CHARACTERIZATIONS AND DECOMPOSITION OF STRONGLY WRIGHT-CONVEX FUNCTIONS OF HIGHER ORDER

Attila Gilányi, Nelson Merentes, Kazimierz Nikodem, and Zsolt Páles

Abstract. Motivated by results on strongly convex and strongly Jensen-convex functions by R. Ger and K. Nikodem in [Strongly convex functions of higher order, Nonlinear Anal. 74 (2011), 661–665] we investigate strongly Wright-convex functions of higher order and we prove decomposition and characterization theorems for them. Our decomposition theorem states that a function \( f \) is strongly Wright-convex of order \( n \) if and only if it is of the form \( f(x) = g(x) + p(x) + cx^{n+1} \), where \( g \) is a (continuous) \( n \)-convex function and \( p \) is a polynomial function of degree \( n \). This is a counterpart of Ng’s decomposition theorem for Wright-convex functions. We also characterize higher order strongly Wright-convex functions via generalized derivatives.

Keywords: generalized convex function, Wright-convex function of higher order, strongly convex function.

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1. INTRODUCTION

Let \( c \) be a positive constant and \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \to \mathbb{R} \) is called
- strongly convex with modulus \( c \) if
  \[
  f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)(x - y)^2
  \]  
  for all \( x, y \in I \) and \( t \in [0, 1] \);
- strongly Wright-convex with modulus \( c \) if
  \[
  f(tx + (1 - t)y) + f((1 - t)x + ty) \leq f(x) + f(y) - 2ct(1 - t)(x - y)^2
  \]  
  for all \( x, y \in I \) and \( t \in [0, 1] \);
– **strongly Jensen-convex with modulus** \(c\) if

\[
f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4} (x - y)^2 \quad (1.3)
\]

for all \(x, y \in I\).

Note that every strongly convex function is also strongly Wright-convex and every strongly Wright-convex function is also strongly Jensen-convex with the same modulus \(c\), but the converse statement is not true (cf. [14, Example 1.1]). Strongly convex functions were introduced in the paper [19] by B. T. Polyak and they play an important role in optimization theory and mathematical economics. Several results on their behaviour and applications can be found in the literature (cf., e.g., [4,8,13,15,19,22–24]). The concept of strongly Wright-convex functions was introduced by N. Merentes, K. Nikodem and S. Rivas in [14] (in connection with their study we also refer to [18]), while strongly Jensen-convex functions were considered, among others, in [1,4,17], and [24].

Obviously, the usual notion of convexity, Wright-convexity and Jensen-convexity can be obtained from the definition above in the case when \(c = 0\). In [16], C. T. Ng proved that each Wright-convex function \(f\) can be represented as the sum of a convex and an additive function (cf. also [9]). A decomposition of real valued strongly Wright-convex functions \(f\) defined on an interval \(I\) of the form \(f(x) = h(x) + a(x) + cx^2, (x \in I)\), where \(h\) is a convex function and \(a\) is an additive function, was obtained in [14]. The aim of this note is to generalize this result to strongly Wright-convex functions of higher order. We also present a characterization of strongly Wright-convex functions of higher order via generalized derivatives.

We note that, throughout this paper, all of our considerations remain valid in the case when the constant \(c\) is negative. Then the results so formulated concern higher order approximate convexity, Wright-convexity, and Jensen-convexity, respectively.

2. NOTATION, TERMINOLOGY AND BASIC PROPERTIES

First we recall and also introduce the basic definitions that we shall use throughout this paper. Let \(n\) be a positive integer and let \(I \subseteq \mathbb{R}\) be an interval.

The \(n^{th}\) **order divided difference** of a function \(f : I \to \mathbb{R}\) with respect to the pairwise distinct points \(x_0, \ldots, x_n \in I\) is defined by

\[
[x_0, \ldots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)}.
\]

(2.1)

It is easy to prove that they satisfy the recursivity property

\[
[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}
\]

(2.2)

for all positive integers \(n\) and \(x_0, \ldots, x_n \in I\), where \([x_0; f] = f(x_0)\).
According to E. Hopf ([7]) and T. Popoviciu ([20, 21]), a function $f : I \to \mathbb{R}$ is called convex of order $n - 1$ on $I$ (or monotone of order $n$) on $I$ if

$$[x_0, \ldots, x_n; f] \geq 0$$

holds for all $x_0 < \cdots < x_n \in I$. By the definition of R. Ger and K. Nikodem ([4]), if $c$ is a positive real number, a function $f : I \to \mathbb{R}$ is called strongly convex of order $n$ with modulus $c$ (or strongly $n$-convex with modulus $c$) if

$$[x_0, \ldots, x_{n+1}; f] \geq c$$

is valid for all $x_0 < \cdots < x_{n+1} \in I$.

The $\Delta_{h_1, \ldots, h_n}$ difference of $f : I \to \mathbb{R}$ with increments $h_1, \ldots, h_n$ is defined recursively by

$$\Delta_{h_1, \ldots, h_n} f(x) = f(x + h_1) - f(x)$$

for each $x \in I$ and $h_1, \ldots, h_n > 0$ such that $x + h_1 + \cdots + h_n \in I$. In the case when $h = h_1 = \cdots = h_n$, we also use the notation $\Delta_n^h$ instead of $\Delta_{h_1, \ldots, h_n}$.

Also based on Hopf’s ([7]) and Popoviciu’s ([20, 21]) definition, a function $f : I \to \mathbb{R}$ is said to be Jensen-convex of order $n$ (or $n$-Jensen-convex) if it satisfies the inequality

$$\Delta_{n+1}^h f(x) \geq 0$$

for all $x \in I$, $h > 0$ such that $x + (n + 1)h \in I$. If $c$ is a positive real number, $f$ is called strongly Jensen-convex of order $n$ with modulus $c$ (or strongly $n$-Jensen-convex with modulus $c$) if it fulfills

$$\Delta_{n+1}^h f(x) \geq c(n+1)!h^{n+1}$$

for all $x \in I$, $h > 0$ such that $x + (n + 1)h \in I$ (cf. [4]).

The function $f$ is said to be Wright-convex of order $n$ (or $n$-Wright-convex) if

$$\Delta_{h_1, \ldots, h_{n+1}} f(x) \geq 0$$

for all $x \in I$, $h_1, \ldots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$. We call $f$ strongly Wright-convex of order $n$ with modulus $c$ (or strongly $n$-Wright-convex with modulus $c$) if

$$\Delta_{h_1, \ldots, h_{n+1}} f(x) \geq c(n+1)!h_1 \cdots h_{n+1}$$

holds for all $x \in I$, $h_1, \ldots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$.

**Remark 2.1.** It is easy to see that the definitions of strongly $n$-convex functions, strongly $n$-Wright-convex functions and strongly $n$-Jensen-convex functions, with $c = 0$, give the concepts of $n$-convex, $n$-Wright-convex and $n$-Jensen-convex functions, respectively.
We will use the following property of the difference operator in the sequel.

**Lemma 2.2** ([6, Lemma 5.1]). Let $n$ be a positive integer, $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then the equation

$$
\Delta_{h_1, \ldots, h_n} f(x) = h_1 \cdots h_n \sum_{(i_1, \ldots, i_n)} [x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_n}; f]
$$

is valid for all $x \in I$, $h_1, \ldots, h_n > 0$ with $x + h_1 + \cdots + h_n \in I$, where the summation is for all permutations $(i_1, \ldots, i_n)$ of the integers $\{1, \ldots, n\}$.

**Remark 2.3.** It is a consequence of the statement above that every function $f : I \to \mathbb{R}$ which is strongly $n$-convex with modulus $c$, is also strongly $n$-Wright-convex with modulus $c$. Indeed, if $f$ is $n$-convex with modulus $c$, then

$$
[x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_{n+1}}; f] \geq c
$$

for all $x \in I$ and $h_1, \ldots, h_{n+1} > 0$, such that $x + h_1 + \cdots + h_{n+1} \in I$, where $(i_1, \ldots, i_{n})$ is an arbitrary permutation of the integers $\{1, \ldots, n\}$. By Lemma 2.2, we have

$$
\Delta_{h_1, \ldots, h_{n+1}} f(x) = h_1 \cdots h_{n+1} \sum_{(i_1, \ldots, i_{n+1})} [x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_{n+1}}; f]
\geq c(n+1)!h_1 \cdots h_{n+1},
$$

which means that $f$ is strongly $n$-Wright-convex with modulus $c$.

It is also easy to see that a strongly $n$-Wright-convex function with modulus $c$ is also $n$-Jensen-convex with modulus $c$.

**Remark 2.4.** In the case when $n = 1$, inequality (2.5) reduces to

$$
\Delta_{h_1, h_2} f(x) \geq 2ch_1 h_2,
$$

that is,

$$
f(x + h_1 + h_2) - f(x + h_1) - f(x + h_2) + f(x) \geq 2ch_1 h_2. \quad (2.6)
$$

Putting $u = x$, $v = x + h_1 + h_2$ and $t = \frac{h_2}{h_1 + h_2}$, we get $x + h_1 = tu + (1 - t)v$, $x + h_2 = (1 - t)u + tv$ and $h_1h_2 = t(1 - t)(u - v)^2$. Thus, property (2.6) gives

$$
f(tu + (1 - t)v) + f((1 - t)u + tv) \leq f(u) + f(v) + 2ct(1 - t)(u - v)^2,
$$

which means that $f$ is strongly Wright-convex with modulus $c$. Note that, if $n = 1$, also (2.3) and (2.4) reduces to (1.1) and (1.3), respectively.

3. MAIN RESULTS

Before formulating our main results, we present two lemmas. They can be proved by a simple calculation (cf. also [10] and [11, Chapter 15]).
Lemma 3.1. The operator $\Delta_{h_1,\ldots,h_n}$ is linear, that is, if $n$ is a positive integer, $h_1,\ldots,h_n$ and $a,b$ are real numbers, $I \subseteq \mathbb{R}$ is an interval and $f,g : I \to \mathbb{R}$ are arbitrary functions, then
\[
\Delta_{h_1,\ldots,h_n}(af + bg) = a\Delta_{h_1,\ldots,h_n}f + b\Delta_{h_1,\ldots,h_n}g.
\]

Lemma 3.2. Let $n$ be a positive integer and let $h_1,\ldots,h_n$ be real numbers. Then
\[
\Delta_{h_1,\ldots,h_n}x^n = n!h_1\cdots h_n.
\]

Now, we characterize higher order strongly Wright-convex functions via Wright-convex functions of higher order.

Theorem 3.3. Let $n$ be a positive integer, $c$ be a positive real number, and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is strongly $n$-Wright-convex with modulus $c$ if and only if the function $g : I \to \mathbb{R}$, $g(x) = f(x) - cx^{n+1}, (x \in I)$ is $n$-Wright-convex.

Proof. Suppose first that $f$ is strongly $n$-Wright-convex with modulus $c$ and let $g(x) = f(x) - cx^{n+1}$. Then, by Lemmas 3.1 and 3.2, we have
\[
\Delta_{h_1,\ldots,h_{n+1}}g(x) = \Delta_{h_1,\ldots,h_{n+1}}f(x) - \Delta_{h_1,\ldots,h_{n+1}}cx^{n+1}
\]
\[
\geq c(n+1)!h_1\cdots h_{n+1} - c(n+1)!h_1\cdots h_{n+1} = 0,
\]
which implies that $g$ is $n$-Wright-convex. Let us assume now that $g$ is $n$-Wright-convex. For $f(x) = g(x) + cx^{n+1}$, using Lemmas 3.1 and 3.2 again, we obtain
\[
\Delta_{h_1,\ldots,h_{n+1}}f(x) = \Delta_{h_1,\ldots,h_{n+1}}g(x) + \Delta_{h_1,\ldots,h_{n+1}}cx^{n+1}
\]
\[
\geq 0 + c(n+1)!h_1\cdots h_{n+1} = c(n+1)!h_1\cdots h_{n+1},
\]
which gives the strong $n$-Wright-convexity of $f$ with modulus $c$.

In the decomposition of $n$-Wright-convex and strongly $n$-Wright-convex functions, polynomial functions are used. A function $f : I \to \mathbb{R}$ is said to be a polynomial function of degree $n$ if it satisfies the equation
\[
\Delta_{h}^{n+1}f(x) = 0
\]
for all $x \in I$, $h > 0$ such that $x + (n+1)h \in I$.

The following generalization of Ng’s Theorem for Wright-convex functions of higher order was proved by Gy. Maksa and Zs. Páles.

Theorem 3.4 ([12]). Let $n$ be a positive integer and $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is $n$-Wright-convex if and only if it is of the form
\[
f(x) = h(x) + p(x) \quad (x \in I),
\]
where $h : I \to \mathbb{R}$ is an $n$-convex function and $p : \mathbb{R} \to \mathbb{R}$ is a polynomial function of degree $n$ with $p(\mathbb{Q}) = \{0\}$. Furthermore, the decomposition in (3.1) is unique.
The following theorem is a counterpart of the statement above for strongly Wright-convex functions of higher order. Note that the above result was proved for open intervals, therefore, the next result is stated also in this setting.

**Theorem 3.5.** Let \( n \) be a positive integer, \( c \) be a positive real number, and \( I \subseteq \mathbb{R} \) be an open interval. A function \( f : I \to \mathbb{R} \) is strongly \( n \)-Wright-convex with modulus \( c \) if and only if it is of the form

\[
f(x) = h(x) + p(x) + cx^{n+1} \quad (x \in I),
\]

where \( h : I \to \mathbb{R} \) is an \( n \)-convex function and \( p : \mathbb{R} \to \mathbb{R} \) is a polynomial function of degree \( n \) with \( p(0) = 0 \). Furthermore, the decomposition in (3.2) is unique.

**Proof.** The statement can be obtained as a combination of Theorems 3.4 and 3.3. \( \square \)

In the last part of the paper, we give a characterization of higher order Wright-convex functions via a generalized derivative introduced by Zs. Páles and A. Gilányi in [5].

If \( n \) is a positive integer, \( I \subseteq \mathbb{R} \) is an interval then the \( n \)th order lower generalized Dinghas interval derivative of a function \( f : I \to \mathbb{R} \) at a point \( \xi \in I \) is defined by

\[
D^n f(\xi) = \lim \inf_{(x \to \xi, h_1 \searrow 0, \ldots, h_n \searrow 0)} \frac{\Delta_{h_1, \ldots, h_n} f(x)}{h_1 \cdots h_n}.
\]

We note that the operator \( D^n \) is superlinear, i.e., superadditive and positively homogeneous.

If the limit exists, we call it the \( n \)th order generalised Dinghas interval derivative of \( f \) at \( \xi \) and we denote it by \( D^n f(\xi) \).

**Remark 3.6.** It is easy to see that, in the case when \( f \) is \( n \) times differentiable at \( \xi \), then \( D^n f(\xi) = f^{(n)}(\xi) \), that is, \( D^n \) is a generalized derivative. We also note that, in the case when \( h_1 = \cdots = h_n \) and the limit in (3.3) exists, the definition above gives the so called Dinghas interval derivative, introduced by A. Dinghas in [2] (cf. also [3,25] and [5]).

The following theorem is a simple consequence of Corollary 1 proved in [5].

**Theorem 3.7.** Let \( n \) be a positive integer and \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \to \mathbb{R} \) is \( n \)-Wright-convex on \( I \) if and only if

\[
D^{n+1} f(\xi) \geq 0
\]

for all \( \xi \in I \).
Finally, we present the characterization theorem for strongly \( n \)-Wright-convex functions via the generalized derivative above and we formulate its consequence for \( n + 1 \) times differentiable functions.

**Theorem 3.8.** Let \( n \) be a positive integer, \( c \) be a positive real number, and \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \to \mathbb{R} \) is strongly \( n \)-Wright-convex with modulus \( c \) if and only if
\[
D^{n+1} f(\xi) \geq c(n+1)!
\]
for all \( \xi \in I \).

**Proof.** Let first \( f \) be an \( n \)-Wright-convex function with modulus \( c \). Then, by theorem 3.3, the function \( g : I \to \mathbb{R}, g(x) = f(x) - cx^{n+1}, (x \in I) \) is \( n \)-Wright-convex. Using Theorem 3.7 and Lemmas 3.1 and 3.2, we obtain that
\[
D^{n+1} f(\xi) = D^{n+1} (g(\xi) + c\xi^{n+1}) \geq D^{n+1} g(\xi) + D^{n+1} c\xi^{n+1} \geq 0 + c(n+1)! = c(n+1)!
\]
for all \( \xi \in I \), which gives the first part of the statement. Assume now that \( f \) satisfies inequality (3.4) with a \( c > 0 \) for all \( \xi \in I \). Let us consider the function \( g : I \to \mathbb{R}, g(x) = f(x) - cx^{n+1}, (x \in I) \). It is easy to see that, by (3.4) and Lemmas 3.1 and 3.2,
\[
D^{n+1} g(\xi) = D^{n+1} (f(\xi) - c\xi^{n+1}) \geq D^{n+1} f(\xi) + D^{n+1} (-c\xi^{n+1}) \\
\geq c(n+1)! - c(n+1)! = 0
\]
for all \( \xi \in I \), which, combined with Theorem 3.7, implies our statement. \( \square \)

**Corollary 3.9.** Let \( n \) be a positive integer, \( c \) be a positive real number, \( I \subseteq \mathbb{R} \) be an interval, \( f : I \to \mathbb{R} \) be a function and suppose that \( f \) is \( n + 1 \) times differentiable on \( I \). Then \( f \) is strongly \( n \)-Wright-convex with modulus \( c \) if and only if \( f^{(n+1)}(\xi) \geq c(n+1)! \) for all \( \xi \in I \).

**Proof.** The statement is a consequence of Theorem 3.8 and Remark 3.6. \( \square \)

**Remark 3.10.** We note, that the corollary above can also be obtained as a consequence of a characterization of strong convex functions of higher order via derivatives given in Theorem 6 in [4], and the fact that in the case of continuous functions, the classes of \( n \)-Wright-convex functions and \( n \)-convex functions coincide.

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REFERENCES


Attila Gilányi
gilanyi@inf.unideb.hu

University of Debrecen
Faculty of Informatics
Pf. 12, 4010 Debrecen, Hungary

Nelson Merentes
nmerucv@gmail.com

Universidad Central de Venezuela
Escuela de Matemáticas
Caracas, Venezuela

Kazimierz Nikodem
knikodem@ath.bielsko.pl

University of Bielsko-Biała
Department of Mathematics and Computer Science
ul. Willowa 2, 43-309 Bielsko-Biała, Poland

Zsolt Páles
pales@science.unideb.hu

University of Debrecen
Institute of Mathematics
Pf. 12, 4010 Debrecen, Hungary
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