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PROBLEMS AND CONJECTURES AROUND SHIFT RADIX SYSTEMS

SHIGEKI AKIYAMA, HORST BRUNOTTE, ATTILA PETHŐ, WOLFGANG STEINER, AND JÖRG M. THUSWALDNER

Abstract. Some basic open problems and conjectures concerning shift radix systems are listed and their relations to well-known concepts and questions are outlined.

Original proposers of the open problem: Shigeki Akiyama, Horst Brunotte, Attila Pethő, Wolfgang Steiner and Jörg M. Thuswaldner.
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1. Introduction and basic definitions

In 2005 Akiyama et al. [2] introduced the notion of a shift radix system and pointed out connections of this simple dynamical system to well-known number systems such as *beta-numeration* and *canonical number systems*. Let us first recall the definitions (here for $y \in \mathbb{R}$ we denote by $\lfloor y \rfloor$ the largest $n \in \mathbb{Z}$ with $n \le y$).

Definition 1.1. *Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{R}^d$.*

(i) *The mapping $\tau_\mathbf{r} : \mathbb{Z}^d \to \mathbb{Z}^d$ given by*

$$\tau_\mathbf{r}(\mathbf{z}) = (z_1, \dots, z_{d-1}, -\lfloor \mathbf{r}\mathbf{z} \rfloor)^t \qquad (\mathbf{z} = (z_0, \dots, z_{d-1})^t \in \mathbb{Z}^d)$$

is called a shift radix system *(SRS for short), where we set $\mathbf{r}\mathbf{z} := r_0 z_0 + \cdots + r_{d-1} z_{d-1}$.*

(ii) *We say that $\tau_\mathbf{r}$ has the* finiteness property *if for each $\mathbf{z} \in \mathbb{Z}^d$ there is $k \in \mathbb{N}$ such that the k-fold iterate of the application of $\tau_\mathbf{r}$ to $\mathbf{z}$ satisfies $\tau_\mathbf{r}^k(\mathbf{z}) = \mathbf{0}$.*

This definition agrees with the one in [11], but the SRS in [2] coincide with our SRS with finiteness property. Our definition is equivalent to the property that $\tau_\mathbf{r}(z_0, \dots, z_{d-1}) = (z_1, \dots, z_{d-1}, z_d)^t$, where z_d is the unique integral solution of the linear inequality

$$0 \le r_0 z_0 + \cdots + r_{d-1} z_{d-1} + z_d < 1\,.$$

Therefore, we can write the mapping $\tau_\mathbf{r}$ as the sum of a linear function and a small error term. More explicitly, we have

$$\tau_\mathbf{r}(\mathbf{z}) = R_\mathbf{r}\,\mathbf{z} + \mathbf{v}(\mathbf{z}) \qquad (\mathbf{z} \in \mathbb{Z}^d),$$

where we put $\mathbf{v}(\mathbf{z}) := (0, \dots, 0, \mathbf{r}\mathbf{z} - \lfloor \mathbf{r}\mathbf{z} \rfloor)^t$ (in particular, $||\mathbf{v}(\mathbf{z})||_\infty < 1$) and $R_\mathbf{r}$ is a companion matrix of the polynomial

$$\chi_\mathbf{r}(X) := X^d + r_{d-1}X^{d-1} + \cdots + r_1 X + r_0 \qquad (\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{R}^d).$$

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Since $\chi_{\mathbf{r}}$ agrees with the characteristic polynomial of the linear recurrence

$$z_n = -r_{d-1}z_{n-1} - \cdots - r_0 z_{n-d},$$

SRS is viewed as an *almost linear recurrence.*

SRS and their relation to beta-numeration seem to have appeared for the first time in Hollander's PhD thesis [15] in 1996; two years earlier Vivaldi [26] had studied similar mappings in his detailed investigation of discretized rotations (see also Reeve-Black and Vivaldi [22]). The dynamical aspects of SRS are described in a broader context by Barat et al. [9].

The aim of the present paper is to provide a concise list of open problems and conjectures concerning SRS thereby extending work of the first [1] and the third authors [21, Section 6]. The reader is referred to [17] for detailed background information, illustrations and algorithms.

2. Problems and conjectures

The following sets play a central role in the study of various aspects of SRS.

Definition 2.1. *For $d \in \mathbb{N}$ set*

$$\mathcal{D}_d := \left\{\mathbf{r} \in \mathbb{R}^d \ : \ \forall \mathbf{z} \in \mathbb{Z}^d \, \exists k \in \mathbb{N} \, \exists \ell \in \mathbb{N}_{>0} : \tau_{\mathbf{r}}^k(\mathbf{z}) = \tau_{\mathbf{r}}^{k+\ell}(\mathbf{z})\right\} \quad \textit{and}$$
$$\mathcal{D}_d^{(0)} := \left\{\mathbf{r} \in \mathbb{R}^d \ : \ \tau_{\mathbf{r}} \textit{ is an SRS with finiteness property}\right\}.$$

Our fundamental open problem can roughly be described in the following way:

Problem 2.2. *Give a complete description of $\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$ for each $d \geq 2$.*

In the sequel we break up Problem 2.2 into several subproblems and conjectures. Computational experiments suggest the following (cf. [21]):

Problem 2.3. *Prove that $\mathbf{r} \in \mathcal{D}_d^{(0)} \cap \mathbb{Q}^d$ cannot be verified in polynomial time. Is it true that this problem does not belong to the NP complexity class?*

SRS are closely related to number systems:

- For an algebraic integer $\beta > 1$ the restriction of the *beta-transformation* T_β to $\mathbb{Z}[\beta]$ is conjugate to an SRS associated to a parameter defined by β.
- Akiyama et al. [5] investigated number systems with rational bases and established relations of these number systems to Mahler's $\frac{3}{2}$-problem (cf. [20]). These number systems can also be regarded as special cases of SRS (see Steiner and Thuswaldner [25]) and there seem to be relations between the $\frac{3}{2}$-problem and the length of SRS tiles (see below) associated with $\tau_{-2/3}$.
- The *backward division mapping* used to define *canonical number systems*[1] is conjugate to $\tau_{\mathbf{r}}$ for certain parameters $\mathbf{r}$; thus, characterizing all bases of canonical number systems is a special case of describing certain vectors $\mathbf{r} \in \mathbb{Q}^d$ giving rise to SRS with finiteness property (cf. Akiyama et al. [2]).

[1]In [16] the term "complete base" was coined.

Problem 2.4. *Characterize all parameters* $\mathbf{r}$ *for which the digits of the underlying number systems gives a language of a sofic shift (see* [18] *for the definition).*

The *Schur-Cohn region* $\mathcal{E}_d$ (see Schur [24]) is the set of all vectors $\mathbf{r} \in \mathbb{R}^d$ which define a contractive polynomial $\chi_{\mathbf{r}}$.

Conjecture 2.5. *We have* $\mathcal{D}_d^{(0)} \subset \mathcal{E}_d$.

This conjecture has only been settled for $d \in \{1, 2, 3\}$ (see [4, 14]).

Problem 2.6. *What is the measure of* $\mathcal{D}_d^{(0)}$*? What can we say about the topology of* $\mathcal{D}_d^{(0)}$*? What is the Hausdorff dimension of the boundary of* $\mathcal{D}_d^{(0)}$*?*

These problems have only been settled for $d = 1$.

Results on the topology of $\mathcal{D}_2^{(0)}$ are given by Weitzer [27], in particular, he showed that $\mathcal{D}_2^{(0)}$ is neither connected nor simply connected, and explicitly exhibited "holes" and components. It would be interesting to prove the following conjecture.

Conjecture 2.7. *The fundamental group of* $\mathcal{D}_3^{(0)}$ *is non-trivial, i.e., it has a handle.*

The following task was put forward and commented in [21].

Problem 2.8. *Given* $\mathbf{r} \in \mathcal{E}_d \cap \mathbb{Q}^d$*, is* $\mathbf{r} \in \mathcal{D}_d$ *algorithmically decidable?*

Periodicity of the orbits on $\partial\mathcal{D}_d$ is an open problem even in dimension 2. For instance, the special case $(1, \lambda) \in \partial\mathcal{D}_2$ can be expressed in the following simple arithmetical form:

Conjecture 2.9. *Let* $\lambda \in \mathbb{R}$ *be such that* $-2 < \lambda < 2$*. Further, let* $a_0, a_1 \in \mathbb{Z}$ *and define*

$$a_{n+1} := -a_{n-1} - \lfloor \lambda a_n \rfloor \qquad (n \geq 1).$$

Then the sequence $a_0, a_1, a_2, \ldots$ *is periodic.*

Some partial results have been obtained in [3] (see also [19, 6]). A weaker conjecture is that for any λ there exist infinitely many non-symmetric periodic orbits.

A remarkable conjecture of Schmidt [23] can be reformulated more generally in terms of the ultimate periodicity of $\tau_{\mathbf{r}}$, where $\mathbf{r}$ belongs to the hypersurface

$$E_d^{(\mathbb{C})} := \{\mathbf{r} \in \partial\mathcal{E}_d \ : \ R_{\mathbf{r}} \text{ has a non-real eigenvalue of modulus } 1\}.$$

Problem 2.10. *For which* $\mathbf{r} \in E_d^{(\mathbb{C})}$ *is each orbit of* $\tau_{\mathbf{r}}$ *ultimately periodic, in particular the orbit of* $(1, 0, \ldots, 0)^t$ *?*

For particular orbits addressed in Problem 2.10 we suspect the following more explicit result.

Conjecture 2.11. *Let* $\mathbf{r} \in E_d^{(\mathbb{C})}$ *be such that* $\chi_{\mathbf{r}}$ *is irreducible. Let* s *be the number of pairs of complex conjugate roots* $(\alpha, \bar{\alpha})$ *of* $\chi_{\mathbf{r}}$ *with* $|\alpha| = 1$*. Then every orbit of* $(1, 0, \ldots, 0)^t$ *under* $\tau_{\mathbf{r}}$ *is periodic if* $s \in \{1, 2\}$*, and there exist* $\mathbf{r}$ *such that the orbit of* $(1, 0, \ldots, 0)^t$ *is aperiodic if* $s \geq 3$ *(see also* [12], [13]*).*

SRS also admit a geometric theory, in particular, it is possible to define so-called *SRS tiles* (see Berthé et al. [11]). In the sequel we assume $r_0 \neq 0$.

Definition 2.12. *Let* $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathcal{E}_d$ *and* $\mathbf{z} \in \mathbb{Z}^d$ *be given. The set*

$$\mathcal{T}_{\mathbf{r}}(\mathbf{z}) = \operatorname*{Lim}_{n \to \infty} R_{\mathbf{r}}^n \, \tau_{\mathbf{r}}^{-n}(\{\mathbf{z}\})$$

is called the SRS tile associated with $\mathbf{r}$*; here the limit is taken with respect to the Hausdorff metric.*

A tiling is a collection of compact sets covering $\mathbb{R}^d$ with zero measure overlaps (see [25]). SRS tiles are conjectured to induce tilings of their representation spaces. A special case of this conjecture implies the *Pisot substitution conjectures* (see e.g. Arnoux and Ito [7], Baker-Barge-Kwapisz [8], Barge [10]) for Pisot beta substitutions. We present two challenging tasks.

Conjecture 2.13. *Let* $\mathbf{r} \in \mathcal{E}_d$*. Then* $\{\mathcal{T}_{\mathbf{r}}(\mathbf{z}) \,:\, \mathbf{z} \in \mathbb{Z}^d\}$ *is a tiling of* $\mathbb{R}^d$*.*

Problem 2.14. *Let* $\mathbf{r} \in \mathcal{E}_d$ *and* $\mathbf{z} \in \mathbb{Z}^d$*. Give criteria for* $\mathcal{T}_{\mathbf{r}}(\mathbf{z})$ *being the closure of its interior.*

Finally, we state a problem related to the connectivity of the so-called central SRS tiles $\mathcal{T}_{\mathbf{r}}(\mathbf{0})$.

Problem 2.15. *Describe the Mandelbrot sets*

$$\{\mathbf{r} \in \mathcal{E}_d \,:\, \mathcal{T}_{\mathbf{r}}(\mathbf{0}) \text{ is connected}\}$$

for $d \geq 2$*.*

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Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki, JAPAN (zip:305-8571)

E-mail address: akiyama@math.tsukuba.ac.jp

Haus-Endt-Strasse 88, D-40593 Düsseldorf, GERMANY

E-mail address: brunoth@web.de

Faculty of Informatics, University of Debrecen, P.O. Box 12, H-4010 Debrecen, HUNGARY

E-mail address: pethoe@inf.unideb.hu

LIAFA, CNRS UMR 7089, Université Paris Diderot – Paris 7, Case 7014, 75205 Paris Cedex 13, FRANCE

E-mail address: steiner@liafa.jussieu.fr

Chair of Mathematics and Statistics, University of Leoben, A-8700 Leoben, AUSTRIA

E-mail address: joerg.thuswaldner@unileoben.ac.at