparrow \Delta$. We can define the family of sets $\overline{\Delta}_i = \Delta_i - \Delta_{i-1}$ with $\Delta_0 = \emptyset$. We have, $\overline{\Delta}_i \cap \overline{\Delta}_j = \emptyset$, $i \neq j$. Moreover, $\Delta_n = \cup_{i=1}^{n} \overline{\Delta}_i$ and $\cup_{i=1}^{\infty} \overline{\Delta}_i = \Delta$. Therefore,

$$\lim_{n\to\infty} \|F(\Delta) - F(\Delta_n)\| = \lim_{n\to\infty} \|F(\Delta) - \sum_{i=1}^{n} F(\overline{\Delta}_i)\| = 0.$$

If F is uniformly continuous, the last reasoning is true for any Δ. □

Proposition 5.1.3 ([21, 23]). *F is uniformly continuous if and only if,*

$$\lim_{i\to\infty} \|F(\Delta_i)\| = 0$$

whenever $\Delta_i \downarrow \emptyset$.

Proof. Suppose that $\lim_{i\to\infty} \|F(\Delta_i)\| = 0$ whenever $\Delta_i \downarrow \emptyset$. Let $\{\Delta_i\}_{i\in\mathbb{N}}$ be a disjoint sequence of sets such that $\cup_{i=1}^{\infty} \Delta_i = \Delta$. We have $\Delta - \cup_{i=1}^{n} \Delta_i \downarrow \emptyset$. Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \|F(\Delta) - \sum_{i=1}^{n} F(\Delta_i)\| \\ &\qquad = \lim_{n\to\infty} \|F(\Delta - \cup_{i=1}^{n} \Delta_i)\| = 0. \end{aligned}$$

Conversely, suppose F is uniformly continuous and $\Delta_i \downarrow \emptyset$. We can define the family of sets $\overline{\Delta}_i = \Delta_1 - \Delta_i$. Clearly, $\overline{\Delta}_i \uparrow \Delta_1$. Therefore, by proposition 5.1.2,

$$\begin{aligned}\lim_{i\to\infty} \|F(\Delta_i)\| &= \lim_{i\to\infty} \|F(\Delta_i) - F(\Delta_1) + F(\Delta_1)\| \\ &= \lim_{i\to\infty} \|F(\Delta_1) - F(\overline{\Delta}_i)\| = 0.\end{aligned}$$

□

Now, we introduce the concept of absolute continuity.

Definition 5.1.4. *Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ be a POVM and $\nu : X \to \mathbb{R}$ a regular measure. F is absolutely continuous with respect to ν if there exists a number c such that*

$$\|F(\Delta)\| \le c\,\nu(\Delta), \quad \forall \Delta \in \mathcal{B}(X).$$

Notice that the absolute continuity with respect to a finite regular measure implies the uniform continuity.

Theorem 5.1.5 ([21, 23])**.** *Let F be absolutely continuous with respect to a finite measure ν. Then, F is uniformly continuous.*

Proof. Suppose $\Delta_i \uparrow \Delta$. We have

$$\begin{aligned}\lim_{i\to\infty} \|F(\Delta) - F(\Delta_i)\| &= \lim_{i\to\infty} \|F(\Delta - \Delta_i)\| \\ &\le c \lim_{i\to\infty} \nu(\Delta - \Delta_i) = 0.\end{aligned}$$

Proposition 5.1.2 ends the proof. □

In the case that F is absolutely continuous with respect to an infinite measure, we have the following weak version of theorem 5.1.5.

Theorem 5.1.6 ([21, 23])**.** *Suppose F is absolutely continuous with respect to a regular measure ν. Suppose, Δ is such that $\nu(\Delta) < \infty$. Then, F is uniformly continuous at Δ.*

Proof. By proposition 5.1.2, F is uniformly continuous at Δ if and only if for any increasing family of sets $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \uparrow \Delta$,

$$\lim_{i\to\infty} \|F(\Delta) - F(\Delta_i)\| = 0.$$

Suppose $\nu(\Delta) < \infty$. Suppose $\Delta_i \uparrow \Delta$. Since $\Delta_i \subset \Delta$, $\nu(\Delta_i) \le \nu(\Delta) < \infty$ for every $i \in \mathbb{N}$. By the continuity of ν, $\lim_{i\to\infty} \nu(\Delta - \Delta_i) = 0$. Hence, by the absolute continuity of F,

$$\begin{aligned}\lim_{i\to\infty} \|F(\Delta) - F(\Delta_i)\| \quad &= \lim_{i\to\infty} \|F(\Delta - \Delta_i)\| \\ &\le c \lim_{i\to\infty} \nu(\Delta - \Delta_i) = 0.\end{aligned}$$

□

Theorem 5.1.7 ([21, 23])**.** *Suppose F absolutely continuous with respect to a regular measure ν. Suppose, Δ is such that $\nu(\Delta)<\infty$. Then, for each decreasing family of sets $\Delta_i\downarrow\Delta$ such that $\nu(\Delta_i)<\infty$,*

$$\lim_{i\to\infty}\|F(\Delta_i)-F(\Delta)\|=0.$$

Proof. Suppose $\nu(\Delta)<\infty$, $\Delta_i\downarrow\Delta$ and $\nu(\Delta_i)<\infty$. By the continuity of ν, $\lim_{i\to\infty}\nu(\Delta_i-\Delta)=0$. Hence, by the absolute continuity of F,

$$\begin{aligned}\lim_{i\to\infty}\|F(\Delta_i)-F(\Delta)\|&=\lim_{i\to\infty}\|F(\Delta_i-\Delta)\|\\&\leq c\lim_{i\to\infty}\nu(\Delta_i-\Delta)=0.\end{aligned}$$

□

5.1.1 On the meaning of uniform continuity

Before we analyze the concept of norm-1 property, we would like to make some observations on the meaning of the uniform continuity. Suppose F is uniformly continuous and $\sigma(F)\subset[0,1]$. Suppose 0 is an accumulation point for the spectrum $\sigma(F)$. Then, $F[(0,t)]\neq\mathbf{0}$ for any interval $(0,t)\subset[0,1]$. The uniform continuity of F implies that, for any $\epsilon>0$, there is a $t_i=\frac{1}{i}$ such that $F[(0,t_i)]\neq\mathbf{0}$ and $\sup_{\|\psi\|=1}\langle\psi,F[(0,t_i)]\psi\rangle=\|F[(0,t_i)]\|\leq\epsilon$. Indeed, $(0,\frac{1}{i})\downarrow\emptyset$ and proposition 5.1.3 applies. Next we recall that, for any unit vector $\psi\in\mathcal{H}$, the expression $\mu_\psi^F(\cdot)=\langle F(\cdot)\psi,\psi\rangle$ defines a probability measure. Therefore, the uniform continuity of F ensures that for any $\epsilon>0$, there is an interval $\Delta=(0,t_i)$ such that

$$\mu_\psi^F(\Delta)\leq\epsilon,\quad\text{for any}\quad\psi\in\mathcal{H},\ \|\psi\|=1.$$

In other words, the probability $\mu_\psi^F(\Delta)$ that a result of the measure of F gives a result in Δ is less than ϵ for any pure state ψ.

That is not true if F is a PVM. Indeed, for any Δ such that $F(\Delta)\neq\mathbf{0}$, there is a unit vector $\psi\in\mathcal{H}$ for which $\langle F(\Delta)\psi,\psi\rangle=1$ (sharp localization) so that the inequality $\mu_\psi^F(\Delta)=\langle F(\Delta)\psi,\psi\rangle\leq\epsilon$ cannot be true for each ψ.

A necessary condition for a PVM E to be uniformly continuous is that $E(\{x\})\neq\mathbf{0}$ for any $x\in\sigma(E)$. That is a consequence of theorem 5.2.3 below since a PVM is a POVM with the norm-1 property.

On the meaning of uniform continuity in the commutative case

As we have seen in section 2.3 the uniform continuity of a commutative POVM F is equivalent to the existence of a strong Feller Markov kernel μ such that

$$F(\Delta) = \int \mu_\Delta(\lambda)\, dE_\lambda = \mu_\Delta(A) \tag{5.1}$$

where, E is the spectral measure corresponding to the sharp version A of F. Since $\langle\psi, E(\Delta)\psi\rangle$ is interpreted as the probability that a sharp measurement gives a result in Δ, F can be interpreted as a random diffusion of E; the random diffusion being realized by the transition probability μ. In other words, we could say that if the sharp value of E is λ, the apparatus produces a reading in Δ with probability $\mu_\Delta(\lambda)$.
For instance (see section 2.5), if $F = Q^f$ is the unsharp position POVM in example 2.4.4 and $E = Q$ is the position operator, the quantity $\langle\psi, Q(\Delta)\psi\rangle$ is the probability that a perfectly accurate measurement of the position gives a result in Δ, while $\mu_\Delta(x)$ is the probability of a reading in Δ if the sharp value of Q is x [78].
When F is uniformly continuous and μ is a Feller Markov kernel we have that if two sharp values λ and λ' are very close to each other then, the corresponding random diffusions are very similar, i.e., the probability to get a result in Δ if the sharp value is λ is very close to the probability to get a result in Δ if the sharp value is λ'. That is exactly the case of the bounded unsharp position observable in example 2.4.4.

5.2 Uniform continuity and norm-1 property

In the present section we give a necessary condition for the norm-1 property of uniformly continuous POVMs. First we recall the definition and the physical meaning of the norm-1 property.

Definition 5.2.1. *A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ has the norm-1-property if $\|F(\Delta)\| = 1$, for each $\Delta \in \mathcal{B}(X)$ such that $F(\Delta) \neq \mathbf{0}$.*

The following proposition explains the physical meaning of the norm-1 property.

Proposition 5.2.2 ([46])**.** *A POVM F has the norm-1 property if and only if, for each $\Delta \in \mathcal{B}(X)$ such that $F(\Delta) \neq \mathbf{0}$, there is a sequence of unit vectors ψ_n such that $\lim_{n\to\infty}\langle\psi_n, F(\Delta)\psi_n\rangle = 1$.*

Proof. Suppose F has the norm-1 property and $\|F(\Delta)\| \neq \mathbf{0}$. Then,

$$1 = \|F(\Delta)\| = \sup_{\|\psi\|=1} \{\langle\psi, F(\Delta)\psi\rangle\}.$$

Hence, there is a sequence $|\psi_n\rangle$ such that

$$\lim_{n\to\infty}\langle\psi_n, F(\Delta)\psi_n\rangle = 1. \tag{5.2}$$

Conversely, since $F(\Delta) \leq \mathbf{1}$, equation (5.2) implies $\|F(\Delta)\| = 1$. □

If an observable is described by a PVM E (sharp observable) then, for any Borel set Δ such that $E(\Delta) \neq \mathbf{0}$, there exists a unit vector ψ for which $\langle\psi, E(\Delta)\psi\rangle = 1$; i.e., the probability that a measure of the observable E in the state ψ gives a result in Δ is one. That is not true if an observable is described by a POVM F (unsharp observable) since there are Borel sets Δ such that $0 < \langle\psi, F(\Delta)\psi\rangle < 1$ for any vector ψ. We have here a relevant difference between sharp and unsharp observables. For example, suppose that E and F are a sharp and an unsharp localization observable respectively (E could refer to a non-relativistic particle and F to the photon). Then, in the sharp case, for any set Δ there is a state ψ such that the system is surely localizable in Δ (sharp localization) while in the unsharp case such a state does not exist in general (unsharp localization). That raises the problem of looking for conditions to be satisfied by the unsharp observables in order to ensure a kind of unsharp localization which is as close as possible to the sharp one. The norm-1 property is a possible answer to such a problem. Indeed, as a consequence of proposition 5.2.2, if F has the norm-1 property then, for any $\epsilon > 0$, there is a pure state ψ such that $\langle\psi, F(\Delta)\psi\rangle > 1 - \epsilon$. In other words, the norm-1 property implies that, for any Δ, there exists a preparation procedure such that the quantum mechanical system can be localized within Δ as accurately as desired although not sharply.

The photon is an example of a not sharply localizable system [76, 88]. In the next section we prove that the localization in phase space of massless relativistic particles does not satisfy the norm-1 property.

In the case of uniformly continuous POVMs we have the following necessary condition for the norm-1 property.

Theorem 5.2.3 ([21])**.** *Let $F : X \to \mathcal{F}(\mathcal{H})$ be uniformly continuous and let $\sigma(F)$ be the spectrum of F. Then, F has the norm-1-property only if $\|F(\{x\})\| \neq 0$ for each $x \in \sigma(F)$.*

Proof. We proceed by contradiction. Suppose that F has the norm-1 property and that there exists $x \in \sigma(F)$, such that $F(\{x\}) = \mathbf{0}$. Let Δ_i be a decreasing family of open sets such that, $\cap_{i=1}^{\infty}\Delta_i = \{x\}$. The existence of such family is assured by the local compactness of X. (See theorem 29.2 in [69].) Since $x \in \sigma(F)$ and $x \in \Delta_i$, we have $F(\Delta_i) \neq \mathbf{0}$ for any $i \in \mathbb{N}$ (see Definition 1.1.7) and, by the norm-1 property, $\|F(\Delta_i)\| = 1$. By the uniform continuity of F and proposition 5.1.3,

$$1 = \lim_{i\to\infty} \|F(\Delta_i)\| \quad = \lim_{i\to\infty} \|F(\Delta_i) - F(\{x\}) + F(\{x\})\| \\ \leq \lim_{i\to\infty} \|F(\Delta_i - \{x\})\| + \|F(\{x\})\| = 0.$$

□

Theorem 5.2.4 ([21]). *Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ be absolutely continuous with respect to a regular measure ν. Then, F has the norm-1 property only if $\|F(\{x\})\| \neq 0$ for each $x \in X$ such that $\nu(\{x\}) < \infty$.*

Proof. We proceed by contradiction. Suppose that F has the norm-1 property. Suppose that $x \in \sigma(F)$ is such that $\nu(\{x\}) < \infty$ and $F(\{x\}) = \mathbf{0}$. Thanks to the regularity of ν, there is a decreasing family of open sets $\{\Delta_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(X)$, $x \in \Delta_i$, $\forall i \in \mathbb{N}$, such that $\lim_{i\to\infty} \nu(\Delta_i) = \nu(\{x\})$. Notice that $\nu(\{x\}) < \infty$ implies that there exists an index n_0 such that $\nu(\Delta_n) < \infty$ for each $n \geq n_0$. By the continuity from above of ν [65],

$$\nu(\{x\}) = \lim_{i\to\infty} \nu(\Delta_i) = \lim_{n\to\infty} \nu(\cap_{i=1}^{n} \Delta_i) = \nu(\Delta)$$

where $\Delta = \cap_{i=1}^{\infty} \Delta_i$. Then, $\nu(\Delta) < \infty$ and, by the absolute continuity of F with respect to ν, $F(\Delta) = F(\{x\})$. Then, by theorem 5.1.7,

$$\lim_{i\to\infty} \|F(\Delta_i) - F(\{x\})\| = \lim_{i\to\infty} \|F(\Delta_i) - F(\Delta)\| = 0$$

By the norm-1 property,

$$1 = \lim_{i\to\infty} \|F(\Delta_i)\| \quad = \lim_{i\to\infty} \|F(\Delta_i) - F(\{x\}) + F(\{x\})\| \\ \leq \lim_{i\to\infty} \|F(\Delta_i - \{x\})\| + \|F(\{x\})\| = 0.$$

□

5.3 Analysis of some relevant physical examples

5.3.1 Bounded position operator

Let us consider the unsharp position operator defined as follows.

$$Q^f(\Delta) := \int_{[0,1]} \mu_\Delta(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{5.3}$$

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x - y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a positive, bounded, continuous function such that $f(x) = 0$, $x \notin [0, 1]$,

$$\int_{[0,1]} f(x)dx = 1,$$

and Q_x is the spectral measure corresponding to the position operator

$$\begin{aligned} Q : L^2([0, 1]) &\to L^2([0, 1]) \\ \psi(x) &\mapsto Q\psi := x\psi(x) \end{aligned}$$

In example 2.4.4, we have already proved that Q^f is absolutely continuous with respect to the measure

$$\nu(\Delta) = M \int_{\Delta\cap[-1,1]} dx.$$

Therefore, by theorem 5.1.5, $Q^f(\Delta)$ is uniformly continuous. Moreover, the continuity of f assures the continuity of μ_Δ for each $\Delta \in \mathcal{B}(\mathbb{R})$ so that μ is a Feller Markov kernel. At last, the norm-1 property of Q^f is forbidden by theorem 5.2.3.

5.3.2 Phase observable

Following Ref. [46], we use the following representation of the phase observable

$$E : \mathcal{B}[0, 2\pi) \to \mathcal{F}(\mathcal{H}),$$

$$E(\Delta) = \sum_{m,n=1}^{\infty} \langle\psi_n|\psi_m\rangle \frac{1}{2\pi} \int_\Delta e^{i(n-m)x}\, dx\, |n\rangle\langle m| \tag{5.4}$$

where $|\psi_n\rangle$ is a sequence of unit vectors in $\mathcal{H}$.

The POVM E just defined is covariant with respect to the phase shift operator; i.e.,

$$e^{iN\theta}E(\Delta)e^{-iN\theta} = E(\Delta \oplus \theta)$$

where, the symbol $\oplus$ denotes addition modulo 2π.

1) First, we analyze the case $\langle\psi_n|\psi_m\rangle = \delta_{n,m}$, $n, m \neq s, t$, $s \neq t$, $|\langle\psi_s|\psi_t\rangle| < 1$. We have

$$\begin{aligned} E_1(\Delta) = \frac{1}{2\pi}|\Delta|\, \mathbf{1} &+ \frac{1}{2\pi}\langle\psi_s|\psi_t\rangle \int_\Delta e^{i(s-t)x}\, dx\, |s\rangle\langle t| \\ &+ \frac{1}{2\pi}\langle\psi_t|\psi_s\rangle \int_\Delta e^{i(t-s)x}\, dx\, |t\rangle\langle s| \end{aligned}$$

where, $|\Delta|$ is the Lebesgue measure of Δ. We notice that E_1 is absolutely continuous with respect to the Lebesgue measure on $[0, 2\pi)$. Indeed,

$$\|E_1(\Delta)\| \leq \frac{1}{2\pi}|\Delta| + 2\frac{1}{2\pi}|\Delta|\,\|(|s\rangle\langle t|)\| \leq \frac{3}{2\pi}|\Delta|. \tag{5.5}$$

Therefore, by theorem 5.1.5, E_1 is uniformly continuous and, by theorem 5.2.3, it cannot have the norm-1 property since $|\{x\}| = 0$ for each $x \in [0, 2\pi)$.

2) If $\psi_n = \psi$, $\forall n \in \mathbb{N}$, we have the canonical phase observable

$$E_{can}(\Delta) = \frac{1}{2\pi}|\Delta|\,\mathbf{1} + \frac{1}{2\pi}\sum_{n\neq m}\int_\Delta e^{i(n-m)x}\,dx\,|n\rangle\langle m|.$$

In [46] it is proved that $E_{can}(\Delta)$ has the norm-1 property. Moreover, we notice that $E_{can}(\{x\}) = \mathbf{0}$ for each $x \in X$. Therefore, by theorem 5.2.3, $E_{can}(\Delta)$ cannot be uniformly continuous.

Finally, we remark that the phase space observables we analyzed in items 1) and 2) are not commutative and therefore there is not a common basis of eigenvectors for the effects $E_1(\Delta)$, $\Delta \in \mathcal{B}([0, 2\pi))$, in the first case and for the effects $E_{can}(\Delta)$, $\Delta \in \mathcal{B}([0, 2\pi))$, in the second one. That implies some difficulties in dealing with such POVMs particularly in the second case since in the first case we have only to deal with a finite numbers of terms and we can easily prove the uniform continuity of E_1. Concerning E_{can} we can apply theorem 5.2.3 only indirectly (once we know that E_{can} satisfy the norm-1 property) in order to prove that the POVM is not uniformly continuous. On the other hand a direct proof that E_{can} is not uniformly continuous would be unhelpful in the present context since theorem 5.2.3 holds only for uniformly continuous POVMs. Finally, we note that one cannot use theorem 2.3.5 since the existence of a Markov kernel would imply the commutativity of E_{can}.

5.3.3 Unsharp number observable

The unsharp number observable describes a photon detector with efficiency ϵ less than 1, and is represented by the commutative POVM

$$F_n^\epsilon := \sum_{m=n}^{\infty}\binom{m}{n}\epsilon^n(1-\epsilon)^{m-n}|m\rangle\langle m|. \tag{5.6}$$

Notice that [17] F_n^ϵ is an unsharp version of the number operator N. That can be seen by introducing the functions

$$\mu_n(m) = \begin{cases} 0 & if \quad n > m \\ \binom{m}{n}\epsilon^n(1-\epsilon)^{m-n} & if \quad n \leq m \end{cases} \tag{5.7}$$

which are such that

$$F_n^\epsilon = \mu_n(N),$$

$$\sum_{n=0}^{\infty} \mu_n(m) = 1, \quad \forall m \in \mathbb{N}, \tag{5.8}$$

For each n, F_n^ϵ is compact.[1] Therefore, the unsharp number observable is not uniformly continuous. Indeed, being the space of compact operators closed with respect to the uniform operator topology, the uniform continuity would imply that $\mathbf{1} = u - \lim_{n\to\infty} \sum_{i=1}^n F_i^\epsilon$ which is not possible since $\mathbf{1}$ is not compact. Moreover, F_n^ϵ has not the norm-1 property. Indeed, if $\|F_n^\epsilon\| = 1$ then (see theorem VI.6 in [79]) $\sup_{\lambda\in\sigma(F_n^\epsilon)} |\lambda| = 1$ and, by the compactness of the spectrum, $1 \in \sigma(F_n^\epsilon)$. On the other hand, the compactness of F_n^ϵ implies that if $\|F_n^\epsilon\| = 1$ then, 1 is an eigenvalue of F_n^ϵ but that contradicts equation (5.6) where the eigenvalues of F_n^ϵ are 0 and

$$\binom{m}{n} \epsilon^n (1-\epsilon)^{m-n}, \quad m \geq n,$$

which, by equation (5.8), are strictly less than 1. The only effect with eigenvalue 1 is F_0^ϵ.

5.3.4 Position and Momentum

In the present subsection, we study an important kind of absolutely continuous POVM, the unsharp position observables obtained as the marginals of a covariant phase space observable. Although we already dealt with the unsharp position observable in example 2.4.4, here we analyze it in the light of the results obtained in the present chapter. For the reader convenience, we prefer to repeat (when necessary) some of the reasoning we already presented in example 2.4.4.

In the following $\mathcal{H} = L^2(\mathbb{R})$, Q and P denote the sharp position and momentum observables respectively. In particular

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$\psi(x) \in L^2(\mathbb{R}) \mapsto (Q\psi)(x) := x\psi(x).$$

---

[1] Since

$$\lim_{m\to\infty} \binom{m}{n} \epsilon^n (1-\epsilon)^{m-n} = \lim_{m\to\infty} \frac{\epsilon^n}{n!} [m(1-\epsilon)^{m-n/n}] \cdots [(m-n)(1-\epsilon)^{m-n/n}] = 0,$$

Notice that F_n^ϵ is compact for every $n \in \mathbb{N}$ (see pages 234-235 in [71]).

while

$$P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$\widetilde{\psi}(k) \in L^2(\mathbb{R}) \mapsto (P\widetilde{\psi})(k) := k\psi(k).$$

where $\widetilde{\psi}$ is the Fourier transform of ψ.
In the following, the symbol $*$ denotes convolution, i.e. $(f * g)(x) = \int f(y)g(x - y)dy$.

Let us consider the joint position-momentum POVM [2, 27, 32, 48, 78, 82]

$$F(\Delta \times \Delta') = \int_{\Delta\times\Delta'} U_{q,p}\, \gamma\, U^*_{q,p}\, dq\, dp$$

where, $U_{q,p} = e^{-iqP}e^{ipQ}$ and $\gamma = |f\rangle\langle f|$, $f \in L^2(\mathbb{R})$, $\|f\|_2 = 1$. The marginal

$$Q^f(\Delta) := F(\Delta \times \mathbb{R}) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |f|^2)(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{5.9}$$

is an unsharp position observable while the map $\mu_\Delta(x) := \mathbf{1}_\Delta * |f(x)|^2$ defines a Markov kernel.
First we recall that Q^f is absolutely continuous with respect to the Lebesgue measure (see example 2.4.4 for the details),

$$Q^f(\Delta) = F(\Delta \times \mathbb{R}) \leq \int_\Delta \mathbf{1}\, dq.$$

Then, by theorem 5.1.6, Q^f is uniformly continuous at each Borel set Δ with finite Lebesgue measure and, by theorem 5.2.4, Q^f cannot have the norm-1 property since $\|Q^f(\{x\})\| \leq |\{x\}| = 0$.

Moreover, it is worth remarking that, the uniform continuity does not hold in general and that there are sets for which the norm-1 property is satisfied. That can be shown by working out the details of the following particular relevant case. Let us set

$$f^2(x) = \frac{1}{l\,\sqrt{2\,\pi}}\, e^{(-\frac{x^2}{2\,l^2})}, \quad l \in \mathbb{R} - \{0\},$$

in (5.9). The corresponding unsharp position POVM is

$$\begin{aligned} Q^f(\Delta) &= \int_{-\infty}^{\infty} \Big(\int_\Delta |f(x-y)|^2)\, dy \Big)\, dQ_x \\ &= \frac{1}{l\,\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \Big(\int_\Delta e^{-\frac{(x-y)^2}{2\,l^2}}\, dy \Big)\, dQ_x = \int_{-\infty}^{\infty} \mu_\Delta(x)\, dQ_x \end{aligned}$$

where,

$$\mu_\Delta(x) = \frac{1}{l\sqrt{2\pi}} \int_\Delta e^{-\frac{(x-y)^2}{2l^2}}\, dy$$

defines a Markov kernel.
Now, we consider the family of sets $\Delta_i = (-\infty, a_i)$, $\lim_{i\to\infty} a_i = -\infty$ such that $\Delta_i \downarrow \emptyset$. In example 2.4.4 we proved that

$$\|Q^f(\Delta_i)\| = 1, \quad \forall i \in \mathbb{N}. \tag{5.10}$$

Therefore,

$$\lim_{i\to\infty} \|Q^f(\Delta_i)\| = 1$$

and, by proposition 5.1.3, Q^f cannot be uniformly continuous.

On the other hand, equation (5.10) implies that Q^f has the norm-1 property at each set $(-\infty, a)$, $a \in \mathbb{R}$.

At last, it is worth noting that μ_Δ is continuous for each interval Δ as we have shown in example 2.4.4.

An analogous reasoning can be made for the momentum observable P.

5.4 Uniform continuity and localization

It is possible to realize quantum experiments where single photons can be detected. For example, a single photon can be detected in a point on a screen (localization of the photon). This phenomenon does not find a theoretical description in the standard formulation of quantum mechanics. Indeed, it not possible to define a PVM which represents the *localization observable* of the photon [88].

Other examples of quantum observables which are not representable by PVMs (and therefore by self-adjoint operators) are the phase observable and the time observable (but in the latter case there are some exceptions [37]). The problem can be overcome by using POVMs in order to describe the *localization observables.*

It is worth remarking that PVMs imply a kind of localization (sharp localization) which is stronger than the one implied by POVMs (unsharp localization). If a localization observable is described by a covariant PVM E (sharp localization) then, for any Borel set Δ such that $E(\Delta) \neq \mathbf{0}$, there exists a unit vector ψ for which $\langle \psi, E(\Delta)\psi \rangle = 1$; i.e., the probability that a measure of the position of the system in the state ψ gives a result in Δ is one. Conversely, if a localization observable is described by a POVM F, there are Borel sets Δ such that $0 < \langle \psi, F(\Delta)\psi \rangle < 1$ for any vector ψ (unsharp localization).

It has been claimed that a POVM F with the norm-1 property; i.e., such that $\|F(\Delta)\| = 1$, for each Δ, would in some sense reduce the gap between the two kinds of localizations. Indeed, if $\|F(\Delta)\| = 1$ then, for each ϵ, it is possible to find a unit vector ψ such that the probability $\langle\psi, F(\Delta)\psi\rangle$ that a measure gives a result in Δ is greater than $1-\epsilon$. In other words, if F has the norm-1 property the quantum system it describes can be localized as accurately as desired (although not sharply). That raises the question of establishing if a given POVM has the norm-1 property.

In the present section we show that the norm-1 property is not satisfied by several important localization observables [21].

A key element in the definition of localization is the concept of covariance with respect to a group G describing the kinematics of the system. We start by considering the general case of localization in an abstract topological space X. Then, we specialize X to $\mathbb{R}^3$ in the case of space localization and to the phase space Γ (symplectic space Γ) in the case of phase space localization. We have two possible definitions of localization; i.e., sharp localization and unsharp localization. Sharp localization is defined by requiring covariance of a PVM under the group G.

Definition 5.4.1. *Let G be a locally compact topological group describing the kinematics of a quantum system. Let $x \mapsto gx$, $g \in G$, be the action of G on a topological space X. Let U be a strongly continuous unitary representation of G in Hilbert space $\mathcal{H}$. A PVM $E : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ represents a sharp localization observable in X (with respect to (G, U)) if*

$$U_g E(\Delta) U_g^\dagger = E(g\Delta).$$

The previous definition can be weakened by replacing the PVM by a POVM. That corresponds to an unsharp localization.

Definition 5.4.2. *Let G be a locally compact topological group describing the kinematics of a quantum system. Let $x \mapsto gx$, $g \in G$, be the action of G on a topological space X. Let U be a strongly continuous unitary representation of G in Hilbert space $\mathcal{H}$. A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ represents an unsharp localization observable in X (with respect to (U, G)) if*

$$U_g F(\Delta) U_g^\dagger = F(g\Delta).$$

Now, we specialize to the case of relativistic localization in $\mathbb{R}^3$. In the relativistic case, the relevant group is the Poincaré group and sharp localization is defined as follows [30, 51]: Let W be a continuous unitary representation of the universal covering of the Poincaré group. Let U be the restriction of W to the universal covering group $ISU(2) = \{(\mathbf{a}, B), \,|\, \mathbf{a} \in \mathbb{R}^3, B \in SU(2)\}$ of the Euclidean group and $\Lambda : SU(2) \to SO(3)$ the universal covering homomorphism.

A quantum system is said to be Wightman localizable if there is a PVM $E : \mathcal{B}(\mathbb{R}^3) \to \mathcal{E}(\mathcal{H})$ such that

$$U(\mathbf{a}, B)E(\Delta)U^\dagger(\mathbf{a}, B) = E(\mathbf{a} + \Lambda(B)\Delta) \quad \text{(sharp localization)}.$$

The covariance ensures that the results of a localization measurement do not depend on the choice of the origin and the orientation of the reference frame. As we have just remarked, in the case of the photon, sharp localization is impossible [2, 30, 52, 42]. Conversely, localization of the photon in $\mathbb{R}^3$ might be described by means of POVMs $F : \mathcal{B}(\mathbb{R}^3) \to \mathcal{F}(\mathcal{H})$ such that

$$U(\mathbf{a}, B)F(\Delta)U^\dagger(\mathbf{a}, B) = F(\mathbf{a} + \Lambda(B)\Delta) \quad \text{(unsharp localization)}.$$

By specializing X to a phase space Γ we get the concept of unsharp localization in phase space which requires the POVM, F, to be defined on $\Gamma = X$ and to be covariant with respect to a group G which characterizes the symmetries of the system [26]. Examples of symmetry groups are the Galilei group in the non relativistic case and the Poincaré group in the relativistic case. (See section 3 for further details.) For the case of the photons in their phase space the relevant group is the Poincaré group. [26]

5.4.1 Localization in phase space

In the present section, we show that localization observables in phase space cannot satisfy the norm-1 property. In the following the phase space is denoted by the symbol Γ. First, we recall some key elements of the phase space approach to quantum mechanics [2, 78, 5, 83]. We follow References [83, 78].

The main idea is that one can represent the state ρ of a quantum system (i.e., a trace class positive operator of trace 1) by means of a distribution function $f_\rho(\mathbf{q}, \mathbf{p})$ on a suitable phase space. At variance with the Wigner approach [89], the distribution functions are positive definite. Wigner's theorem [89] forbids that the marginals of the distribution functions satisfy the following relations:

$$\begin{aligned} \int_\Delta d\mathbf{q} \int f_\rho(\mathbf{q}, \mathbf{p})\, d\mathbf{p} &= \mathrm{Tr}(\rho\, Q(\Delta)), \\ \int_\Delta d\mathbf{p} \int f_\rho(\mathbf{q}, \mathbf{p})\, d\mathbf{q} &= \mathrm{Tr}(\rho\, P(\Delta)), \end{aligned} \tag{5.11}$$

where $Q(\Delta)$ and $P(\Delta)$ are the spectral measures corresponding to the position and momentum operators respectively. Relations (5.11) might be replaced by

$$\begin{aligned} &\textstyle\int_\Delta d\mathbf{q} \int f_\rho(\mathbf{q}, \mathbf{p})\, d\mathbf{p} = \mathrm{Tr}(\rho F^Q(\Delta)), \\ &\textstyle\int_\Delta d\mathbf{p} \int f_\rho(\mathbf{q}, \mathbf{p})\, d\mathbf{q} = \mathrm{Tr}(\rho F^P(\Delta)). \end{aligned}$$

where $F^Q(\Delta)$ and $F^P(\Delta)$ are POVMs corresponding to Q and P respectively. In particular, $F^Q(\Delta)$ and $F^P(\Delta)$ are the smearing of the position and momentum operators Q and P

$$\begin{aligned} F^Q(\Delta) &= \int_{\mathbb{R}} \omega_\Delta(x)\, dQ_x, \\ F^P(\Delta) &= \int_{\mathbb{R}} \nu_\Delta(x)\, dP_x, \end{aligned}$$

and are called unsharp position and momentum observables [32, 10, 11, 12, 13, 14]. The maps ω and ν are such that $\omega_\Delta(\cdot)$ and $\nu_\Delta(\cdot)$ are measurable functions for each Δ and $\omega_{(\cdot)}(x)$ and $\nu_{(\cdot)}(x)$ are probability measures for each x. They are Markov kernels which describe a stochastic diffusion of the standard observables Q and P [16, 19]. That is why F^Q and F^P are usually called the unsharp version of the sharp observables Q and P respectively. All that shows that POVMs play a key role in the phase space formulation. Moreover, it is worth remarking that a derivation of classical and quantum mechanics in a unique mathematical framework is possible [19, 20]. The phase space approach seems to be also fruitful in the applications to solid state physics [7].

One of the main steps in this approach is the construction of the phase space. In brief, we can say that there is a procedure that starting from a Lie group G allows the classification of all the closed subgroups $H \subset G$ such that G/H is a simplectic space (i.e, a phase space). For example, in the case of the Galilei group, a possible choice for H is the group $H = SO_3$. Then, $\Gamma = G/H = \mathbb{R}^3 \times \mathbb{R}^3$, which coincides with the phase space of classical mechanics. A different choice of H generates a different phase space. In other words, the procedure allows the calculation of all the phase spaces corresponding to a locally compact Lie group, G, with a finite dimensional Lie algebra. Once we have the phase space, we can look for a strongly continuous unitary representation of G in a Hilbert space $\mathcal{H}$ and then we can define the localization observable [83].

Definition 5.4.3 (See [83].). *Let G be a locally compact topological group, H a closed subgroup of G, U a strongly continuous unitary irreducible representation of G in a complex Hilbert space $\mathcal{H}$ and μ a volume measure on G/H. A localization observable is represented by a POVM*

$$A^\eta(\Delta) = \int_\Delta |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| \, d\mu(x).$$

where, $\sigma : G/H \mapsto G$ is a measurable map and η is a unit vector such that

$$\int_{G/H} |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| \, d\mu(x) = \mathbf{1}. \tag{5.12}$$

As we said above, from an operational viewpoint, the POVM $A^\eta : \mathcal{B}(G/H) \to \mathcal{F}(\mathcal{H})$ represents a measurement procedure. The novelty of the phase space approach is that A^η is defined on the space of Borel subsets of the phase space $\Gamma = G/H$ and therefore the measurement procedure it describes is of a quite general kind. For example, if G is the Galilei group and $H = \mathbb{R}^2 \times \mathbb{R}^2 \times SO(3)$, then $G/H = \mathbb{R}_q \times \mathbb{R}_p$, μ is the Lebesgue measure, $U(\sigma(x)) = U_{q,p} = e^{-iqP}e^{ipQ}$, $\eta \in L^2(\mathbb{R})$ and the localization observable reads

$$A(\Delta_q \times \Delta_p) = \int_{\Delta_q \times \Delta_p} |U_{q,p}\eta\rangle\langle U_{q,p}\eta| \, d\mathbf{q} \, d\mathbf{p}.$$

The marginals

$$F^Q(\Delta) := A^\eta(\Delta \times \mathbb{R}_p) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\eta|^2)(x) \, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}),$$
$$F^P(\Delta) := A^\eta(\mathbb{R}_q \times \Delta) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\widetilde{\eta}|^2)(x) \, dP_x, \quad \Delta \in \mathcal{B}(\mathbb{R})$$

where, $\widetilde{\eta}$ is the Fourier transform of η, are the unsharp position and momentum observables respectively [32]. Notice that the map $\mu_\Delta(x) := \mathbf{1}_\Delta * |\eta(x)|^2$ defines a Markov kernel.

Therefore, $A^\eta(\Delta_q \times \Delta_p)$ is a joint measurement for the unsharp position and momentum observables. In other words, the phase space approach allows the description of joint measurements of unsharp position and momentum observables also if they do not commute; i.e., $[F^Q(\Delta_q), F^P(\Delta_p)] \neq \mathbf{0}$. That is one of the main advantages from the physical viewpoint of using the phase space approach outlined above. A second relevant aspect we would like to note is that Definition 5.4.3 allows the introduction of a quantization procedure [83, 84]. Indeed, for any real-valued Borel function f,

$$A^\eta(f) = \int_{G/H} f(x) \, |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| \, d\mu(x)$$

defines a self-adjoint operator which is positive whenever f is positive. Moreover, for any pure state $\rho = |\psi\rangle\langle\psi|$,

$$Tr(A^\eta(f)\rho) = \int_{G/H} f(x) \, f_\rho(x) \, d\mu(x)$$

where $f_\rho(x) = |\langle U(\sigma(x))\eta, \psi\rangle|^2$. The last relevant aspect on which we would like to remark is that in the phase space approach a relativistically conserved positive current can be defined [5, 78, 84].

Before we proceed with the analysis of the norm-1 property of the phase space localization observables, it is worth stressing the dependence of the localization observable $A^\eta(\cdot)$ on the vector η. The physical relevance of such a dependence has been analyzed in Ref.s [78, 83, 84]. We limit ourselves to remark that $\{|U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|,\, x\in G/H\}$ defines a coherent state basis and that η can be interpreted as characterizing the measuring device in the sense that the interaction of the measuring device (described by $A^\eta(\cdot)$) with the system prepared in the state $\rho = P_\psi$ depends on the transition probability $|\langle U(\sigma(x))\eta, \psi\rangle|^2$ and hence on η.

Theorem 5.4.4. *[83] The POVM A^η defined in Definition 5.4.3 is covariant with respect to U.*

A general property of the localization observables in Definition 5.4.3 is that they are absolutely continuous with respect to the measure μ.

Theorem 5.4.5. *[83] The POVM in Definition 5.4.3 is absolutely continuous with respect to μ.*

Proof. For each $\psi \in \mathcal{H}$,

$$\begin{aligned}\langle\psi, A^\eta(\Delta)\psi\rangle &= \int_\Delta \langle\psi, U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta, \psi\rangle\, d\mu(x)\\ &= \int_\Delta |\langle\psi, U(\sigma(x)\eta\rangle|^2\, d\mu(x) \le \int_\Delta d\mu(x).\end{aligned}$$

□

The localization of the photon in phase space was introduced in Ref. [26] with the same procedure we just described. Therefore, at variance with the usual definition of localization (where the covariance under the Euclidean group is required), localization in phase space requires that F is covariant with respect to the group which describes the symmetry of the system (the Galilei group in the non-relativistic case and the Poincaré group in the relativistic case).

Before we prove that the norm-1 property is not possible for localization observables in phase space, we want to give a physical motivation which is based on the Uncertainty Relations. Let F be a phase space localization observable covariant with respect to the Galilei group. In this case the phase space Γ corresponding to the system can be chosen to be $\Gamma = \mathbb{R}^3_q \times \mathbb{R}^3_p = \mathbb{R}^3 \times \mathbb{R}^3$ (See [83], page 425.) Then, suppose that the norm-1 property holds, i.e., for each Borel set $\Delta_{\mathbf{q}} \times \Delta_{\mathbf{p}} \in \Gamma$ with $F(\Delta_{\mathbf{q}} \times \Delta_{\mathbf{p}}) \neq \mathbf{0}$, there exists a family of unit vectors ψ_n such that

$$\lim_{n\to\infty} \langle\psi_n, F(\Delta_{\mathbf{q}} \times \Delta_{\mathbf{p}})\psi_n\rangle = 1 \tag{5.13}$$

where, $\langle \psi_n, F(\Delta_{\mathbf{q}} \times \Delta_{\mathbf{p}})\psi_n \rangle$ is interpreted as the probability that an outcome of a joint measurement of the unsharp position and momentum observables is in $\Delta_{\mathbf{q}} \times \Delta_{\mathbf{p}}$ when the state is $P_{\psi_n} = |\psi_n\rangle\langle\psi_n|$. Therefore, the violation of Uncertainty Relations comes from the fact that (5.13) holds for any Borel set $\Delta_{\mathbf{q}} \times \Delta_{\mathbf{p}}$.

In the following, we apply Theorem 5.2.3 to the case of the Galilei group $G = \{(t, \mathbf{q}, \mathbf{p}, R) \,|\, t \in \mathbb{R}, \mathbf{q}, \mathbf{p} \in \mathbb{R}^3, R \in SO(3)\}$ with $H = \{(t, \mathbf{0}, \mathbf{0}, R) \,|\, t \in \mathbb{R}, R \in SO(3)\}$. Therefore, $G/H = \mathbb{R}^3_q \times \mathbb{R}^3_p = \mathbb{R}^3 \times \mathbb{R}^3$. In that case, the invariant measure is the Lebesgue measure. In the following we set $\mathbf{x} = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$. Theorem 5.2.3 implies that the POVM A^η in Definition 5.4.3 does not have the norm-1 property.

Theorem 5.4.6 ([21])**.** *The localization observable represented by the POVM A^η with $G/H = \mathbb{R}^3 \times \mathbb{R}^3$ and μ the Lebesgue measure does not have the norm-1 property.*

Proof. Let $\mathbf{x} \in G/H$. By Theorem 5.4.5,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}).$$

Since μ is the Lebesgue measure on G/H,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}) = 0.$$

Theorem 5.2.3 completes the proof. □

An analogous result can be proved in the case of massless relativistic particles. In the relativistic case G is the double cover $\mathcal{P} = T^4 \oslash SL(2, \mathbb{C})$ of the Poincaré group. In particular, T^4 is the Minkowski space and $SL(2, \mathbb{C})$ is the double cover of the Lorentz group. The symbol $\oslash$ denotes the semidirect product. In the massless relativistic case ([83], page 454) the relevant subgroup is $H = \mathbb{R}\mathbf{p}_0 \oslash SL(2, \mathbb{C})_{\mathbf{p}_0} = \mathbb{R}\mathbf{p}_0 \oslash (\mathbb{R}^2 \oslash \widetilde{O}(2))$ where, $\mathbf{p}_0 = (1, 0, 0, 1)$, $\mathbb{R}\mathbf{p}_0 = \{\lambda\, \mathbf{p}_0, \lambda \in \mathbb{R}\}$, $\widetilde{O}(2)$ is the double cover of the group of rotations in $\mathbb{R}^2$, and $SL(2, \mathbb{C})_{\mathbf{p}_0}$ is the set of matrices $A \in SL(2, \mathbb{C})$ such that $A[\mathbf{p}_0] = \mathbf{p}_0$. The phase space for the photon is then $\mathcal{P}/H$ which is the space of cosets

$$(\mathbf{a}, A)(\mathbb{R}\mathbf{p}_0 \oslash SL(2, \mathbb{C})_{\mathbf{p}_0}) = (\mathbf{a} + \mathbb{R}A[\mathbf{p}_0],\; ASL(2, \mathbb{C})_{\mathbf{p}_0})$$

where, $(\boldsymbol{a}, A) \in \mathcal{P}$, $SL(2, \mathbb{C})_{\mathbf{p}_0}$ is isomorphic to $\mathbb{R}^2 \oslash \widetilde{O}(2)$ and the quotient $SL(2, \mathbb{C})/SL(2, \mathbb{C})_{\mathbf{p}_0}$ is homeomorphic to $\mathbb{R}^+ \times S^2$. The invariant measure on $\mathcal{P}/H$ is (see equation (344), page 463, in Ref.[83])

$$d\mu = d(\alpha)d(\gamma)d(\delta) \times (p^0 + p^3)^{-1} d(p^0 + p^3) \wedge dp^1 \wedge dp^2 \tag{5.14}$$

where $\alpha = a_\mu(A[\mathbf{p}_0])^\mu$, $\gamma = a_\mu(A[\mathbf{u}_0])^\mu$, $\delta = a_\mu(A[\mathbf{v}_0])^\mu$, with $\mathbf{u}_0 = (0,1,0,0)$, $\mathbf{v}_0 = (0,0,1,0)$.
Thus α, γ, δ are in $\mathbb{R}$. Hence, we have a representation of the zero mass particles. Moreover, μ is zero in each single point subset of the phase space so that the reasoning in the proof of theorem 5.4.6 can be used.

Theorem 5.4.7 ([21])**.** *If* $G/H = T^4 \oslash SL(2,\mathbb{C})/\mathbb{R}\mathbf{p}_0 \oslash (\mathbb{R}^2 \oslash \widetilde{O}(2))$ *with the measure* μ *in equation (5.14), the POVM* A^η *in Definition 5.4.3 does not have the norm-1 property.*

Proof. Let $\mathbf{x} \in G/H$. By Theorem 3.13,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}).$$

Since $\mu(\{\mathbf{x}\}) = 0$,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}) = 0.$$

Theorem 5.2.3 completes the proof. □

Remark 5.4.8. *It is worth remarking that in using the measure (5.14) one must necessarily integrate with respect to the position coordinates* α*,* γ*, and* δ *first. (See Ref. [83], page 464 and Ref. [26].) That is due to the fact that* $\mathcal{P}/H$ *is diffeomorphic to the bundle* $\cup_{\mathbf{p}\in V^+}\mathbb{R}^4/\mathbb{R}\mathbf{p}$ *(where* $V^+ = \{\mathbf{p} \in \mathbb{R}^4\,;\, p^\mu p_\mu = 0\} = \{A[\mathbf{p}_0],\, A \in SL(2,\mathbb{C})\}$*) which is not homeomorphic to* $\mathbb{R}^n \times \mathbb{R}^n$ *for any integer* n*. (See [26], page 5961, item c)*
The measure $d\mu(\mathbf{a}+\mathbb{R}\mathbf{p},\mathbf{p}) = d\lambda_{\mathbf{p}}(\mathbf{a})d\nu(\mathbf{p})$ *on* $\cup_{\mathbf{p}\in V^+}\mathbb{R}^4/\mathbb{R}\mathbf{p}$ *is such that* $d\lambda_{\mathbf{p}}(\mathbf{a})$ *depends on* $\mathbf{p} = A[\mathbf{p}_0] \in V^+$ *and has the form* $d(\alpha)d(\gamma)d(\delta)$ *for any fixed* $\mathbf{p}$*. (See equation (5.14))*

We have a different situation if we consider the case of a particle of mass m *and spin* $\mathbf{s} = (0, s_1, s_2, s_3)$*. In such case, (see [83], pages 454, 455)* $\mathbf{p}_0 = m(1,0,0,0)$*,* $H = \mathbb{R}\mathbf{p}_0 \oslash \widetilde{O}(2)$ *and* $\mathcal{P}/H = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$*.*

5.4.2 Localization in Configuration Space

Now, we study the marginals of A^η in the non-relativistic case and prove that they cannot have the norm-1 property. We limit ourselves to the marginal $F^Q_\eta(\Delta_{\mathbf{q}}) := A^\eta(\Delta_{\mathbf{q}} \times \mathbb{R}_{\mathbf{p}})$ which represents the unsharp position observable. Clearly what we prove applies also to the marginal $F^P_\eta(\Delta_{\mathbf{p}}) := A^\eta(\mathbb{R}_{\mathbf{q}} \times \Delta_{\mathbf{p}})$ which represents the unsharp momentum observable.

Theorem 5.4.9 ([21])**.** *The POVM* $F^Q_\eta(\Delta_{\mathbf{q}})$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}_{\mathbf{q}}$*.*

Proof.

$$\begin{aligned} F_\eta^Q(\Delta_{\mathbf{q}}) = A^\eta(\Delta \times \mathbb{R}_{\mathbf{p}}) \quad &= \textstyle\int_{\Delta\times\mathbb{R}_{\mathbf{p}}} U(\sigma(\mathbf{q},\mathbf{p}))\,|\eta\rangle\langle\eta|\,U^\dagger(\sigma(\mathbf{q},\mathbf{p}))\,d\mathbf{q}\,d\mathbf{p} \\ &= \textstyle\int_\Delta\, d\mathbf{q} \int_{\mathbb{R}_{\mathbf{p}}} U(\sigma(\mathbf{q},\mathbf{p}))\,|\eta\rangle\langle\eta|\,U^\dagger(\sigma(\mathbf{q},\mathbf{p}))\,d\mathbf{p} \\ &= \textstyle\int_\Delta \widehat{Q}_\eta(\mathbf{q})\,d\mathbf{q} \le \int_\Delta \mathbf{1}\,d\mathbf{q}, \end{aligned}$$

where

$$\widehat{Q}_\eta(\mathbf{q}) = \int_{\mathbb{R}_{\mathbf{p}}} U(\sigma(\mathbf{q},\mathbf{p}))\,|\eta\rangle\langle\eta|\,U^\dagger(\sigma(\mathbf{q},\mathbf{p}))\,d\mathbf{p}$$

and equation (5.12) in definition 5.4.3 has been used. □

Theorems 5.2.3 implies the following corollary.

Corollary 5.4.10 ([21])**.** *F_η^Q cannot have the norm-1 property*

Remark 5.4.11. *Remark 5.4.8 implies that the reasoning in Theorem 5.4.9 and Corollary 5.4.10 cannot be applied in the massless relativistic case where, there is no hope of even defining localization in configuration space at all.*

In Ref. [28] it is shown that in order for a localization observable to satisfy Einstein causality, the localization observable must be commutative. It is worth remarking that, although A^η is not commutative, F_η^Q is commutative and can be characterized as the smearing of the position operator [10]-[19], [17, 15, 18]. Unfortunately, as we have just proved, F_η^Q does not satisfy the norm-1 property. It would be interesting to analyze in general the relationships between causality and norm-1 property. That will be the topic of a future work.

Summary

The aim of the present thesis is the analysis of the mathematical structure and physical meaning of POVMs. In particular we answered the following questions:

1) What are the relationships between POVMs and spectral measures? What is the physical meaning of the smearing which connects a commutative POVM to its sharp version? (Chapter 2).

2) A commutative POVM is the smearing of its sharp version. What are the relationships between this characterization of commutative POVMs and Naimark's dilation theorem? Is there a universal one-to-one function f such that the sharp version A^F of any commutative POVM F can be written in the form $A^F = \int f(t)\,dF_t$? (Chapter 3).

3) Is there any loss of "information" during the smearing from the sharp version A of F to F? (Chapter 4).

4) Can we give conditions for a POVM to have the norm-1 property? Is the norm-1 property relevant to localization observables? (Chapter 5).

It follows a summary of the results I got for each of the questions above.

1) Characterization of commutative POVMs. (See Ref. [22, 11])

The problem of the relationships between POVMs and PVMs can be answered in the commutative case [11, 22]. Indeed, in chapter 2, we prove that a POVM F is commutative (see theorem 2.2.3) if and only if there exist a spectral measure E and a Markov kernel (transition probability) $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(X) \to [0,1]$, $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, such that

(5.15) $$F(\Delta) = \int_\Gamma \mu_\Delta(\lambda)\,dE_\lambda$$

and $\mu_\Delta(\cdot)$ is continuous for each $\Delta \in R$ where, $R \subset \mathcal{B}(X)$ is a ring which generates the Borel σ-algebra $\mathcal{B}(X)$. It turns out that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(X) \to [0,1]$ is a Feller Markov kernel [67, 80]. Therefore, F is commutative if and only if there exists a Feller Markov kernel μ such that equation (5.15) is satisfied.
We also prove that the family of functions $\{\mu_\Delta\}_{\Delta\in\mathcal{B}(X)}$ separates the points of $\sigma(A)$ up to a null set (see theorem 2.1.1) so that the probability measures $\mu_{(\cdot)}(\lambda)$ and $\mu_{(\cdot)}(\lambda')$ corresponding to λ and λ' are different.

As an example we consider the unsharp position observable defined as follows. For any $\Delta \in \mathcal{B}(\mathbb{R})$ and $\psi \in L^2([0,1])$, we set

$$\langle \psi, Q^f(\Delta)\psi\rangle := \int_{[0,1]} \mu_\Delta(x)\, d\langle \psi, Q_x\psi\rangle, \tag{5.16}$$
$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a positive, bounded, Borel function such that $f(y) = 0$, $y \notin [0,1]$, and $\int_{[0,1]} f(y)dy = 1$, while Q_x is the spectral measure corresponding to the position operator

$$Q : L^2([0,1]) \to L^2([0,1])$$
$$\psi(x) \mapsto Q\psi := x\psi(x)$$

We recall that $\langle \psi, Q(\Delta)\psi\rangle$ is interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in Δ. Then, a possible interpretation of equation (5.16) is that Q^f is a randomization of Q. Indeed [78], the outcomes of the measurement of the position of a particle depend on the measurement imprecision so that, if the sharp value of the outcome of the measurement of Q is x then the apparatus produces with probability $\mu_\Delta(x)$ a reading in Δ.

It is worth noting that (see example 2.3.6) the Markov kernel

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

in equation (5.16) above is such that the function $x \mapsto \mu_\Delta(x)$ is continuous for each $\Delta \in \mathcal{B}(\mathbb{R})$. The continuity of μ_Δ means that if two sharp values x and x' are very close to each other then, the corresponding random diffusions are very similar, i.e., the probability to get a result in Δ if the sharp value is x is very close to the probability to get a result in Δ if the sharp value is x'.

As a consequence of theorem 2.2.3, it is always possible to choose the Markov kernel μ in equation (5.15) to be continuous on a ring which generates the Borel σ-algebra $\mathcal{B}(X)$. Anyway, that is the most we can do in the general case. Indeed, we prove (see theorem 2.3.5) that the continuity for each Borel set is equivalent to the uniform continuity of F which in its turn is equivalent to require that the smearing in equation (5.16) can be realized by a strong Feller Markov kernel. It is worth remarking that although in the general case the continuity holds only for a ring of subsets which generates $\mathcal{B}(X)$, that is sufficient to prove the weak convergence of $\mu_{(\cdot)}(x)$ to $\mu_{(\cdot)}(x')$.

Finally, we prove (see section 2.4) that a POVM F which is absolutely continuous with respect to a regular finite measure ν is uniformly continuous (theorem

2.4.2). We give some examples of absolutely continuous POVMs (see example 2.4.4) and analyze the unsharp position observable which is obtained as the marginal of a phase space observable (see subsection 2.4.1).

Naimark's dilation and sharp version of a POVM F. (See Ref.s [16, 17, 18]

Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ be a real commutative POVM. In section 2.2, we pointed out that there exist a spectral measure E and a Feller Markov kernel $\mu_{(\cdot)}(\lambda)$ such that

$$F(\Delta) = \int_{\Gamma} \mu_{\Delta}(\lambda)\, dE_{\lambda} \tag{5.17}$$

Since by the spectral theorem there is a one-to-one correspondence between spectral measures and self-adjoint operators, equation (5.17) can equivalently be written as follows

$$F(\Delta) = \mu_{\Delta}(A), \quad \Delta \in \mathcal{B}(\mathbb{R})$$

where, $A := \int \lambda\, dE_{\lambda}$ is the self adjoint operator corresponding to the spectral measure E. It is worth remarking that A is a generator of the von Neumann algebra generated by F; i.e., the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta \in \mathcal{B}(\mathbb{R})}$, and is called the sharp version of F.

On the other hand, Naimark's dilation theorem [73, 71] ensures us that each POVM F in a Hilbert space $\mathcal{H}$ can be extended to a PVM E^{+} in an extended Hilbert space $\mathcal{H}^{+}$ in such a way that F is the projection of E^{+}. That raises the following question: what are the relationships between the self adjoint operators $A^{+} = \int \lambda\, dE^{+}_{\lambda}$ and A?

In order to answer such question, we prove [16, 17] (section 3.2) that, if F is a POVM with spectrum in $[0, 1]$ such that, for any $\Delta \in \mathcal{B}([0, 1])$, $F(\Delta)$ is a discrete operator then, A is a linear combination of the moments of F, i.e.,

$$A = \sum_{i=1}^{\infty} \alpha_i F[i] \tag{5.18}$$

where, $\alpha_i \geq 0$, $\sum_{i=1}^{\infty} \alpha_i < \infty$, and

$$F[i] = \int_{[0,1]} t^i\, dF_t$$

This result can be used to characterize the relationships between A and A^{+} for each $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ such that $F(\Delta)$ is a discrete operator. In particular, we prove [16, 17] that A coincides with $Pf(A^{+})$ where the symbol P denotes the projection from $\mathcal{H}^+$ to $\mathcal{H}$ and f is a one-to-one measurable function. Moreover, we use this last result [15, 16, 17] in order to show that

$$A = \int f(t)\, dF_t$$

with f one-to-one. In other words, the sharp version A of F can be recovered from F by means of f. At the same time, F is the smearing of A. In section 4.1 we explain how all that suggests the equivalence of A and F from the informational point of view [15].

It is worth remarking that in order to prove the result in equation (5.18) we prove the following properties of the sequences of linear functionals [16]. Consider an infinite sequence of linear functionals $\{T_i\}_{i\in\mathbb{N}}$, $T_i f = \int f(t)\, d\mu_t(i)$, corresponding to an infinite sequence of probability measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$, on the Borel σ-algebra $\mathcal{B}([0,1])$ such that, $\mu_{(\cdot)}(i) \neq \mu_{(\cdot)}(j)$, $i,j \in \mathbb{N}$, $i \neq j$. There exists a real, bounded, one-to-one function f such that

$$T_i f = \int f(t)\, d\mu_t(i) \neq \int f(t)\, d\mu_t(j) = T_j f, \quad i,j \in \mathbb{N},\, i \neq j.$$

In section 3.4 a constructive proof of the existence of f is given [17].

At last, in section 3.3 we show [18] how an extension of the last result to the case of an uncountable family of measures $\{\mu_{(\cdot)}(\lambda)\}_{\lambda\in[0,1]}$ would imply the existence of a universal one-to-one function f such that, for any commutative POVM F,

$$A^F = \int f(t)\, dF_t$$

where A^F is the sharp version of F. Anyway, that is an open problem.

Informational content of POVMs. (See Ref. [15])

As we have seen in chapter 2, for any commutative POVM F there exists a PVM E such that F can be interpreted as a random diffusion of E. In its turn, the self-adjoint operator $A = \int \lambda\, dE_\lambda$ corresponding to E, can be interpreted [13, 14] as the projection of a Naimark operator corresponding to the Naimark dilation E^+ of F. Moreover E can be algorithmically reconstructed by F [31, 10]. All that suggests that, in some sense, the observables represented by E and F should have the same informational content. We introduce an equivalence relation on the set of observables which we compare with other well known equivalence relations and prove that it is the only one for which E is always equivalent to F [15].

Uniformly continuous POVMs, norm-1 property and localization. (See Ref. [21, 23])

It is possible to realize quantum experiments where single photons can be detected. For example, a single photon can be detected in a point on a screen (localization of the photon). This phenomenon does not find a theoretical description in the standard formulation of quantum mechanics. Indeed, it not possible to define a PVM which represents the *localization observable* of the photon [88].

Other examples of quantum observables which are not representable by PVMs (and therefore by self-adjoint operators) are the phase observable and the time observable (but in the latter case there are some exceptions). The problem can be overcome by using POVMs in order to describe the *localization observables.*

It is worth remarking that PVMs imply a kind of localization (sharp localization) which is stronger than the one implied by POVMs (unsharp localization). If a localization observable is described by a covariant PVM E (sharp localization) then, for any Borel set Δ such that $E(\Delta) \neq \mathbf{0}$, there exists a unit vector ψ for which $\langle \psi, E(\Delta)\psi \rangle = 1$; i.e., the probability that a measure of the position of the system in the state ψ gives a result in Δ is one. Conversely, if a localization observable is described by a POVM F, there are Borel sets Δ such that $0 < \langle \psi, F(\Delta)\psi \rangle < 1$ for any vector ψ (unsharp localization).

It has been claimed that a POVM F with the norm-1 property; i.e., such that $\|F(\Delta)\| = 1$ whenever $F(\Delta) \neq \mathbf{0}$, would in some sense reduce the gap between the two kinds of localizations. Indeed, if $\|F(\Delta)\| = 1$ then, for each ϵ, it is possible to find a unit vector ψ such that the probability $\langle \psi, F(\Delta)\psi \rangle$ that a measure gives a result in Δ is greater than $1 - \epsilon$. In other words, if F has the norm-1 property the quantum system it describes can be localized as accurately as desired (although not sharply). That raises the question of establishing if a given POVM has the norm-1 property. In order to answer such question it is helpful [21, 23] to introduce the concept of uniform continuity for POVMs.

A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ defined on a topological space X is said to be uniformly continuous if, for any Δ and for any disjoint decomposition $\Delta = \cup_{i=1}^{\infty} \Delta_i$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(\Delta_i) = F(\Delta)$$

in the uniform operator topology. I proved [23] that a uniformly continuous POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ has the norm-1 property only if $F(\{x\}) \neq \mathbf{0}$ for each $x \in X$. Then, I used this last result in order to prove that there is a wide class of POVMs which describe localization observables that cannot have the norm-1 property [23]. That is the case for example of the localization in phase space of massless relativistic particles. That seems to suggest that the concept of unsharp localizability is not generally compatible with the possibility of a localization as accurate as desired (although not sharp).

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