Egyetemi doktori (PhD) értekezés

Diophantine equations with separable variables

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Természettudományi Doktori Tanács
Matematika- és Számítástudományok Doktori Iskola

Debrecen, 2016

Ezen értekezést a Debreceni Egyetem Természettudományi Doktori Tanács Matematika- és Számítástudományok Doktori Iskola *Diofantikus és konstruktív számelmélet* programja keretében készítettem a Debreceni Egyetem természettudományi doktori (PhD) fokozatának elnyerése céljából.

Debrecen, 2016. március 10.

Péter Gyöngyvér
jelölt

Tanúsítom, hogy Péter Gyöngyvér doktorjelölt 2009-2013 között a fent megnevezett Doktori Iskola *Diofantikus és konstruktív számelmélet* programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javasolom.

Debrecen, 2016. március 10.

Pintér Ákos
témavezető

Diophantine equations with separated variables

értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
matematika tudományágban

Írta: Péter Gyöngyvér okleveles matematikus

Készült a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskolája (Diofantikus és konstruktív számelmélet programja) keretében.

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A doktori szigorlati bizottság:

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A doktori szigorlat időpontja: 2014. szeptember 26.

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Köszönetnyilvánítás

Ezúton szeretném megköszönni témavezetőmnek, Pintér Ákosnak, akinek legfőképp szakmai fejlődésemet köszönhetem, segítőkészségét és a dolgozatom elkészítése alatt nyújtott állandó támogatását, továbbá Győry Kálmán professzor úrnak a biztatást és támogatást, amivel elősegítette munkám sikerét.

Szeretnék továbbá köszönetet mondani Tanáraimnak mindazért a tudásért, amit tőlük kaptam és az Algebra és Számelmélet Tanszék minden munkatársának a tőlük kapott szakmai segítségért és támogatásért, hozzájuk bármikor fordulhattam munkám során.

Köszönöm édesapámnak a sok támogatást és kisfiamnak, hogy segített megtanítani arra, ami fontos a hivatásom terén is.

Contents

Chapter 1

Introduction

The question of the Diophantine equations can be found everywhere across mathematical history. Already the ancient Babylonians could solve simple equations and systems of equations in integers. Mostly because of philosophical backgrounds integer numbers and the Diophantine equations were in the center of interest and therefore flourishing in Ancient Greece with many classical results. Also in the other great civilizations like China, India and the Islam world the questions of Diophantine problems remained in interest without any tools for solving general Diophantine equations.

This is Hilbert's tenth problem (from 1900), namely the determination of the solvability of a Diophantine equation. Whether there exists a general finite algorithm for any Diophantine equation with any number of variables which determines whether the equation has rational integer solutions. In 1970 Matijasevic [59] proved that such an algorithm does not exist.

As a result, since there is no universal algorithm, methods for solving certain classes of Diophantine equations gained big interest since then.

Such an efficient tool for solving different types of Diophantine equations is the Baker method, based on Baker's inequality giving a non-trivial lower bound for linear logarithmic forms. Baker gained his famous result in 1966. Based on this result since then also several further improvements and applications were born. For further details in this topic we refer to [3], [5], [6] and [7].

Next we introduce a sharper version of the original theorem, the Baker-Wüstholz Theorem [9].

Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers, not 0 or 1, and by $\log \alpha_1, \dots,$ $\log \alpha_n$ we mean fixed determinations of the logarithms. Let K be the field generated by $\alpha_1, \dots, \alpha_n$ over the rationals $\mathbb{Q}$ and let d be the degree of K.

Set $A_j = \max\left(H(\alpha_j), e\right)$, where $H(\alpha_j)$ denotes the classical height of α_j, i. e. the maximum of the absolute values of the coefficients of the minimal defining polynomial of α_j and $e = 2,718....$

Theorem A *Let $b_1, \dots, b_n$ be rational integers, not all 0 and suppose that $B \geq \max |b_j|$. If $\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + \dots + b_k \log \alpha_k \neq 0$ then*

$$\log |\Lambda| > -(16nd)^{2(n+2)} \log A_1 \dots \log A_n \log B.$$

In the following we present some applications of the Baker method for special families of equations.

Now suppose that $f(X, Y)$ is a binary form with rational integer coefficients and with at least three pairwise non-proportional linear factors in its factorisation over $\mathbb{C}$. Let k be a non-zero rational integer. We consider the solutions of the equation

$$f(x, y) = k, \tag{1.1}$$

called the Thue equation in rational integers x and y. Thue proved an ineffective finiteness theorem on equation 1.1 however, by the Baker method we get an effective result.

Theorem B *If x and y are rational integers satisfying equation 1.1, then*

$$\max(|x|,|y|) \leq C_1|k|^{C_2}$$

for some computable numbers C_1 and C_2 depending only on f.

See [77, Chapter 5].

Two other important types of equations are the hyper- and superelliptic equations in integers x and y. Let

$$f(x) = by^m \tag{1.2}$$

where $f \in \mathbb{Z}[x]$, $\deg f \geq 2$ and $m \geq 2$ fixed and further $b \in \mathbb{Z}$, $b \neq 0$. The equation is called hyperelliptic in case $m = 2$, and called superelliptic in case $m \geq 3$. Applying his method Baker reached the following results in some special cases.

Theorem C *Let $m \geq 3$. Suppose that $f(X)$ has at least two simple roots. If x and y are rational integers satisfying equation 1.2, then*

$$\max(|x|,|y|) \leq C_3$$

for some computable C_3 depending only on b, m and f.

See [4].

Theorem D *Suppose that $m = 2$ and $f(X)$ has at least three simple roots. Then all the solutions of equation 1.2 in rational integers x and y satisfy*

$$\max(|x|,|y|) \leq C_4$$

where C_4 is a computable number depending only on b and f.

See [4].

In their theorem Schinzel and Tijdeman used the Baker method to get an effective result for the exponent as a variable.

Theorem E *Let $f(X)$ be a polynomial with rational integer coefficients and with at least two distinct roots. Suppose $b \neq 0$, $m \geq 0$, x and y with $|y| > 1$ are rational integers satisfying*

$$f(x) = by^m$$

Then m is bounded by a computable number depending only on b and f.

This is the main result of [76].

Let us have the form of the equation where the polynomial $f(x)$ has the factorisation $f(x) = (x - \alpha_1)^{r_1} \cdots (x - \alpha_n)^{r_n}$.

Also based on the Baker method the last result here in the theory of superelliptic equations is the following theorem of Brindza. He also gave in addition a quantitative version of this theorem.

Theorem F *Let $m \geq 2$ and $n \geq 2$. Put*

$$q_i = \frac{m}{(m, r_i)} \quad (i = 1, \ldots, n).$$

Suppose that $(q_1, \ldots, q_n)$ is not a permutation of either of the n-tuples $(q, 1, 1, \ldots, 1)$, $t \in \mathbb{N}$ or $(2, 2, 1, 1, \ldots, 1)$. Let x and y rational inters satisfy 1.2. There exists a computable number C_5 depending only on b, m and f such that

$$\max(|x|, |y|) \leq C_5.$$

See [77, Chapter 8].

There are several other effective results based on different results from the Baker Theorem.

The generalisation of the Thue equation is the Thue-Mahler equation. Based on the result of van der Poorten and Yu, see [77] on the p-adic analogue of the inequality of Baker, there exists also for this equation an effective theorem.

Let $f(X,Y)$ be a binary form of degree n with rational integer coefficients and with at least three pairwise non-proportional linear factors in its factorisation over $\mathbb{Q}$.

The upper bound here is due to Győry.

Theorem G *Let k and s be rational integers with $k \neq 0$ and $s > 0$. Let $p_1, \ldots, p_s$ be primes with $p_1 < p_2 < \cdots < p_s =: P$. All solutions of the equation*

$$f(x,y) = kp_1^{z_1} \cdots p_s^{z_s} \qquad \text{in } x, y, z_1, \ldots, z_s \in \mathbb{Z}$$

with $(x,y) = 1$ and $z_1 \geq 0, \ldots, z_s \geq 0$, satisfy

$$\max(|x|, |y|, z_j) \leq C_6 \quad (1 \leq j \leq s)$$

where C_6 is a computable number depending only on f, k, n, s and P.

See [77, Chapter 7].

The previous theorem implies the following application.

Theorem H *Let $f(X,Y)$ and $g(X,Y)$ be binary forms with rational integer coefficients. Suppose f has at least three pairwise non-proportional linear factors in its factorisation over $\mathbb{C}$ which do not divide g over $\mathbb{Q}$. Suppose* $\deg(f) > \deg(g)$. *Then all solutions of the equation*

$$f(x,y) = g(x,y) \qquad \text{in rational integers } x, y$$

with $f(x,y) \neq 0$ are such that $\max(|x|,|y|)$ *is bounded by a computable number depending only on f and g.*

See [77, Chapter 7].

For several classes of separable Diophantine equations of the form $f(x) = g(y)$ with $f, g \in \mathbb{Z}[x]$ in x, y rational integers such *effective* results do not exist. For the very specific superelliptic equations we have such result as seen above.

However there are theorems which present general results for the equations of the type $f(x) = g(y)$. There are two key results concerning separable Diophantine equations. The first one is due to Davenport, Lewis and Schinzel [29].

Theorem I *Let $f(x)$ be a polynomial with integral coefficients of degree $n > 1$ and $g(y)$ be a polynomial with integral coefficients of degree $m > 1$. Let $D(\lambda) = \mathrm{disc}(f(x) + \lambda)$ and $E(\lambda) = \mathrm{disc}(g(y) + \lambda)$. Suppose that there are at least $\lceil n/2 \rceil$ distinct roots of $D(\lambda) = 0$ for which $E(\lambda) \neq 0$. Then $f(x) - g(y)$ is irreducible over the complex numbers. Further, the genus of the equation $f(x) - g(y) = 0$ is strictly positive except possibly when $m = 2$ or $m = n = 3$. Apart from these possible exceptions, the equation has at most a finite number of integral solutions.*

The last part of the theorem is based on Siegels famous result about the number of integral points on irreducible algebraic curves. The Bilu-Tichy Theorem [16] is an improvement of the previous theorem.

To formulate the theorem, we define *five kinds of standard pairs of polynomials.*

In the sequel α and β denote non-zero rational numbers, q, s and t are positive integers, r is a non-negative integer and $v(X) \in \mathbb{Q}[X]$ is a non-zero polynomial, which may be constant.

A *standard pair of the first kind* is

$$(X^q, \alpha X^r v(X)^q), \text{ or switched, } (\alpha X^r v(X)^q, X^q)$$

where $0 \leq r < q$, $(r, q) = 1$ and $r + \deg v(X) > 0$.

A *standard pair of the second kind* is

$$(X^2, (\alpha X^2 + \beta)\nu(X)^2) \text{ (or switched)}.$$

Denote by $D_s(X, \alpha)$ the sth Dickson polynomial, defined by, for example, the explicit formula

$$D_s(X, \alpha) = \sum_{i=0}^{[s/2]} \frac{s}{s-i} \binom{s-i}{i} (-\alpha)^i X^{s-2i}.$$

A *standard pair of the third kind* is

$$\bigl(D_s(X, \alpha^t), D_t(X, \alpha^s)\bigr)$$

where $\gcd(s, t) = 1$.

A *standard pair of the fourth kind* is

$$\left(\alpha^{-s/2} D_s(X, \alpha), -\beta^{-t/2} D_t(X, \beta)\right),$$

where $\gcd(s, t) = 2$.

A *standard pair of the fifth kind* is

$$\bigl((\alpha X^2 - 1)^3, 3X^4 - 4X^3\bigr) \text{ (or switched)}.$$

Theorem J *Let $P(X)$, $Q(X) \in \mathbb{Q}[X]$ be non-constant polynomials such that the equation $P(x) = Q(y)$ has infinitely many solutions x, y with a bounded denominator. Then we have $P = \phi \circ f \circ \lambda$ and $Q = \phi \circ g \circ \mu$, where $\lambda(X)$, $\mu(X) \in \mathbb{Q}[X]$ are linear polynomials, $\phi(X) \in \mathbb{Q}[X]$ and $(f(X), g(X))$ is a standard pair.*

This result relies on Siegel's theorem.

In our dissertation we are going to investigate some specific types of separable Diophantine equations. In our research we were focusing to find and give ineffective and effective theorems on certain classes of separable Diophantine equations based on classical results like the Bilu-Tichy Theorem or the Baker Theorem. Our goal was also to be able to give ineffective general results, effective results for specific classes just as to give numerical results for known parameters.

First we investigate the arithmetic properties of repdigit numbers. Namely we study the equal values of repdigit and lth order k dimensional polygonal numbers.

Second we are going to examine the question whether one can give general conditions for two trinomials of the form $ax^m + bx^n + c = dy^p + ey^q$ to have infinitely many equal values.

Last we are going to deal with the question of separable Diophantine equations of discrete geometrical background. Namely we are going to investigate the equal values of standard counting polynomials i.e. for m, n positive integers the equally many integer points of an m-dimensional and an n-dimensional unit cube, simplex, pyramid or octahedron.

Chapter 2

On some polynomial values of repdigit numbers

2.1 Introduction

Let

$$f_{k,l}(x) = \frac{x(x+1)\cdots(x+k-2)((l-2)x+k+2-l)}{k!} \tag{2.1}$$

be the lth order k dimensional polygonal number, where $k \geq 2$ and $l \geq 3$ are fixed integers. As special cases for $f_{k,3}(x)$ we get the binomial coefficient $\binom{x+k-1}{k}$, for $f_{2,l}(x)$ and $f_{3,l}(x)$ we have the corresponding polygonal and pyramidal numbers, respectively. These figural numbers have already been investigated from several aspects and therefore have a rich literature, see Dickson [30]. For example, the question whether a perfect square is a binomial coefficient, i.e., if $f_{k,3}(x) = f_{2,4}(y)$ and also the more general question on the power values of binomial coefficients was resolved by Győry [38]. The equation $\binom{x}{n} = \binom{y}{2}$ has been investigated by several authors, for general effective finiteness statements we refer to Kiss [54] and Brindza [18]. In the special cases $l = 3, 4, 5$ and 6, the corresponding Diophantine equations were resolved by Avanesov [2], Pintér [66] and de Weger [86] (independently), Bugeaud, Mignotte, Stoll, Siksek, Tengely [25] and Hajdu, Pintér [47], respectively. The equal values of polygonal and pyramidal numbers were studied by Brindza, Pintér, Turjányi [23] and Pintér, Varga [69].

Another important class of combinatorial numbers is the numbers of the form $d \cdot \frac{10^n-1}{10-1}$, $1 \leq d \leq 9$. They are called repdigits and for $d = 1$, repunits. Various results and conjectures have been stated concerning prime repunits and certain Diophantine problems related to repdigits, see [35] and [77, Chapter 12], respectively. For example, Ballew and Weger [10] proved earlier that there are only six numbers, namely $1, 3, 6, 55, 66, 666$ that are both triangular and repdigit numbers. Recently, Jaroma [50] gave an elementary proof of the fact that 1 is the only triangular repunit number. Keith [52] investigated the problem to determine which polygonal numbers are repdigits and solved it for numbers less than 10^7. He also introduced an efficient algorithm for finding repdigit polygonal numbers and gave a complete characterization of all such numbers up to 50 digits.

One can also define the so-called generalized repunits with the formula

$$\frac{b^n - 1}{b - 1} \tag{2.2}$$

for an integer $b \geq 2$. Dubner [31] gave a table of generalized repunit primes and probable primes for b up to 99 and for large values of n.

In our work we study the equal values of repdigits and the k dimensional polygonal numbers. We state some effective finiteness theorems, and for small parameter values we completely solve the corresponding equations.

2.2 New results

A common generalization of repdigits and generalized repunits are numbers of the form

$$d \cdot \frac{b^n - 1}{b - 1}, \tag{2.3}$$

i.e., taking repdigits with repeating digit d in the number system of base b, where $1 \leq d < b$ and $b \geq 2$ integers.

We consider equation

$$d \cdot \frac{b^n - 1}{b - 1} = f_{k,l}(x) \tag{2.4}$$

and its special cases

$$d \cdot \frac{10^n - 1}{10 - 1} = f_{k,l}(x) \tag{2.5}$$

and

$$\frac{b^n - 1}{b - 1} = f_{k,l}(x). \tag{2.6}$$

In our first result we represent an effective finiteness statement concerning the most general equation 2.4.

Theorem 2.1 *Suppose that $k \geq 3$ or $k = 2$ and $l = 4$ or $l > 13$. Then equation 2.4 has only finitely many integer solutions in x and n, further,*

$$\max(|x|, n) < c_1,$$

where c_1 is an effectively computable constant depending on k, l, b and d. For $k = 2$ and $l \in \{3, 5, 6, 7, 8, 9, 10, 11, 12\}$ equation 2.4 has infinitely many solutions for infinitely many values of the parameters b, d.

In the following two theorems we consider the special cases of equation 2.4 with repdigits or generalized repunits.

Theorem 2.2 *Equation 2.5 with $k \geq 2$ has only finitely many integer solutions x, n except for the values $(d, l) = (3, 8)$. In these cases the equation has infinitely many solutions that can be given explicitly.*

Theorem 2.3 *Equation 2.6 with $k \geq 2$ has only finitely many integer solutions x, n except for the values $(b, l) = (4, 8), (9, 3), (9, 6), (25, 5)$. In these cases the equation has infinitely many solutions that can be given explicitly.*

In our numerical investigations we take those polynomials $f_{k,l}(x)$, where $k \in \{2, 3\}$. For both cases we let $d \in \{1, 2, \ldots, 9\}$ and $l \in \{3, 4, \ldots, 20\}$ and solve completely the corresponding equation. To state our numerical results, we need the following concept. A solution to equation 2.5 is called trivial if it yields $0 = 0$ or $1 = 1$. This concept is needed because of the huge number of trivial solutions; on the other hand, such solutions of 2.5 can be listed easily for any fixed k.

Theorem 2.4 *All nontrivial solutions of equation 2.5 in case of $k = 2, 3$, respectively, are exactly those contained in Tables 2.2 and 2.1 respectively.*

Remark. We considered some other related equations, corresponding to larger values of the parameter k of the polynomial $f_{k,l}(x)$, that lead to genus 2 equations. However, because of certain technical difficulties, we could not solve them by the Chabauty method.

2.3 Proofs of the Theorems

Our proofs are based on the previously introduced theorem of Schinzel and Tijdeman.

Lemma 2.1 *Let $f(X)$ be a polynomial with rational integer coefficients and with at least two distinct roots. Suppose $b \neq 0$, $m \geq 0$, x and y with*

$|y| > 1$ *are rational integers satisfying*

$$f(x) = by^m.$$

Then m is bounded by a computable number depending only on b and f.

Proof of Theorem 2.1. Equation 2.4 is equivalent to

$$k!db^n = (b-1)x(x+1)\cdots(x+(k-2))((l-2)x+k+2-l)+dk!. \quad (2.7)$$

Let us assume first that $k \geq 4$. Our aim is to show that the polynomial on the right-hand side of 2.7 is never an almost perfect power. On supposing the contrary we have

$$(b-1)x(x+1)\cdots(x+(k-2))((l-2)x+k+2-l)+dk! = c(x-\alpha)^k, \quad (2.8)$$

with $c, \alpha \in \mathbb{Q}$. Substituting $x = 0, -1, -2$ in equation 2.8, we obtain the equalities

$$dk! = c(-\alpha)^k, \quad (2.9)$$

$$dk! = c(-1-\alpha)^k, \quad (2.10)$$

$$dk! = c(-2-\alpha)^k. \quad (2.11)$$

From 2.9 and 2.10 we get that

$$c(-\alpha)^k = c(-1-\alpha)^k,$$

which yields that

$$\left(\frac{1+\alpha}{\alpha}\right)^k = 1.$$

Therefore $(1+\alpha)/\alpha$ is a rational root of unity, i.e., ± 1 which means that $\alpha = -1/2$. On the other hand, considering 2.9 and 2.11, we obtain that

$$c(-\alpha)^k = c(-2-\alpha)^k.$$

Following a similar calculation we get that $\alpha = -1$, which is a contradiction. Therefore, our theorem follows from Lemma 2.1 for the case $k \geq 4$.

Now, let $k = 3$. Then equation 2.7 has the form

$$6db^n = (b-1)x(x+1)((l-2)x+5-l)+6d.$$

After carrying out the multiplications on the right-hand side we obtain that

$$6db^n = (b-1)(-2)x^3 + 3(b-1)x^2 + (b-1)(5-l)x + 6d. \qquad (2.12)$$

Let us again assume that the right-hand side is an almost perfect power, i.e., equals $c(x-\alpha)^3$, with $c, \alpha \in \mathbb{Q}$. Then the original coefficients have the form

$$(b-1)(l-2) = c, \qquad 3(b-1) = -3c\alpha,$$
$$(b-1)(5-l) = 3c\alpha^2, \qquad 6d = -c\alpha^3.$$

From the first and second equation we get that $\alpha = \frac{1}{2-l}$. At the same time from the second and third equation we get that $\alpha = \frac{l-5}{3}$. This yields that $l \in \mathbb{C}\setminus\mathbb{R}$. Hence we derived a contradiction again. As in the previous case, Lemma 2.1 completes the proof for $k = 3$.

In the remaining case let $k = 2$. Then equation 2.7 has the form

$$2db^n = (b-1)x\left((l-2)x+4-l\right)+2d. \qquad (2.13)$$

If the right-hand side of 2.13 is an almost perfect square then

$$(b-1)(l-2)x^2 + (b-1)(4-l)x + 2d = cx^2 - 2cx\alpha + c\alpha^2$$

with rational c and α, further, on comparing the corresponding coefficients we have

$$(b-1)(l-2)=c, \quad (b-1)(4-l)=-2c\alpha, \quad 2d=c\alpha^2.$$

Hence we get that $\alpha=\frac{4-l}{4-2l}$ and so $\frac{b-1}{d}=\frac{8(l-2)}{(4-l)^2}\geq 1$. This yields that $3\leq l\leq 13$ integer and $l\neq 4$ and

$$\frac{b-1}{d}\in\left\{\frac{88}{81},\frac{5}{4},\frac{72}{49},\frac{16}{9},\frac{56}{25},3,\frac{40}{9},8,24\right\}.$$

This is satisfied by infinitely many pairs b,d. Therefore for infinitely many parameter values b,d the right-hand side of equation 2.13 can be an almost perfect square which yields infinitely many integer solutions n,x of equation 2.4. Otherwise, Lemma 2.1 gives our statement for $k=2$ and $l=4$ or $l>13$.

Proof of Theorem 2.2. For $k\geq 3$ the statement follows from Theorem 2.1. Now, let $k=2$. By a similar argument as in the proof of Theorem 2.1, case $k=2$, we obtain that

$$\frac{9}{d}=\frac{8(l-2)}{(4-l)^2}>0.$$

Since d and l are integers, their only possible value is $(d,l)=(3,8)$.

Apart from this case the right-hand side of 2.13 cannot be a perfect square. Hence by Lemma 2.1 the theorem follows for $k=2$. In addition, in the exceptional case we show that equation 2.7 has infinitely many integer solutions n,x. Our equation is

$$6\cdot 10^n=54x^2-36x+6=54\left(x-\frac{1}{3}\right)^2.$$

This yields that for arbitrary $k \in \mathbb{N}$ we have a solution $n = 2k$ and $x = -\frac{10^k - 1}{3}$.

Proof of Theorem 2.3. For $k \geq 3$ the statement follows from Theorem 2.1. In case of $k = 2$ a similar calculation has to be carried out as in the proof of Theorem 2.2. This yields the exceptional cases: $(b, l) = (4, 8), (9, 3), (9, 6), (25, 5)$. Showing that for these parameters the original equation has infinitely many solutions can be done similarly as in the previous proof.

Proof of Theorem 2.4. Let $k = 2$. Then $f_{2,l}(x) = \frac{(l-2)x^2 + (4-l)x}{2}$. Since the right-hand side of equation 2.5 is of degree 2 by reducing the left-hand side to a polynomial of degree 3 we obtain an elliptic equation which can further be solved by the program package MAGMA [17]. We illustrate these computations by an example. Set $(d, l) = (3, 11)$. Then equation 2.5 is

$$3 \cdot \frac{10^n - 1}{9} = \frac{9x^2 - 7x}{2}. \tag{2.14}$$

The left-hand side of this equation can be reduced to polynomials of degree 3 by considering n mod 3. If $n \equiv i \pmod 3$, $(i = 0, 1, 2)$ then $10^n = 10^{3k+i}$ for some $k \in \mathbb{Z}$, $(i = 0, 1, 2)$. Then substituting $y = 10^k$, we get the following three distinct equations:

$$2y^3 - 2 = 27x^2 - 21x, \tag{2.15}$$

$$20y^3 - 2 = 27x^2 - 21x, \tag{2.16}$$

$$200y^3 - 2 = 27x^2 - 21x. \tag{2.17}$$

Multiplying both hand sides of equations 2.15,2.16,2.17 by 108, 10800, 1080000, respectively, and introducing the new variables $X_1 = 54x$, $Y_1 = 6y$; $X_2 = 540x$, $Y_2 = 60y$; $X_3 = 5400x$, $Y_3 = 600y$; respectively, we obtain

$$Y_1^3 - 216 = X_1^2 - 42X_1,$$

$$Y_2^3 - 21600 = X_2^2 - 420X_2,$$

$$Y_3^3 - 2160000 = X_3^2 - 4200X_3,$$

respectively. With the procedure `IntegralPoints` of MAGMA one can compute the integer points of these curves, and then determine the solutions n, x of equation 2.14. The solutions are exactly the ones listed in Table 2.2.

Now let $k = 3$. Then $f_{3,l}(x) = \frac{(l-2)x^3+3x^2+(5-l)x}{6}$. Since the right-hand side of equation 2.5 is of degree 3 by reducing the left-hand side to a polynomial of degree 2 we obtain an elliptic equation again which can be solved by Magma. We illustrate these computations by an example. Set $(d, l) = (4, 3)$. Then equation 2.5 is

$$4 \cdot \frac{10^n - 1}{9} = \frac{x^3 + 3x^2 + 2x}{6}. \qquad (2.18)$$

The left-hand side of this equation can be reduced to polynomials of degree 2 by considering n modulo 2. If $n \equiv i \pmod 2$, $(i = 0, 1)$ then $10^n = 10^{2k+i}$ for some $k \in \mathbb{Z}$, $(i = 0, 1)$. Then substituting $y = 10^k$, we get the following two distinct equations:

$$8y^2 - 8 = 3x^3 + 9x^2 + 6x, \qquad (2.19)$$

$$80y^2 - 8 = 3x^3 + 9x^2 + 6x. \qquad (2.20)$$

Multiplying both hand sides of equations 2.19,2.20 by 72, 72000, respectively, and introducing the new variables $X_1 = 6x$, $Y_1 = 24y$; $X_2 = 60x$, $Y_2 = 2400y$; respectively, we obtain

$$Y_1^2 - 576 = X_1^3 + 18X_1^2 + 72X_1,$$

and

$$Y_2^2 - 576000 = X_2^3 + 180X_2^2 + 7200X_2,$$

respectively. With the procedure `IntegralPoints` of MAGMA one can compute the integer points of these curves, and then determine the solutions n, x of equation 2.18. The solutions are exactly the ones listed in Table 2.1.

(d, l)	solutions (n, x)	$f_{k,l}(x)$
$(1, 10)$	$(2, 2)$	11
$(2, 6)$	$(2, 3)$	22
$(4, 3)$	$(1, 2)$	4
$(5, 4)$	$(1, 2)$	5
$(5, 4)$	$(2, 5)$	55
$(6, 5)$	$(1, 2)$	6
$(6, 17)$	$(2, 3)$	66
$(7, 6)$	$(1, 2)$	7
$(8, 7)$	$(1, 2)$	8
$(9, 8)$	$(1, 2)$	9

Table 2.1: The case of $f_{3,l}(x)$

(d,l)	solutions (n,x)	$f_{k,l}(x)$	(d,l)	solutions (n,x)	$f_{k,l}(x)$
$(1,9)$	$(3,6)$	111	$(6,3)$	$(2,11)$	66
$(1,11)$	$(2,2)$	11	$(6,3)$	$(2,-12)$	66
$(1,14)$	$(2,-1)$	11	$(6,3)$	$(3,36)$	666
$(1,19)$	$(4,-11)$	1111	$(6,3)$	$(3,-37)$	666
$(2,5)$	$(1,-1)$	2	$(6,6)$	$(1,2)$	6
$(2,5)$	$(2,4)$	22	$(6,6)$	$(2,6)$	66
$(2,5)$	$(3,-12)$	222	$(6,6)$	$(3,-18)$	666
$(2,10)$	$(2,-2)$	22	$(6,9)$	$(1,-1)$	6
$(3,3)$	$(1,2)$	3	$(6,9)$	$(2,-4)$	66
$(3,3)$	$(1,-3)$	3	$(6,9)$	$(4,44)$	6666
$(3,6)$	$(1,-1)$	3	$(6,17)$	$(3,-9)$	666
$(3,11)$	$(3,9)$	333	$(7,5)$	$(1,-2)$	7
$(3,12)$	$(2,3)$	33	$(7,5)$	$(2,-7)$	77
$(4,4)$	$(1,2)$	4	$(7,7)$	$(1,2)$	7
$(4,4)$	$(1,-2)$	4	$(7,10)$	$(1,-1)$	7
$(4,7)$	$(1,-1)$	4	$(8,8)$	$(1,2)$	8
$(5,3)$	$(2,-11)$	55	$(8,11)$	$(1,-1)$	8
$(5,3)$	$(2,10)$	55	$(8,16)$	$(2,4)$	88
$(5,5)$	$(1,2)$	5	$(9,4)$	$(1,3)$	9
$(5,6)$	$(2,-5)$	55	$(9,4)$	$(1,-3)$	9
$(5,7)$	$(2,5)$	55	$(9,7)$	$(2,-6)$	99
$(5,8)$	$(1,-1)$	5	$(9,9)$	$(1,2)$	9
$(6,3)$	$(1,3)$	6	$(9,12)$	$(1,-1)$	9
$(6,3)$	$(1,-4)$	6	$(9,19)$	$(2,-3)$	99

Table 2.2: The case of $f_{2,l}(x)$

Chapter 3

On equal values of trinomials

3.1 Introduction

One of the classical results concerning equal values of trinomials is the determination of rational integers which can be represented as a product of two and three consecutive integers simultaneously. In other words, the problem is to solve the Diophantine equation

$$x^3 - x = y^2 - y$$

in integers x and y. Using tools from algebraic number theory Mordell [62] resolved this problem. By an elementary approach, one can prove that the unique integer solution x, y of the $x^4 - x = y^2 - y$ with $|xy| > 1$ is $(x, y) = (-1, 2)$. Indeed a straightforward calculation yields that

$$(2x^2 - 1)^2 < 4x^4 - 4x + 1 = (2y - 1)^2 < (2x^2)^2$$

for $x > 1$ and similarly,

$$(2x^2)^2 < 4x^4 - 4x + 1 = (2y - 1)^2 < (2x^2 + 1)^2$$

for $x < -1$. These inequalities imply that $|x| \leq 1$. Recently, Bugeaud et. al. [25] obtained all solutions of the five-degree equation $x^5 - x = y^2 - y$. Their technique is based on some new methods of modern number theory.

Let m and p be fixed positive integers with $m > p \geq 2$. As a general result, Mignotte and Pethő [61] proved a finiteness statement on solutions x, y to the equation

$$x^m - x = y^p - y.$$

The proof depends on Theorem I.

3.2 New results

Let a, b, c, d, e, m, n, p and q be fixed rational integers. In this chapter we prove

Theorem 3.1 *The Diophantine equation*

$$ax^m + bx^n + c = dy^p + ey^q \tag{3.1}$$

where

$$m > n > 0, p > q > 0, (m, n) = (p, q) = 1, ab \neq 0, de \neq 0$$
$$\text{and either } m > p \geq 3 \text{ or } m = p \geq 3, n \geq q \tag{3.2}$$

has infinitely many solutions x, y with a bounded denominator if and only if either

$$m = p, n = q, a = dt^m, b = et^n, t \in \mathbb{Q}, c = 0 \qquad (3.3)$$

or

$$m = p = 3, n = q = 2, a^2e^3 + b^3d^2 = 0, c = -\frac{4b^3}{27a^2}, \qquad (3.4)$$

or

$$m = p = 3, n = 2, q = 1, 27a^4e^3 + b^6d = 0, c = \frac{2a^2e^3}{b^3d^2}. \qquad (3.5)$$

The main ingredient in the proof of Theorem 3.1 is the Bilu-Tichy Theorem, which is an ineffective result, so Theorem 3.1 provides the finiteness of the number of solutions to (3.1), only.

In the special case $p = 2$ we give an upper bound for the solutions x and y. Set $H = \max(|a|, |b|, |c|, |d|, |e|, m, n)$.

Theorem 3.2 *Suppose that $m \geq 5$, $m > n > 0$, $abd \neq 0$, $m \neq 2n$ and*

$$(m, n) \notin \{(6, 2), (6, 4)\},$$

further, if $4dc + e^2 = 0$ then assume that $m - n \geq 3$ or $m - n = 2$ and n is odd. The Diophantine equation

$$ax^m + bx^n + c = dy^2 + ey \text{ in integers } x \text{ and } y$$

implies $max(|x|, |y|) < c_2$, where c_2 is an effectively computable constant depending only on H.

A result on solutions of hyperelliptic equations obtained by Brindza [19], see Lemma 3.1, plays an important role in the proof. We give some families of Diophantine equations with infinitely many integer solutions x and y

for the exponential pairs (m, n) in the following table (here t, u and v are integer parameters).

(m,n)	equation	solutions
$(2n,n)$	$x^{2n}+2bx^n+b^2-1=y^2+2y$	$y=x^n+b-1$
$(3,1)$	$x^3-3t^4x+t^6=y^2+2t^3y$	$x=u^2-2t^2$, $y=u(u^2-3t^2)-t^3$
$(3,2)$	$x^3+3t^2x^2-5t^6=y^2+2t^3y$	$x=u^2+t^2$, $y=u(u^2+3t^2)-t^3$
$(4,1)$	$x^4+4t^3x+t^4=2y^2+4t^2y$	$u^2+2t^2=2v^2$, $x=u+t$, $y=v(u+2t)-t^2$
$(4,3)$	$x^4+4tx^3+25t^4=2y^2+4t^2y$	$u^2+2t^2=2v^2$, $x=u+t$, $y=v(u+4t)-t^2$
$(6,2)$	$x^6-3t^4x^2=2y^2+4t^3y$	$u^2+2t^2=2v^2$, $x=u$, $y=v(u^2-t^2)-t^3$
$(6,4)$	$x^6-3t^2x^4+2t^6=2y^2+4t^3y$	$u^2+t^2=2v^2$, $x=u$, $y=v(u^2-2t^2)-t^3$

Now suppose that $d=1$, $4c+e^2=0$, $m-n=2$, n is even, and we choose the values of a and b, such that the Pellian equation $s^2-at^2=b$ has infinitely many integer solutions s and t. One can check that equation

$$4x^n(ax^2+b)=(2y+e)^2$$

possesses infinitely many solutions in integers x and y. For $d=1$, $4c+e^2=0$, $m-n=1$, n even and $b\equiv f^2 \pmod{a}$ we take $x=(f^2-b)/a$, $y=fx^{n/2}-e/2$. Under the same conditions for odd values of n, $a>0$, a is not a square, the Pellian equation $s^2-at^2=1$ has infinitely many solutions in integers s, t such that

$$s\equiv 1 \pmod{2a}, \quad t\equiv 0 \pmod{2}.$$

Then taking $x=b(s-1)/2a$, $y=btx^{(n-1)/2}/2-e/2$, we provide infinitely many solutions as well.

3.3 Auxiliary results

In the following we are going to use the previously introduced Bilu-Tichy Theorem and the five kinds of standard pairs of polynomials.

Lemma 3.1 *Let f be a polynomial with rational coefficients and suppose that it possesses at least three zeros of odd multiplicity. Then the equation $f(x) = y^2$ in unknown x, y implies $\max\{|x|, |y|\} \leq C_7$ where C_7 is an effectively computable constant depending only on the parameters of the polynomial f.*

Proof. This is a corollary of the main result in [19]. For further generalizations and improvements see [24].

The following lemmata describe multiplicity of the zeros of trinomials and connections between standard pairs and trinomials.

Lemma 3.2 *Multiplicity of every zero of a polynomial $AX^m + BX^n + C$, $AC \neq 0$ is at most two and if $B = 0$ than one.*

Proof. This is an easy consequence of Hajós' [48] result.

Lemma 3.3 *Let $f \in \mathbb{C}[X] \setminus \mathbb{C}$ and $f(X)^2 | AX^m + BX^n + C$, where $m > n > 0$, $ABC \neq 0$. Then $\deg f \leq (m, n)$.*

Proof. By Lemma 3.2, every zero of f is simple. Let ζ be such a zero, $AX^m + BX^n + C = T(X)$. Since $T(\zeta) = T'(\zeta) = 0$ we obtain

$$\zeta^m = \frac{C_n}{(m-n)A}, \quad \zeta^n = \frac{C_m}{(n-m)B},$$

thus $\zeta = \zeta^{(m,n)}$ is uniquely determined and $\deg f \leq (m, n)$.

Lemma 3.4 *If $A(\alpha x + \beta)^m + B(\alpha x + \beta)^n + C = D_m(X, g)$, where m, n satisfy 3.1, further, $A\alpha g \neq 0$ and $D_m(X, g)$ is the mth Dickson polynomial, then $m \leq 3$.*

Proof. Put $\beta = \alpha\beta_1$, $B = B_1\alpha^{-m}$. Clearly $A = \alpha^{-m}$, and we obtain

$$\begin{aligned}(X+\beta_1)^m + B_1(X+\beta_1)^n + C &= D_m(X,g) \\ &= \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{m}{m-i}\binom{m-i}{i}(-g)^i X^{m-2i}.\end{aligned} \tag{3.6}$$

The coefficient of X^{m-1} on the right-hand side vanishes, hence it does on the left-hand side and if $\beta_1 \neq 0$ we obtain $n = m-1$

$$m\beta_1 + B_1 = 0$$

and, unless $m = 2$,

$$\binom{m}{2}\beta_1^2 + nB_1\beta_1 = \frac{m}{m-1}\binom{m-1}{1}(-g) = -mg.$$

It follows that

$$\binom{m}{2}\beta_1^2 - mn\beta_1^2 = -mg, \quad \beta_1^2 = \frac{2g}{m-1}$$

and unless $m \leq 3$

$$\binom{m}{3}\beta_1^3 + \binom{n}{2}B_1\beta_1^2 = 0,$$

hence $\binom{m}{3}\beta_1^3 - \binom{n}{2}m\beta_1^3 = 0$; $\binom{m}{3} = \binom{n}{2}m = 3\binom{m}{3}$, which is a contradiction.

Assume now that $\beta_1 = 0$. Equation 3.6 gives either $m \leq 3$ or $n = m-2$ and $0 = m-4$. However, in the latter case $(m,n) = 2$, contrary to our assumptions.

Lemma 3.5 *If m, n, p, q satisfy 3.2 and $AB\alpha \neq 0 \neq DE\gamma$, then*

$$< A(\alpha X+\beta)^m+B(\alpha X+\beta)^n+C,\ D(\gamma X+\delta)^p+E(\gamma X+\delta)^q+F > \quad (3.7)$$

is not a standard pair.

Proof. Suppose first that 3.7 is a standard pair of the first kind. If

$$A(\alpha X+\beta)^m + B(\alpha X+\beta)^n + C = X^m. \quad (3.8)$$

then β is a zero of AX^m+BX^n+C of multiplicity m, hence by Lemma 3.2 either $m \leq 2 < p$, contrary to 3.2, or $C = 0$. In the latter case $\alpha X + \beta | X$, $\beta = 0$ and 3.7 contradicts $B \neq 0$. If

$$D(\gamma X+\delta)^p + E(\gamma X+\delta)^q + F = X^p. \quad (3.9)$$

then a similar argument leads to a contradiction with $p \geq 3$.

Suppose next that 3.7 is a standard pair of the second kind. Then either $m = 2$ or $p = 2$, however this is impossible by 3.2.

Suppose next that 3.7 is a standard pair of the third or the fourth kind. Then we have

$$A(\alpha X+\beta)^m + B(\alpha X+\beta)^n + C = D_m(X, a^p) \ \textit{or}\ a^{-m/2}D_m(X, a),$$

$$D(\gamma X+\delta)^p + E(\gamma X+\delta)^q + F = D_p(X, a^m) \ \textit{or}\ -b^{-p/2}D_p(X, b),$$

respectively, where $(m,p) = 1$ or $(m,p) = 2$, respectively. By Lemma 3.4 $m \leq 3$, $p \leq 3$, thus by 3.2, $m = p = 3$ and $(m,p) = 3$, we get a contradiction again.

Suppose finally that 3.7 is a standard pair of the fifth kind. Then one of the polynomials $A(\alpha X+\beta)^m+B(\alpha X+\beta)^n+C$, $D(\gamma X+\delta)^p+E(\gamma X+\delta)^q+F$ has two zeros of multiplicity three, which contradicts Lemma 3.2.

Lemma 3.6 *Under the assumption 3.2 we have*

$$aX^m + bX^n + c = d(\varepsilon X + \xi)^p + e(\varepsilon X + \xi)^q \tag{3.10}$$

for some $\varepsilon, \xi \in \mathbb{Q}$ if and only if 3.3, 3.4 or 3.5 hold.

Proof. Assume first that we have 3.10. Clearly $m = p$ and either $c = \xi = 0$ or $\xi \neq 0$. In the former case we obtain 3.3 with $t = \epsilon$. In the latter case, by Lemma 3.2, $q \leq 2$. However, the case $m = p = 4$, $q = 2$ is excluded by 3.2, hence if $m = p \geq 4$ we have $p - q \geq 3$, thus X^{m-1} and X^{m-2} occur on the right-hand side of 3.10 with the coefficients $dm\varepsilon^{m-1}\xi \neq 0$ and $d\binom{m}{2}\varepsilon^{m-2}\xi^2 \neq 0$. On the left-hand side of 3.10 X^{m-1} and X^{m-2} cannot occur both with non-zero coefficients. The obtained contradiction shows that $m = p = 3$. If $n = q = 2$, then 3.10 gives $a = d\epsilon^3$, $b = 3d\epsilon^2\xi + e\epsilon^2$, $0 = 3d\epsilon\xi^2 + 2e\epsilon\xi$, $c = d\xi^3 + e\xi^2$, which on elimination of ϵ and ξ leads to 3.4. If $n = 2$, $q = 1$, then 3.10 gives $a = d\epsilon^3$, $b = 3d\epsilon^2\xi$, $0 = 3d\epsilon\xi^2 + e\epsilon$, $c = d\xi^3 + e\xi$, which on elimination of ϵ and ξ leads to 3.5. Finally, since $\xi \neq 0$ we cannot have $n = q = 1$, because the coefficient of X^2 on the left-hand side of 3.10 is 0, on the right-hand side is $3d\varepsilon^2\xi \neq 0$.

Assume now that we have 3.3, 3.4 or 3.5. If 3.3 holds, we get 3.8 with $\varepsilon = t$, $\xi = 0$; if 3.4 holds we obtain 3.8 with

$$\varepsilon = -\frac{ae}{bd}, \quad \xi = -\frac{2e}{3d}$$

and finally if 3.5 holds we have 3.8 with

$$\varepsilon = -\frac{b^2}{3ae} \quad \xi = \frac{3a^2e^2}{b^3d}.$$

Lemma 3.7 *If $aX^m+bX^n+c = f_1 \circ f_2(X)$, where 3.2 holds and $f_i \in \mathbb{Q}[x]$, then for a suitable linear function $h \in \mathbb{Q}[x]$ we have either $f_1 \circ h = X$, $h^{-1} \circ f_2 = aX^m + bX^n + c$ or $f_1 \circ h = aX^m + bX^n + c$, $h^{-1} \circ f_2 = X$.*

Proof. This follows from Lemma 3 in [36] with $h \in \mathbb{C}[X]$. Since f_2 is a polynomial with rational coefficients we have $h \in \mathbb{Q}[X]$.

3.4 Proofs of the Theorems

Proof of Theorem 3.1. Assume first that equation (3.1) has infinitely many rational solutions with a bounded denominator. The Bilu-Tichy Theorem gives

$$aX^m + bX^n + c = \varphi \circ f \circ \lambda, \quad dX^p + eX^q = \varphi \circ g \circ \mu,$$

where λ, μ are linear polynomials, $\varphi \in \mathbb{Q}[x]$ and (f, g) is a standard pair of the ith kind ($1 \leq i \leq 5$).

By Lemma 3.7 there exist linear functions h_1, h_2 in $\mathbb{Q}[x]$ such that either

$$\varphi \circ h_1 = X,\ h_1^{-1} \circ f \circ \lambda = aX^m + bX^n + c,\ \varphi \circ h_2 = X, h_2^{-1} \circ g \circ \mu = dX^p + eX^q \tag{3.11}$$

or

$$\varphi \circ h_1 = aX^m + bX^n + c,\ h_1^{-1} \circ f \circ \lambda = X \tag{3.12}$$

and

$$\varphi \circ h_2 = dX^p + eX^q,\ h_2^{-1} \circ g \circ \mu = X. \tag{3.13}$$

In case 3.11 we have $h_1 = \varphi^{-1} = h_2$ and putting $h_1(X) = \varepsilon X + \xi$, $h_1(aX^m + bX^n + c) = AX^m + BX^n + C$, $h_1(dX^p + eX^q) = DX^p + EX^q + F$, $\lambda^{-1}(X) = \alpha X + \beta$, $\mu^{-1}(X) = \gamma X + \delta$ we obtain that

$$< A(\alpha X+\beta)^m + B(\alpha X+\beta)^n + C,\ D(\gamma X+\delta)^p + E(\gamma X+\delta)^q + F >$$

is a standard pair. However, by Lemma 3.5, this is impossible.

In case 3.12 taking $h_2^{-1} \circ h_1 = \varepsilon X + \xi$ we obtain

$$aX^m + bX^n + c = d(\varepsilon X+\xi)^p + e(\varepsilon X+\xi)^q. \tag{3.14}$$

By Lemma 3.6 we have 3.3, 3.4 or 3.5. Conversely, if 3.3, 3.4 or 3.5 holds, we get 3.14 for some rational ε, ξ. Taking an arbitrary integer x, we obtain infinitely many solutions $(x, \varepsilon x+\xi)$ of 3.1 with a bounded denominator.

Proof of Theorem 3.2. Using Lemma 3.1 it is enough to show that the trinomial

$$F(X) = 4adX^m + 4bdX^n + 4cd + e^2$$

has at least three zeros of odd multiplicity. First we consider the case when the constant term $4cd+e^2$ is non-zero. On supposing the contrary we have

$$F(X) = f(X)^2 g(X),$$

where degree of polynomial $g(X)$ is at most two, and, as a consequence of Lemma 3.2 polynomial $f(X)$ possesses only simple zeros. Let ζ be one of them. Similarly to the proof of Lemma 3.3 we obtain $\zeta^{(m,n)}$ is uniquely determined and $\deg f \leq (m,n)$. It follows that $m < 2(m,n)+3$, hence either $m = 2n$ or $(m,n) = (3,1), (3,2), (4,1), (4,3), (6,2), (6,3)$.

If $4cd+e^2$ is vanishing then $F(X) = 4dX^n(aX^{m-n}+b)$ and apart from the cases listed in Theorem 3.2 we cannot guarantee the existence of at least three zeros of odd multiplicity.

Finally, we would like to remark here that Schinzel [75] omitted the assumption $(m,n) = (p,q) = 1$ from Theorem 3.1.

Chapter 4

Equal values of standard counting polynomials

4.1 Introduction

The most fundamental polynomials counting integer points are X^n in an n-dimensional unit cube, $\binom{X+n}{n}$ in a standard n-simplex,

$$S_{n-1}(X) = 1^{n-1} + 2^{n-1} + \ldots + X^{n-1}$$

in an n-dimensional pyramid, and

$$P_n(X) = \sum_{j=0}^{n} \binom{n}{j} \binom{X+n-j}{n}$$

for octahedron in dimension n, see [11, Chapter 2]. Our purpose is to consider the possible equal values of these polynomials in case of integral variables. In other words, for given positive integers m, n, how often can two bodies (unit cube, simplex, pyramid, octahedron) of dimensions m and n, respectively, contain equally many integral points? It is a bit surprising that this discrete geometrical question is the common background of some classical Diophantine problems. One can see that the above problems lead to 9 nontrivial families of Diophantine equations, see Table 4.1. We give a survey of known results concerning these equations. Further, we prove some new theorems for the solutions. For each family of solutions, the following three types of results can be established. An ineffective finiteness theorem for the general case obtained by the Bilu-Tichy Theorem, an effective result based on Baker's theory when one of the dimensions involved is small, and the resolution by computer algebraic packages if both dimensions are small.

No	equation	remark
1	$S_m(x) = S_n(y)$	$n > m \geq 1$
2	$S_m(x) = y^n$	$m \geq 1, n \geq 2, (m,n) \notin \{(1,2),(3,2),(3,4),(5,2)\}$
3	$S_m(x) = \binom{y}{n}$	$m \geq 1, n \geq 2, (m,n) \neq (1,2)$
4	$S_m(x) = P_n(y)$	$m \geq 1, n \geq 2, (m,n) \neq (1,2)$
5	$\binom{x}{m} = y^n$	$m \geq 2, n \geq 2, (m,n) \neq (2,2)$
6	$\binom{x}{m} = \binom{y}{n}$	$n > m \geq 2$
7	$\binom{x}{m} = P_n(y)$	$m \geq 2, n \geq 2, (m,n) \neq (2,2)$
8	$P_m(x) = y^n$	$m \geq 2, n \geq 2, (m,n) \neq (2,2)$
9	$P_m(x) = P_n(y)$	$n > m \geq 2$

Table 4.1: The investigated families of Diophantine equations

4.2 Lemmas and auxiliary results

First we note that $S_{n-1}(X)$ can be expressed in the form

$$S_{n-1}(X) = \frac{1}{n}(B_n(X+1) - B_n(0)), \tag{4.1}$$

where $B_n(X)$ denotes the n-th Bernoulli polynomial which is of degree n and has its coefficients in $\mathbb{Q}$.

We now collect some lemmas to prove our new results. The first one deals with the simple zeros of a family of polynomials. Let n be a positive integer, $f(X)$ an integer-valued polynomial with $\deg f(X) \leq n-1$, and $g(X)$ a polynomial with rational integer coefficients.

Lemma 4.1 *Suppose that $n \geq 6$ and let p denote a prime for which*

$$\frac{2}{3}n < p \leq n.$$

If a_n is an integer not divisible by p then the polynomial

$$F(X) = a_n \binom{X}{n} + f(X) + g(X)$$

has at least $\left[\frac{n}{3}\right] + 1$ simple zeros.

Proof. This is the Theorem in [65].

The following previously introduced result provides an effective upper bound for the solutions to the hyperelliptic equations.

Lemma 4.2 *Let f be a polynomial with rational coefficients and suppose that it possesses at least three simple zeros. Then the equation $f(x) = y^2$ in unknown integers x, y implies* $\max(|x|, |y|) < c_4$*, where c_4 is an effectively computable constant depending on the degree and the maximum height of the coefficients of f.*

There is a similar result for superelliptic equations.

Lemma 4.3 *Let f be a polynomial with rational coefficients and suppose that it possesses at least two simple zeros. Then the equation $f(x) = y^m$ in unknown integers $x, y, m \geq 2$ implies* $\max(|x|, |y|, m) < c_5$*, where c_5 is an* *effectively computable constant depending on the degree and the maximum height of the coefficients of the polynomial f.*

The next results are used in the proofs of our effective statements.

Lemma 4.4 *Let $m > 1, r, s \neq 0$ be fixed integers. Then apart from the cases when $m = 3, r = 0$ or $s + 64r = 0$; $m = 5, r = 0$ or $s - 324r = 0$, the equation*

$$s(1^m + 2^m + \ldots + x^m) + r = y^n$$

in integers $x > 0$, y with $|y| \geq 2$, and $n \geq 2$ has only finitely many solutions which can be effectively determined.

Proof. This is Theorem 2.2 in [72].

Lemma 4.5 *Let a, b, c and m be given integers with $ab \neq 0$ and $m \geq 3$. Apart from the cases when $m = 4, c/a = -1/24$ or $3/128$, $n = 2$ and b/a is not a square, the Diophantine equation*

$$a\binom{x}{m} = by^n + c$$

has only finitely many solutions in $x, y > 1, n \geq 2$ and all these solutions can be effectively bounded in terms of a, b, c and m.

Proof. This is the main result of [88].

Lemma 4.6 *Let a, b, m, n be integers with $a \neq 0$, $m \geq 1$, $n > 2$. The equation*

$$S_m(x) = a\binom{y}{n} + b$$

in integers x and y has only finitely many solutions apart from the following possible exceptions

$$(m, n) \in \{(1, 4), (2, 3), (3, 4)\}.$$

Proof. This is a special case of Theorem 2 in [71].

The next result will be useful for the application of the previous lemma (cf. [78] and [68]).

Lemma 4.7 *The product of two or more consecutive positive integers is never a perfect power.*

Proof. For the proof we refer to [34].

We need the following technical lemma. Let $a, b, \tilde{a}, \tilde{b}, \bar{a}, \bar{b}$ be rational numbers with $a\tilde{a}\bar{a} \neq 0$.

Lemma 4.8 *None of the polynomials $\binom{aX+b}{m}$ and $P_m(\tilde{a}X + \tilde{b})$ is of the form $e_1X^m + e_0$ with $e_1 \in \mathbb{Q} \setminus \{0\}$ and $m \geq 3$ or $e_1D_m(X, \alpha) + e_0$ with $e_1, \alpha \in \mathbb{Q} \setminus \{0\}$ and $m \geq 5$. The polynomial $S_m(\bar{a}X + \bar{b})$ is not of the form $e_1X^q + e_0$ with $q \geq 3$ or $e_1D_\nu(X, \alpha) + e_0$ with $\nu > 4$, where α, e_1, e_0 are rational numbers with $e_1 \neq 0$.*

Proof. For the fact that $\binom{aX+b}{m}$ is not of the form $e_1X^m + e_0$ with $m \geq 3$ we refer to [15, Lemma 5.2].

Now suppose that

$$\binom{aX+b}{m} = e_1D_m(X, \alpha) + e_0$$

for an integer $m \geq 5$ and $\alpha \in \mathbb{Q} \setminus \{0\}$ and set

$$\binom{aX+b}{m} = \sum_{i=0}^{m} c_iX^i.$$

On comparing the corresponding coefficients, an easy calculation shows that

$$c_m = \frac{a^m}{m!} = e_1,$$

$$c_{m-1} = \frac{a^{m-1}\left(b - \frac{m-1}{2}\right)}{(m-1)!} = 0,$$

$$c_{m-2} = \frac{a^{m-2}(12b^2 + 12(1-m)b + 3m^2 - 7m + 2)}{24(m-2)!} = -e_1\alpha m,$$

and

$$c_{m-4} = \frac{a^{m-4}(240b^4 + 480(1-m)b^3 + f_1(m)b^2 + f_2(m)b + f_3(m))}{5760(m-4)!} =$$

$$= \frac{e_1 m(m-3)\alpha^2}{2},$$

where $f_1(m) = 120(3m^2 - 7m + 2), f_2(m) = 120(-m^3 + 4m^2 - 3m)$ and $f_3(m) = 15m^4 - 90m^3 + 125m^2 - 18m - 8$. Using the second equation, we have $b = \frac{m-1}{2}$ and thus

$$c_{m-2} = -\frac{a^{m-2}(m+1)}{24(m-2)!} = -e_1\alpha m$$

and

$$c_{m-4} = \frac{a^{m-4}(5m^2 + 12m + 7)}{5760(m-4)!} = \frac{e_1 m(m-3)\alpha^2}{2}.$$

From these relations with $c_m = \frac{a^m}{m!} = e_1$ we get

$$\frac{(m-1)(m+1)}{24} = a^2\alpha$$

and

$$\frac{(m+1)(5m+7)(m-1)(m-2)}{2880} = a^4\alpha^2,$$

that is

$$(m+1)(m-1) = \frac{(5m+7)(m-2)}{5}$$

and $m = 3$, a contradiction.

The proof of the corresponding statements for the polynomials $P_m(\tilde{a}X+\tilde{b})$ and $S_m(\bar{a}X + \bar{b})$ can be found in [81].

4.3 New and known results

Family 1: equation

$$S_m(x) = S_n(y) \tag{4.2}$$

where $n > m \geq 1$ are fixed and x, y are unknown integers.

For (m, n) with $m = 1$ and $m = 3$, Brindza and Pintér [21] proved some effective finiteness results for the solutions x and y. Their proof is based on the structure of zeros of the corresponding shifted Bernoulli polynomials. In the same paper they obtained an ineffective finiteness result for an infinite class of pairs (m, n) using Davenport-Lewis-Schinzel Theorem. Later, applying Bilu-Tichy Theorem, the authors of [15] extended this statement to every pair (m, n). For small values of m and n the problem leads to certain elliptic curves. For the resolution of the special cases $(m, n) = (1, 2), (1, 3), (1, 5), (1, 7)$ we refer to [1] and [83], [26] and [58], [47], [55], respectively. We propose the following

Conjecture 1 *All the solutions to the equation* (4.2) *in integers* $n > m \geq 1$ *and* x, y *are*

$$(m, n, x, y) = (1, 2, 10, 5), (1, 2, 13, 6), (1, 3, 8, 3), (1, 5, 23, 3), (1, 5, 353, 9).$$

This conjecture is based upon an extensive numerical investigation. However, its proof seems well beyond the reach of current techniques.

Family 2: equation

$$S_m(x) = y^n \tag{4.3}$$

where $m \geq 1, n \geq 2, x \geq 1, y \geq 1$ **are unknown integers and** $S_m(X) = 1^m + 2^m + \ldots + X^m$**.**

Equation (4.3) has the solution $(x, y) = (1, 1)$ which is called *trivial.* For $m = n = 2$, (4.3) has only the nontrivial solution $(x, y) = (24, 70)$. This was proved by Watson [85]. In 1956, Schäffer [74] proved that for fixed $m \geq 1$ and $n \geq 3$, (4.3) has at most finitely many solutions in x, y, unless

$$(m, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\}, \tag{4.4}$$

where in each case, there are infinitely many such solutions.

Schäffer's proof is ineffective. Using Baker's method, Győry, Tijdeman and Voorhoeve [44] proved a more general and effective result in which the exponent n is also unknown. A special case of their result is the following

Theorem 4.1 *For given* $m \geq 2$ *with* $m \notin \{3, 5\}$*, all solutions* $x, y \geq 1, n \geq 2$ *of* (4.3) *satisfy* $\max(x, y, n) \leq c_6(m)$*, where* $c_6(m)$ *is an effectively computable number which depends only on* m*.*

Later, Győry, Tijdeman and Voorhoeve [84] showed that for any fixed polynomial $R(X)$ with integral coefficients, the equation

$$S_m(x) + R(x) = y^n$$

has only finitely many solutions in integers $x, y \geq 1, n \geq 2$ provided that $m \geq 2$ is fixed such that $m \neq \{3, 5\}$. The proof furnishes an effective upper bound for n, but not for x and y. An effective version was obtained in a more general form by Brindza [20].

Pintér [67] proved that for fixed $m > 2$, all solutions of (4.3) with $y > 1$ satisfy $n < c_7 m \log m$, where c_7 is an effectively computable absolute constant.

For fixed $m \geq 2$ with $m \notin \{3, 5\}$, Theorem 4.1 makes it possible, at least in principle, to determine all solutions of (4.3). However, the bound $c_6(m)$ in Theorem 4.1 is not given explicitly. Moreover, even an explicit value obtained by Baker's method would be too large for practical use. Schäffer [74] was able to prove that for some special pairs (m, n) with small m, n, (4.3) has only the trivial solution. Further, he formulated the following

Theorem 4.2 *For $m \geq 1$ and $n \geq 2$ with (m, n) not in (4.4), equation (4.3) has only one nontrivial solution, namely $(m, n, x, y) = (2, 2, 24, 70)$.*

Recently, a considerable progress has been made in this direction. Jacobson, Pintér and Walsh [49] confirmed the conjecture for $n = 2$ and for even m with $m \leq 58$. Further, Bennett, Győry and Pintér [13] proved completely Schäffer's conjecture for $m \leq 11$ and for arbitrary n.

For fixed m and $(m, n) \neq (3, 4)$, Brindza and Pintér [22] gave the upper bound $\max(c_8, e^{3m})$ for the number of solutions of (4.3) with $x, y > 1, n > 2$, where c_8 is an effectively computable absolute constant.

In the proofs of the above presented results the first step is to express $S_m(X)$ in the form (4.1). This implies that $S_m(X)$ is divisible by $X^2(X+1)^2$ in $\mathbb{Q}[X]$ if $m > 1$ is odd, and by $X(X+1)(2X+1)$ if $m \geq 2$ is even. Then (4.3) can be reduced both to superelliptic equations and to finitely many binomial Thue equations of the form $AX^n - BY^n = 1$ in non-zero $X, Y \in \mathbb{Z}$ with fixed non-zero integers A, B. Finally, various deep theorems and techniques can be applied to these equations to establish the desired results for equation (4.3).

For more details and related results we refer to the survey paper [42] of Győry and Pintér.

Family 3: equation

$$S_m(x) = \binom{y}{n}, \tag{4.5}$$

where $m \geq 1, n \geq 2$ **are fixed integers with** $(m, n) \neq (1, 2)$ **and** x, y **are unknown integers.**

As an easy consequence of Lemma 4.6 we have

Theorem 4.3 *If* $m \geq 1, n \geq 2$ *and* $(m, n) \neq (1, 2)$ *then the equation* (4.5) *has only finitely many solutions in integers* x *and* y*.*

Proof. In view of Lemma 4.6 we have to check the possible exceptional cases $(m, n) \in \{(1, 4), (2, 3), (3, 4)\}$ only. For $(m, n) = (1, 4)$, we get the classical equation

$$\binom{x+1}{2} = \binom{y}{4},$$

and for the resolution of this equation see [86] and [66]. In the case $(m, n) = (2, 3)$ we obtain

$$x(x+1)(2x+1) = y(y-1)(y-2).$$

By using MAPLE one can verify that the genus of the corresponding curve is 1, so it has only finitely many solutions in integers x and y. Finally, if $(m, n) = (3, 4)$, our equation takes the form

$$\left(\frac{x(x+1)}{2}\right)^2 = \binom{y}{4}$$

and, by [33], there is no integer solution of this problem.

If m or n is small then we have an effective result.

Theorem 4.4 *Let $n \in \{2, 4\}$ and $m \geq 1$ with $(m, n) \neq (1, 2)$ or $m \in \{1, 3\}$ and $n \geq 2$. Then all the solutions of the equation* (4.5) *in integers x and y are bounded by an effectively computable constant depending only on m or n, respectively. Further, if $m = 3$ and $n \geq 2$, then there is no solution.*

Proof. In the first case $n = 2$ or 4. Now, our equation (4.5) leads to the equations

$$8S_m(x) + 1 = (2y - 1)^2,$$

or

$$24S_m(x) + 1 = (y(y - 3) + 1)^2,$$

respectively, and Lemma 4.4 completes the proof. If $m = 1$ or $m = 3$ we have the equations

$$(2x + 1)^2 = 8\binom{y}{n} + 1,$$

or

$$\left(\frac{x(x + 1)}{2}\right)^2 = \binom{y}{n},$$

respectively. Our statements follow from Lemma 4.5 and Theorem 4.7 below, respectively.

Family 4: equation

$$S_m(x) = P_n(y), \tag{4.6}$$

where $m \geq 1, n \geq 2$ are fixed integers and x, y are unknown integers.

For small values of m or n we prove the following

Theorem 4.5 *If $m \in \{1, 3\}$ and $n \geq 2$ or $n \in \{2, 4\}$ and $m \geq 1$ then the equation* (4.6) *implies that* $\max(x, y) < c_9$*, where c_9 is an effectively computable constant depending only on n or m, respectively.*

Proof. If $(m, n) = (1, 2)$ or $(3, 2)$ we have the equations

$$\binom{x}{2} = 2y^2 + 2y + 1$$

and

$$\binom{x}{2}^2 = 2y^2 + 2y + 1,$$

respectively. One can check that in the first case there is no integer solution in x and y, further the second equation represents a genus one curve, so it possesses only finitely many and effectively determinable solutions in x and y.

In the sequel we suppose that $m \in \{1, 3\}$ and $n \geq 3$. Then we have the following families of equations

$$(2x - 1)^2 = 8P_n(y) + 1$$

and

$$\left(\frac{x(x-1)}{2}\right)^2 = P_n(y),$$

respectively. Since the leading coefficient of the polynomial $P_n(X)$ is $\frac{2^n}{n!}$, Lemmata 4.1 and 4.2 give the proof of our theorem for $n \geq 6$. In the remaining cases a simple calculation shows that the corresponding polynomials have only simple zeros.

Now assume that $n \in \{2, 4\}$ and $m \geq 2$. We have the Diophantine equations

$$2S_m(x) = (2y + 1)^2$$

and

$$3S_m(x)+5=2(y^2+y+2)^2,$$

respectively, and Lemma 4.4 proves the statement of our theorem.

Theorem 4.6 *Assume that* $m \geq 2, n > 2$ *and* $\gcd(m+1,n)=1$*. Then equation* (4.6) *has only finitely many solutions in integers* x *and* y*.*

We conjecture that Theorem 4.6 is true omitting the condition for the greatest common divisor of $m+1$ and n, cf. [71].

Proof. On supposing the contrary and using the Bilu-Tichy Theorem we have

$$S_m(aX+b)=\phi(f(X)), P_n(\tilde{a}X+\tilde{b})=\phi(g(X)),$$

where $a,\tilde{a},b,\tilde{b}\in\mathbb{Q}$ with $a\tilde{a}\neq 0, \phi(X)\in\mathbb{Q}[X]$ and (f,g) is a standard pair. Since the greatest common divisor of $m+1$ and n is 1, we have that $\deg\phi=1, \phi(X)=e_0X+e_1$, say, where e_0,e_1 are rational numbers and $e_0\neq 0$. Now applying the conditions for m and n we get

$$\deg f>2, \deg g>2, \gcd(\deg f, \deg g)=1,$$

and this excludes the standard pairs of the second, fourth and fifth kind. From Lemma 4.8 we obtain $\max\{m,n\}\leq 5$, and by the conditions for m,n and Theorem 4.5, the remaining cases are $(m,n)=(2,5),(4,3)$ and $(5,5)$. However, using MAPLE, one can check that the genus of the corresponding three curves is 4,4 and 10, respectively, so there are only finitely many integral points on these curves.

Family 5: equation

$$\binom{x}{m}=y^n, \tag{4.7}$$

where $m \geq 2, n \geq 2, x > m, y \geq 2$ are unknown integers.

For $m = n = 2$, equation (4.7) can be written in the form

$$(2x-1)^2 - 8y^2 = 1$$

which has infinitely many solutions, and all these can be given in a recursive way. For $m = 3, n = 2$, Meyl [60, x odd] and Watson [85, x even] proved that

$$\binom{50}{3} = 140^2 \tag{4.8}$$

is the only solution of (4.7).

It was conjectured by Erdős [32] that for $n > 2$, equation (4.7) has no solution. Erdős [32] proved this for $n = 3$ and for $n \geq 2^m$, and Obláth [63] for $n = 4$ and 5.

By means of an ingenious elementary method Erdős [33] confirmed his conjecture for $m \geq 4$. For $m < 4$, the method of Erdős does not work.

Using Baker's method, Tijdeman [82] proved that for $m = 2$ and 3 equation (4.7) has only finitely many solutions, and all of them can be, at least in principle, determined. Later, Terai [80] showed that for $m = 2$ and 3, (4.7) implies $n < 4250$.

Finally, Győry [38] proved Erdős' conjecture for $m = 2, 3$ and $n > 2$, and hence completed the proof of the following

Theorem 4.7 *Apart from the case* $(m, n) = (2, 2)$, (4.8) *gives the only solution of equation* (4.7).

Győry's proof combines some results of Győry [37] and Darmon and Merel [28] on generalized Fermat equations, and a theorem of Bennett and de Weger [12] on binomial Thue equations.

There are several related results in the literature, see e.g. the survey papers [40] and [41] and the references given there. For example, Theorem 4.7 has been extended to the equation

$$x(x-1)\cdots(x-m+1) = by^n \tag{4.9}$$

by Saradha [73, $m \geq 4$] and Győry [39, $m < 4$], where $b \geq 1$ is also unknown, but has only prime factors not exceeding m. For $b = m!$, the results of [73] and [39] imply Theorem 4.7, while for $b = 1$, they give the celebrated theorem of Erdős and Selfridge [34] which states that the product of consecutive positive integers is never a power.

Family 6: equation

$$\binom{x}{m} = \binom{y}{n}, \tag{4.10}$$

where $n > m \geq 2$ are fixed integers and $x \geq m, y \geq n$ are unknown integers.

This equation possesses a very extensive literature. There are several scattered computational results for special pairs (m, n). For the resolution of the corresponding equation in the cases $(m, n) = (2,3), (2,4), (2,5), (2,6), (3,4)$ we refer to [2], [86] and [66], [25], [47], [87], respectively. For a nice survey on certain numerical problems and for the cases $(m, n) = (2,8), (3,6), (4,6), (4,8)$ see [79]. Generalizing an earlier result by Kiss [54], Brindza [18] proved an effective finiteness statement for the solutions to the equation (4.10) with $m = 2$. Using some elementary considerations, de Weger [87] dealt with equal values of binomial coefficients and proposed the following general conjecture.

Conjecture 2 *All solutions of equation* (4.10) *in positive integers* m,n,x,y *with* $n > m \geq 2, x > m, y > n$ *are*

$$\binom{16}{2} = \binom{10}{3}, \binom{56}{2} = \binom{22}{3}, \binom{153}{2} = \binom{19}{5}, \binom{221}{2} = \binom{17}{8}$$

$$\binom{78}{2} = \binom{15}{5} = \binom{14}{6}, \binom{21}{2} = \binom{10}{4}, \binom{120}{2} = \binom{36}{3},$$

and an infinite family

$$\binom{F_{2i+2}F_{2i+3}}{F_{2i}F_{2i+3}} = \binom{F_{2i+2}F_{2i+3} - 1}{F_{2i}F_{2i+3} + 1}$$

for $i = 1, 2, \ldots$, *where* F_n *denotes the nth Fibonacci number defined by* $F_0 = 0, F_1 = 1$ *and* $F_{n+1} = F_n + F_{n-1}$ *for* $n = 1, 2, \ldots$.

For general, however, ineffective finiteness results see [14] and [70].

Family 7: equation

$$\binom{x}{m} = P_n(y), \tag{4.11}$$

where $m \geq 2, n \geq 2$ **are fixed integers and** $x \geq m, y$ **are unknown integers.**

In the special case $(m, n) = (2, 2)$ we have the equation

$$\binom{x}{2} = 2y^2 + 2y + 1$$

and a straightforward calculation gives that the transformed equation

$$(2x - 1)^2 - (4y + 2)^2 = 5$$

has no solution in integers $x \geq 2$ and y.

For small values of m or n we prove the following

Theorem 4.8 *If $m \in \{2,4\}$ and $n \geq 3$ or $n \in \{2,4\}$ and $m \geq 3$ then equation* (4.11) *implies that* $\max(x,y) < c_{10}$*, where c_{10} is an effectively computable constant depending only on n or m, respectively.*

Proof. First suppose that $m \in \{2,4\}$ and $n \geq 3$. We have the equations

$$8P_n(y) + 1 = (2x-1)^2$$

and

$$24P_n(y) + 1 = (x^2 - 3x - 1)^2,$$

respectively. Using the fact that

$$P_n(X) = 2^n \binom{X}{n} + f(X),$$

where $f(X)$ is an integer-valued polynomial of degree $< n$, and Lemmata 4.1 and 4.2 give our statement for $n \geq 6$. If $n = 3, 4, 5$ then an easy calculation shows that the corresponding polynomials have at least three simple zeros, and the proof is completed in these cases as well.

Now assume that $n \in \{2,4\}$ and $m \geq 3$. We get the equations

$$2\binom{x}{m} - 1 = (2y+1)^2$$

and

$$3\binom{x}{m} + 5 = 2(y^2 + y + 2)^2,$$

respectively. Our Lemmata 4.5 and 4.2 completes the proof for $m \geq 3$.

Theorem 4.9 *Suppose that* $\min\{m,n\} \geq 3$*. Then* (4.11) *has only finitely many solutions in integers x and y.*

Proof. On supposing the contrary and using the Bilu-Tichy Theorem we have

$$\binom{aX+b}{m} = \phi(f(X))$$

and

$$P_n(\tilde{a}X+\tilde{b}) = \phi(g(X)),$$

where (f,g) is a standard pair, $\phi(X) \in \mathbb{Q}[X]$ and $a, b, \tilde{a}, \tilde{b} \in \mathbb{Q}$ with $a\tilde{a} \neq 0$. We will prove that $k := \deg \phi = 1$. Indeed, it is clear that the ratio of the leading coefficients of the polynomials $\binom{aX+b}{m}$ and $P_n(\tilde{a}X+\tilde{b})$ is a kth power in $\mathbb{Q}$. On the other hand, this ratio is

$$\frac{a^m \cdot n!}{2^n \cdot \tilde{a}^n \cdot m!}.$$

Since $m = k \cdot \deg f$ and $n = k \cdot \deg g$ are divisible by k, then the number $n!/m!$ is a kth power in $\mathbb{Q}$. Lemma 4.7 gives that $k = 1$ or $k \geq 2, |n-m| = 1$. However, in the second case, $2 \leq k \leq \gcd(m,n) = 1$ and we have a contradiction. Thus we obtain

$$\binom{aX+b}{m} = e_1 f(X) + e_0$$

and

$$P_n(\tilde{a}X+\tilde{b}) = f_1 g(X) + f_0,$$

where e_0, e_1, f_0, f_1 are rational numbers with $e_1 f_1 \neq 0$. By the condition $\min\{m,n\} \geq 3$, (f,g) is not a standard pair of the second kind, further by Theorem 4.8, we get that (f,g) is not a standard pair of the fifth kind. Using Lemma 4.8 and Theorem 4.8 our theorem is proved apart from the case $(m,n) = (3,3)$. In this case the corresponding curve is

$$\frac{x(x-1)(x-2)}{6} - \frac{4}{3}y^3 - 2y^2 - \frac{8}{3}y - 1 = 0,$$

its genus determined by MAPLE is one, so we have only finitely many integer solutions.

Family 8: equation

$$P_m(x) = y^n, \tag{4.12}$$

where $m \geq 2$ is fixed and $x, y, n \geq 2$ are unknown positive integers with $(m,n) \neq (2,2)$.

In the trivial case $(m,n) = (2,2)$ we have $P_2(x) = 2x^2 + 2x + 1$ so the corresponding Diophantine equation is

$$2x^2 + 2x + 1 = y^2,$$

or equivalently,

$$(2x+1)^2 - 2y^2 = -1$$

which is a Pellian equation with infinitely many solutions. We can rewrite the polynomial $P_n(X)$ as

$$P_n(X) = \sum_{j=0}^{n} \binom{n}{j} \binom{X+n-j}{n} = 2^n \binom{X}{n} + f(X),$$

where $f(X)$ is an integer-valued polynomial of degree $< n$. So from Lemma 4.1 we get that $P_n(X)$ has at least three simple zeros for $n \geq 6$. In the remaining cases we obtain

$$P_2(X) = 2X^2 + 2X + 1, P_3(X) = \frac{4}{3}X^3 + 2X^2 + \frac{8}{3}X + 1,$$

$$P_4(X) = \frac{2}{3}X^4 + \frac{4}{3}X^3 + \frac{10}{3}X^2 + \frac{8}{3}X + 1,$$

and

$$P_5(X) = \frac{4}{15}X^5 + \frac{2}{3}X^4 + \frac{8}{3}X^3 + \frac{13}{3}X^2 + \frac{46}{15}X + 1,$$

and one can calculate their non-zero discriminants showing that these polynomials possess only simple zeros. Thus the following statement follows from Lemmata 4.2 and 4.3.

Theorem 4.10 *Let m, n be integers with $m \geq 2, n \geq 2$ and suppose that $(m, n) \neq (2, 2)$. The equation (4.12) in integers x, y and n implies $\max\{|x|, |y|, n\} < c_1 1$ where $c_1 1$ is an effectively computable constant depending only on m.*

Cohn [27] resolved the equation $x^2 + 1 = y^n$ and proved that all the solutions of this equation in integers x, y, n with $n > 1$ are $x = y = 1$ and $x = 239, y = 13, n = 4$. Using this result we have

Theorem 4.11 *All the solutions of the equation $P_2(x) = y^n$ in integers x, y and $n > 2$ are $x = 0$, $y = 1$ and $x = 119$, $y = 13$, $n = 4$.*

We note that Theorems 4.10 and 4.11 are new.

Family 9: equation

$$P_m(x) = P_n(y), \tag{4.13}$$

where $n > m \geq 2$ are fixed integers and x, y are unknown integers.

Hajdu studied the equation (4.13) for small values of m and n and resolved the corresponding elliptic type Diophantine equations, see [45] and [46]. Further, he conjectured that the equation has only finitely many solutions for $n > m = 2$. This conjecture was confirmed by Kirschenhofer, Pethő and Tichy [53]. Later, using the Bilu-Tichy Theorem, Bilu, Stoll and Tichy [81] extended their result to the general case by proving an ineffective finiteness statement for the number of solutions x and y for every pair (m, n).

Summary

Our dissertation consists of four chapters. Since there is no universal finite algorithm for the solvability of any types of Diophantine equations in general, methods for solving certain classes of Diophantine equations became in the center of interest. One of the main tools for solving these Diophantine equations is Baker's inequality giving a non-trivial lower bound for linear logarithmic forms. Based on these results there were born further improvements and applications. See [3], [5], [6] and [7]. A sharper version of the original theorem is the Baker-Wüstholz Theorem, see Theorem A.

In their theorem Schinzel and Tijdeman used the Baker method to get an effective result with also the exponent as a variable for the hyper- and superelliptic type equations, see Theorem E.

There are also theorems which present general results for the equations of the type $f(x) = g(y)$. One of the two key results is the Bilu-Tichy Theorem, see Theorem J. We introduced here the notion of the five standard pairs as well.

In the second chapter we were dealing with the polynomial values of repdigit numbers. On one hand we have the lth order k dimensional polygonal numbers of the form (2.1) with special values like $l = 3$ the binomial coefficients or $k = 2$ or $k = 3$ the corresponding polygonal or pyramidal numbers respectively. Equal values of polynomial numbers have already been widely investigated. See Dickson [30], Győry [38], Kiss [54], Brindza

[18], Avanesov [2], Pintér [66], de Weger [86], Bugeaud, Mignotte, Stoll, Siksek, Tengely [25] and Hajdu, Pintér [47], Brindza, Pintér, Turjányi [23] and Pintér, Varga [69].

On the other hand we have an other important class of combinatorial numbers i.e. the repdigits, generalised repdigits (2.3), repunits and generalized repunits (2.2). Results concerning these numbers can be found in [35] and in [77, Chapter 12], Ballew and Weger [10] and Keith [52].

Using the effective finiteness criterion in the theorem of Schinzel and Tijdeman and with elementary tools we proved effective finiteness theorems in the general case for the equal values of polygonal numbers and (generalized) repdigits i.e. for equations (2.4), (2.5) and (2.6). These are the results in Theorems 2.1, 2.2 and 2.3.

In our numerical investigations we took the polygonal numbers $f_{k,l}(x)$ in (2.5) with $k \in \{2, 3\}$, and $l \in \{3, 4, \ldots, 20\}$ and repdigits with $d \in \{1, 2, \ldots, 9\}$. The right-hand side of the equation is of degree 2 or 3 respectively and by reducing the left-hand side to a polynomial of degree 3 or 2 respectively we obtain an elliptic equation which can further be solved by the program package MAGMA [17]. This way we solved these equations completely. These results are contained in Theorem 2.4.

Other related equations, corresponding to larger values of the parameter k could not be solved because of certain technical difficulties.

In the third chapter we introduced some new results concerning equal values of trinomials in the most general case.

There have already been several previous partial results concerning some special and classical cases of equal values of trinomials. See Mordell [62], Bugeaud et. al. [25] and Mignotte and Pethő [61].

Let a, b, c, d, e, m, n, p, q be fixed rational integers. As a new result we managed to give an ineffective finiteness criterion for the general equation $ax^m + bx^n + c = dy^p + ey + q$. The result is contained in Theorem 3.1. The proof is mainly based on the ineffective Bilu-Tichy Theorem, on the decomposition properties of trinomials due to Fried and Schinzel and on the theorem of Hajós on the multiplicity of the zeros of trinomials.

In the special case $ax^m + bx^n + c = dy^2 + ey$ where $p = 2$ in integers x and y we obtained an effective upper bound for the size of solutions.

The proof was mainly based on Brindza's theorem on hyperelliptic equations and on the fact that apart from the exceptional cases the corresponding trinomial possesses at least three zeros of odd multiplicity. Our result is presented in Theorem 3.2.

In the fourth chapter we introduced an extended examination of the equal values of different standard counting polynomials. A standard counting polynomial gives back by definition the number of integral points contained in the body.

The four bodies in question are the unit cube, simplex, pyramid and octahedron. The discrete geometrical problem behind the equations is that for given positive integers m and n respectively when do two bodies with dimensions m and n contain equally many integer points. This problem leads to 9 nontrivial families of Diophantine equations, see Table 4.1. We gave a survey of known results concerning these equations and we also introduced new results.

For the 3rd, 4th, 7th and 8th families of equations both effective and ineffective new results were introduced. These results are based on the Bilu-Tichy Theorem and on the Baker Theorem respectively. These new results are presented in Theorems 4.4, 4.5, 4.6, 4.8, 4.9, 4.10 and 4.11. We proposed a conjecture on all solutions based on extensive numerical investigations for Family 1, see Conjecture 1.

The proofs further relied on Pintér's result on the number of simple zeros in polynomials [65], on Rakaczki's ineffective results on equation $S_m(x) = g(y)$ [71], on Schinzel and Tijdeman's effective result [76], on Rakaczki's effective result on equation $s(1^m + 2 + m + \cdots + x^m) + r = y^n$ [72], on Yuan's effective finiteness theorem on $a\binom{x}{m} = by^n + c$ [88], on

Győry's effective result on the equation $\binom{n}{k} = x^l$ [38] and on the theorem of Erdős and Selfridge on the product of consecutive integers [34].

Összefoglaló

Disszertációnk négy fejezetből áll, az első fejezet a Bevezető.

Ismeretes Hilbert 10. problémája alapján, hogy tetszőleges diofantikus egyenlet megoldhatósága általánosan nem eldönthető. Így a diofantikus egyenletek megodhatóságát és megoldási módszereit vizsgálva a figyelem az egyes speciális és így már kezelhető egyenlettípusok vizsgálata felé fordult. Bevezetőnkben bemutattuk a legfontosabb, egyenletek bizonyos, különböző típusai illetve osztályai esetén alkalmazható eredményeket.

Ilyen, nagy jelentőségű eredmény a Baker-módszer, amely diofantikus egyenletek több osztálya esetén is alkalmazható. Lásd [77]. Alapja a Baker-egyenlőtlenség, amely lineáris logaritmikus formák egy nem triviális alsó becslését adja meg. Ebben a témakörben további részletek a [3], [5], [6] és [7] cikkekben és jegyzetekben találhatók.

A következő tételben bemutatjuk az eredeti tétel egy élesítését, a Baker-Wüstholz tételt.

Legyenek $\alpha_1, \ldots, \alpha_n$ 0-tól és 1-től különböző algebrai számok és jelölje $\log \alpha_1, \ldots, \log \alpha_n$ a logaritmusok egy-egy rögzített értékét. Legyen K a $\mathbb{Q}$ racionális számtest $\alpha_1, \ldots, \alpha_n$ általi algebrai bővítése, melynek fokát jelölje d.

Legyen továbbá $A_j = \max(H(\alpha_j), e)$, ahol jelölje $H(\alpha_j)$ az α_j klasszikus magasságát, amely alatt az α_j algebrai szám definiáló főpolinomjában szereplő együtthatók abszolutértékének maximumát értjük és legyen $e =$

$2,718\ldots.$

Tétel A. *Legyenek* $b_1, \ldots, b_n$ *nem mind azonosan* 0 *racionális egészek és tegyük föl, hogy* $B \geq \max |b_j|$*. Ha* $\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_k \log \alpha_k \neq 0$ *teljesül, akkor*

$$\log |\Lambda| > -(16nd)^{2(n+2)} \log A_1 \ldots \log A_n \log B.$$

A Baker-módszert felhasználva Schinzel és Tijdeman effektív végességi állítást nyert az x, y egész változójú

$$f(x) = by^m,$$

($f \in \mathbb{Z}[x]$, $\deg f \geq 2$, $m \geq 2$ és $b \in \mathbb{Z}$, $b \neq 0$ rögzített) hiper- illetve szuper-elliptikus egyenletekre abban az esetben is, amikor a kitevőt is változónak tekintjük.

Tétel E. *Legyen* $f(X)$ *racionális egész együtthatós polinom, melynek van legalább két különböző gyöke. Legyenek* $b \neq 0$*,* $m \geq 0$*, továbbá* x *és* y *olyan racionális egészek, melyekre* $|y| > 1$*, és legyen*

$$f(x) = by^m.$$

Ekkor m *értéke felülről korlátos, felső korlátja kiszámítható konstans, mely csak* b*-től és* f*-től függ.*

Tudományos munkánk során bizonyos szeparábilis diofantikus egyenletek vizsgálatával foglalkoztunk. Általánosan az $f(x) = g(y)$ típusú szeparábilis diofantikus egyenletekre effektív eredmények nem léteznek.

Léteznek ugyanakkor az $f(x) = g(y)$ típusú egyenletekre általános esetben is ineffektív, végességi állítások. Az egyik ilyen csúcseredmény Davenport, Lewis és Schinzel tétele [29], a másik, mely ennek egy élesítése, a Bilu-Tichy tétel.

Tétel J. *Legyenek $P(X)$, $Q(X) \in \mathbb{Q}[X]$ olyan nem konstans polinomok, melyekre a $P(x) = Q(y)$ egyenletnek végtelen sok x, y korlátos nevezőjű megoldása van. Ekkor $P = \phi \circ f \circ \lambda$ és $Q = \phi \circ g \circ \mu$, ahol $\lambda(X)$, $\mu(X) \in \mathbb{Q}[X]$ lineáris polinomok, $\phi(X) \in \mathbb{Q}[X]$ és $(f(X), g(X))$ pedig egy standard pár.*

Ennek kapcsán szükség volt a dolgozatunkban ismertetett *standard párok öt típusá*nak definíciójára is.

A második fejezetben repdigit számok figurális értékeit vizsgáltuk. Egy felől vizsgáltuk az

$$f_{k,l}(x) = \frac{x(x+1)\cdots(x+k-2)((l-2)x+k+2-l)}{k!},$$

l-ed rendű k dimenziójú figurális számokat melyek speciálisan $l = 3$ esetén a binomiális együtthatók, $k = 2$ illetve $k = 3$ esetén a megfelelő poligonális illetve piramidális számok. Polinomiális számok egyenlő értékeit már sokszor vizsgálták. Lásd Dickson [30], Győry [38], Kiss [54], Brindza [18], Avanesov [2], Pintér [66], de Weger [86], Bugeaud, Mignotte, Stoll, Siksek, Tengely [25] és Hajdu, Pintér [47], Brindza, Pintér, Turjányi [23] és Pintér, Varga [69].

Másfelől komibnatorikus számok egy fontos osztályának, a repdigiteknek $d \cdot \frac{10^n-1}{10-1}$, $1 \leq d \leq 9$, továbbá tetszőleges $b \geq 2$ esetén általánosított repdigiteknek $d \cdot \frac{b^n-1}{b-1}$, illetve $d = 1$ esetén repunitoknak és általánosított repunitoknak $\frac{b^n-1}{b-1}$ a poligonális számokkal egyenlő értékeit vizsgáltuk. Lásd repdigitekkel kapcsolatos eredmények [35] és [77, 12. fejezet], Ballew és Weger [10] és Keith [52].

Schinzel és Tijdeman effektív végességi kritériumának felhasználásával illetve elemi módszerek segítségével általános esetben poligonális számok és (általánosított) repdigitek egyenlő értékeire vonatkozó három effektív végességi tételt mondtunk ki.

1. Tétel. *Tegyük fel, hogy $k \geq 3$ vagy $k = 2$ és $l = 4$ vagy $l > 13$. Ekkor a*

$$d \cdot \frac{b^n - 1}{b - 1} = f_{k,l}(x)$$

egyenletnek csak véges sok x és n egész megoldása van, továbbá

$$\max(|x|, n) < c_1,$$

ahol c_1 effektíven kiszámolható konstans, amely csak k, l, b és d értékétől függ. $k = 2$ és $l \in \{3, 5, 6, 7, 8, 9, 10, 11, 12\}$ esetén a 2.4 egyenletnek végtelen sok megoldása van a b, d paraméterek végtelen sok értéke mellett.

2. Tétel. *A*

$$d \cdot \frac{10^n - 1}{10 - 1} = f_{k,l}(x)$$

egyenletnek $k \geq 2$ esetén csak véges sok egész n, x megoldása van kivéve a $(d, l) = (3, 8)$ esetet. Ebben az esetben az egyenletnek végtelen sok expliciten megadható megoldása van.

3. Tétel. *A*

$$\frac{b^n - 1}{b - 1} = f_{k,l}(x)$$

egyenletnek $k \geq 2$ esetén csak véges sok egész n, x megoldása van, kivéve, ha $(b, l) = (4, 8), (9, 3), (9, 6), (25, 5)$. Ezekben az esetekben az egyenletnek végtelen sok expliciten megadható megoldása van.

Numerikusan az $f_{k,l}(x)$ alakú figurális számok illetve a repdigitek azonos értékeit vizsgáltuk $k \in \{2, 3\}$ és $l \in \{3, 4, ..., 20\}$, illetve $d \in \{1, 2, ...9\}$ esetén. A bal oldali polinomot 2 illetve 3 fokúra redukálva elliptikus egyenletet kaptunk, melyet a MAGMA programcsomaggal oldottunk meg. Így az összes egyenletet sikerült megoldanunk.

4. Tétel. *A*

$$d \cdot \frac{10^n - 1}{10 - 1} = f_{k,l}(x)$$

egyenlet összes nemtriviális megoldását $k = 2, 3$ esetén rendre a 2.2 illetve az 2.1 táblázat tartalmazza.

Megjegyezzük, hogy a fenti egyenletekkel azonos típusú egyenletek vizsgálata nagyobb k paraméter esetén technikai nehézségekbe ütközött.

A harmadik fejezetben trinomok egyenlő értékeivel kapcsolatos új eredményeinket mutattuk be a legáltalánosabb esetben.

Trinomok egyenlő értékeire vonatkozó eredmények már születtek korábban is néhány speciális és klasszikus esetben. Az x és y egész változójú $x^3 - x = y^2 - y$ diofantikus egyenlet összes megoldását Mordell [62] határozta meg algebrai számelméleti eszközök segítségével. Bugeaud és társai [25] az $x^5 - x = y^2 - y$ egyenlet összes megoldását megtalálták modern algebrai számelméleti módszerek segítségével. Az $x^m - x = y^p - y$ alakú általános egyenletre, ahol m és p, $m > p \geq 2$ rögzített egészek, Mignotte és Pethő [61] bizonyított ineffektív végességi állítást Davenport, Levis és Schinzel tételének segítségével.

Legyenek adva $a, b, c, d, e, m, n, p, q$ rögzített racionális egészek. Új eredményként ineffektív végességi kritériumot tudtunk adni az általános $ax^m + bx^n + c = dy^p + ey + q$ egyenlet esetén.

5. Tétel. *Az*

$$ax^m + bx^n + c = dy^p + ey^q$$

diofantikus egyenletnek, ahol $m > n > 0$, $p > q > 0$, $(m, n) = (p, q) = 1$, $ab \neq 0$, $de \neq 0$ és vagy $m > p \geq 3$ vagy $m = p \geq 3$, $n \geq q$ pontosan akkor van végtelen sok x, y korlátos nevezőjű megoldása, ha

$$m = p, n = q, a = dt^m, b = et^n, t \in \mathbb{Q}, c = 0$$

vagy

$$m = p = 3, n = q = 2, a^2e^3 + b^3d^2 = 0, c = -\frac{4b^3}{27a^2},$$

vagy

$$m = p = 3, n = 2, q = 1, 27a^4e^3 + b^6d = 0, c = \frac{2a^2e^3}{b^3d^2}.$$

feltételek közül valamelyik teljesül.

A $p = 2$ speciális esetben felső korlátot is tudtunk adni x és y lehetséges értékeire. Legyen $H = \max(|a|, |b|, |c|, |d|, |e|, m, n)$.

A bizonyítás az ineffektív Bilu-Tichy tételen, Fried és Schinzel trinomok dekompozíciós tulajdonságaira vonatkozó tételén illetve Hajós trinomok gyökeinek multiplicitására vonatkozó tételén alapszik.

A speciális, x és y egész változójú $ax^m + bx^n + c = dy^2 + ey$ egyenlet esetén, ahol $p = 2$, effektív felső korlátot sikerült adni a megoldások abszolut értékére.

6. Tétel. *Legyen* $m \geq 5$*,* $m > n > 0$*,* $abd \neq 0$*,* $m \neq 2n$ *és* $(m, n) \notin \{(6, 2), (6, 4)\}$*, továbbá, ha* $4dc + e^2 = 0$*, akkor legyen* $m - n \geq 3$ *vagy* $m - n = 2$ *illetve* n *páratlan. Ekkor az*

$$ax^m + bx^n + c = dy^2 + ey$$

diofantikus egyenlet x *és* y *egész megoldásaira* $\max(|x|, |y|) < c_2$*, ahol* c_2 *olyan effektíven kiszámolható konstans, amely csak* H *értékétől függ.*

A bizonyítás Brindza hiperelliptikus egyenletekre vonatkozó tételén, illetve azon a tényen alapszik, hogy a kivételes esetektől eltekintve a megfelelő trinom legalább három páratlan multiplicitású gyökkel rendelkezik.

A negyedik fejezetben különöbző számláló polinomok egyenlő értékeit vizsgáltuk és bizonyos esetekben ilyen típusú egyenletekre vonatkozó új eredményeket is megfogalmaztunk. A standard számlálópolinom definíció szerint egy adott dimenziójú test egész pontjainak a számát határozza meg.

A négy vizsgált test az egységkocka, szimplex, gúla és az oktahedron. Az egyenletekhez kapcsolódó diszkrét geometriai probléma a következő. Adott m és n pozitív egészek esetén mikor tartalmaz egy n és egy m dimenziójú test ugyanannyi egész pontot? Ez a probléma diofantikus egyenletek 9 (nemtriviális) osztályát határozza meg, lásd alábbi táblázat.

No	egyenlet	megjegyzés
1	$S_m(x) = S_n(y)$	$n > m \geq 1$
2	$S_m(x) = y^n$	$m \geq 1, n \geq 2, (m, n) \notin \{(1,2),(3,2),(3,4),(5,2)\}$
3	$S_m(x) = \binom{y}{n}$	$m \geq 1, n \geq 2, (m, n) \neq (1,2)$
4	$S_m(x) = P_n(y)$	$m \geq 1, n \geq 2, (m, n) \neq (1,2)$
5	$\binom{x}{m} = y^n$	$m \geq 2, n \geq 2, (m, n) \neq (2,2)$
6	$\binom{x}{m} = \binom{y}{n}$	$n > m \geq 2$
7	$\binom{x}{m} = P_n(y)$	$m \geq 2, n \geq 2, (m, n) \neq (2,2)$
8	$P_m(x) = y^n$	$m \geq 2, n \geq 2, (m, n) \neq (2,2)$
9	$P_m(x) = P_n(y)$	$n > m \geq 2$

A vizsgált diofantikus egyenletek 9 osztálya

Munkánkban összegyűjtöttük és összegeztük az ezzel a kilenc egyenlőséggel kapcsolatos korábbi eredményeket illetve új eredményeket is megfogalmaztunk. Széleskörű numerikus vizsgálatot követően az összes megoldásra vonatkozó sejtést is megfogamaztunk az egyenletek 1. osztálya esetén.

1. Sejtés. *Sejtésünk, hogy az $S_m(x) = S_n(y)$ egyenlet összes egész $n > m \geq 1$ és x, y megoldása az alábbi számnégyesek valamelyike.*

$$(m, n, x, y) \in \{(1,2,10,5),(1,2,13,6),(1,3,8,3),(1,5,23,3),(1,5,353,9)\}.$$

Az alább szereplő új eredményekben az általános esetre vonatkozó ineffektív végességi állításokat a Bilu-Tichy tétel segítségével kaptuk meg, míg effektív eredményeink alapja a Baker-módszer.

7. Tétel. *Legyen* $m \geq 1, n \geq 2$ *és* $(m,n) \neq (1,2)$. *Ekkor az* $S_m(x) = \binom{y}{n}$ *egyenletnek csak véges sok* x *és* y *egész megoldása van.*

8. Tétel. *Legyen* $n \in \{2,4\}$ *és* $m \geq 1$, *ahol* $(m,n) \neq (1,2)$ *vagy* $m \in \{1,3\}$ *és* $n \geq 2$. *Ekkor az* $S_m(x) = \binom{y}{n}$ *összes egész* x *és* y *megoldásaira effektív kiszámítható felső korlát adható, amely rendre csak* m*-től illetve* n*-től függ. Továbbá, ha* $m = 3$ *és* $n \geq 2$, *akkor nincs megoldás.*

9. Tétel. *Legyen* $m \in \{1,3\}$ *és* $n \geq 2$ *vagy* $n \in \{2,4\}$ *és* $m \geq 1$. *Ekkor az* $S_m(x) = P_n(y)$ *egyenlet megoldásaira teljesül, hogy* $\max(x,y) < c_9$, *ahol* c_9 *rendre csak* n*-től illetve* m*-től függö effektíven kiszámolható konstans.*

10. Tétel. *Legyen* $m \geq 2, n > 2$ *és* $(m+1,n) = 1$. *Ekkor az* $S_m(x) = P_n(y)$ *egyenletnek csak véges sok* x *és* y *egész megoldása van.*

Sejtésünk, hogy ezen tétel állításából elhagyva a $(m+1,n) = 1$ feltételt az állítás érvényben marad.

11. Tétel. *Legyen* $m \in \{2,4\}$ *és* $n \geq 3$ *vagy* $n \in \{2,4\}$ *és* $m \geq 3$. *Ekkor az* $\binom{x}{m} = P_n(y)$ *egyenlet megoldásaira* $\max(x,y) < c_{10}$, *ahol* c_{10} *rendre csak* n*-től vagy* m*-től függö effektíven kiszámítható konstans.*

12. Tétel. *Legyen* $\min\{m,n\} \geq 3$. *Ekkor az* $\binom{x}{m} = P_n(y)$ *egyenletnek csak véges sok egész* x *és* y *megoldása van.*

13. Tétel. *Legyenek* m,n *egészek, ahol* $m \geq 2, n \geq 2$ *és legyen* $(m,n) \neq (2,2)$. *Ekkor a* $P_m(x) = y^n$ *egyenlet egész* x,y *és* n *megoldásaira* $\max\{|x|, |y|, n\} < c_{11}$, *ahol* c_{11} *effektíven kiszámítható konstans, amely csak* m*-től függ.*

14. Tétel. *A* $P_2(x) = y^n$ *egyenletnek az* x, y *és* $n > 2$ *egészek körében az* $x = 0$*,* $y = 1$ *illetve* $x = 119$*,* $y = 13$*,* $n = 4$ *értékeken kívül nincs más megoldása.*

A bizonyítások során felhasznált további tételek Pintér polinomok egyszeres gyökeinek számára vonatkozó tétele [65], Rakaczki ineffektív eredménye az $S_m x = g(y)$ típusú egyenletre [71], továbbá Schinzel és Tijdeman effektív ereménye [76], Rakaczki effektív eredménye az $s(1^m + 2^m + \cdots + x^m) + r = y^n$ típusú egyenletre [72], Yuan effektív végességi tétele az $a\binom{x}{m} = by^n + c$ típusú egyenletre [88], Győry effektív eredménye az $\binom{n}{k} = x^l$ egyenletre [38] illetve Erdős és Selfridge egymást követő egészek szorzatáról szóló tétele [34].

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