On some polynomial values of repdigit numbers

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Abstract

We study the equal values of repdigit numbers and the k dimensional polygonal numbers. We state some effective finiteness theorems, and for small parameter values we completely solve the corresponding equations.

Keywords: Polygonal numbers, repdigit numbers, elliptic equations.

1 Introduction

Let
$$f_{k,m}(x) = \frac{x(x+1)\cdots(x+k-2)((m-2)x+k+2-m)}{k!} \tag{1}$$
be the mth order k dimensional polygonal number, where $k \geq 2$ and $m \geq 3$ are fixed integers. As special cases for $f_{k,3}$ we get the binomial coefficient $\binom{x+k-1}{k}$, for $f_{2,m}(x)$ and $f_{3,m}(x)$ we have the corresponding polygonal and pyramidal numbers, respectively.
Another important class of combinatorial numbers is the numbers of the form $d \cdot \frac{10^n-1}{10-1}$, $1 \leq d \leq 9$. They are called repdigits and for $d = 1$, repunits.

In this extended abstract we study the equal values of repdigits and the k dimensional polygonal numbers. We state some effective finiteness theorems, and for small parameter values we completely solve the corresponding equations. The full paper based on the results mentioned below will be published in Periodica Mathematica Hungarica [19].

2 New results

A common generalization of repdigits and generalized repunits are numbers of the form
$$d \cdot \frac{b^n-1}{b-1},$$
i.e., taking repdigits with repeating digit d in the number system of base b, where $1 \leq d < b$ and $b \geq 2$ integers.

We consider equation
$$d \cdot \frac{b^n-1}{b-1} = f_{k,m}(x) \tag{2}$$

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and its special cases

$$d \cdot \frac{10^n - 1}{10 - 1} = f_{k,m}(x) \tag{3}$$

and

$$\frac{b^n - 1}{b - 1} = f_{k,m}(x). \tag{4}$$

In our first result we represent an effective finiteness statement concerning the most general equation (2).

Theorem 2.1 *Suppose that $k \geq 3$ or $k = 2$ and $m = 4$ or $m > 13$. Then equation (2) has only finitely many integer solutions in x and n, further,*

$$\max(|x|, n) < c,$$

where c is an effectively computable constant depending on k, m, b and d. For $k = 2$ and $m \in \{3, 5, 6, 7, 8, 9, 10, 11, 12\}$ equation (2) has infinitely many solutions for infinitely many values of the parameters b, d.

In the following two theorems we consider the special cases of equation (2) with repdigits or generalized repunits.

Theorem 2.2 *Equation (3) with $k \geq 2$ has only finitely many integer solutions n, x except for the values $(d, m) = (3, 8)$. In these cases the equation has infinitely many solutions that can be given explicitly.*

Theorem 2.3 *Equation (4) with $k \geq 2$ has only finitely many integer solutions n, x except for the values $(b, m) = (4, 8), (9, 3), (9, 6), (25, 5)$. In these cases the equation has infinitely many solutions that can be given explicitly.*

In our numerical investigations, we take those polynomials $f_{k,m}(x)$, where $k \in \{2, 3, 4\}$. For each of these cases, let $d \in \{1, 2, \ldots, 9\}$ and $m \in \{3, 4, \ldots, 20\}$ and we solve completely the corresponding equation. To state our numerical results, we need the following concept. A solution to equation (3) is called trivial if it yields $0 = 0$ or $1 = 1$. This concept is needed because of the huge number of trivial solutions; on the other hand, such solutions of (3) can be listed easily for any fixed k.

Theorem 2.4 *All nontrivial solutions of equation (3) in case of $k = 2, 3$, respectively, can be given. If $k = 4$ equation (3) has only trivial solutions.*

Remarks. In the original paper [19] all the solutions are determined. We considered some other related equations, corresponding to larger values of the parameter k of the polynomial $f_{k,m}(x)$, that lead to genus 2 equations.

However, because of certain technical difficulties, we could not solve them by the Chabauty method.

3 Sketch of the proofs

Lemma 3.1 *Let $f(X)$ be a polynomial with rational integer coefficients and with at least two distinct roots. Suppose $b \neq 0$, $m \geq 0$, x and y with $|y| > 1$ are rational integers satisfying*

$$f(x) = by^m.$$

Then m is bounded by a computable number depending only on b and f.

Proof. This is the main result of [22]. □

Proof of Theorem 2.1 Equation (2) is equivalent to

$$k!db^n = (b-1)x(x+1)\cdots(x+(k-2))((m-2)x+k+2-m)+dk!. \quad (5)$$

Let us assume first that $k \geq 4$. Our aim is to show that the polynomial on the right-hand side of (5) is never an almost perfect power. On supposing the contrary we have

$$(b-1)x(x+1)\cdots(x+(k-2))((m-2)x+k+2-m)+dk! = c(x-\alpha)^k, \quad (6)$$

with $c, \alpha \in \mathbb{Q}$. Substituting $x = 0, -1, -2$ in equation (6), we get that $\alpha = -1/2$ and on the other hand $\alpha = -1$, which is a contradiction. Therefore, our theorem follows from Lemma 3.1 for the case $k \geq 4$.

Now, let $k = 3$. Then equation (5) has the form

$$6db^n = (b-1)x(x+1)((m-2)x+5-m)+6d.$$

After carrying out the multiplications on the right-hand side we obtain that

$$6db^n = (b-1)(m-2)x^3+3(b-1)x^2+(b-1)(5-m)x+6d. \quad (7)$$

Let us again assume that the right-hand side is an almost perfect power, i.e., equals $c(x-\alpha)^3$, with $c, \alpha \in \mathbb{Q}$. On comparing the corresponding coefficients we have $\alpha = \dfrac{1}{2-m}$ and $\alpha = \dfrac{m-5}{3}$. This yields that $m \in \mathbb{C} \setminus \mathbb{R}$. Hence we derived a contradiction again. As in the previous case, Lemma 3.1 completes the proof for $k = 3$.

In the remaining case let $k = 2$. Then equation (5) has the form

$$2db^n = (b-1)x\left((m-2)x + 4 - m\right) + 2d. \tag{8}$$

If the right-hand side of (8) is an almost perfect square then

$$(b-1)(m-2)x^2 + (b-1)(4-m)x + 2d = cx^2 - 2cx\alpha + c\alpha^2$$

with rational c and α, further, on comparing the corresponding coefficients we get that $\alpha = \dfrac{4-m}{4-2m}$ and so $\dfrac{b-1}{d} = \dfrac{8(m-2)}{(4-m)^2} \geq 1$. This yields that $3 \leq m \leq 13$ integer and $m \neq 4$ and

$$\frac{b-1}{d} \in \left\{\frac{88}{81}, \frac{5}{4}, \frac{72}{49}, \frac{16}{9}, \frac{56}{25}, 3, \frac{40}{9}, 8, 24\right\}.$$

This is satisfied by infinitely many pairs b, d. Therefore for infinitely many parameter values b, d the right-hand side of equation (8) can be an almost perfect square which yields infinitely many integer solutions n, x of equation (2). Otherwise, Lemma 3.1 gives our statement for $k = 2$ and $m = 4$ or $m > 13$. $\square$

Proof of Theorem 2.2 For $k \geq 3$ the statement follows from Theorem 2.1. Now, let $k = 2$. By a similar argument as in the proof of Theorem 2.1, case $k = 2$, we obtain that

$$\frac{9}{d} = \frac{8(m-2)}{(4-m)^2} > 0.$$

Since d and m are integers, their only possible value is $(d, m) = (3, 8)$. Apart from this case the right-hand side of (8) cannot be a perfect square. Hence by Lemma 3.1 the theorem follows for $k = 2$. In addition, in the exceptional case the equation (5) has infinitely many integer solutions n, x. $\square$

Proof of Theorem 2.3 For $k \geq 3$ the statement follows from Theorem 2.1. In case of $k = 2$ a similar calculation has to be carried out as in the proof of Theorem 2.2. This yields the exceptional cases: $(b, m) = (4, 8), (9, 3), (9, 6), (25, 5)$. Showing that for these parameters the original equation has infinitely many solutions can be done similarly as in the previous proof. $\square$

Proof of Theorem 2.4 Let $k = 2$. Then $f_{2,m}(x) = \frac{(m-2)x^2+(4-m)x}{2}$. Since the right-hand side of equation (3) is of degree 2 by reducing the left-hand side to a polynomial of degree 3 we obtain an elliptic equation which can further be solved by the program package Magma [5].

Let $k = 3$. Then $f_{3,m}(x) = \frac{(m-2)x^3+3x^2+(5-m)x}{6}$. Since the right-hand side of equation (3) is of degree 3 by reducing the left-hand side to a polynomial of degree 2 we obtain an elliptic equation again which can be solved by Magma. Finally, let $k = 4$. Now

$$f_{4,m}(x) = \frac{(m-2)x^4 + 2mx^3 + (14-m)x^2 + (12-2m)x}{24}.$$

Since the right-hand side of equation (3) is of degree 4 by reducing the left-hand side to a polynomial of degree 2 we obtain a genus 1 equation which can be solved by Magma and some elementary considerations. $\square$

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