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Mathematical Structure of Positive Operator Valued Measures and Applications

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Chapter 1

Introduction

The present dissertation is devoted to the study of positive operator valued measures (POVMs) which were introduced in the 40's [73, 74, 71] in order to study self-adjoint extensions of symmetric operators.

A POVM is a σ-additive map $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ from the Borel σ-algebra of a topological space X to the space $\mathcal{F}(\mathcal{H})$ of positive operators less than the identity (effects) on a Hilbert space $\mathcal{H}$. This generalizes the concept of spectral resolution of the identity, which is a σ-additive map $E : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$ from the Borel σ-algebra of the reals to the space of orthogonal projections $\mathcal{E}(\mathcal{H})$.

In the 70's several scholars [32, 45, 48, 50, 49, 66, 2, 78, 27] used POVMs as the main tool in the description of the quantum measurement process and to formulate a general theory of statistical decision. Moreover POVMs suggested an extension of the concept of quantum observable which is usually represented by a spectral measure or, equivalently, by a self-adjoint operator. Such an extension turned out to be very fruitful since it permitted a mathematical representation of the time observable, the photon localization observable and the phase observable which resulted to be impossible in the old framework (spectral measures). Another improvement allowed by POVMs consists in the possibility of building a representation of standard quantum mechanics in a phase space by mapping density operators to density probability functions on a symplectic space [78, 83, 27, 85]. Nowdays, POVMs are a standard tool in quantum information theory and quantum optics [50, 87, 90].

It was then natural both from the mathematical and the physical viewpoint to ask what are the relationships between POVMs and spectral measures. Four possible answers have been given each one corresponding to a different characterizations of POVMs [73, 48, 2, 31, 10, 11, 22, 53]. Although the answer given by Naimark [73, 71, 1, 83] is the most powerful since it refers to general POVMs and is not limited to the commutative case, it is to be confronted with the problem of the physical interpretation of the extended Hilbert space it introduces. As we shall see, if one avoids the commitment with an extended Hilbert space, a clear answer to our question can be given in the commutative case [48, 2, 31, 10, 11, 12, 53], the commutative POVMs being the most similar to the spectral measures.

The present dissertation is based on the author's contribution to the formulation of one of the possible characterizations of commutative POVMs (chapter 2). The analysis of the relationships between such characterization and Naimark's theorem is the topic of chapter 3. Chapter 4 is devoted to the analysis of its relevance to the concept of "informational content" of an observable. The last chapter is devoted to the characterization of the uniform continuity of a general POVM (not necessarily commutative), to the analysis of the POVMs with the norm-1 property and to the analysis of the relevance of norm-1 property and uniform continuity to the localization problem in relativistic quantum mechanics.

Next we outline the main properties of POVMs and show how they emerge in the quantum context.

Main properties of POVMs

In the present section we recall the main properties of POVMs. We restrict ourselves to the case of POVMs defined on the Borel σ-algebra of a topological set X. For a more general exposition we refer to the book by Berberian [24]. In the following $\mathcal{H}$ denotes a complex separable Hilbert space, $\mathcal{L}_s(\mathcal{H})$ the space of linear self-adjoint operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ the subspace of positive operators less than the identity, i.e., $B \in \mathcal{F}(\mathcal{H})$ if $\mathbf{0} \leq B \leq \mathbf{1}$.

Definition 1.0.1. *Let X be a topological space and $\mathcal{B}(X)$ the Borel σ-algebra on X. A normalized POVM is a map $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ such that*

$F(X) = \mathbf{1}$ *and:*

$$F\Big(\bigcup_{n=1}^{\infty} \Delta_n\Big) = \sum_{n=1}^{\infty} F(\Delta_n)$$

where, $\{\Delta_n\}$ *is a countable family of disjoint sets in* $\mathcal{B}(X)$ *and the series converges in the weak operator topology.*

Definition 1.0.2. *A POVM is said to be commutative if*

$$[F(\Delta_1), F(\Delta_2)] = \mathbf{0} \quad \forall\, \Delta_1\,, \Delta_2 \in \mathcal{B}(X). \tag{1.1}$$

It is said to be orthogonal if

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \ \text{ if } \Delta_1 \cap \Delta_2 = \emptyset. \tag{1.2}$$

Definition 1.0.3. *A PVM is an orthogonal, normalized POVM. A real PVM* $E : \mathcal{B}(R) \to \mathcal{F}(\mathcal{H})$ *is said to be a spectral measure.*

Proposition 1.0.4. *A PVM* E *on* X *is a map* $E : \mathcal{B}(X) \to \mathcal{E}(\mathcal{H})$ *from the Borel* σ*-algebra of* $\mathcal{B}(X)$ *to the space of projection operators on* $\mathcal{H}$*.*

Definition 1.0.5. *The von Neumann algebra* $\mathcal{A}^W(F)$ *generated by the POVM* F *is the von Neumann algebra generated by the set* $\{F(\Delta)\}_{\Delta\in\mathcal{B}(X)}$*.*

Definition 1.0.6. *The spectrum* $\sigma(F)$ *of a POVM* F *is the set of points* $x \in X$ *such that* $F(\Delta) \neq \mathbf{0}$*, for any open set* Δ *containing* x*.*

The spectrum $\sigma(F)$ of a POVM F is a closed set since its complement $X - \sigma(F)$ is the union of all the open sets $\Delta \subset X$ such that $F(\Delta) = \mathbf{0}$.

We can introduce integration with respect to a POVM. Indeed, for any $\psi \in \mathcal{H}$, the expression $\langle F(\cdot)\psi, \psi\rangle$ defines a probability measure and we will use the symbol $d\langle F_\lambda\psi, \psi\rangle$ to mean integration with respect to the measure $\langle F(\cdot)\psi, \psi\rangle$. For any real, bounded and measurable function f and for any POVM F, there is a unique [24] bounded self-adjoint operator $B \in \mathcal{L}_s(\mathcal{H})$ such that

$$\langle B\psi, \psi\rangle = \int f(\lambda) d\langle F_\lambda\psi, \psi\rangle, \quad \text{for each} \quad \psi \in \mathcal{H}. \tag{1.3}$$

If equation (1.3) is satisfied, we write $B = \int f(\lambda) dF_\lambda$ or $B = \int f(\lambda) F(d\lambda)$ equivalently.

By the spectral theorem [35, 79], real PVMs E (spectral measures) are in a one-to-one correspondence with self-adjoint operators A, the correspondence being given by

$$A = \int \lambda dE_\lambda.$$

Moreover in this case, a functional calculus can be developed. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable real-valued function, we can define the self-adjoint operator [79]

$$f(A) = \int f(\lambda) dE_\lambda, \tag{1.4}$$

where E is the PVM corresponding to A. If f is bounded, then $f(A)$ is bounded [79]. In particular,

$$E[i] := \int t^i \, dE_t = \Big(\int t \, dE_t \Big)^i = A^i \tag{1.5}$$

and $A = \int t \, dE_t$ is the generator of the von Neumann algebra generated by E.

We point out that if F is not projection valued, equations (1.5) and (1.4) do not hold [59] and, in order to recover the generator of the von Neumann algebra $\mathcal{A}^W(F)$, we need all the moments of F. In particular, in the case of a real commutative POVM F with bounded spectrum and such that $F(\Delta)$ is discrete for any Δ (see chapter 3 for the details), we have

$$A = \sum_{i=0}^{\infty} \alpha_i \, F[i], \quad \alpha_i \geq 0, \quad \sum_{i=0}^{\infty} \alpha_i < \infty$$

where, A is a generator of the von Neumann algebra $\mathcal{A}^W(F)$.

The following result due to Naimark shows that a POVM F in a Hilbert space $\mathcal{H}$ can always be interpreted as the restriction to $\mathcal{H}$ of a PVM E defined in an extended Hilbert space $\mathcal{H}^+$.

Theorem 1.0.7. (Naimark [72, 1, 50, 71]) *Let F be a POVM of the Hilbert space $\mathcal{H}$. Then, there exist a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a PVM E^+ of the space $\mathcal{H}^+$ such that*

$$F(\Delta) = P^+ E^+(\Delta)_{|\mathcal{H}}$$

where P^+ is the operator of projection onto $\mathcal{H}$.

Naimark's theorem is a powerful result on the relationships between PVMs and POVMs but from the physical viewpoint its interpretation is not clear. That is due to the difficulties in interpreting the Hilbert space $\mathcal{H}^+$. As we shall see in chapter 3, a relationships between the Naimark's extension of F and the sharp version A of F can be established in the commutative case. That could provide new insights in the problem of the interpretation of the Naimark's extension.

Positive operator valued measures in the quantum mechanical framework

As we have already seen the set of PVMs is a subset of the set of POVMs. Moreover, real PVMs (spectral measures) are in one-to-one correspondence with self-adjoint operators (spectral theorem) [79] and are used in standard quantum mechanics to represent quantum observables. It was pointed out [2, 27, 32, 50, 78, 82] that POVMs are more suitable than spectral measures in representing quantum observables.

From a general theoretical viewpoint, the introduction of POVMs can be justified by analyzing the statistical description of a measurement. Indeed, a measurement procedure can be described as an affine map from the set of states $\mathcal{S}$ into the set of probability measures on $\mathcal{B}(X)$. The set of states represents the set of possible preparation procedures of the system while the set of probability measures represents the statistical distribution of the results of the possible measurements. It was shown [50, 51] that there exists a one-to-one correspondence between POVMs $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ and affine maps $S \mapsto \mu_S^F(\cdot)$ from the set of states $\mathcal{S}$ into the set of probability measures on $\mathcal{B}(X)$. Moreover, this correspondence is determined by the relation $\mu_S^F(\Delta) = Tr[SF(\Delta)]$. That allows one to interpret the number

$$\mu_S^F(\Delta) = Tr[SF(\Delta)]$$

as the probability that the outcomes of a measurement of the observable $\mathcal{F}$ (corresponding to a POVM F) is in Δ when the physical system is prepared in the state $S \in \mathcal{S}$. We recall that an analogous relation holds for standard observables which are represented by real PVMs $E : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$, that is:

$$\mu_S^E(\Delta) = Tr[SE(\Delta)].$$

That shows again why the quantum observables described by POVMs are a generalization of the standard quantum observables. They are called generalized observables or unsharp observables.

Chapter 2

Characterization of commutative POVMs.

The problem of the relationships between POVMs and PVMs can be answered in the commutative case [11, 22]. Indeed, we can prove that a POVM is commutative if and only if it is the smearing of a PVM. In particular, in the present chapter, we generalize the results I proved in Ref.s [22, 11].

Before we state the theorem we need to introduce the concept of Markov kernel and Feller Markov kernel. In the following Λ and X denotes topological spaces.

Definition 2.0.8. *A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0,1]$ such that,*

1. *$\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,*
2. *$\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.*

A Feller Markov kernel is a Markov kernel $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(X) \to [0,1]$ such that the function

$$G(\lambda) = \int_X f(x)\, d\mu_x(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever f is continuous and bounded.

$$F(\Delta) = \int \mu_{\Delta}(\lambda)\, dE_{\lambda} \tag{2.1}$$

Characterization by means of Feller Markov kernels

Now, we can give the characterization of commutative POVMs by means of Feller Markov kernels. We also explain the concept of smearing which is crucial for such a characterization.

Theorem 2.0.9 (Beneduci, [22]). *A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if, there exist a bounded self-adjoint operator $A = \int \lambda\, dE_{\lambda}$ with spectrum $\sigma(A) \subset [0,1]$, a subset $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$ and a Feller Markov Kernel $\mu : \Gamma \times \mathcal{B}(X) \to [0,1]$ such that:*

1) $\mu_{\Delta}(\cdot) : \Gamma \to [0,1]$ is continuous for each $\Delta \in \mathcal{R}$,

2) $F(\Delta) = \int_{\Gamma} \mu_{\Delta}(\lambda)\, dE_{\lambda}, \quad \Delta \in \mathcal{B}(X)$.

3) $\mathcal{A}^W(F) = \mathcal{A}^W(A)$.

4) μ separates the points in Γ.

where, $\mathcal{R} \subset \mathcal{B}(X)$ is a ring of subsets which generates $\mathcal{B}(X)$.

Definition 2.0.10. *The operator A in theorem 2.0.9 is called the sharp version of F.*

Definition 2.0.11. *Whenever F, A, and μ are such that $F(\Delta) = \mu_{\Delta}(A)$, $\Delta \in \mathcal{B}(X)$, we say that (F, A, μ) is a von Neumann triplet.*

We note that the sharp version A of F is unique up to almost everywhere bijections.

Theorem 2.0.12 (Beneduci, [14]). *Let $(F, A; \mu)$ be a von Neumann triplet such that A is the sharp version of F. Than, i) for any for von Neumann triplet $\big(F, B,\ \mu)$, there exists a real function g such that $A = g(B)$, ii) for any von Neumann triplet (F, A', ν) satisfying item i) there exists an almost everywhere one-to-one function h such that $A' = h(A)$.*

Item 2) in theorem 2.0.9 expresses F as a smearing of E. In order to illustrate the concept of smearing, we consider the unsharp position observable defined as follows.

Example 2.0.13. *For any $\Delta \in \mathcal{B}(\mathbb{R})$ and $\psi \in L^2([0,1])$, we set*

$$\langle \psi, Q^f(\Delta)\psi \rangle := \int_{[0,1]} \mu_\Delta(x)\, d\langle \psi, Q_x \psi \rangle, \tag{2.2}$$
$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a positive, bounded, Borel function such that $f(y) = 0$, $y \notin [0,1]$, and $\int_{[0,1]} f(y)dy = 1$, while Q_x is the spectral measure corresponding to the position operator

$$Q : L^2([0,1]) \to L^2([0,1])$$
$$\psi(x) \mapsto Q\psi := x\psi(x)$$

We recall that $\langle \psi, Q(\Delta)\psi \rangle$ is interpreted as the probability that a perfectly accurate measurement (sharp measurement) of the position gives a result in Δ. Then, a possible interpretation of equation (2.2) is that Q^f is a randomization of Q. Indeed [78], the outcomes of the measurement of the position of a particle depend on the measurement imprecision represented by the Markov kernel μ so that, if the sharp value of the outcome of the measurement of Q is x, the apparatus produces with probability $\mu_\Delta(x)$ a reading in Δ.

Characterization by means of strong Feller Markov kernels

It is worth noting that the Markov kernel

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

in equation (2.2) above is such that the function $x \mapsto \mu_\Delta(x)$ is continuous for each $\Delta \in \mathcal{B}(\mathbb{R})$. The continuity of μ_Δ means that if two sharp values x and x' are very close to each other then, the corresponding random diffusions are very similar, i.e., the probability to get a result in Δ if the sharp value is x is very close to the probability to get a result in Δ if the sharp value is x'.

Theorem 2.0.9 assures that, in the general case, the Markov kernel μ_Δ in item 2) is continuous for each $\Delta \in \mathcal{R}$ where $\mathcal{R}$ is a ring which generates the Borel σ-algebra $\mathcal{B}(X)$. That is the most we can have in the general case. Indeed, we prove (see theorem 2.0.16) that the continuity of μ_Δ for each Borel set is equivalent to the uniform continuity of F which in its turn is equivalent to require that the smearing in item 2) can be realized by a strong Feller Markov kernel.

Definition 2.0.14. *Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$. Let $\Delta = \cup_{i=1}^{\infty} \Delta_i$, $\Delta_i \cap \Delta_j = \emptyset$. If*

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(\Delta_i) = F(\Delta)$$

in the uniform operator topology then we say that F is uniformly continuous.

Definition 2.0.15. *A Markov kernel $\mu_{(\cdot)}(\cdot) : [0,1] \times \mathcal{B}(X) \to [0,1]$ is said to be strong Feller if μ_Δ is a continuous function for each $\Delta \in \mathcal{B}(X)$.*

Theorem 2.0.16 (Beneduci, [22])**.** *A commutative POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ admits a strong Feller Markov kernel if and only if it is uniformly continuous.*

It is worth remarking that although in the general case the continuity holds only for a ring of subsets which generates $\mathcal{B}(X)$ (theorem 2.0.9), that is sufficient to prove the weak convergence of $\mu_{(\cdot)}(x)$ to $\mu_{(\cdot)}(x')$.

Finally, we prove that a POVM F which is absolutely continuous with respect to a regular finite measure ν is uniformly continuous.

Definition 2.0.17. *[82, 83] A POVM $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ is absolutely continuous with respect to a measure $\nu : \mathcal{B}(X) \to [0,1]$ if there exists a positive number c such that $\|F(\Delta)\| \leq c\,\nu(\Delta)$, for each $\Delta \in \mathcal{B}(X)$.*

Theorem 2.0.18 (Beneduci, [22])**.** *Let F be absolutely continuous with respect to a finite measure ν. Then, F is uniformly continuous.*

Corollary 2.0.19 (Beneduci, [22]). *Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ be absolutely continuous with respect to a finite measure ν. Then, F is commutative if and only if there exist a self-adjoint operator A and a strong Feller Markov kernel $\mu : \sigma(A) \times \mathcal{B}(X) \to [0, 1]$ such that:*

$$F(\Delta) = \mu_\Delta(A), \quad \Delta \in \mathcal{B}(X). \tag{2.3}$$

Example 2.0.20. *Let us consider the unsharp position operator defined in example 2.2 and show that Q^f is absolutely continuous with respect to the measure*

$$\nu(\Delta) = M \int_{\Delta\cap[-1,1]} dx.$$

Indeed, for each $\psi \in \mathcal{H}$, $|\psi|^2 = 1$,

$$\langle \psi, Q^f(\Delta)\psi \rangle = \int_{[0,1]} \mu_\Delta(x)\, \psi^2(x)\, dx \leq M \int_{\Delta\cap[-1,1]} dx$$

where, the inequality

$$\mu_\Delta(x) = \int_\Delta f(x - y)\, dy \leq M \int_{\Delta\cap[-1,1]} dx$$

has been used.
Therefore, by theorem 2.0.18, $Q^f(\Delta)$ is uniformly continuous. Moreover, the continuity of f assures the continuity of μ_Δ for each $\Delta \in \mathcal{B}(\mathbb{R})$ so that μ is a strong Feller Markov kernel.

Chapter 3

POVMs and Naimark's dilations.

Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ be a real commutative POVM. In chapter 2, we pointed out that there exist a spectral measure E and a Feller Markov kernel $\mu_{(\cdot)}(\lambda)$ such that

$$F(\Delta) = \int_{\Gamma} \mu_{\Delta}(\lambda)\, dE_{\lambda} = \mu_{\Delta}(A) \tag{3.1}$$

where, $A = \int \lambda\, dE_{\lambda}$ is the self adjoint operator corresponding to the spectral measure E. It is worth remarking (item 3 in theorem 2.0.9) that A is a generator of the von Neumann algebra generated by F and is called the sharp version of F.

On the other hand, Naimark's dilation theorem [73, 71] ensures us that each POVM F in a Hilbert space $\mathcal{H}$ can be extended to a PVM E^{+} in an extended Hilbert space $\mathcal{H}^{+}$ in such a way that F is the projection of E^{+}. That raises the following question: what are the relationships between the self adjoint operators $A^{+} = \int \lambda\, dE^{+}_{\lambda}$ and A?

In the present chapter we answer such a question and analyze some open problems. The result of the chapter are published in Ref.s [16, 17, 18].

The main result of the chapter is based on the following preliminary result.

Theorem 3.0.21 (Beneduci, [16])**.** *Let $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ be a sequence of distinct probability measures on $\mathcal{B}([0,1])$. Let us consider the infinite sequence* *of linear functionals* $\{T_i\}_{i\in\mathbb{N}}$ *defined as follows*

$$T_i f := \int f(t)\, d\mu_t(i) =: G_f(i), \quad i \in \mathbb{N} \tag{3.2}$$

where, $f : [0,1] \to \mathbb{R}$*, is a bounded Borel function and the integration is in the sense of Lebesgue-Stieltjes.*
There exists a continuous one-to-one function $f(t)$ *such that* G_f *is one-to-one*

$$G_f(i) = \int f(t)\, d\mu_t(i) \neq \int f(t)\, d\mu_t(j) = G_f(j), \quad i, j \in \mathbb{N},\, i \neq j.$$

Moreover,

$$f(t) = \sum_{i=1}^{\infty} \alpha_i f_i \tag{3.3}$$

where, $\alpha_i \geq 0$, $\sum_i \alpha_i < \infty$ *and* $f_i = t^i$ *for* $i \geq 0$.

As we already said, Theorem 3.0.21 is the key of the proof of the main theorem of the present section [16, 17].

Theorem 3.0.22 (Beneduci, [16])**.** *Let* $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ *be a commutative POVM such that the operators in the range of* F *are discrete. Then, the sharp version* A *of* F *is a linear combination of the moments of* F*.*

In particular, we have

$$A = \sum_{i=1}^{\infty} \alpha_i F[i] \tag{3.4}$$

where, $\alpha_i \geq 0$, $\sum_{i=1}^{\infty} \alpha_i < \infty$, and

$$F[i] = \int_{[0,1]} t^i\, dF_t.$$

Equivalently,

$$A = \int f(t)\, dF_t \tag{3.5}$$

where, $f(t) = \sum_{i=1}^{\infty} \alpha_i t^i$.

Theorems 3.0.21 and 3.0.22 can be proved by construction [17]. In particular, one can prove the following theorem (see section 3.4 of the dissertation).

Theorem 3.0.23 (Beneduci, [17]). *Let $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ be a commutative POVM such that the operators in the range of F are discrete. Then, one can build a real, one-to-one, continuous from the left function f such that the function*

$$G(i) := \int_{[0,1]} f(t)\, d\mu_t(i).$$

is injective, i.e., $G(i) \neq G(j)$, $i \neq j$, and

$$G(A) = \int_{[0,1]} f(t)\, dF_t. \tag{3.6}$$

Now, let us go back to equation 3.5 and consider a Naimark's extension E^+ of F. We have

$$A = \int f(t)\, dF_t = P^+ f(A^+) P^+, \tag{3.7}$$

which expresses A as the projection of $f(A^+)$. That is assured by the following theorem.

Theorem 3.0.24 (Beneduci, [16, 17]). *Let E^+ be a Naimark's extension of F and $A^+ = \int \lambda dE^+_\lambda$. Let f be a measurable function which is bounded with respect to E^+. Then*

$$P^+ f(A^+)_{|\mathcal{H}} = \int f(t) dF_t$$

and $P^+ f(A^+)_{|\mathcal{H}}$ is a bounded self-adjoint operator.

Definition 3.0.25. *Let A be the sharp version of F. Suppose there exist f and G one-to-one and bounded such that*

$$G(A) = \int f(t)\, dF_t = P^+ f(A^+)_{|\mathcal{H}}.$$

Then we say that the sharp version A is equivalent to the projection of a Naimark operator and write $A \leftrightarrow \operatorname{Pr} A^+$.

Since the function f in equation (3.7) is one-to-one, as a consequence of theorem 3.0.22 we have that A coincides with the projection of the Naimark operator $f(A^+)$ and then $A \leftrightarrow \operatorname{Pr} A^+$. The equivalence between sharp versions and projections of Naimark operators can be generalized to the case of POVM with spectrum in $\mathbb{R}$.

Theorem 3.0.26 (Beneduci, [16])**.** *Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ be a commutative POVM such that the operators in the range of F are discrete. Let A be the sharp version of F, E^+ an extension of F whose existence is asserted by Naimark's theorem and A^+ the Naimark operator $\int \lambda\, dE^+_\lambda$. Then, $A \leftrightarrow \operatorname{Pr} A^+$.*

Another consequence of theorems 3.0.22 and 3.0.26 is that the sharp version A of F can be recovered by integrating f over F, i.e.,

$$A = \int f(t)\, dF_t.$$

At the same time, F is the smearing of A, i.e,

$$F(\Delta) = \int \mu_\Delta(\lambda)\, dE_\lambda. \tag{3.8}$$

All that suggests the equivalence of A and F from the informational point of view [15]. That will be the topic of the next chapter.

Open problems

In the previous section we characterized the sharp version A of a POVM F as the projection of a Naimark operator $f(A^+)$. In order for such characterization to be complete, it is necessary to extend theorem 3.0.26 to the general case. That seems not to be straitforward. In the present section we introduce two conjectures (conjecture 3.0.28) which suggest a possible path to such extension and generalize theorem 3.0.26.

Definition 3.0.27. *Whenever F, A, and μ are such that $F(\Delta) = \mu_\Delta(A)$, $\Delta \in \mathcal{B}(X)$, we say that (F, A, μ) is a von Neumann triplet.*

Conjecture 3.0.28. *For each von Neumann triplet (F, A, μ) there exist a one-to-one measurable function $f : \mathbb{R} \to [0, 1]$ and a real E-a.e. one-to-one measurable function $G(\lambda)$ such that*

$$G(A) = \int f(t)\, dF_t \tag{3.9}$$

where, E is the spectral measure corresponding to A.

It is clear that if conjecture 3.0.28 is true, theorem 3.0.26 can be extended to any commutative POVM.
Now, we give a necessary and sufficient condition for the conjecture to be true.

Theorem 3.0.29 (Beneduci, [18])**.** *Conjecture 3.9 is true if and only if, for any von Neumann triplet (F, A, μ), there exists a measurable, one-to-one function $f : \mathbb{R} \to [0, 1]$ such that the function*

$$G(\lambda) := \int f(t)\, d\mu_t(\lambda)$$

is E-a.e. one-to-one on $\sigma(A)$.

Now, we introduce a second conjecture which is stronger than conjecture 3.0.28. It corresponds to an extension of theorem 3.0.21 to the continuous case but, at variance with conjecture 1, we require that G is one-to-one. Some of its consequences are analyzed below.

Conjecture 3.0.30. *For each family of probability measures $\{\mu_{(\cdot)}(\lambda)\}_{\lambda \in B}$, $B \subseteq [0, 1]$, $\mu_{(\cdot)}(\lambda) : \mathcal{B}(\mathbb{R}) \to [0, 1]$, there exists a real, measurable, one-to-one function $f : \mathbb{R} \to [0, 1]$ such that,*

$$G(\lambda) := \int f(t)\, d\mu_t(\lambda) \neq \int f(t)\, d\mu_t(\lambda') =: G(\lambda')$$

whenever $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$.

Theorem 3.0.31 (Beneduci, [18]). *Conjecture 3.0.30 implies conjecture 3.0.28*

The following theorem shows an important consequence of conjecture 3.0.30.

Theorem 3.0.32 (Beneduci, [18]). *If the conjecture 3.0.30 is true, the functions f and G in the equation*

$$G(A) = \int f(t)\, dF_t$$

do not depend on the POVM F.

In other words, f, G and μ are fixed and, for any F, there is a self-adjoint operator A_F such that (F, A^F, μ) is a von Neumann triplet and $G(A^F) = \int f(t)\, dF_t$. Therefore, A^F is the only object which changes when F changes.

Theorem 3.0.23 suggests a procedure for extending theorem 3.0.21 to the general case as well as to prove conjecture 3.0.30 constructively. It is worth remarking that it would be very helpful to have a procedure for the construction of f in the general case. Indeed, as we already said, if conjecture 3.0.30 is true, the function f is universal, i.e., it does not depend on F. Therefore, once f is constructed, it can be used to recover the sharp version of any commutative POVM. That is why f was called the universal antismearing function.

Chapter 4

On the Informational content of a POVM.

As we have seen in chapter 2, for any commutative POVM F there exists a PVM E such that F can be interpreted as a random diffusion of E. In particular,

$$F(\Delta) = \int \mu_{\Delta}(\lambda)\, dE_{\lambda} \tag{4.1}$$

where μ is a Markov kernel. At the same time, as a consequence of theorems 3.0.22 and 3.0.26, the sharp version A of F can be represented as the integral of f over F, i.e.,

$$A = \int f(t)\, dF_t.$$

where f is a measurable one-to-one function. It is also worth remarking that E can be algorithmically reconstructed by F [31, 10]. All that suggests that, in some sense, the observables represented by E and F should have the same informational content. In the present chapter, we recall some well known partial order relations on the set of observables and show that no one of them satisfies this condition. Then, we propose a partial ordering for which E and F are always equivalent and establish some relationships between this one and the others well known partial order relations. The results we present here have been published in Ref. [15].

Definition 4.0.33 (**Smearing**). *Let F_1 and F_2 be two observables. If there exists a Markov kernel $\mu_{(\cdot)}(\lambda)$ such that,*

$$F_1(\Delta) = \int \mu_\Delta(\lambda) F_2(d\lambda),$$

we say that F_1 is a smearing of F_2 and write $F_1 \preceq_f F_2$. If $F_1 \preceq_f F_2 \preceq_f F_1$, we say that F_1 and F_2 are $\sim_f$ equivalent and write $F_1 \sim_f F_2$.

In the following, we denote by $\mathcal{T}_1^+(\mathcal{H})$ the space of trace class operators with trace one on the Hilbert space $\mathcal{H}$. The states of a system are represented by operators in $\mathcal{T}_1^+(\mathcal{H})$.

Definition 4.0.34. *Let ρ_1 and ρ_2 be two states. Let F be an observable. If there exists a set $\Delta \in \mathcal{B}(\mathbb{R})$ such that $Tr[F(\Delta)\rho_1] \neq Tr[F(\Delta)\rho_2]$ we say that F can distinguish between the states ρ_1 and ρ_2.*

Definition 4.0.35 (**State distinction**). *If for all ρ_1, ρ_2*

$$Tr[F_2(\Delta)\rho_1] = Tr[F_2(\Delta)\rho_2], \quad \forall \Delta \in \mathcal{B}(\mathbb{R})$$

$$\Downarrow$$

$$Tr[F_1(\Delta)\rho_1] = Tr[F_1(\Delta)\rho_2], \quad \forall \Delta \in \mathcal{B}(\mathbb{R})$$

we say that the state distinction power of F_2 is greater than or equal to F_1 and write $F_1 \preceq_i F_2$. If $F_1 \preceq_i F_2 \preceq_i F_1$, we say that F_1 and F_2 are $\sim_i$ equivalent or that they have the same informational content and write $F_1 \sim_i F_2$.

Definition 4.0.36. *The set of the states determined by the observable F is $\mathcal{O}_F := \{\rho |\ \forall \rho' \neq \rho,\ \exists\, \Delta,\ Tr[F(\Delta)(\rho - \rho')] \neq 0\}$*

Definition 4.0.37 (**State determination**). *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the sets of states determined by F_1 and F_2 respectively. If $\mathcal{O}_1 \subset \mathcal{O}_2$ we say that F_2 has a state determination power greater or equal than F_1 and write $F_1 \preceq_d F_2$. If $F_1 \preceq_d F_2 \preceq_d F_1$, we say that F_1 and F_2 are $\sim_d$ equivalent and write $F_1 \sim_d F_2$.*

Theorem 4.0.38 ([43]). $F_1 \preceq_f F_2 \Rightarrow F_1 \preceq_i F_2 \Rightarrow F_1 \preceq_d F_2$

By means of counterexamples we proved that no one of the equivalence relations we just introduced are such that the sharp version E of a commutative POVM F is equivalent to F. We can summarize our result in the following theorem.

Theorem 4.0.39 (Beneduci, [15]). *Let F be a commutative POVM and E be the spectral measure corresponding to the sharp version A of F. In general $E \sim_f F$, $E \sim_i F$ and $E \sim_d F$ are false.*

Now, we introduce an equivalence relation between observables which should capture the meaning of equivalence between a commutative POVM and its sharp version outlined above. Moreover, the relation we are going to introduce is not restricted to commutative POVMs. The analysis of its meaning in the general case will be the aim of a future work.

Definition 4.0.40. *Let F_1 and F_2 be two POVMs. We say that $F_1 \preceq_a F_2$ if, for each real, bounded, measurable function f there exists a real, bounded, measurable function g_f such that*

$$B_1 := \int f \, dF_1 \preceq_f \int g_f \, dF_2 =: B_2.$$

If $F_1 \preceq_a F_2 \preceq_a F_1$ we say that F_1 and F_2 are $\sim_a$ equivalent and write $F_1 \sim_a F_2$.

Notice that, B_1 and B_2 are self-adjoint operators, so that $B_1 \preceq_f B_2$ means that there exists a measurable function h such that $B_1 = h(B_2)$ [36]. The following theorem shows that, at least in the hypothesis of Theorem 3.0.22 in chapter 3, a commutative POVM is always equivalent to its sharp version in the sense of Definition 4.0.40.

Theorem 4.0.41 (Beneduci, [15]). *Let F be a commutative POVM with discrete spectrum such that the operators in the range of F are discrete. Let E be its sharp version. Then, $E \sim_a F$.*

Moreover, we have the following property of the equivalence relation we introduced.

Theorem 4.0.42 (Beneduci, [15]). *Let F_1 and F_2 be two POVMs. If $F_1 \preceq_f F_2$, then $F_1 \preceq_a F_2$.*

This proves that $F_1 \preceq_f F_2$ implies $F_1 \preceq_a F_2$ so that, smearing is stronger than definition 4.0.40.

We end this chapter by observing that neither state distinction nor state determination are weaker than definition 4.0.40 (see section 4.2 of the dissertation for the details).

Chapter 5

Uniform continuity and localization.

It is possible to realize quantum experiments where single photons can be detected. For example, a single photon can be detected in a point on a screen (localization of the photon). This phenomenon does not find a theoretical description in the standard formulation of quantum mechanics. Indeed, it not possible to define a PVM which represents the *localization observable* of the photon [88].

Other examples of quantum observables which are not representable by PVMs (and therefore by self-adjoint operators) are the phase observable and the time observable (but in the latter case there are some exceptions [37, 68]). The problem can be overcome by using POVMs in order to describe the *localization observables*.

It is worth remarking that PVMs imply a kind of localization (sharp localization) which is stronger than the one implied by POVMs (unsharp localization). We recall that in quantum mechanics, $\langle\psi, F(\Delta)\psi\rangle$ is interpreted as the probability that a measurement of the observable F when the system is in the state ψ gives a result in Δ. If a localization observable is described by a covariant PVM E (sharp localization) then, for any Borel set Δ such that $E(\Delta) \neq \mathbf{0}$, there exists a unit vector ψ for which $\langle\psi, E(\Delta)\psi\rangle = 1$; i.e., the probability that a measure of the position of the system in the state ψ gives a result in Δ is one. Conversely, if a localization observable is described by a POVM F, there are Borel sets Δ such that $0 < \langle \psi, F(\Delta)\psi \rangle < 1$ for any vector ψ (unsharp localization).

It has been claimed that a POVM F with the norm-1 property; i.e., such that $\|F(\Delta)\| = 1$ whenever $F(\Delta) \neq \mathbf{0}$, would in some sense reduce the gap between the two kinds of localizations. Indeed, if $\|F(\Delta)\| = 1$ then, for each ϵ, it is possible to find a unit vector ψ such that the probability $\langle \psi, F(\Delta)\psi \rangle$ that a measure gives a result in Δ is greater than $1-\epsilon$. In other words, if F has the norm-1 property the quantum system it describes can be localized as accurately as desired (although not sharply). That raises the question of establishing if a given POVM has the norm-1 property. In order to answer such question it is helpful [21, 23] to characterize the uniform continuity for POVMs. The results in the present section are published in Ref.s [21, 23].

Definition 5.0.43. *[21, 23] Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ be a POVM. F is said to be uniformly continuous at Δ if, for any disjoint decomposition $\Delta = \cup_{i=1}^{\infty} \Delta_i$,*

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(\Delta_i) = F(\Delta)$$

in the uniform operator topology. F is said uniformly continuous if it is uniformly continuous at each $\Delta \in \mathcal{B}(X)$.

Proposition 5.0.44 (Beneduci, Schroeck, [21, 23])**.** *A POVM F is uniformly continuous at Δ if and only if, it is uniformly continuous from below at Δ, i.e., for any increasing sequence $\Delta_i \uparrow \Delta$,*

$$\lim_{n\to\infty} \|F(\Delta) - F(\Delta_i)\| = 0.$$

F is uniformly continuous if and only if it is uniformly continuous from below at each Δ.

Proposition 5.0.45 (Beneduci, Schroeck, [21, 23])**.** *F is uniformly continuous if and only if,*

$$\lim_{i\to\infty} \|F(\Delta_i)\| = 0$$

whenever $\Delta_i \downarrow \emptyset$.

Now, we can prove the following necessary conditions for the norm-1 property.

Theorem 5.0.46 (Beneduci, Schroeck, [21, 23]). *Let $F : X \to \mathcal{F}(\mathcal{H})$ be uniformly continuous and let $\sigma(F)$ be the spectrum of F. Then, F has the norm-1-property only if $\|F(\{x\})\| \neq 0$ for each $x \in \sigma(F)$.*

Theorem 5.0.47 (Beneduci, Schroeck, [21, 23]). *Let $F : \mathcal{B}(X) \to \mathcal{F}(\mathcal{H})$ be absolutely continuous with respect to a regular measure ν. Then, F has the norm-1 property only if $\|F(\{x\})\| \neq 0$ for each $x \in X$ such that $\nu(\{x\}) < \infty$.*

We used this result to prove that there is a wide class of POVMs which cannot have the norm-1 property [23]. That is the case for example of the phase observable and the unsharp number observable (see section 5.3 of the dissertation). Moreover, we can also prove that a wide class of POVMs which describe localization observables cannot have the norm-1 property [23]. That is the case, for example, of the localization in phase space of massless relativistic particles and the localization in phase space and in configuration space of non relativistic particles (see below).

Before we can proceed, we need to recall the main step in the formulation of quantum mechanics on phase space, i.e., the construction of the phase space Γ. In brief, we can say that there is a procedure that starting from a Lie group G allows the classification of all the closed subgroups $H \subset G$ such that G/H is a simplectic space (i.e, a phase space). For example, in the case of the Galilei group, a possible choice for H is the group $H = SO_3$. Then, $\Gamma = G/H = \mathbb{R}^3 \times \mathbb{R}^3$, which coincides with the phase space of classical mechanics. A different choice of H generates a different phase space. In other words, the procedure allows the calculation of all the phase spaces corresponding to a locally compact Lie group, G, with a finite dimensional Lie algebra. Once we have the phase space, we can look for a strongly continuous unitary representation of G in a Hilbert space $\mathcal{H}$ and then we can define the localization observable [83].

Definition 5.0.48 (See [83]). *Let G be a locally compact topological group, H a closed subgroup of G, U a strongly continuous unitary irreducible representation of G in a complex Hilbert space $\mathcal{H}$ and μ a volume measure on G/H. A localization observable is represented by a POVM*

$$A^\eta(\Delta) = \int_\Delta |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| \, d\mu(x), \quad \Delta \in \mathcal{B}(G/H).$$

where, $\sigma : G/H \mapsto G$ is a measurable map and η is a unit vector such that

$$\int_{G/H} |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|\, d\mu(x) = \mathbf{1}. \tag{5.1}$$

For example, if G is the Galilei group and $H = \mathbb{R}^2 \times \mathbb{R}^2 \times SO(3)$, then $G/H = \mathbb{R}_q \times \mathbb{R}_p$, μ is the Lebesgue measure, $U(\sigma(x)) = U_{q,p} = e^{-iqP}e^{ipQ}$, $\eta \in L^2(\mathbb{R})$ and the localization observable reads

$$A(\Delta_q \times \Delta_p) = \int_{\Delta_q \times \Delta_p} |U_{q,p}\eta\rangle\langle U_{q,p}\eta|\, d\mathbf{q}\, d\mathbf{p}.$$

The marginals

$$F^Q(\Delta) : = A^\eta(\Delta \times \mathbb{R}_p) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\eta|^2)(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}),$$
$$F^P(\Delta) : = A^\eta(\mathbb{R}_q \times \Delta) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\widetilde{\eta}|^2)(x)\, dP_x, \quad \Delta \in \mathcal{B}(\mathbb{R})$$

where, $\widetilde{\eta}$ is the Fourier transform of η, are the unsharp position and momentum observables respectively [32]. Notice that the map $\mu_\Delta(x) := \mathbf{1}_\Delta * |\eta(x)|^2$ defines a Markov kernel.

Therefore, $A^\eta(\Delta_q \times \Delta_p)$ is a joint measurement for the unsharp position and momentum observables. In other words, the phase space approach allows the description of joint measurements of unsharp position and momentum observables also if they do not commute; i.e., $[F^Q(\Delta_q), F^P(\Delta_p)] \neq \mathbf{0}$. That is one of the main advantages from the physical viewpoint of using the phase space approach outlined above. A second relevant aspect we would like to note is that Definition 5.0.48 allows the introduction of a quantization procedure [83, 84]. Indeed, for any real-valued Borel function f,

$$A^\eta(f) = \int_{G/H} f(x)\, |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|\, d\mu(x)$$

defines a self-adjoint operator which is positive whenever f is positive. Moreover, for any pure state $\rho = |\psi\rangle\langle\psi|$,

$$Tr(A^\eta(f)\rho) = \int_{G/H} f(x)\, f_\rho(x)\, d\mu(x)$$

where $f_\rho(x) = |\langle U(\sigma(x))\eta, \psi\rangle|^2$.

A general property of the localization observables in Definition 5.0.48 is that they are absolutely continuous with respect to the measure μ.

Theorem 5.0.49. *[83] The POVM in Definition 5.0.48 is absolutely continuous with respect to μ.*

Another important property of A^η the covariance with respect to U.

Theorem 5.0.50. *[83] The POVM A^η defined in Definition 5.0.48 is covariant with respect to U, i.e.,*

$$U_g F(\Delta) U_g^\dagger = F(g\Delta), \quad g \in G, \quad \Delta \in \mathcal{B}(G/H).$$

We point out that covariance is a key element in the definition of localization since it ensures that the results of a localization measurement do not depend on the choice of the origin and the orientation of the reference frame.

Now, we use theorem 5.0.46 to establish whether the POVM A^η in Definition 5.0.48 has the norm-1 property.

Theorem 5.0.51 (Beneduci, Schroeck, [21])**.** *The localization observable represented by the POVM A^η with $G/H = \mathbb{R}^3 \times \mathbb{R}^3$ and μ the Lebesgue measure does not have the norm-1 property.*

An analogous result can be proved in the case of massless relativistic particles. In the relativistic case G is the double cover $\mathcal{P} = T^4 \oslash SL(2,\mathbb{C})$ of the Poincaré group. In particular, T^4 is the Minkowski space and $SL(2,\mathbb{C})$ is the double cover of the Lorentz group. The symbol $\oslash$ denotes the semidirect product. In the massless relativistic case ([83], page 454) the relevant subgroup is $H = \mathbb{R}\mathbf{p}_0 \oslash SL(2,\mathbb{C})_{\mathbf{p}_0} = \mathbb{R}\mathbf{p}_0 \oslash (\mathbb{R}^2 \oslash \widetilde{O}(2))$ where, $\mathbf{p}_0 = (1,0,0,1)$, $\mathbb{R}\mathbf{p}_0 = \{\lambda\,\mathbf{p}_0, \lambda \in \mathbb{R}\}$, $\widetilde{O}(2)$ is the double cover of the group of rotations in $\mathbb{R}^2$, and $SL(2,\mathbb{C})_{\mathbf{p}_0}$ is the set of matrices $A \in SL(2,\mathbb{C})$ such that $A[\mathbf{p}_0] = \mathbf{p}_0$. The phase space for the photon is then $\mathcal{P}/H$ which is the space of cosets

$$(\mathbf{a}, A)(\mathbb{R}\mathbf{p}_0 \oslash SL(2,\mathbb{C})_{\mathbf{p}_0}) = (\mathbf{a} + \mathbb{R}A[\mathbf{p}_0],\, ASL(2,\mathbb{C})_{\mathbf{p}_0})$$

where, $(\boldsymbol{a}, A) \in \mathcal{P}$, $SL(2,\mathbb{C})_{\mathbf{p}_0}$ is isomorphic to $\mathbb{R}^2 \oslash \widetilde{O}(2)$ and the quotient $SL(2,\mathbb{C})/SL(2,\mathbb{C})_{\mathbf{p}_0}$ is homeomorphic to $\mathbb{R}^+ \times S^2$. The invariant measure on $\mathcal{P}/H$ is (see equation (344), page 463, in Ref.[83])

$$d\mu = d(\alpha)d(\gamma)d(\delta) \times (p^0 + p^3)^{-1} d(p^0 + p^3) \wedge dp^1 \wedge dp^2 \tag{5.2}$$

where $\alpha = a_\mu(A[\mathbf{p}_0])^\mu$, $\gamma = a_\mu(A[\mathbf{u}_0])^\mu$, $\delta = a_\mu(A[\mathbf{v}_0])^\mu$, with $\mathbf{u}_0 = (0, 1, 0, 0)$, $\mathbf{v}_0 = (0, 0, 1, 0)$.
Thus α, γ, δ are in $\mathbb{R}$. Hence, we have a representation of the zero mass particles. Moreover, μ is zero in each single point subset of the phase space so that the reasoning in the proof of theorem 5.0.51 can be used.

Theorem 5.0.52 (Beneduci, Schroeck, [21]). *If $G/H = T^4 \oslash SL(2,\mathbb{C})/\mathbb{R}\mathbf{p}_0 \oslash (\mathbb{R}^2 \oslash \widetilde{O}(2))$ with the measure μ in equation (5.2), the POVM A^η in Definition 5.0.48 does not have the norm-1 property.*

Now, we study the marginals of A^η in the non-relativistic case and prove that they cannot have the norm-1 property. We limit ourselves to the marginal $F^Q_\eta(\Delta_{\mathbf{q}}) := A^\eta(\Delta_{\mathbf{q}} \times \mathbb{R}_{\mathbf{p}})$ which represents the unsharp position observable. Clearly what we prove applies also to the marginal $F^P_\eta(\Delta_{\mathbf{p}}) := A^\eta(\mathbb{R}_{\mathbf{q}} \times \Delta_{\mathbf{p}})$ which represents the unsharp momentum observable.

Theorem 5.0.53 (Beneduci, Schroeck, [21]). *The POVM $F^Q_\eta(\Delta_{\mathbf{q}})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_{\mathbf{q}}$.*

Theorems 5.0.46 implies the following corollary.

Corollary 5.0.54 (Beneduci, Schroeck [21]). *F^Q_η cannot have the norm-1 property*

That forbids localization in configuration space to have the norm-1 property.

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