

ON THE EXISTENCE OF VAN DER WAERDEN TYPE NUMBERS FOR LINEAR RECURRENCE SEQUENCES WITH CONSTANT COEFFICIENTS

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ABSTRACT. We prove some existence and non-existence results on van der Waerden type numbers for linear recurrence sequences with constant positive integer coefficients.

1. INTRODUCTION

Let X be a non-empty set and C a set of colours with r elements where r is a positive integer. Then a function $\chi : X \rightarrow C$ is called an r -colouring of X . A non-empty subset $Y \subseteq X$ is monochromatic under this colouring if χ is constant on Y . Theorems of Ramsey theory typically say that if X is large enough in some sense, then for any r -colouring of X , there exists a monochromatic subset with some given properties.

Ramsey type results in the area of number theory usually deal with colourings of the positive integers. One of the oldest Ramsey type theorems is due to I. Schur [12]. It states that for any integer $r \geq 1$, there exists a least positive integer $s(r)$ such that for any r -colouring of $[1, s(r)] \cap \mathbb{Z}$, there exists a monochromatic solution of the equation $x + y = z$.

Schur's result was later generalized to systems of linear equations. A homogeneous system \mathcal{E} of linear equations with integer coefficients is said to be regular if for any integer $r \geq 1$, there exists a least positive integer $g(\mathcal{E}, r)$ such that for any r -colouring of $[1, g(\mathcal{E}, r)] \cap \mathbb{Z}$, there exists a monochromatic solution of \mathcal{E} . Let C be an $m \times n$ integer matrix with $n \geq 2$ and denote its j th column by c_j ($j = 1, \dots, n$). R. Rado [11] proved that the system $Cx = 0$ is regular if and only if C satisfies the so-called columns condition: there exists a partition $\{I_1, \dots, I_t\}$ of $\{1, \dots, n\}$ such that $\sum_{j \in I_1} c_j = 0$ and $\sum_{j \in I_i} c_j$ is a linear combination of columns

c_k ($k \in I_1 \cup \dots \cup I_{i-1}$) for $i = 2, \dots, t$. For a single equation $\sum_{i=1}^n c_i x_i = 0$ with coefficients $c_i \in \mathbb{Z} \setminus \{0\}$ ($i = 1, \dots, n$), this means equivalently that there exists $I \subseteq \{1, \dots, n\}$, $I \neq \emptyset$ such that $\sum_{i \in I} c_i = 0$.

Instead of monochromatic solutions of diophantine equations, one can search for monochromatic sequences. The classical theorem of B. L. van der Waerden [13] asserts that for any integers $k \geq 2$ and $r \geq 1$, there exists a least positive integer $w(k, r)$ such that for any r -colouring of $[1, w(k, r)] \cap \mathbb{Z}$, there exists a monochromatic, strictly increasing finite arithmetic progression of length k .

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The similar problem arises when we replace arithmetic progressions with another family of sequences. Let \mathcal{S} be a non-empty set of sequences of positive integers. On a finite sequence of \mathcal{S} of length k , we mean the first k elements of a sequence from \mathcal{S} . For integers $k \geq 2$ and $r \geq 1$, let $w(\mathcal{S}, k, r)$ be the least positive integer if it exists, such that for any r -colouring of $[1, w(\mathcal{S}, k, r)] \cap \mathbb{Z}$, there exists a monochromatic finite sequence of \mathcal{S} of length k . Obviously, $w(\mathcal{S}, k, 1)$ exists for any $k \geq 2$. Moreover, if $k \geq k' \geq 2$, $r \geq r' \geq 1$ and $w(\mathcal{S}, k, r)$ exists, then $w(\mathcal{S}, k', r)$, $w(\mathcal{S}, k, r')$ also exist and $w(\mathcal{S}, k', r) \leq w(\mathcal{S}, k, r)$, $w(\mathcal{S}, k, r') \leq w(\mathcal{S}, k, r)$.

For example, for the family of strictly increasing geometric progressions, the existence of van der Waerden type numbers can be deduced from van der Waerden's theorem. Several other families of sequences are investigated, for example, in [9].

However, there are only a few results for linear recurrence sequences. Since strictly increasing arithmetic progressions are exactly those sequences $(x_i)_{i=1}^\infty$ which satisfy the recurrence relation $x_i = 2x_{i-1} - x_{i-2}$ ($i \geq 3$) and the inequality $x_1 < x_2$ for the initial values, van der Waerden's original theorem can be viewed as the first such result.

Denote by \mathcal{F} the set of sequences $(x_i)_{i=1}^\infty$ of positive integers which satisfy the Fibonacci recurrence $x_i = x_{i-1} + x_{i-2}$ ($i \geq 3$). Schur's theorem guarantees the existence of $w(\mathcal{F}, 3, r) = s(r)$ for any $r \geq 1$.

B. M. Landman [8] (see also [9], Section 3.6) gave an upper bound on van der Waerden type numbers in the case of $r = 2$, among others for the family of sequences satisfying the linear recurrence $x_i = b_i x_{i-1} + (1 - b_i)x_{i-2}$ ($i \geq 3$), where $x_1 < x_2$ and $b_i \in [2, +\infty[\cap \mathbb{Z}$ are non-fixed coefficients. His results were recently generalized by G. Nyul and B. Rauf [10].

Second order linear recurrence sequences with constant positive integer coefficients were studied by H. Harborth and S. Maasberg [5], [6], with a special emphasis on the Fibonacci recurrence. In [6] they proved that for $k \geq 3$ and $r \geq 2$, $w(\mathcal{F}, k, r)$ exists if and only if $k = 3$ or $k = 4$. Moreover, if $\tilde{\mathcal{F}} = \{(x_i)_{i=1}^\infty \in \mathcal{F} \mid x_1 \leq x_2\}$, then $w(\tilde{\mathcal{F}}, k, r)$ exists if and only if $k = 3$. This latter assertion was rediscovered by H. Ardal, D. S. Gunderson, V. Jungić, B. M. Landman and K. Williamson [2]. We notice that H. Harborth, S. Maasberg [7] and H. Ardal, Z. Dvořák, V. Jungić, T. Kaiser [1] investigated similar problems for second order linear recurrence sequences with rational coefficients, but these results do not fit to our case, since only the first few elements of these sequences are certainly positive integers.

However, higher order linear recurrence sequences have not been studied yet. In Section 2, our purpose is to give some existence and non-existence theorems for linear recurrence sequences with constant positive integer coefficients of any order. After these general results, as a special case, we turn our attention to multibonacci sequences in Section 3.

2. LINEAR RECURRENCE SEQUENCES WITH POSITIVE INTEGER COEFFICIENTS

In the sequel, we study linear recurrence sequences, where the coefficients in the recurrence equation are fixed positive integers.

Let $s \geq 2$ and a_1, \dots, a_s be positive integers. Denote by $\mathcal{R}_{a_1, \dots, a_s}$ the family of those sequences $(x_i)_{i=1}^\infty$ which consist of positive integers and satisfy the linear recurrence $x_i = a_1 x_{i-s} + \dots + a_s x_{i-1}$ ($i \geq s + 1$) of order s .

We notice that for $k \leq s$ and arbitrary $r \geq 1$, the first k terms of the sequence from $\mathcal{R}_{a_1, \dots, a_s}$ with initial values $x_1 = \dots = x_s = 1$ give a monochromatic finite sequence of $\mathcal{R}_{a_1, \dots, a_s}$ of length k , independently from the choice of the r -colouring. Therefore, it is enough to consider the case when $k \geq s + 1$.

To our knowledge, there has been no investigation into this direction yet. Although, for the smallest interesting values of k and r , a result of S. Guo and Z.-W. Sun [4] can be interpreted

this way. Namely, if $k = s + 1$, then the existence of $w(\mathcal{R}_{a_1, \dots, a_s}, k, 2)$ follows from a less-known theorem of R. Rado [11], since the linear equation $a_1x_1 + \dots + a_sx_s - x_{s+1} = 0$ with three or more unknowns has both positive and negative coefficients. The exact value of this van der Waerden type number is $w(\mathcal{R}_{a_1, \dots, a_s}, k, 2) = ab^2 + b - a$, where $a = \min \{a_1, \dots, a_s\}$ and $b = a_1 + \dots + a_s$.

We prove our first result using Rado's theorem.

Theorem 2.1. *Let $s \geq 2$ and a_1, \dots, a_s be positive integers.*

1. *For $k = s + 1$, $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ exists for all $r \geq 1$ if and only if $a_1 = 1$ or ... or $a_s = 1$.*
2. *For $k = s + 2$, $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ exists for all $r \geq 1$ if and only if $a_1 = a_s = 1$.*
3. *For $k \geq s + 3$, $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ does not exist for all $r \geq 1$.*

Proof. **1.** If $k = s + 1$, then in fact we look for a monochromatic positive integer solution of $a_1x_1 + \dots + a_sx_s - x_{s+1} = 0$. The sum of some coefficients can be 0 if and only if $a_i = 1$ for some $1 \leq i \leq s$, hence applying Rado's theorem we conclude that this equation is regular precisely in this case.

2. If $k = s + 2$, then we have to check the regularity of the homogeneous system of linear equations with coefficient matrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{s-1} & a_s & -1 & 0 \\ 0 & a_1 & a_2 & \dots & a_{s-1} & a_s & -1 \end{pmatrix}.$$

Denote by c_i the i th column of this matrix ($i = 1, \dots, k$). We verify the columns condition for it.

If there exist columns with zero vector sum, then the sum of all of their coordinates is 0, as well. The first, the $(s + 1)$ st and the $(s + 2)$ nd column sums are $a_1 \geq 1$, $a_s - 1 \geq 0$ and -1 , respectively, while the i th column sum is $a_i + a_{i-1} \geq 2$ ($i = 2, \dots, s$). The sum of these column sums can be 0 only when we choose the columns

- c_{s+1} and $a_s = 1$,
- c_1, c_{s+2} and $a_1 = 1$,
- c_{s+1}, c_{s+2} and $a_s = 2$,
- c_1, c_{s+1}, c_{s+2} and $a_1 = a_s = 1$.

The sums of the chosen columns are

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

respectively. Hence only the last case remains, when the sum of the other columns (i.e. $c_2 + \dots + c_s$) can be written as a linear combination of the chosen ones by

$$(a_2 + \dots + a_s) \cdot c_1 + 0 \cdot c_{s+1} - (a_1 + \dots + a_{s-1}) \cdot c_{s+2}.$$

This means that the above system of linear equations is regular if and only if $a_1 = a_s = 1$ by Rado's theorem.

THE FIBONACCI QUARTERLY

3. If $k \geq s + 3$, then the coefficient matrix of the homogeneous system of linear equations under investigation is

$$\begin{pmatrix} a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & \dots & a_{s-1} & a_s & -1 \end{pmatrix}.$$

Denote its i th column by c_i ($i = 1, \dots, k$). Similarly to the above case, we consider again the column sums.

If $s \geq 3$, or if $s = 2$ and a_1, a_2 are not both 1, then the first, the $(k - 2)$ nd, the $(k - 1)$ st and the k th column sums are $a_1 \geq 1$, $a_{s-1} + a_s - 1 \geq 1$, $a_s - 1 \geq 0$ and -1 , respectively. Finally, the i th column sum is at least 2 for $2 \leq i \leq k - 3$. The sum of these column sums can be 0 only in those cases when we choose columns

- c_{k-1} and $a_s = 1$,
- c_1, c_k and $a_1 = 1$,
- c_{k-2}, c_k and $a_{s-1} = a_s = 1$,
- c_{k-1}, c_k and $a_s = 2$,
- c_1, c_{k-1}, c_k and $a_1 = a_s = 1$,
- c_{k-2}, c_{k-1}, c_k and $a_{s-1} = a_s = 1$.

If $s = 2$ and $a_1 = a_2 = 1$, then the first, the second, the i th ($3 \leq i \leq k - 2$), the $(k - 1)$ st and the k th column sums are 1, 2, 1, 0 and -1 , respectively. Now the sum of these column sums can be 0 only if we choose c_{k-1} , or one of $c_1, c_3, c_4, \dots, c_{k-2}$ and c_k , or one of $c_1, c_3, c_4, \dots, c_{k-2}$ and c_{k-1} and c_k .

But none of these cases is suitable, because the sum of the chosen columns is never the zero vector. Thus it follows from Rado's theorem that our system of linear equations is not regular. □

In those cases, when $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ does not exist for all $r \geq 1$ by the above theorem, we give a concrete number of colours from which this van der Waerden type number does not exist.

Theorem 2.2. *Let $s \geq 2$ and a_1, \dots, a_s be positive integers. Further, let p be the smallest prime such that $a_1 + \dots + a_s + 1 \leq p$.*

1. *Suppose that $a_1, \dots, a_s \geq 2$. If $k \geq s + 1$ and $r \geq p - 1$, then $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ does not exist.*
2. *Suppose that $a_1 \geq 2$ or $a_s \geq 2$. If $k \geq s + 2$ and $r \geq p - 1$, then $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ does not exist.*
3. *If $k \geq s + 3$ and $r \geq p - 1$, then $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ does not exist.*

Proof. Throughout this proof consider the following $(p - 1)$ -colouring of positive integers: Every positive integer x can be uniquely written in the form $x = p^\alpha(p\beta + \gamma)$ where α, β are nonnegative integers and $\gamma \in \{1, \dots, p - 1\}$. Then let the colour of x be γ .

1. First we prove that there does not exist a monochromatic finite sequence of $\mathcal{R}_{a_1, \dots, a_s}$ of length $s + 1$, or in other words, a monochromatic solution of equation $a_1x_1 + \dots + a_sx_s = x_{s+1}$.

ON THE EXISTENCE OF VAN DER WAERDEN TYPE NUMBERS...

Indirectly suppose that $x_i = p^{\alpha_i}(p\beta_i + \gamma)$ ($i = 1, \dots, s+1$) are solutions of the equation with nonnegative integers α_i, β_i ($i = 1, \dots, s+1$) and $\gamma \in \{1, \dots, p-1\}$, that is

$$a_1 p^{\alpha_1}(p\beta_1 + \gamma) + \dots + a_s p^{\alpha_s}(p\beta_s + \gamma) = p^{\alpha_{s+1}}(p\beta_{s+1} + \gamma).$$

Let $\alpha_{\min} = \min\{\alpha_1, \dots, \alpha_s\}$ and $I = \{i \in \{1, \dots, s\} \mid \alpha_i = \alpha_{\min}\}$. Since $p^{\alpha_{\min}}$ divides the left-hand side of the equation, it also divides the right-hand side, hence $\alpha_{s+1} \geq \alpha_{\min}$.

After dividing by $p^{\alpha_{\min}}$ and reducing modulo p , we obtain $\sum_{i \in I} a_i \gamma \equiv 0$ or $\gamma \pmod{p}$ depending on whether $\alpha_{s+1} > \alpha_{\min}$ or $\alpha_{s+1} = \alpha_{\min}$. By the assumptions, we know that $2 \leq \sum_{i \in I} a_i \leq p-1$, therefore the previous congruence is equivalent to $\gamma \equiv 0 \pmod{p}$, which is a contradiction.

2. Now we show that there does not exist a monochromatic finite sequence of $\mathcal{R}_{a_1, \dots, a_s}$ of length $s+2$, that is the system of equations

$$\begin{aligned} a_1 x_1 + & & a_2 x_2 + & \dots + & & a_s x_s & = & x_{s+1} \\ a_1 a_s x_1 + & (a_1 + a_2 a_s) x_2 + & \dots + & (a_{s-1} + a_s^2) x_s & = & x_{s+2} \end{aligned}$$

has no monochromatic solution. Similarly to the proof of the first part, suppose that there exists a monochromatic solution and use the same notation. Then we have $\alpha_{s+1}, \alpha_{s+2} \geq \alpha_{\min}$.

If $I = \{i\}$ and $a_i \geq 2$ ($1 \leq i \leq s$), or $|I| \geq 2$, then the statement follows the same way as in the first part.

If $I = \{1\}$ and $a_1 = 1$, then dividing the second equation by $p^{\alpha_{\min}}$ and reducing modulo p give $a_s \gamma \equiv 0$ or $\gamma \pmod{p}$. This leads similarly to a contradiction since $2 \leq a_s \leq p-1$.

If $I = \{i\}$ ($2 \leq i \leq s$) and $a_i = 1$, then by the above argument we obtain from the second equation that $(a_{i-1} + a_s) \gamma \equiv 0$ or $\gamma \pmod{p}$, which implies again a contradiction because $2 \leq a_{i-1} + a_s \leq p-1$.

3. Finally, we prove that there does not exist a monochromatic finite sequence of $\mathcal{R}_{a_1, \dots, a_s}$ of length $s+3$, which means that the following system of equations has no monochromatic solution:

$$\begin{aligned} a_1 x_1 + & & a_2 x_2 + & & a_3 x_3 + & \dots + & & a_s x_s & = & x_{s+1} \\ a_1 a_s x_1 + & (a_1 + a_2 a_s) x_2 + & (a_2 + a_3 a_s) x_3 + & \dots + & (a_{s-1} + a_s^2) x_s & = & x_{s+2} \\ b_1 x_1 + & & b_2 x_2 + & & b_3 x_3 + & \dots + & & b_s x_s & = & x_{s+3} \end{aligned}$$

$$\text{(where } b_1 = a_1 a_{s-1} + a_1 a_s^2, \quad b_2 = a_2 a_{s-1} + a_1 a_s + a_2 a_s^2,$$

$$b_3 = a_1 + a_3 a_{s-1} + a_2 a_s + a_3 a_s^2, \dots, b_s = a_{s-2} + 2a_{s-1} a_s + a_s^3).$$

Suppose that we have a monochromatic solution and use the same notation as in the previous parts. Then we have again $\alpha_{s+1}, \alpha_{s+2}, \alpha_{s+3} \geq \alpha_{\min}$.

If $I = \{1\}$ and $a_1 \geq 2$, or $I = \{1\}$ and $a_s \geq 2$, or $I = \{i\}$ ($2 \leq i \leq s$), or $|I| \geq 2$, then the proof goes along the lines of the proof of the second part.

Moreover, if $I = \{1\}$ and $a_1 = a_s = 1$, then divide the third equation by $p^{\alpha_{\min}}$ and reduce it modulo p to get $(a_{s-1} + a_s) \gamma \equiv 0$ or $\gamma \pmod{p}$, which gives a contradiction since $2 \leq a_{s-1} + a_s \leq p-1$. \square

Remark 2.3. Apart from the cases $s = 2, a_1 = 1$, and $s = 2, a_2 = 1$, it could be shown that in the second and third parts of the theorem p can be chosen to be the smallest prime for which $a_1 + \dots + a_s \leq p$, because in the case of $a_1 + \dots + a_s = p$ and $I = \{1, \dots, s\}$ we can use the second equation instead of the first one.

If k is large enough, then a similar theorem can be proved for arbitrary number of colours.

Theorem 2.4. *Let $s \geq 2$ and a_1, \dots, a_s be positive integers. If $k \geq 2s + 1$ and $r \geq 2$, then $w(\mathcal{R}_{a_1, \dots, a_s}, k, r)$ does not exist.*

Proof. For $j \geq 0$, let $I_j = [(a_1 + \dots + a_s)^j, (a_1 + \dots + a_s)^{j+1} - 1] \cap \mathbb{Z}$. Then $\{I_j \mid j \geq 0\}$ is a partition of the positive integers. Colour the elements of I_j with 0 if j is even, and colour them with 1 if j is odd.

Indirectly suppose that there exists a sequence $(x_i)_{i=1}^\infty$ of $\mathcal{R}_{a_1, \dots, a_s}$ such that the first $2s + 1$ elements are monochromatic.

Let $m \in \{1, \dots, s\}$ be defined by $x_m = \max\{x_1, \dots, x_s\}$, and t be the exponent such that

$$(a_1 + \dots + a_s)^t \leq x_m < (a_1 + \dots + a_s)^{t+1}.$$

Then we can derive the inequalities

$$(a_1 + \dots + a_s)^t \leq x_m \leq a_m x_m \leq a_1 x_1 + \dots + a_s x_s = x_{s+1},$$

$$x_{s+1} = a_1 x_1 + \dots + a_s x_s \leq (a_1 + \dots + a_s) x_m < (a_1 + \dots + a_s)^{t+2}.$$

Since x_m and x_{s+1} share their colours, it follows that $(a_1 + \dots + a_s)^t \leq x_{s+1} < (a_1 + \dots + a_s)^{t+1}$. Now, from $x_2, \dots, x_s \leq x_m \leq x_{s+1}$ we obtain the inequalities

$$(a_1 + \dots + a_s)^t \leq x_{s+1} \leq a_s x_{s+1} \leq a_1 x_2 + \dots + a_s x_{s+1} = x_{s+2},$$

$$x_{s+2} = a_1 x_2 + \dots + a_s x_{s+1} \leq (a_1 + \dots + a_s) x_{s+1} < (a_1 + \dots + a_s)^{t+2}.$$

Since x_{s+1} and x_{s+2} have the same colour, we get $(a_1 + \dots + a_s)^t \leq x_{s+2} < (a_1 + \dots + a_s)^{t+1}$. Similarly, we can deduce that $x_{s+1} \leq x_{s+2} \leq \dots \leq x_{2s}$ and $(a_1 + \dots + a_s)^t \leq x_j < (a_1 + \dots + a_s)^{t+1}$ ($j = s + 1, \dots, 2s$).

Finally, the inequalities

$$(a_1 + \dots + a_s)^{t+1} \leq (a_1 + \dots + a_s) x_{s+1} \leq a_1 x_{s+1} + \dots + a_s x_{2s} = x_{2s+1},$$

$$x_{2s+1} = a_1 x_{s+1} + \dots + a_s x_{2s} \leq (a_1 + \dots + a_s) x_{2s} < (a_1 + \dots + a_s)^{t+2}$$

hold, whence $(a_1 + \dots + a_s)^{t+1} \leq x_{2s+1} < (a_1 + \dots + a_s)^{t+2}$, which means that the colour of x_{2s+1} differs from the colour of the previous elements. \square

Remark 2.5. Denote by $\tilde{\mathcal{R}}_{a_1, \dots, a_s}$ the set of those sequences $(x_i)_{i=1}^\infty$ of $\mathcal{R}_{a_1, \dots, a_s}$ for which $x_s = \max\{x_1, \dots, x_s\}$. For this family, a stronger result can be proved by the above argument: Let $s \geq 2$ and a_1, \dots, a_s be positive integers. If $k \geq 2s$ and $r \geq 2$, then $w(\tilde{\mathcal{R}}_{a_1, \dots, a_s}, k, r)$ does not exist.

3. MULTIBONACCI SEQUENCES

For $s \geq 2$, denote by $\mathcal{M}_s = \mathcal{R}_{1, \dots, 1}$ the family of multibonacci sequences of order s , that is the set of sequences $(x_i)_{i=1}^\infty$ containing positive integers and satisfying the linear recurrence $x_i = x_{i-s} + \dots + x_{i-1}$ ($i \geq s + 1$). For example, $\mathcal{M}_2 = \mathcal{F}$.

We notice that for $k = s + 1$, $w(\mathcal{M}_s, k, 2)$ exists and $w(\mathcal{M}_s, k, 2) = s^2 + s - 1$, which was proved by A. Beutelspacher and W. Brestovansky [3] before the general case in [4].

In addition, for $s \geq 2$, let $\tilde{\mathcal{M}}_s = \tilde{\mathcal{R}}_{1, \dots, 1}$, that is the family of sequences $(x_i)_{i=1}^\infty$ of \mathcal{M}_s for which $x_s = \max\{x_1, \dots, x_s\}$. Especially, $\tilde{\mathcal{M}}_2 = \tilde{\mathcal{F}}$.

Our Theorems 2.1, 2.2 (together with Remark 2.3), 2.4 and Remark 2.5 in the previous section give the following consequences for \mathcal{M}_s and $\tilde{\mathcal{M}}_s$.

Theorem 3.1.

1. Let $s \geq 2$ and $k \geq s + 1$. Then $w(\mathcal{M}_s, k, r)$ exists for all $r \geq 1$ if and only if $k = s + 1$ or $s + 2$.
2. Let $s \geq 3$ and p be the smallest prime for which $s \leq p$. If $k \geq s + 3$ and $r \geq p - 1$, then $w(\mathcal{M}_s, k, r)$ does not exist.
3. Let $s \geq 2$. If $k \geq 2s + 1$ and $r \geq 2$, then $w(\mathcal{M}_s, k, r)$ does not exist.
4. Let $s \geq 2$. If $k \geq 2s$ and $r \geq 2$, then $w(\widetilde{\mathcal{M}}_s, k, r)$ does not exist.

In our final remark, we make a few observations about the colouring used in the proof of Theorem 2.2 through multibonacci sequences.

Remark 3.2. Following the lines of the proof of Theorem 2.2, for \mathcal{M}_s the system of equations is

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_s &= x_{s+1} \\ x_1 + 2x_2 + 2x_3 + \cdots + 2x_s &= x_{s+2} \\ 2x_1 + 3x_2 + 4x_3 + \cdots + 4x_s &= x_{s+3} \end{aligned}$$

The second part of Theorem 3.1 is the best possible which can be achieved by this colouring in the following sense: We cannot choose a smaller prime p , since in case of $s = p + 1$ and $\alpha_1 = \cdots = \alpha_s$ we would obtain $(p + 1)\gamma \equiv 0$ or $\gamma \pmod{p}$, $(2p + 1)\gamma \equiv 0$ or $\gamma \pmod{p}$, $(4p + 1)\gamma \equiv 0$ or $\gamma \pmod{p}$ from the above equations, respectively, but these give no contradiction.

From a different point of view, we have used all the three equations to obtain the second part of Theorem 3.1. Now suppose that $s \geq 4$. If $I = \{2\}$, or $I = \{i\}$ ($3 \leq i \leq s$), or $s = p$ and $I = \{1, \dots, s\}$, then we could use the third equation instead of the second one to obtain $3\gamma \equiv 0$ or $\gamma \pmod{p}$, $4\gamma \equiv 0$ or $\gamma \pmod{p}$, $(4p - 3)\gamma \equiv 0$ or $\gamma \pmod{p}$, respectively. All of these cases result in a contradiction, since $p \geq 5$ when $s \geq 4$. It means that for $s \geq 4$, the second equation can be omitted, hence $x_1, \dots, x_s, x_{s+1}, x_{s+3}$ cannot be monochromatic under this colouring.

In a similar way, for $s = 2$, with the choice of $p = 3$, we can show that if $k \geq 5$ and $r \geq 2$, then $w(\mathcal{M}_2, k, r)$ does not exist. Moreover, we can omit the first equation, and x_1, x_2, x_4, x_5 cannot be monochromatic.

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