Integral bases and monogenity of composite fields

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Abstract

We consider infinite parametric families of high degree number fields composed of quadratic fields with pure cubic, pure quartic, pure sextic fields and with the so called simplest cubic, simplest quartic fields. We explicitly describe an integral basis of the composite fields. We construct the index form, describe their factors and prove that the monogenity of the composite fields imply certain divisibility conditions on the parameters involved. These conditions usually can not hold, which implies the non-monogenity of the fields.

The fields that we consider are higher degree number fields, of degrees 6 up to 12. The non-monogenity of the number fields is stated very often as a consequence of the non-existence of the solutions of the index form equation. Up to our knowlegde it is not at all feasible to solve the index form equation in these high degree fields, especially not in a parametric form.

On the other hand our method implies directly the non-monogenity in almost all cases. We obtain our results in a parametric form, characterizing these infinite parametric families of composite fields.

1 Introduction

Monogenity of number fields and the existence of **power integral bases** of type $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a classical topic of algebraic number theory. The coefficients of the generators of power integral bases are obtained as solutions of the corresponding **index form equations** cf. Section 2.

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There are algorithms for the resolution of index form equations in **given specific** low degree number fields (for degrees 3 and 4 and some tedious methods for degrees 5 and 6) and in some special type of higher degree number fields see [6].

We also succeeded to solve the index form equation in certain **infinite parametric families** of number fields, using the solutions of a corresponding family of Thue equations or using congruence considerations. Remark that very often we considered parametric families of number fields, whose integral bases were not known in a parametric form. In those cases we considered the problem of monogenity in the corresponding equation order.

We considered **composites of number fields** of coprime discriminants in [5] showing a suitable factorization of the index form in this case. In [8], [7], [15] the authors considered composites of equation orders of number fields and proved that under certain congruence conditions on the defining polynomial these orders are not monogenic.

In some recent papers [9], [11] the authors developed a new and efficient technics to consider monogenity in **infinite parametric families of higher degree number fields**. The most important features of this method are that

–the integral bases are determined in a parametric form,
–the factors of the index form are explicitly calculated,
–some linear combinations of these factors are shown to have some non-trivial divisors.

In case a power integral basis (that is a solution of the index form equation) exists, these imply some divisibility conditions on the parameters. These divisibility conditions are usually not satisfied, whence the fields are not monogenic.

Note that our algorithm to determine an integral basis is based on standard methods cf. I.Stewart and D.Tall [17], J.Cook [4]. We use this algorithm for parametric families of number fields. The explicit calculation of the factors of the index form requires a very careful procedure because of the high degree, the parameters and the large number of variables of the index form.

In our former results [9], [11] we used this method to pure fields (up to degree 8) and to the family of simplest sextic number fields. Here we considerably extend our method and apply to composits of number fields up to degree 12. Up to our knowlegde this is the first time that monogenity of such high degree fields are completely characterized.

In the present paper **we present an integral basis and obtain conditions on the monogenity in composites of**

- quadratic fields and the simplest cubic fields (degree 6)
- quadratic fields and pure cubic fields (degree 6)
- quadratic fields and pure quartic fields (degree 8)
- quadratic fields and the simplest quartic fields (degree 8)
- the field $\mathbb{Q}(i\sqrt{3})$ and pure sextic fields (degree 12)

In each case we consider monogenity in the ring of integers of the composite field.

2 Power integral bases and monogenity of number fields

Here we shortly recall the concepts connected with monogenity of fields [6], what we shall use throughout. Let α be a primitive integral element of the number field K (that is $K = \mathbb{Q}(\alpha)$) of degree n with ring of integers $\mathbb{Z}_K$. The **index** of α is

$$I(\alpha) = (\mathbb{Z}_K^+ : \mathbb{Z}[\alpha]^+) = \sqrt{\left|\frac{D(\alpha)}{D_K}\right|} = \frac{1}{\sqrt{|D_K|}} \prod_{1\le i<j\le n} \left|\alpha^{(i)} - \alpha^{(j)}\right| \ ,$$

where D_K is the discriminant of K and $\alpha^{(i)}$ denote the conjugates of α.

If $B = \{b_1 = 1, b_2, \ldots, b_n\}$ is an integral basis of K, then the **index form** corresponding to this integral basis is defined by

$$I(X_2, \ldots, X_n) = \frac{1}{\sqrt{|D_K|}} \prod_{1\le i<j\le n} \left((b_2^{(i)} - b_2^{(j)})X_2 + \ldots + (b_n^{(i)} - b_n^{(j)})X_n\right)$$

(where $b_j^{(i)}$ denote the conjugates of b_j). This is a homogeneous polynomial with integral coefficients of degree $n(n-1)/2$. For the integral element

$$\alpha = x_1 + b_2 x_2 + \ldots + b_n x_n$$

(with $x_1, \ldots, x_n \in \mathbb{Z}$) we have

$$I(\alpha) = |I(x_2, \ldots, x_n)|$$

independently of x_1. The element α generates a **power integral basis** $\{1, \alpha, \ldots, \alpha^{n-1}\}$ if and only if $I(\alpha) = 1$ that is $(x_2, \ldots, x_n) \in \mathbb{Z}^{n-1}$ is a solution of the **index form equation**

$$I(x_2, \ldots, x_n) = \pm 1 \quad \text{in} \quad (x_2, \ldots, x_n) \in \mathbb{Z}^{n-1}. \tag{1}$$

In this case the ring of integers of K is a simple ring extension of $\mathbb{Z}$, that is $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and K is called **monogenic**.

In our following statements and tables for brevity we do not display the discriminants of the number fields involved but they can be easily calculated from the discriminants of the generating elements and the structure of the integral basis.

3 Composites of quadratic fields and the simplest cubic fields

Throughout this section we assume that

$$n, m \ \text{are integers}, n \neq 0, 1, \ \text{such that} \ n, m^2 + 3m + 9 \ \text{are squarefree and coprime.} \tag{2}$$

In this section we consider the composite field $K = \mathbb{Q}(\alpha, \beta)$, where

$$\begin{array}{lll} \alpha & \text{is a root of} & f(x) = x^2 - n, \\ \beta & \text{is a root of} & g(x) = x^3 - mx^2 - (m+3)x - 1. \end{array}$$

The fields $M = \mathbb{Q}(\beta)$ are called *simplest cubic fields*, see [16].

Theorem 1. *An integral basis of K is given by*

$$\left\{1, \beta, \beta^2, \frac{\alpha+1}{2}, \frac{\alpha\beta+\beta}{2}, \frac{\alpha\beta^2+\beta^2}{2}\right\}, \quad \text{if } n \equiv 1 \pmod 4,$$

$$\left\{1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2\right\}, \quad \text{if } n \equiv 2, 3 \pmod 4.$$

Proof.
It is well known that an integral basis $\{1, \omega\}$ and the discriminant D_L of $L = \mathbb{Q}(\alpha)$ are

$$\begin{array}{rl} \omega = (1+\sqrt{n})/2,\ D_L = n & \text{if } n \equiv 1 \pmod 4, \\ \omega = \sqrt{n},\ D_L = 4n & \text{if } n \equiv 2, 3 \pmod 4. \end{array} \tag{3}$$

More over, if $m^2 + 3m + 9$ is squarefree then an integral basis $\{1, \delta_1, \delta_2\}$ and the discriminant D_M of $M = \mathbb{Q}(\beta)$ are

$$\delta_1 = \beta,\ \delta_2 = \beta^2,\ D_M = (m^2 + 3m + 9)^2.$$

The discriminants D_L, D_M are coprime, hence the composite field $K = LM$ has integral basis $\{1, \delta_1, \delta_2, \omega, \delta_1\omega, \delta_2\omega\}$ with discriminant $D_K = D_L^3 \cdot D_M^2$.
□

Theorem 2. *If K is monogenic then*

$$n \mid (m^2 + 3m + 9) \pm 1 \ \text{ and } \ (m^2 + 3m + 9) \mid n^3 \pm 1, \quad \text{if } n \equiv 1 \pmod 4,$$

$$n \mid (m^2 + 3m + 9) \pm 1 \ \text{ and } \ (m^2 + 3m + 9) \mid 64n^3 \pm 1, \quad \text{if } n \equiv 2, 3 \pmod 4.$$

Throughout the paper the $\pm$ signs in the divisibility relations mean that the condition must hold either with $+$ or with $-$.

Proof.
The conjugates of α and β are

$$\alpha^{(1)} = \sqrt{n},\ \alpha^{(2)} = -\sqrt{n},\ \beta^{(1)} = \beta,\ \beta^{(2)} = \frac{-1}{1+\beta},\ \beta^{(3)} = \frac{-1-\beta}{\beta}.$$

Set

$$L^{(i,j)} = L^{(i,j)}(X_1, \ldots, X_6) = X_1 + X_2\delta_1^{(j)} + X_3\delta_2^{(j)} + X_4\omega^{(i)} + X_5\omega^{(i)}\delta_1^{(j)} + X_6\omega^{(i)}\delta_2^{(j)}.$$

for $i=1,2; j=1,2,3$. Let

$$\begin{array}{rcl} F_1 & = & \left(L^{(1,1)}-L^{(1,2)}\right)\left(L^{(1,1)}-L^{(1,3)}\right)\left(L^{(1,2)}-L^{(1,3)}\right)\cdot \\ & & \left(L^{(2,1)}-L^{(2,2)}\right)\left(L^{(2,1)}-L^{(2,3)}\right)\left(L^{(2,2)}-L^{(2,3)}\right) \\ F_2 & = & \left(L^{(1,1)}-L^{(2,1)}\right)\left(L^{(1,2)}-L^{(2,2)}\right)\left(L^{(1,3)}-L^{(2,3)}\right) \\ F_3 & = & \left(L^{(1,1)}-L^{(2,2)}\right)\left(L^{(1,1)}-L^{(2,3)}\right)\left(L^{(1,2)}-L^{(2,1)}\right)\cdot \\ & & \left(L^{(1,2)}-L^{(2,3)}\right)\left(L^{(1,3)}-L^{(2,1)}\right)(L^{(1,3)}-L^{(2,2)}\right). \end{array} \tag{4}$$

We find that

$$F_i(X_2,\dots,X_6)=f_i\cdot G_i(X_2,\dots,X_6)\ \ (i=1,2,3)$$

where

$$f_1=m^2+3m+9=\sqrt{|D_M^2|},\ \ f_2=\sqrt{|D_L^3|},\ \ f_3=1$$

and $G_i=G_i(X_2,\dots,X_6)$ $(i=1,2,3)$ are primitive polynomials with integer coefficients. By

$$f_1f_2f_3=\sqrt{|D_K|},$$

the index form equation corresponding to the given integral basis of K is just

$$G_1(x_2,\dots,x_6)\cdot G_2(x_2,\dots,x_6)\cdot G_3(x_2,\dots,x_6)=\pm 1\ \text{ in }\ x_2,\dots,x_6\in\mathbb{Z}.$$

If K admits a power integral basis, then there exist $x_2,\dots,x_6\in\mathbb{Z}$ satisfying this equation, that is

$$G_i(x_2,\dots,x_6)=\pm 1\ \ (i=1,2,3),$$

or equivalently

$$F_i(x_2,\dots,x_6)=\pm f_i\ \ (i=1,2,3).$$

Direct calculation of the factors show that the polynomials F_1+F_3 and $F_2^2-F_3$ have integer coefficients and

$$n|F_1(X_2,\dots,X_6)+F_3(X_2,\dots,X_6)\ \text{ and }\ (m^2+3m+9)|F_2^2(X_2,\dots,X_6)-F_3(X_2,\dots,X_6).$$

This immediately gives

$$n|f_1\pm f_3\ \text{ and }\ (m^2+3m+9)|f_2^2\pm f_3.$$

in case of a solution, which implies the assertion of Theorem 2.
□

4 Composites of quadratic and pure cubic fields

Throughout this section we assume that

$$\begin{array}{l} n,m\ \text{ are integers }\ n,m\neq 0,1, \\ n\ \text{ is squarefree, } m \text{ is cubefree and }\ \gcd(n,m)\in\{1,2,3\}, \\ m=uv^2 \text{ with squarefree, integers } u,v, \text{ with } (u,v)=1, \text{ and } 2,3\not| v. \end{array} \tag{5}$$

In this section we consider the composite field $K=\mathbb{Q}(\alpha,\beta)$, where

$$\begin{aligned} \alpha \quad & \text{is a root of} \quad f(x) = x^2 - n, \\ \beta \quad & \text{is a root of} \quad g(x) = x^3 - m. \end{aligned}$$

Theorem 3. *According to the behaviour of m, n mod 36, an integral basis of K is given by the following table.*

n mod 36	m mod 36	integral basis
1,5,13,17,25,29	1,10,17,19,26,35	$\left\{1, \beta, \frac{\beta^2 + uv^2\beta + v^2}{3v}, \frac{\alpha+1}{2}, \frac{(\alpha+1)\beta}{2}, \frac{(\alpha+1)(\beta^2 + uv^2\beta + v^2}{6v}\right\}$
1,5,13,17,25,29	2,3,5,6,7,11,13,14, 15,21,22,23,25,29, 30,31,33,34	$\left\{1, \beta, \frac{\beta^2}{v}, \frac{\alpha+1}{2}, \frac{(\alpha+1)\beta}{2}, \frac{(\alpha+1)\beta^2}{2v}\right\}$
2,7,10,11,14,19,22, 23,26,31,34,35	1,17,19,35	$\left\{1, \beta, \frac{\beta^2 + uv^2\beta + v^2}{3v}, \alpha, \alpha\beta, \frac{\alpha(\beta^2 + uv^2\beta + v^2)}{3v}\right\}$
2,7,10,11,14,19,22, 23,26,31,34,35	3,5,7,11,13,15,21, 23,25,29,31,33	$\left\{1, \beta, \frac{\beta^2}{v}, \alpha, \alpha\beta, \frac{\alpha\beta^2}{v}\right\}$
2,10,14,22,26,34	2,6,14,22,30,34	$\left\{1, \beta, \frac{\beta^2}{v}, \alpha, \alpha\beta, \frac{\alpha\beta^2}{2v}\right\}$
2,10,14,22,26,34	10,26	$\left\{1, \beta, \frac{\beta^2 + uv^2\beta + v^2}{3v}, \alpha, \alpha\beta, \frac{\alpha(\beta^2 + uv^2\beta + 4v^2)}{6v}\right\}$
7,11,19,23,31,35	2,6,14,22,30,34	$\left\{1, \beta, \frac{\beta^2}{v}, \alpha, \alpha\beta, \frac{(\alpha+1)\beta^2}{2v}\right\}$
7,11,19,23,31,35	10,26	$\left\{1, \beta, \frac{\beta^2 + uv^2\beta + v^2}{3v}, \alpha, \alpha\beta, \frac{(\alpha+1)(\beta^2 + uv^2\beta + 4v^2)}{6v}\right\}$
3,6,15,30	1,17,19,35	$\left\{1, \beta, \frac{\beta^2 + uv^2\beta + v^2}{3v}, \alpha, \frac{\alpha(\beta - u)}{3}, \frac{\alpha(\beta^2 + uv^2\beta + v^2)}{3v}\right\}$
3,6,15,30	5,7,11,13,23,25,29,31	$\left\{1, \beta, \frac{\beta^2}{v}, \alpha, \alpha\beta, \frac{\alpha(\beta^2 + uv^2\beta + v^2)}{3v}\right\}$
3,6,15,30	3,15,21,33	$\left\{1, \beta, \frac{\beta^2}{v}, \alpha, \alpha\beta, \frac{\alpha\beta^2}{3v}\right\}$

n mod 36	m mod 36	integral basis
21,33	1,10,17,19,26,35	$\left\{1,\beta,\frac{\beta^2+uv^2\beta+v^2}{3v},\frac{\alpha+1}{2},\frac{(\alpha+3)(\beta-u)}{6},\frac{(\alpha+1)(\beta^2+uv^2\beta+v^2)}{6v}\right\}$
21,33	2,5,7,11,13,14, 22,23,25,29,31,34	$\left\{1,\beta,\frac{\beta^2}{v},\frac{\alpha+1}{2},\frac{(\alpha+1)\beta}{2},\frac{(\alpha+3)(\beta^2+uv^2\beta+v^2)}{6v}\right\}$
21,33	3,6,15,21,30,33	$\left\{1,\beta,\frac{\beta^2}{v},\frac{\alpha+1}{2},\frac{(\alpha+1)\beta}{2},\frac{(\alpha+3)\beta^2}{6v}\right\}$
6,30	10	$\left\{1,\beta,\frac{\beta^2+v^2\beta+v^2}{3v},\alpha,\frac{\alpha(\beta+2)}{3},\frac{\alpha(\beta^2+2v^2\beta+6v^2)}{6v}\right\}$
6,30	2,14,22,34	$\left\{1,\beta,\frac{\beta^2}{v},\alpha,\alpha\beta,\frac{\alpha(\beta^2+uv^2\beta+4v^2)}{6v}\right\}$
6,30	6,30	$\left\{1,\beta,\frac{\beta^2}{v},\alpha,\alpha\beta,\frac{\alpha\beta^2}{6v}\right\}$
6,30	26	$\left\{1,\beta,\frac{\beta^2+2v^2\beta+v^2}{3v},\alpha,\frac{\alpha(\beta+1)}{3},\frac{\alpha(\beta^2+2v^2\beta+4v^2)}{6v}\right\}$
3,15	10	$\left\{1,\beta,\frac{\beta^2+v^2\beta+v^2}{3v},\alpha,\frac{\alpha(\beta+2)}{3},\frac{\alpha(\beta^2+2v^2\beta+6)+\beta^2+4v^2\beta+4v^2}{6v}\right\}$
3,15	2,14,22,34	$\left\{1,\beta,\frac{\beta^2}{v},\alpha,\alpha\beta,\frac{\alpha(\beta^2+uv^2\beta+4v^2)+3\beta^2}{6v}\right\}$
3,15	6,30	$\left\{1,\beta,\frac{\beta^2}{v},\alpha,\alpha\beta,\frac{(\alpha+3)\beta^2}{6v}\right\}$
3,15	26	$\left\{1,\beta,\frac{\beta^2+2v^2\beta+v^2}{3v},\alpha,\frac{\alpha(\beta+1)}{3},\frac{(\alpha+1)(\beta^2+2v^2\beta+4v^2)}{6v}\right\}$

Proof.
An integral basis $\{1,\omega\}$ and discriminant D_L of $L=\mathbb{Q}(\alpha)$ are given in (3). An integral basis $(1,\delta_1,\delta_2)$ and discriminant D_M of $M=\mathbb{Q}(\beta)$ are given (cf. [3], Theorem 6.4.13) by:

$$\begin{aligned} &\left\{1,\beta,\frac{\beta^2}{v}\right\}, \quad D_M=-27u^2v^2 \quad \text{if } u^2\not\equiv v^2 \pmod 9,\\ &\left\{1,\beta,\frac{v^2+uv^2\beta+\beta^2}{3v}\right\}, \quad D_M=-3u^2v^2 \quad \text{if } u^2\equiv v^2 \pmod 9. \end{aligned} \tag{6}$$

Denote by $D_{K/L}$ and $D_{K/M}$ the relative discriminants of K over L and M, respectively. We have (cf. [14])

$$D_K=N_{L/\mathbb{Q}}(D_{K/L})\cdot D_L^3, \quad D_K=N_{M/\mathbb{Q}}(D_{K/M})\cdot D_M^2. \tag{7}$$

We have $\gcd(m,n)=1,2,3$. Denote by $\nu_p(k)$ the exponent of the prime p in an integer k. Set $u_0=u/(2^{\nu_2(u)}3^{\nu_3(u)})$, $n_0=n/(2^{\nu_2(n)}3^{\nu_3(n)})$. The above discriminant relations imply

$$3n_0^3(u_0v)^4|D_K.$$

On the other hand $\{1,\delta_1,\delta_2,\omega,\delta_1\omega,\delta_2\omega\}$ is a basis in K (not necessarily integral basis), whence D_K divides the discriminant of this basis:

$$D_K|D_L^3D_M^2.$$

Therefore D_K must be of the form $\pm 2^r 3^s n_0^3 u_0^4 v^4$. Following the algorithm of [4] in order to obtain an integral basis, we start with the initial basis $\{b_1, \ldots, b_6\} = \{1, \delta_1, \delta_2, \omega, \delta_1\omega, \delta_2\omega\}$ and we test if its discriminant can be reduced by a 2-factor or by a 3-factor by interchanging one of its elements by a new element.

For $p = 2$ and $p = 3$ we perform the following procedure separately. Let

$$\mu = \frac{\lambda_1 b_1 + \ldots + \lambda_6 b_6}{p}. \tag{8}$$

We let λ_i $(i = 1, \ldots, 6)$ run through $\{0, 1, \ldots, p-1\}$ and calculate the defining polynomial of μ:

$$F(x) = \prod_{i=1}^{6} \left(x - \frac{\lambda_1 b_1^{(i)} + \ldots + \lambda_6 b_6^{(i)}}{p} \right).$$

Here $b_j^{(i)}$ denote the conjugates of b_j. For each λ_i $(i = 1, \ldots, 6)$ this polynomial can be written as

$$F(x) = x^6 + \frac{e_5}{p} x^5 + \ldots + \frac{e_1}{p^5} x + \frac{e_0}{p^6}$$

with integers $e_5 \ldots, e_1, e_0$ depending on m and n. We calculated for which values $m(\mathrm{mod} p^6)$ and $n(\mathrm{mod} p^6)$ does this polynomial have integer coefficients, that is for which values $m(\mathrm{mod} p^6)$ and $n(\mathrm{mod} p^6)$ we have

$$\begin{aligned} e_5 &\equiv 0 \ (\mathrm{mod} p), \\ &\ldots \\ e_1 &\equiv 0 \ (\mathrm{mod} p^5), \\ e_0 &\equiv 0 \ (\mathrm{mod} p^6). \end{aligned}$$

For the selected pairs $m(\bmod p^6)$ and $n(\bmod p^6)$ the basis element having coefficient $\lambda_i = 1$ can be replaced by the above element μ to diminish the discriminant by a p^2 factor. (Any non-zero λ_i can be transformed into $\lambda_i = 1$ by multiplying (8) by the inverse of λ_i modulo p and by subtracting a suitable integer element.)

This procedure is continued with p until no reduction of the discriminant of the basis is possible. Then the same procedure is performed for the other value of p, as well.

Finally we combine the basis $\{b_1, \ldots, b_6\}$ of the 2-maximal order and the basis $\{f_1, \ldots, f_6\}$ of the 3-maximal order of K into a basis of the maximal order of K.

First we remark that we choose the basis elements b_i and f_i so that in the numerator of b_i and f_i the coefficient of x^{i-1} is equal to 1 and the coefficients of $x^i, x^{i+1}, \ldots$ are equal to 0. We construct the integral basis $h_1 = 1, h_2, \ldots, h_n$ of K with the same property. Assume that b_i has denominator 2^{k_2} and f_i has denominator 3^{k_3}. Calculate y_2 and y_3 with $2^{k_2} y_2 \equiv 1(\bmod 3^{k_3})$ and $3^{k_3} y_3 \equiv 1(\bmod 2^{k_2})$. Then $x = 2^{k_2} y_2 + 3^{k_3} y_3$ is a solution of the system $x \equiv 1(\bmod 2^{k_2})$, $x \equiv 1(\bmod 3^{k_3})$. We also have $x \equiv 1(\bmod 2^{k_2} 3^{k_3})$. We set

$$h_i' = y_3 b_i + y_2 f_i = \frac{y_3 3^{k_3} (2^{k_2} b_i) + y_2 2^{k_2} (3^{k_3} f_i)}{2^{k_2} 3^{k_3}}$$

which is an algebraic integer. The coefficient of x^{i-1} in h_i' is

$$c_{i-1} = \frac{y_3 3^{k_3} + y_2 2^{k_2}}{2^{k_2} 3^{k_3}}.$$

The numerator can be written as $1 + \ell \cdot 2^{k_2} 3^{k_3}$ (with an integer ℓ), hence

$$c_{i-1} = \frac{1}{2^{k_2} 3^{k_3}} + \ell.$$

We set $h_i = h_i' - \ell x^{i-1}$ which is also an algebraic integer. In the numerator of h_i the coefficient of x^{i-1} is equal to 1 and the coefficients of $x^i, x^{i+1}, \ldots$ are equal to 0.

We show that b_i can be expressed as a linear combination of $(1, x, \ldots, x^{i-2}, h_i)$ (the same holds for f_i) which proves that $\{h_1 = 1, h_2, \ldots, h_n\}$ is indeed an integral basis, being a 2-maximal order and a 3-maximal order of K.

We make use of $3^{k_3} y_3 \equiv 1 (\bmod\ 2^{k_2})$ here, which implies $3^{k_3} y_3 = 1 + q 2^{k_2}$ with an integer q. We have

$$3^{k_3} h_i' = y_3 3^{k_3} b_i + y_2 3^{k_3} f_i = (1 + q 2^{k_2}) b_i + y_2 3^{k_3} f_i = b_i + q 2^{k_2} b_i + y_2 3^{k_3} f_i.$$

Observe that $2^{k_2} b_i$ and $3^{k_3} f_i$ is a linear combination of $(1, x, \ldots, x^{i-1})$ with integer coefficients. We have

$$b_i \in \mathcal{L}(1, x, \ldots, x^{i-1}, h_i') \subseteq \mathcal{L}(1, x, \ldots, x^{i-2}, h_i)$$

since the coefficient of x^{i-1} in the numerator of h_i is equal to 1. ($\mathcal{L}$ denotes the set of the linear combinations with integer coefficients of the elements involved.) This implies our assertion.

Finally we collect those pairs m, n for which we obtained the same type of integral basis.

□

Remark. The above method obviously works for any distinct primes, as well.

Theorem 4. *If K admits a power integral basis, then the following divisibility conditions must hold:*

n mod 36	m mod 36	1	2
$1,5,13,17,25,29$	$1,10,17,19,26,35$	$n \mid 3m^2 \pm 1$	$m \mid n^3 \pm 1$
$1,5,13,17,25,29$	$2,3,5,6,7,11,13,14,$ $15,21,22,23,25,29,$ $30,31,33,34$	$n \mid 27m^2 \pm 1$	$9m \mid n^3 \pm 1$
$2,7,10,11,14,19,22,$ $23,26,31,34,35$	$1,17,19,35$	$4n \mid 3m^2 \pm 1$	$m \mid 64n^3 \pm 1$
$2,7,10,11,14,19,22,$ $23,26,31,34,35$	$3,5,7,11,13,15,21,$ $23,25,29,31,33$	$4n \mid 27m^2 \pm 1$	$9m \mid 64n^3 \pm 1$
$2,7,10,11,14,19,$ $22,23,26,31,34,35$	$2,6,14,22,30,34$	$4n \mid \frac{27m^2}{2} \pm 2$	$9m \mid 16n^3 \pm 2$
$2,7,10,11,14,19,$ $22,23,26,31,34,35$	$10,26$	$4n \mid \frac{3m^2}{2} \pm 2$	$m \mid 16n^3 \pm 2$
$3,6,15,30$	$1,17,19,35$	$\frac{4n}{3} \mid m^2 \pm 3$	$3m \mid \frac{64n^3}{9} \pm 3$
$3,6,15,30$	$3,5,7,11,13,15,$ $21,23,25,29,31,33$	$4n \mid 9m^2 \pm 3$	$9m \mid \frac{64n^3}{9} \pm 3$
$21,33$	$1,10,19$	$\frac{n}{3} \mid m^2 \pm 3$	$3m \mid \frac{n^3}{9} \pm 3$
$21,33$	$2,3,5,6,7,11,13,$ $14,15,21,22,23,25,$ $29,30,31,33,34$	$n \mid 9m^2 \pm 3$	$9m \mid \frac{n^3}{9} \pm 3$
$21,33$	$17,26,35$	$\frac{n}{3} \mid m^2 \pm 3$	$m \mid \frac{n^3}{9} \pm 3$
$3,6,15,30$	$10,26$	$\frac{4n}{3} \mid \frac{m^2}{2} \pm 6$	$3m \mid \frac{16n^3}{9} \pm 6$
$3,6,15,30$	$2,6,14,22,30,34$	$4n \mid \frac{9m^2}{2} \pm 6$	$9m \mid \frac{16n^3}{9} \pm 6$

Proof.
Let

$$\alpha^{(1)} = \sqrt{n},\ \alpha^{(2)} = -\sqrt{n}\ \beta^{(1)} = \sqrt[3]{m},\ \beta^{(2)} = \varepsilon\sqrt[3]{m},\ \beta^{(3)} = \varepsilon^2\sqrt[3]{m}$$

with $\varepsilon = \exp(2\pi i/3)$. Assume that $\{b_1 = 1, b_2, \ldots, b_6\}$ is an integral basis of K. The elements of the integral basis are composed of α and β hence it is unique to denote the conjugate of b_k corresponding to $\alpha^{(i)}$ and $\beta^{(j)}$ by $b_k^{(i,j)}$ $(i = 1,2; j = 1,2,3)$. Set

$$L^{(i,j)} = L^{(i,j)}(X_1, X_2, \ldots, X_6) = X_1 + b_2^{(i,j)}X_2 + \ldots + b_6^{(i,j)}X_6.$$

Using the $L^{(i,j)}$ we construct the same polynomials F_1, F_2, F_3 like in (4). We write these polynomials in the form

$$F_i(X_2, \ldots, X_6) = f_i \cdot G_i(X_2, \ldots, X_6) \ \ (i = 1,2,3)$$

where f_i are integers or square roots of integers with $(f_1 f_2 f_3)^2 = |D_K|$ depending on the parameters only and G_i are primitive polynomials with integer coefficients. Then the index form equation corresponding to the basis $\{b_1 = 1, b_2, \ldots, b_6\}$ can be written as

$$G_1(x_2, \ldots, x_6) \cdot G_2(x_2, \ldots, x_6) \cdot G_3(x_2, \ldots, x_6) = \pm 1, \text{ in } x_2, \ldots, x_6 \in \mathbb{Z},$$

that is

$$G_i(x_2, \ldots, x_6) = \pm 1 \ \ (i = 1, 2, 3),$$

or equivalently

$$F_i(x_2, \ldots, x_6) = \pm f_i \ \ (i = 1, 2, 3).$$

If a power integral basis in K exists then there exist $x_2, \ldots, x_6 \in \mathbb{Z}$ satisfying the above equations. Calculating the explicit form of the F_i in each case of the integral basis we always have

$$n | F_1(X_2, \ldots, X_6) + F_3(X_2, \ldots, X_6) \text{ and } m | F_2^2(X_2, \ldots, X_6) - F_3(X_2, \ldots, X_6),$$

where again the polynomials $F_1 + F_3$ and $F_2^2 - F_3$ have integer coefficients. These imply the divisibility conditions of the Theorem.
□

5 Composites of quadratic and pure quartic fields

Throughout this section we assume that

$$\begin{aligned} & n, m \text{ are squarefree integers}, n, m \neq 0, 1, \\ & \gcd(n, m) \in \{1, 2\}, \\ & \text{if } m = n, \text{ then } m = n \notin \{-2, -1, 2\}. \end{aligned} \tag{9}$$

In this section we consider the composite field $K = \mathbb{Q}(\alpha, \beta)$, where

$$\begin{array}{lll} \alpha & \text{is a root of} & f(x) = x^2 - n, \\ \beta & \text{is a root of} & g(x) = x^4 - m. \end{array}$$

Theorem 5. *According to the behaviour of m, n mod 8, an integral basis of K is given by the following table:*

n mod 8	m mod 8	integral basis
1,5	2,3,6,7	$\left\{1, \beta, \beta^2, \beta^3, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2}{2}, \frac{\beta^3\alpha+\beta^3}{2}\right\}$
1,5	1	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2+\alpha+1}{4}, \frac{\beta^3\alpha+\beta^3+\beta^2\alpha+\beta^2+\beta\alpha+\beta+\alpha+1}{8}\right\}$
1,5	5	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta}{2}, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2+\alpha+1}{4}, \frac{\beta^3\alpha+\beta^3+\beta\alpha+\beta}{4}\right\}$
2,6	1	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \alpha, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^2\alpha+\alpha}{2}, \frac{\beta^3\alpha+\beta^2\alpha+\beta\alpha+\alpha}{4}\right\}$
2,6	5	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta}{2}, \alpha, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^2\alpha+\alpha}{2}, \frac{\beta^3\alpha+\beta^2\alpha+\beta\alpha+\alpha}{4}\right\}$
3,7	2,6	$\left\{1, \beta, \beta^2, \beta^3, \alpha, \frac{\beta^3+\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2}{2}, \frac{\beta^3\alpha+\beta^3}{2}\right\}$
3,7	3,7	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2+\alpha}{2}, \frac{\beta^3+\beta\alpha}{2}, \frac{\beta^2\alpha+1}{2}, \frac{\beta^3\alpha+\beta}{2}\right\}$
3,7	1	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \alpha, \frac{\beta\alpha+\beta+\alpha+1}{2}, \frac{\beta^2\alpha+\beta^2+2\beta+3\alpha+1}{4}, \frac{\beta^3\alpha+\beta^2\alpha+\beta\alpha+\alpha}{4}\right\}$
3,7	5	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta}{2}, \alpha, \frac{\beta\alpha+\beta+\alpha+1}{2}, \frac{\beta^2\alpha+\alpha}{2}, \frac{\beta^3\alpha+\beta^3+\beta^2\alpha+\beta^2+\beta\alpha+\beta+\alpha+1}{4}\right\}$
2	2	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2+\alpha}{2}, \frac{\beta^3+\beta\alpha}{2}, \frac{\beta^2\alpha+2}{4}, \frac{\beta^3\alpha+2\beta}{4}\right\}$
2	3	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2\alpha+\alpha}{2}, \frac{\beta^2+\beta\alpha+1}{2}, \frac{\beta^3+\beta^2\alpha+\beta}{2}, \frac{\beta^3\alpha+2\beta^3+\beta^2\alpha+\beta\alpha+\alpha+2}{4}\right\}$
2	6	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2+\alpha}{2}, \frac{\beta^3+\beta\alpha}{2}, \frac{\alpha\beta^2}{2}, \frac{\beta^3\alpha+2\beta^3+2\beta}{4}\right\}$
2	7	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2\alpha+2\beta+\alpha}{4}, \frac{\beta^2+\beta\alpha+1}{2}, \frac{\beta^3+\beta^2\alpha+\beta}{2}, \frac{\beta^3\alpha+2\beta^2+\beta\alpha}{4}\right\}$
6	2	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2+\alpha}{2}, \frac{\beta^3+\beta\alpha}{2}, \frac{\alpha\beta^2}{2}, \frac{\beta^3\alpha+2\beta^3+2\beta}{4}\right\}$
6	3	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2\alpha+\alpha}{2}, \frac{\beta^2+\beta\alpha+1}{2}, \frac{\beta^3+\beta^2\alpha+\beta}{2}, \frac{\beta^3\alpha+\beta^2\alpha+2\beta^2+\beta\alpha+2\beta+\alpha}{4}\right\}$
6	6	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^2+\alpha}{2}, \frac{\beta^3+\beta\alpha}{2}, \frac{\beta^2\alpha+2}{4}, \frac{\beta^3\alpha+2\beta}{4}\right\}$
6	7	$\left\{1, \beta, \beta^2, \beta^3, \frac{2\beta^3+\beta^2\alpha+\alpha}{4}, \frac{\beta^2+\beta\alpha+1}{2}, \frac{\beta^3+\beta^2\alpha+\beta}{2}, \frac{\beta^3\alpha+\beta\alpha+2}{4}\right\}$

Proof.
An integral basis $\{1,\omega\}$ and discriminant D_L of $L=\mathbb{Q}(\alpha)$ are given in (3). An integral basis $\{1,\delta_2,\delta_3,\delta_4\}$ and discriminant D_M of $M=\mathbb{Q}(\beta)$ are given by (cf. [9]):

$$\{1,\beta,\beta^2,\beta^3\},\quad D_M=-256m^3,\quad \text{if}\ \ m\equiv 2,3 \pmod 4,$$

$$\left\{1,\beta,\frac{1+\beta^2}{2},\frac{1+\beta+\beta^2+\beta^3}{4}\right\},\qquad D_M=-4m^3,\quad \text{if}\ \ m\equiv 1 \pmod 8,$$

$$\left\{1,\beta,\frac{1+\beta^2}{2},\frac{\beta+\beta^3}{2}\right\},\qquad D_M=-16m^3,\quad \text{if}\ \ m\equiv 5 \pmod 8.$$

Starting from the initial basis $\{1,\delta_2,\delta_3,\delta_4,\omega,\delta_2\omega,\delta_3\omega,\delta_4\omega\}$ we obtain an integral basis of K using the same procedure like in the proof of Theorem 3.

Theorem 6. *If K admits a power integral basis, then the following divisibility conditions must hold:*

n mod 8	m mod 8	1	2	3	4	5	6
1,5	2,3,6,7	$n \mid 16m^2 \pm 1$	$n \mid 16m \pm 1$	$8m \mid n^4 \pm 1$	$8m \mid n^2 \pm 1$	$8m \mid 1 \pm 1$	$1024m^3 \mid m^2(256 \pm 256)$
1,5	1	$n \mid m^2 \pm 1$	$n \mid 4m \pm 1$	$m \mid n^4 \pm 1$	$m \mid n^2 \pm 1$	$m \mid 1 \pm 1$	$16m^3 \mid m^2(16 \pm 16)$
1,5	5	$n \mid m^2 \pm 1$	$n \mid 16m \pm 1$	$m \mid n^4 \pm 1$	$4m \mid n^2 \pm 1$	$m \mid 1 \pm 1$	$16m^3 \mid m^2(16 \pm 256)$
2,6	1	$4n \mid m^2 \pm 1$	$2n \mid 2m \pm 2$	$m \mid 64n^4 \pm 1$	$2m \mid 8n^2 \pm 2$	$m \mid 1 \pm 4$	$4m^3 \mid m^2(16 \pm 4)$
2,6	5	$4n \mid m^2 \pm 1$	$4n \mid 4m \pm 4$	$m \mid 16n^4 \pm 1$	$4m \mid 4n^2 \pm 4$	$m \mid 1 \pm 16$	$16m^3 \mid m^2(16 \pm 16)$
3,7	2,6	$4n \mid 4m^2 \pm 4$	$2n \mid 8m \pm 2$	$16m \mid 4n^4 \pm 4$	$8m \mid 2n^2 \pm 2$	$16m \mid 4 \pm 4$	$256m^3 \mid m^2(64 \pm 64)$
3,7	3,7	$n \mid m^2 \pm 16$	$n \mid 16m \pm 1$	$m \mid n^4 \pm 16$	$8m \mid n^2 \pm 1$	$m \mid 16 \pm 1$	$16m^3 \mid m^2(16 \pm 256)$
3,7	1	$4n \mid m^2 \pm 1$	$n \mid m \pm 4$	$m \mid 16n^4 \pm 1$	$4m \mid 4n^2 \pm 4$	$m \mid 1 \pm 16$	$m^3 \mid m^2(16 \pm 1)$
3,7	5	$4n \mid m^2 \pm 1$	$8n \mid 4m \pm 4$	$m \mid 16n^4 \pm 1$	$8m \mid 4n^2 \pm 4$	$m \mid 1 \pm 16$	$16m^3 \mid m^2(16 \pm 16)$
2	2	$\frac{n}{2} \mid \frac{m^2}{4} \pm 64$	$\frac{n}{2} \mid 16m \pm 1$	$\frac{m}{2} \mid \frac{n^4}{16} \pm 64$	$8m \mid \frac{n^2}{4} \pm 1$	$\frac{m}{2} \mid 64 \pm 1$	$2m^3 \mid m^2(4 \pm 256)$
2	3	$8n \mid 4m^2 \pm 4$	$n \mid 2m \pm 8$	$8m \mid \frac{n^4}{4} \pm 4$	$2m \mid \frac{n^2}{2} \pm 8$	$4m \mid 4 \pm 64$	$4m^3 \mid m^2(64 \pm 4)$
2	6	$2n \mid m^2 \pm 16$	$n \mid 8m \pm 2$	$2m \mid \frac{n^4}{4} \pm 16$	$8m \mid \frac{n^2}{2} \pm 2$	$2m \mid 16 \pm 4$	$8m^3 \mid m^2(16 \pm 64)$
2	7	$8n \mid 4m^2 \pm 4$	$\frac{n}{2} \mid m \pm 16$	$m \mid \frac{n^4}{16} \pm 4$	$m \mid \frac{n^2}{4} \pm 16$	$4m \mid 4 \pm 256$	$m^3 \mid m^2(64 \pm 1)$
6	2	$2n \mid m^2 \pm 16$	$n \mid 8m \pm 2$	$2m \mid \frac{n^4}{4} \pm 16$	$8m \mid \frac{n^2}{2} \pm 2$	$2m \mid 16 \pm 4$	$8m^3 \mid m^2(16 \pm 64)$
6	3	$8n \mid 4m^2 \pm 4$	$n \mid 2m \pm 8$	$8m \mid \frac{n^4}{4} \pm 4$	$2m \mid \frac{n^2}{2} \pm 8$	$4m \mid 4 \pm 64$	$4m^3 \mid m^2(64 \pm 4)$
6	6	$\frac{n}{2} \mid \frac{m^2}{4} \pm 64$	$\frac{n}{2} \mid 16m \pm 1$	$\frac{m}{2} \mid \frac{n^4}{16} \pm 64$	$8m \mid \frac{n^2}{4} \pm 1$	$\frac{m}{2} \mid 64 \pm 1$	$2m^3 \mid m^2(4 \pm 256)$
6	7	$8n \mid 4m^2 \pm 4$	$\frac{n}{2} \mid m \pm 16$	$m \mid \frac{n^4}{16} \pm 4$	$m \mid \frac{n^2}{4} \pm 16$	$4m \mid 4 \pm 256$	$m^3 \mid m^2(64 \pm 1)$

Note that the formulas like $m|0$ do not yield any restriction for m, but in all other situations columns 5 and 6 yield only a few possible values for m, and also for n by columns 1 and 2.

In most of the cases column 5 implies that there are only a few possible values of m and n if K is monogenic.

Corollary 7. *If K is monogenic and $(n \pmod 8, m \pmod 8)$ is contained in one of the sets*

$$\{1,5\} \times \{5\}, \quad \{2,6\} \times \{1,2,3,5,6,7\}, \quad \{3,7\} \times \{1,3,5,7\},$$

then

$$|m| \leq 130 \text{ and } |n| \leq 32|m| + 32.$$

This yields that the statement is valid in all cases of the above table up to lines 1, 2 and 6.

Note that our Corollary above uses only columns 2,5,6 of the above table.

In a recent paper [10] the authors considered monogenity of fields of type $\mathbb{Q}(i, \sqrt[4]{m})$ for squarefree integers $m \equiv 2, 3 \pmod 4$. Our theorem allows us to extend this result to the case $m \equiv 1 \pmod 4$, since in these cases $m | 1 \pm 16$ must be satisfied:

Corollary 8. *If m is squarefree integer, $|m| \neq 1, 3, 5, 15, 17$ then $\mathbb{Q}(i, \sqrt[4]{m})$ not monogenic.*

We conjecture that the octic fields $\mathbb{Q}(i, \sqrt[4]{m})$ with $m = \pm 3, \pm 5, \pm 15, \pm 17$ are not monogenic, either. (The fields with $m = \pm 1$ are not octic fields.)

Proof of Theorem 6.
Let

$$\alpha^{(1)} = \sqrt{n},\ \alpha^{(2)} = -\sqrt{n},\ \beta^{(1)} = \sqrt[4]{m},\ \beta^{(2)} = i\sqrt[4]{m},\ \beta^{(3)} = -\sqrt[4]{m},\ \beta^{(4)} = -i\sqrt[4]{m}.$$

Assume that $\{b_1 = 1, b_2, \ldots, b_8\}$ is an integral basis of K. The elements of an integral basis are composed of α and β hence it is unique to denote the conjugate of b_k corresponding to $\alpha^{(i)}$ and $\beta^{(j)}$ by $b_k^{(i,j)}$ $(i = 1, 2; j = 1, 2, 3)$. Set

$$L^{(i,j)} = L^{(i,j)}(X_1, X_2, \ldots, X_6) = X_1 + b_2^{(i,j)} X_2 + \ldots + b_8^{(i,j)} X_8.$$

Using the $L^{(i,j)}$ we construct the polynomials F_1, F_2, F_3, F_4, F_5 in the following way:

$$\begin{aligned} F_1 &= \left(L^{(1,1)} - L^{(1,2)}\right)\left(L^{(1,2)} - L^{(1,3)}\right)\left(L^{(1,3)} - L^{(1,4)}\right)\left(L^{(1,4)} - L^{(1,1)}\right) \\ &\quad \left(L^{(2,1)} - L^{(2,2)}\right)\left(L^{(2,2)} - L^{(2,3)}\right)\left(L^{(2,3)} - L^{(2,4)}\right)\left(L^{(2,4)} - L^{(2,1)}\right), \end{aligned}$$

$$F_2 = \left(L^{(1,1)} - L^{(1,3)}\right)\left(L^{(1,2)} - L^{(1,4)}\right)\left(L^{(2,1)} - L^{(2,3)}\right)\left(L^{(2,2)} - L^{(2,4)}\right),$$

$$F_3 = \left(L^{(1,1)} - L^{(2,1)}\right)\left(L^{(1,2)} - L^{(2,2)}\right)\left(L^{(1,3)} - L^{(2,3)}\right)\left(L^{(1,4)} - L^{(2,4)}\right),$$

$$\begin{aligned} F_4 &= \left(L^{(1,1)} - L^{(2,2)}\right)\left(L^{(1,1)} - L^{(2,4)}\right)\left(L^{(1,2)} - L^{(2,1)}\right)\left(L^{(1,2)} - L^{(2,3)}\right) \\ &\quad \left(L^{(1,3)} - L^{(2,2)}\right)\left(L^{(1,3)} - L^{(2,4)}\right)\left(L^{(1,4)} - L^{(2,1)}\right)\left(L^{(1,4)} - L^{(2,3)}\right), \end{aligned}$$

$$F_5 = \left(L^{(1,1)} - L^{(2,3)}\right)\left(L^{(1,2)} - L^{(2,4)}\right)\left(L^{(1,3)} - L^{(2,1)}\right)\left(L^{(1,4)} - L^{(2,2)}\right).$$

We find that

$$F_i(X_2, \ldots, X_8) = f_i \cdot G_i(X_2, \ldots, X_8) \ \ (i = 1, \ldots, 5),$$

where f_i are integers or square roots of integers with $(f_1 \ldots f_5)^2 = |D_K|$ depending on the parameters only and $G_i(X_2, \ldots, X_8)$ $(i = 1, \ldots, 5)$ are primitive polynomials with integer coefficients. Then the index form equation corresponding to the basis $\{1, b_2, \ldots, b_8\}$ as given in Theorem 5 is equivalent to

$$G_1(x_2, \ldots, x_8) \ldots G_5(x_2, \ldots, x_8) = \pm 1, \ \text{ in } \ x_2, \ldots, x_8 \in \mathbb{Z},$$

that is

$$G_i(x_2, \ldots, x_8) = \pm 1 \ \ (i = 1, \ldots, 5),$$

or equivalently

$$F_i(x_2, \ldots, x_8) = \pm f_i \ \ (i = 1, \ldots, 5).$$

If a power integral basis in K exists then there exist $x_2,\dots,x_8\in\mathbb{Z}$ satisfying the above equations. Calculating the explicit form of the F_i in each case of the integral basis we always have

$$n|F_1-F_4,\;\; n|F_2-F_5,\;\; m|F_3^2-F_4,\;\; m|F_3-F_5,\;\; m|F_4-F_5^2,\;\; m^3|16F_1-F_2^2,$$

where the polynomials involved have integer coefficients. These imply the divisibility conditions of the Theorem.
□

6 Composites of quadratic fields and the simplest quartic fields

Throughout this section we assume that

$$\begin{array}{l} n,m \text{ are integers}, n\neq 0,1, m\neq 0,\pm 3 \text{ such that } n \text{ is squarefree},\\ m_0=m^2+16 \text{ is not divisible by an odd square, } \gcd(n,m_0)\in\{1,2\}. \end{array} \qquad (10)$$

In this section we consider the composite field $K=\mathbb{Q}(\alpha,\beta)$, where

$$\begin{array}{lll} \alpha & \text{is a root of} & f(x)=x^2-n,\\ \beta & \text{is a root of} & g(x)=x^4-mx^3-6x^2+mx+1. \end{array}$$

The fields $M=\mathbb{Q}(\beta)$ are called *simplest quartic fields*, see [12].

Theorem 9. *According to the behaviour of n mod 8, and m mod 16 an integral basis of K is given by the following table:*

n mod 8	m mod 16	integral basis
1,5	1,3,5,7,9,11,13,15	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2}{2}, \frac{\beta^3\alpha+\beta^3+\alpha+1}{4}\right\}$
1,5	2,6,10,14	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta}{2}, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2+\alpha+1}{4}, \frac{\beta^3\alpha+\beta^3+\beta\alpha+\beta}{4}\right\}$
1,5	4,12	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2+\alpha+1}{4}, \frac{\beta^3\alpha+\beta^3+\beta^2\alpha+\beta^2+\beta\alpha+\beta+\alpha+1}{8}\right\}$
1,5	0,8	$\left\{1, \beta, \frac{\beta^2+2\beta+3}{4}, \frac{\beta^3+3\beta+2}{4}, \frac{\alpha+1}{2}, \frac{\beta\alpha+\beta}{2}, \frac{\beta^2\alpha+\beta^2+2\beta\alpha+2\beta+3\alpha+3}{8}, \frac{\beta^3\alpha+\beta^3+3\beta\alpha+3\beta+2\alpha+2}{8}\right\}$
2,6	1,3,5,7,9,11,13,15	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \alpha, \alpha\beta, \alpha\beta^2, \frac{\beta^3\alpha+\alpha}{2}\right\}$
2,6	2,6,10,14	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta}{2}, \alpha, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^2\alpha+\alpha}{2}, \frac{\beta^3\alpha+\beta^2\alpha+\beta\alpha+\alpha}{4}\right\}$
2	4	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \frac{\beta^2+2\beta+2\alpha+3}{4}, \frac{\beta^3+\beta^2+2\beta\alpha+\beta+2\alpha+5}{8}, \frac{\beta^3+\beta^2\alpha+\beta^2+5\beta+\alpha+5}{8}, \frac{\beta^3\alpha+\beta^2\alpha+2\beta^2+\beta\alpha+\alpha+2}{8}\right\}$
2	12	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \frac{\beta^2+2\beta+2\alpha+3}{4}, \frac{\beta^3+\beta^2+2\beta\alpha+5\beta+2\alpha+1}{8}, \frac{\beta^3+\beta^2\alpha+3\beta^2+5\beta+\alpha+7}{8}, \frac{\beta^3\alpha+\beta^2\alpha+2\beta^2+\beta\alpha+\alpha+2}{8}\right\}$
2	0	$\left\{1, \beta, \frac{\beta^2+2\beta+3}{4}, \frac{\beta^3+3\beta+2}{4}, \frac{1+\beta+\alpha}{2}, \frac{\beta^3+\beta^2+2\beta\alpha+5\beta+2\alpha+5}{8}, \frac{\beta^3+\beta^2\alpha+\beta^2+2\beta\alpha+9\beta+3\alpha+5}{8}, \frac{\beta^3\alpha+2\beta^3+\beta^2\alpha+9\beta\alpha+14\beta+5\alpha+4}{16}\right\}$
2	8	$\left\{1, \beta, \frac{\beta^2+2\beta+3}{4}, \frac{\beta^3+3\beta+2}{4}, \frac{1+\beta+\alpha}{2}, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^3+\beta^2\alpha+\beta^2+2\beta\alpha+9\beta+3\alpha+5}{8}, \frac{\beta^3\alpha+\beta^2\alpha+9\beta\alpha+5\alpha}{8}\right\}$
6	4	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \frac{\beta^2+2\beta+2\alpha+3}{4}, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^3+\beta^2\alpha+3\beta^2+5\beta+\alpha+7}{8}, \frac{\beta^3\alpha+\beta^2\alpha+2\beta^2+\beta\alpha+\alpha+2}{8}\right\}$
6	12	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \frac{\beta^2+2\beta+2\alpha+3}{4}, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^3+\beta^2\alpha+\beta^2+5\beta+\alpha+5}{8}, \frac{\beta^3\alpha+\beta^2\alpha+2\beta^2+\beta\alpha+\alpha+2}{8}\right\}$
6	0	$\left\{1, \beta, \frac{\beta^2+2\beta+3}{4}, \frac{\beta^3+3\beta+2}{4}, \frac{1+\beta+\alpha}{2}, \frac{\beta\alpha+\alpha}{2}, \frac{\beta^3+\beta^2\alpha+\beta^2+2\beta\alpha+9\beta+3\alpha+5}{8}, \frac{\beta^3\alpha+\beta^2\alpha+9\beta\alpha+5\alpha}{8}\right\}$
6	8	$\left\{1, \beta, \frac{\beta^2+2\beta+3}{4}, \frac{\beta^3+3\beta+2}{4}, \frac{1+\beta+\alpha}{2}, \frac{\beta^3+\beta^2+2\beta\alpha+5\beta+2\alpha+5}{8}, \frac{\beta^3+\beta^2\alpha+\beta^2+2\beta\alpha+9\beta+3\alpha+5}{8}, \frac{\beta^3\alpha+\beta^2\alpha+2\beta^2+13\beta\alpha+4\beta+9\alpha+14}{16}\right\}$
3,7	1,3,5,7,9,11,13,15	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \alpha, \alpha\beta, \alpha\beta^2, \frac{\beta^3\alpha+\alpha}{2}\right\}$
3,7	2,6,10,14	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta}{2}, \frac{\beta^3+\beta^2+3\beta+2\alpha+1}{4}, \frac{\beta^3+\beta^2+2\beta\alpha+\beta+3}{4}, \frac{\beta^2\alpha+\beta^2+2\beta+\alpha+3}{4}, \frac{\beta^3\alpha+\beta^3+\beta\alpha+3\beta+2}{4}\right\}$
3,7	4,12	$\left\{1, \beta, \frac{\beta^2+1}{2}, \frac{\beta^3+\beta^2+\beta+1}{4}, \alpha, \frac{\beta\alpha+\beta+\alpha+1}{2}, \frac{\beta^2\alpha+\beta^2+\alpha+1}{4}, \frac{\beta^3\alpha+\beta^3+\beta^2\alpha+3\beta^2+\beta\alpha+\beta+\alpha+3}{8}\right\}$
3,7	0,8	$\left\{1, \beta, \frac{\beta^2+2\beta+3}{4}, \frac{\beta^3+3\beta+2}{4}, \alpha, \frac{\beta\alpha+\beta+\alpha+1}{2}, \frac{\beta^2\alpha+2\beta\alpha+3\alpha}{4}, \frac{\beta^3\alpha+\beta^3+\beta^2\alpha+\beta^2+5\beta\alpha+5\beta+9\alpha+9}{8}\right\}$

Proof.
An integral basis $\{1, \omega\}$ and discriminant D_L of $L = \mathbb{Q}(\alpha)$ are given in (3). According to [13], under the conditions of our theorem an integral basis $\{1, \delta_2, \delta_3, \delta_4\}$ of the simplest quartic fields are given by

$$\begin{aligned}
\left\{1, \beta, \beta^2, \frac{1+\beta^3}{2}\right\}, &\quad \text{if } \nu_2(m) = 0,\\
\left\{1, \beta, \frac{1+\beta^2}{2}, \frac{\beta+\beta^3}{2}\right\}, &\quad \text{if } \nu_2(m) = 1,\\
\left\{1, \beta, \frac{1+\beta^2}{2}, \frac{1+\beta+\beta^2+\beta^3}{4}\right\}, &\quad \text{if } \nu_2(m) = 2,\\
\left\{1, \beta, \frac{1+2\beta-\beta^2}{4}, \frac{1+\beta+\beta^2+\beta^3}{4}\right\}, &\quad \text{if } \nu_2(m) > 2.
\end{aligned}$$

The discriminant of $g(x)$ is $4m_0^3$. The conditions on n and m_0 imply that the discriminant of K is divisible by $n_1^4 m_1^2$ where $m_1 = m_0/2^{\nu_2(m_0)}, n_1 = n/2^{\nu_2(n)}$ (similarly as in (7)). Starting from the initial basis $\{1, \delta_2, \delta_3, \delta_4, \omega, \delta_2\omega, \delta_3\omega, \delta_4\omega\}$ we obtain an integral basis of K using the same procedure like in the proof of Theorem 3. Reducing the discriminant of the above initial basis we only have to deal with 2-factors.
□

Theorem 10. *If K admits a power integral basis, then the following divisibility conditions must hold:*

n mod 8	m mod 16	1	2	3	4	5	6
1,5	1,3,5,7,11,13,15	$n \mid m_0^2 \pm 1$	$n \mid m_0 \pm 1$	$m_0 \mid n^4 \pm 1$	$m_0 \mid n^2 \pm 1$	$m_0 \mid 1 \pm 1$	$m_0^3 \mid m_0^2(16 \pm 1)$
1,5	2,6,10,14	$n \mid \frac{m_0^2}{16} \pm 1$	$n \mid 4m_0 \pm 1$	$\frac{m_0}{4} \mid n^4 \pm 1$	$m_0 \mid n^2 \pm 1$	$\frac{m_0}{4} \mid 1 \pm 1$	$\frac{m_0^3}{4} \mid m_0^2(1 \pm 16)$
1,5	4,12	$n \mid \frac{m_0^2}{16} \pm 1$	$n \mid m_0 \pm 1$	$\frac{m_0}{2} \mid n^4 \pm 1$	$m_0 \mid n^2 \pm 1$	$\frac{m_0}{2} \mid 1 \pm 1$	$\frac{m_0^3}{4} \mid m_0^2(1 \pm 1)$
1,5	0,8	$n \mid \frac{m_0^2}{256} \pm 1$	$n \mid 4m_0 \pm 1$	$\frac{m_0}{16} \mid n^4 \pm 1$	$m_0 \mid n^2 \pm 1$	$\frac{m_0}{16} \mid 1 \pm 1$	$\frac{m_0^3}{256} \mid m_0^2\left(\frac{1}{16} \pm 16\right)$
2,6,3,7	1,3,5,7,9,11,13,15	$4n \mid m_0^2 \pm 1$	$n \mid m_0 \pm 1$	$m_0 \mid 256n^4 \pm 1$	$m_0 \mid 16n^2 \pm 1$	$m_0 \mid 1 \pm 1$	$m_0^3 \mid m_0^2(16 \pm 1)$
2,6	2,6,10,14	$4n \mid \frac{m_0^2}{16} \pm 1$	$n \mid m_0 \pm 4$	$\frac{m_0}{4} \mid 16n^4 \pm 1$	$m_0 \mid 4n^2 \pm 4$	$\frac{m_0}{4} \mid 1 \pm 16$	$\frac{m_0^3}{4} \mid m_0^2(1 \pm 1)$
2	4,12	$\frac{n}{2} \mid \frac{m_0^2}{1024} \pm 64$	$\frac{n}{2} \mid m_0 \pm 1$	$\frac{m_0}{32} \mid \frac{n^4}{16} \pm 64$	$m_0 \mid \frac{n^2}{4} \pm 1$	$\frac{m_0}{32} \mid 64 \pm 1$	$\frac{m_0^3}{2048} \mid m_0^2\left(\frac{1}{64} \pm 1\right)$
2	0	$4n \mid \frac{m_0^2}{256} \pm 1$	$\frac{n}{2} \mid \frac{m_0}{16} \pm 64$	$\frac{m_0}{16} \mid \frac{n^4}{16} \pm 1$	$\frac{m_0}{16} \mid \frac{n^2}{4} \pm 64$	$\frac{m_0}{16} \mid 1 \pm 4096$	$\frac{m_0^3}{4096} \mid m_0^2\left(\frac{1}{16} \pm \frac{1}{256}\right)$
2	8	$4n \mid \frac{m_0^2}{256} \pm 1$	$n \mid \frac{m_0}{4} \pm 16$	$\frac{m_0}{16} \mid n^4 \pm 1$	$m_0 \mid n^2 \pm 16$	$\frac{m_0}{16} \mid 1 \pm 256$	$\frac{m_0^3}{4096} \mid m_0^2\left(\frac{1}{16} \pm \frac{1}{16}\right)$
6	4 ,12	$2n \mid \frac{m_0^2}{256} \pm 16$	$n \mid \frac{m_0}{2} \pm 2$	$\frac{m_0}{8} \mid \frac{n^4}{4} \pm 16$	$m_0 \mid \frac{n^2}{2} \pm 2$	$\frac{m_0}{8} \mid 16 \pm 4$	$\frac{m_0^3}{512} \mid m_0^2\left(\frac{1}{16} \pm \frac{1}{4}\right)$
6	0	$4n \mid \frac{m_0^2}{256} \pm 1$	$n \mid \frac{m_0}{4} \pm 16$	$\frac{m_0}{16} \mid n^4 \pm 1$	$m_0 \mid n^2 \pm 16$	$\frac{m_0}{16} \mid 1 \pm 256$	$\frac{m_0^3}{4096} \mid m_0^2\left(\frac{1}{16} \pm \frac{1}{16}\right)$
6	8	$4n \mid \frac{m_0^2}{256} \pm 1$	$\frac{n}{2} \mid \frac{m_0}{16} \pm 64$	$\frac{m_0}{16} \mid \frac{n^4}{16} \pm 1$	$\frac{m_0}{16} \mid \frac{n^2}{4} \pm 64$	$\frac{m_0}{16} \mid 1 \pm 4096$	$\frac{m_0^3}{4096} \mid m_0^2\left(\frac{1}{16} \pm \frac{1}{256}\right)$
3,7	2,6,10,14	$4n \mid \frac{m_0^2}{16} \pm 1$	$n \mid \frac{m_0}{4} \pm 16$	$\frac{m_0}{4} \mid n^4 \pm 1$	$\frac{m_0}{4} \mid n^2 \pm 16$	$\frac{m_0}{4} \mid 1 \pm 256$	$\frac{m_0^3}{64} \mid m_0^2\left(1 \pm \frac{1}{16}\right)$
3,7	4,12	$4n \mid \frac{m_0^2}{64} \pm 4$	$n \mid \frac{m_0}{2} \pm 2$	$m_0 \mid 4n^4 \pm 4$	$m_0 \mid 2n^2 \pm 2$	$m_0 \mid 4 \pm 4$	$\frac{m_0^3}{8} \mid m_0^2\left(\frac{1}{4} \pm \frac{1}{4}\right)$
3,7	0,8	$8n \mid \frac{m_0^2}{256} \pm 1$	$n \mid m_0 \pm 4$	$\frac{m_0}{16} \mid 16n^4 \pm 1$	$m_0 \mid 4n^2 \pm 4$	$\frac{m_0}{16} \mid 1 \pm 16$	$\frac{m_0^3}{256} \mid m_0^2\left(\frac{1}{16} \pm 1\right)$

If the right hand term in the divisibility conditions of columns 5 and 6 does not reduce to 0, there remain only a few possible values for m_0.

Corollary 11. *If* $(n \mod 8,\ m \mod 16) \notin \{1,3,5,7\} \times \{4,12\}$ *and* K *is monogenic, then*

$$|m| \leq 64 \ \text{ and } \ |n| \leq 4m^2 + 192.$$

Note that our Corollary uses only columns 2,5,6 of the above table.

Proof of Theorem 10.
Let

$$\alpha^{(1)} = \sqrt{n},\ \alpha^{(2)} = -\sqrt{n},\ \beta^{(1)} = \beta,\ \beta^{(2)} = \frac{\beta - 1}{\beta + 1},\ \beta^{(3)} = \frac{-1}{\beta},\ \beta^{(4)} = \frac{-\beta - 1}{\beta - 1}.$$

Assume that $\{b_1 = 1, b_2, \ldots, b_8\}$ is the integral basis of K as given in Theorem 9. The elements of the integral basis are composed of α and β hence it is unique to denote the conjugate of b_k corresponding to $\alpha^{(i)}$ and $\beta^{(j)}$ by $b_k^{(i,j)}$ $(i = 1, 2; j = 1, 2, 3, 4)$. We construct the same $L^{(i,j)}$ and use the analogous factorization of the index form into polynomials F_1, F_2, F_3, F_4, F_5 like in the proof of Theorem 6. Similarly, the properties

$$n|F_1 - F_4,\ \ n|F_2 - F_5,\ \ m_0|F_3^2 - F_4,\ \ m_0|F_3 - F_5,\ \ m_0|F_4 - F_5^2,\ \ m_0^3|16F_1 - F_2^2$$

(where the polynomials involved have integer coefficients) imply the assertions of our Theorem. □

7 The composite fields $\mathbb{Q}(i\sqrt{3}, \sqrt[6]{m})$

M.-L. Chang [1] considered the normal closure of pure cubic fields, the number fields $\mathbb{Q}(\omega, \sqrt[3]{m})$, where $\omega = e^{2\pi i/3}$ and m is not a complete cube. He showed that these fields are not monogenic, except for $m = 2$.

In a recent paper [10] the authors studied the analogous problem in the fields $\mathbb{Q}(i, \sqrt[4]{m})$ and proved that if m is a square-free integer, $m \equiv 2, 3 \pmod 4$, then the field $\mathbb{Q}(i, \sqrt[4]{m})$ is not monogenic. This assertion was extended in Corollary 8 of the present paper.

Using the technics developed in our paper we shall now continue the series of the above mentioned results and consider the fields $K = \mathbb{Q}(\omega, \sqrt[6]{m})$ where $\omega = e^{2\pi i/3}$.
Throughout this section we assume that

$$m \text{ is a square free integer}, m \neq 0, \pm 1, -3. \qquad (11)$$

Theorem 12. *According to the behaviour of m mod 36 an integral basis of K is given by the following table*

m mod 36	integral basis
1	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+4\beta^2+3\beta+4}{6}, \frac{\beta^5+\beta^3+3\beta^2+4\beta+3}{6}, \omega, \omega\beta, \frac{\beta^2\omega+\beta^2+2\omega+2}{3}, \frac{\beta^3\omega+\beta^3+2\beta\omega+2\beta+3\omega+3}{6}, \frac{\beta^4\omega+4\beta^2\omega+3\beta\omega+4\omega}{6}, \frac{\beta^5\omega+\beta^3\omega+3\beta^2\omega+4\beta\omega+3\omega}{6}\right\}$
2	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+2\beta^2\omega+2\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+2\beta^3\omega+2\beta^3+\beta\omega+\beta}{3}\right\}$
3	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\beta^3}{3}, \frac{\beta^4\omega+\beta^4}{3}, \frac{\beta^5\omega+\beta^5}{3}\right\}$
5	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+\beta}{2}, \frac{\beta^5+\beta^2}{2}, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\omega}{2}, \frac{\beta^4\omega+\beta^4+2\beta^2\omega+2\beta^2+3\beta\omega+3\beta+4\omega+4}{6}, \frac{\beta^5\omega+\beta^5+2\beta^4\omega+2\beta^4+2\beta^3\omega+2\beta^3+\beta^2\omega+\beta^2+4\beta\omega+4\beta+2\omega+2}{6}\right\}$
6	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\beta^3}{3}, \frac{\beta^4\omega+\beta^4}{3}, \frac{\beta^5\omega+\beta^5}{3}\right\}$
7	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+\beta^3\omega+\beta^3+\beta\omega+\beta}{3}\right\}$
10	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^4+\beta^2+1}{3}, \frac{\beta^5+\beta^3+\beta}{3}, \omega, \omega\beta, \frac{\beta^2\omega+\beta^2+2\omega+2}{3}, \frac{\beta^3\omega+\beta^3+2\beta\omega+2\beta}{3}, \frac{\beta^4\omega+\beta^2\omega+\omega}{3}, \frac{\beta^5\omega+\beta^3\omega+\beta\omega}{3}\right\}$
11	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+2\beta^2\omega+2\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+2\beta^3\omega+2\beta^3+\beta\omega+\beta}{3}\right\}$
13	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+\beta}{2}, \frac{\beta^5+\beta^2}{2}, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\omega}{2}, \frac{\beta^4\omega+\beta^4+4\beta^2\omega+4\beta^2+3\beta\omega+3\beta+4\omega+4}{6}, \frac{\beta^5\omega+\beta^5+\beta^4\omega+\beta^4+\beta^3\omega+\beta^3+\beta^2\omega+\beta^2+\beta\omega+\beta+\omega+1}{6}\right\}$
14	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+2\beta^2\omega+2\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+2\beta^3\omega+2\beta^3+\beta\omega+\beta}{3}\right\}$
15	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\beta^3}{3}, \frac{\beta^4\omega+\beta^4}{3}, \frac{\beta^5\omega+\beta^5}{3}\right\}$
17	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+2\beta^2+3\beta+4}{6}, \frac{\beta^5+2\beta^3+3\beta^2+4\beta}{6}, \omega, \omega\beta, \frac{\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^3\omega+\beta^3+4\beta\omega+4\beta+3\omega+3}{6}, \frac{\beta^4\omega+2\beta^2\omega+3\beta\omega+4\omega}{6}, \frac{\beta^5\omega+2\beta^3\omega+3\beta^2\omega+4\beta\omega}{6}\right\}$
19	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^4+\beta^2+1}{3}, \frac{\beta^5+\beta^3+\beta}{3}, \omega, \omega\beta, \frac{\beta^2\omega+\beta^2+2\omega+2}{3}, \frac{\beta^3\omega+\beta^3+2\beta\omega+2\beta}{3}, \frac{\beta^4\omega+\beta^2\omega+\omega}{3}, \frac{\beta^5\omega+\beta^3\omega+\beta\omega}{3}\right\}$
21	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+\beta}{2}, \frac{\beta^5+\beta^2}{2}, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\beta^3+3\omega+3}{6}, \frac{\beta^4\omega+\beta^4+3\beta\omega+3\beta}{6}, \frac{\beta^5\omega+\beta^5+3\beta^2\omega+3\beta^2}{6}\right\}$
22	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+\beta^3\omega+\beta^3+\beta\omega+\beta}{3}\right\}$
23	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+2\beta^2\omega+2\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+2\beta^3\omega+2\beta^3+\beta\omega+\beta}{3}\right\}$
25	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+\beta}{2}, \frac{\beta^5+\beta^2}{2}, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\omega}{2}, \frac{\beta^4\omega+\beta^4+4\beta^2\omega+4\beta^2+3\beta\omega+3\beta+4\omega+4}{6}, \frac{\beta^5\omega+\beta^5+\beta^4\omega+\beta^4+\beta^3\omega+\beta^3+\beta^2\omega+\beta^2+\beta\omega+\beta+\omega+1}{6}\right\}$
26	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^4+2\beta^2+1}{3}, \frac{\beta^5+2\beta^3+\beta}{3}, \omega, \omega\beta, \frac{\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^3\omega+\beta^3+\beta\omega+\beta}{3}, \frac{\beta^4\omega+2\beta^2\omega+\omega}{3}, \frac{\beta^5\omega+2\beta^3\omega+\beta\omega}{3}\right\}$
29	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+\beta}{2}, \frac{\beta^5+\beta^2}{2}, \omega, \omega\beta, \frac{\beta^4\omega+\beta^4+2\beta^2\omega+2\beta^2+3\beta\omega+3\beta+4\omega+4}{6}, \frac{\beta^5\omega+\beta^5+2\beta^4\omega+2\beta^4+2\beta^3\omega+2\beta^3+\beta^2\omega+\beta^2+4\beta\omega+4\beta+2\omega+2}{6}, \frac{\beta^4\omega+\beta\omega}{2}, \frac{\beta^5\omega+\beta^2\omega}{2}\right\}$
30	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\beta^3}{3}, \frac{\beta^4\omega+\beta^4}{3}, \frac{\beta^5\omega+\beta^5}{3}\right\}$
31	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+\beta^3\omega+\beta^3+\beta\omega+\beta}{3}\right\}$
33	$\left\{1, \beta, \beta^2, \frac{\beta^3+1}{2}, \frac{\beta^4+\beta}{2}, \frac{\beta^5+\beta^2}{2}, \omega, \omega\beta, \omega\beta^2, \frac{\beta^3\omega+\beta^3+3\omega+3}{6}, \frac{\beta^4\omega+\beta^4+3\beta\omega+3\beta}{6}, \frac{\beta^5\omega+\beta^5+3\beta^2\omega+3\beta^2}{6}\right\}$
34	$\left\{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \omega, \omega\beta, \omega\beta^2, \omega\beta^3, \frac{\beta^4\omega+\beta^4+\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^5\omega+\beta^5+\beta^3\omega+\beta^3+\beta\omega+\beta}{3}\right\}$
35	$\left\{1, \beta, \beta^2, \beta^3, \frac{\beta^4+2\beta^2+1}{3}, \frac{\beta^5+2\beta^3+\beta}{3}, \omega, \omega\beta, \frac{\beta^2\omega+\beta^2+\omega+1}{3}, \frac{\beta^3\omega+\beta^3+\beta\omega+\beta}{3}, \frac{\beta^4\omega+2\beta^2\omega+\omega}{3}, \frac{\beta^5\omega+2\beta^3\omega+\beta\omega}{3}\right\}$

Proof.
An integral basis of $L=\mathbb{Q}(\omega)$ is $\{1,\omega\}$ and $D_L=-3$. Set $\beta=\sqrt[6]{m}$. An integral basis of $M=\mathbb{Q}(\sqrt[6]{m})$ is given in [9]:

$$\begin{array}{rl}
\{1,\beta,\beta^2,\beta^3,\beta^4,\beta^5\}, & \text{if } m\equiv 2,3,6,7,11,14,15,22,23,30,31,34 \pmod{36},\\
\left\{1,\beta,\beta^2,\frac{1+\beta^3}{2},\frac{\beta+\beta^4}{2},\frac{\beta^2+\beta^5}{2}\right\}, & \text{if } m\equiv 5,13,21,25,29,33 \pmod{36},\\
\left\{1,\beta,\beta^2,\beta^3,\frac{1+\beta^2+\beta^4}{3},\frac{\beta+\beta^3+\beta^5}{3}\right\}, & \text{if } m\equiv 10,19 \pmod{36},\\
\left\{1,\beta,\beta^2,\beta^3,\frac{1+2\beta^2+\beta^4}{3},\frac{\beta+2\beta^3+\beta^5}{3}\right\}, & \text{if } m\equiv 26,35 \pmod{36},\\
\left\{1,\beta,\beta^2,\frac{1+\beta^3}{2},\frac{4+3\beta+2\beta^2+\beta^4}{6},\frac{4\beta+3\beta^2+2\beta^3+\beta^5}{6}\right\}, & \text{if } m\equiv 17 \pmod{36},\\
\left\{1,\beta,\beta^2,\frac{1+\beta^3}{2},\frac{4+3\beta+4\beta^2+\beta^4}{6},\frac{3+4\beta+3\beta^2+\beta^3+\beta^5}{6}\right\}, & \text{if } m\equiv 1 \pmod{36}.
\end{array}$$

The discriminant of $g(x)=x^6-m$ is $2^6 3^6 m^5$. Let $m_1=m/(2^{\nu_2(m)}3^{\nu_3(m)})$. Similarly as in the previous proofs m_1^{10} divides D_K. If $\{1,\delta_2,\ldots,\delta_6\}$ is an integral basis of M, then we start from the initial basis $\{1,\delta_2,\ldots,\delta_6,\omega,\delta_2\omega,\ldots,\delta_6\omega\}$ of K and consider possible reductions of the discriminant of this basis by 2-factors and 3-factors.
□

Theorem 13. *If K admits a power integral basis, then the following divisibility conditions must hold:*

m mod 36	1	2	3	4	5
				6	7
1,17	$m \mid 6561\pm 1$	$m \mid 6561\pm 9$	$m \mid 81\pm 1$	$m \mid 9\pm 1$	$m^3 \mid m^2(729\pm 1)$
				$m^3 \mid m^2(4096\pm 1)$	$m^3 \mid m^2(4096\pm 729)$
2,7,11,14,22,23,31,34	$m \mid 6561\pm 1$	$m \mid 6561\pm 9$	$m \mid 81\pm 1$	$m \mid 9\pm 1$	$m^3 \mid m^2(729\pm 81)$
				$m^3 \mid m^2(4096\pm 4096)$	$m^3 \mid m^2(331776\pm 2985984)$
3,6,15,30	$\frac{m}{3} \mid 729\pm 9$	$\frac{m}{3} \mid 729\pm 1$	$\frac{m}{3} \mid 27\pm 3$	$\frac{m}{3} \mid 9\pm 1$	$\frac{m^3}{27} \mid m^2(81\pm 729)$
				$\frac{m^3}{27} \mid \left(\frac{4096m^2}{9}\pm\frac{4096m^2}{9}\right)$	$\frac{m^3}{27} \mid m^2(2985984\pm 331776)$
5,13,25,29	$m \mid 6561\pm 1$	$m \mid 6561\pm 9$	$m \mid 81\pm 1$	$m \mid 9\pm 1$	$m^3 \mid m^2(729\pm 81)$
				$m^3 \mid m^2(4096\pm 1)$	$m^3 \mid m^2(331776\pm 729)$
10,19,26,35	$m \mid 6561\pm 1$	$m \mid 6561\pm 9$	$m \mid 81\pm 1$	$m \mid 9\pm 1$	$m^3 \mid m^2(729\pm 1)$
				$m^3 \mid m^2(4096\pm 4096)$	$m^3 \mid m^2(4096\pm 2985984)$
21,33	$\frac{m}{3} \mid 729\pm 9$	$\frac{m}{3} \mid 729\pm 1$	$\frac{m}{3} \mid 27\pm 3$	$\frac{m}{3} \mid 9\pm 1$	$\frac{m^3}{27} \mid m^2(81\pm 729)$
				$\frac{m^3}{27} \mid \left(\frac{4096m^2}{9}\pm\frac{4096m^2}{9}\right)$	$\frac{m^3}{27} \mid m^2(2985984\pm 81)$

Obviously the conditions can only be satisfied by a few values of m, cf. eg. column 4 of the table. Testing the possible values we find:

Corollary 14. *If* $|m| \neq 2, 3, 5, 6, 10, 15, 30$ *then* K *is not monogenic.*

We conjecture that the field K is not monogenic for the above values of m, either. Note that our Corollary uses only column 4 of the above table.

Proof of Theorem 13.
Let

$$\omega^{(1)} = \frac{1 + i\sqrt{3}}{2} = \omega, \quad \omega^{(2)} = \frac{1 - i\sqrt{3}}{2},$$

and

$$\beta^{(j)} = \sqrt[6]{m} \cdot \omega^{j-1}, \quad j = 1, \ldots, 6.$$

Assume that $\{b_1 = 1, b_2, \ldots, b_{12}\}$ is the integral basis of K as given in Theorem 12. The elements of the integral basis are composed of ω and β hence it is unique to denote the conjugate of b_k corresponding to $\omega^{(i)}$ and $\beta^{(j)}$ by $b_k^{(i,j)}$ $(i = 1, 2; j = 1, \ldots, 6)$. Set

$$L^{(i,j)} = L^{(i,j)}(X_1, \ldots, X_{12}) = X_1 + b_2^{(i,j)} X_2 + \ldots + b_{12}^{(i,j)} X_{12}.$$

Using the $L^{(i,j)}$ we construct the polynomials $F_1, \ldots, F_7$ in the following way:

$$\begin{aligned}
F_1 &= \left(L^{(1,1)} - L^{(1,2)}\right)\left(L^{(1,2)} - L^{(1,3)}\right)\left(L^{(1,3)} - L^{(1,4)}\right)\left(L^{(1,4)} - L^{(1,5)}\right)\left(L^{(1,5)} - L^{(1,6)}\right)\left(L^{(1,6)} - L^{(1,1)}\right)\\
&\times \left(L^{(2,1)} - L^{(2,2)}\right)\left(L^{(2,2)} - L^{(2,3)}\right)\left(L^{(2,3)} - L^{(2,4)}\right)\left(L^{(2,4)} - L^{(2,5)}\right)\left(L^{(2,5)} - L^{(2,6)}\right)\left(L^{(2,6)} - L^{(2,1)}\right),\\
F_2 &= \left(L^{(1,1)} - L^{(1,3)}\right)\left(L^{(1,2)} - L^{(1,4)}\right)\left(L^{(1,3)} - L^{(1,5)}\right)\left(L^{(1,4)} - L^{(1,6)}\right)\left(L^{(1,5)} - L^{(1,1)}\right)\left(L^{(1,6)} - L^{(1,2)}\right)\\
&\times \left(L^{(2,1)} - L^{(2,3)}\right)\left(L^{(2,2)} - L^{(2,4)}\right)\left(L^{(2,3)} - L^{(2,5)}\right)\left(L^{(2,4)} - L^{(2,6)}\right)\left(L^{(2,5)} - L^{(2,1)}\right)\left(L^{(2,6)} - L^{(2,2)}\right),\\
F_3 &= \left(L^{(1,1)} - L^{(1,4)}\right)\left(L^{(1,2)} - L^{(1,5)}\right)\left(L^{(1,3)} - L^{(1,6)}\right)\left(L^{(2,1)} - L^{(2,4)}\right)\left(L^{(2,2)} - L^{(2,5)}\right)\left(L^{(2,3)} - L^{(2,6)}\right),\\
F_4 &= \left(L^{(1,1)} - L^{(2,1)}\right)\left(L^{(1,2)} - L^{(2,2)}\right)\left(L^{(1,3)} - L^{(2,3)}\right)\left(L^{(1,4)} - L^{(2,4)}\right)\left(L^{(1,5)} - L^{(2,5)}\right)\left(L^{(1,6)} - L^{(2,6)}\right),\\
F_5 &= \left(L^{(1,1)} - L^{(2,2)}\right)\left(L^{(1,2)} - L^{(2,3)}\right)\left(L^{(1,3)} - L^{(2,4)}\right)\left(L^{(1,4)} - L^{(2,5)}\right)\left(L^{(1,5)} - L^{(2,6)}\right)\left(L^{(1,6)} - L^{(2,1)}\right)\\
&\times \left(L^{(1,1)} - L^{(2,6)}\right)\left(L^{(1,2)} - L^{(2,1)}\right)\left(L^{(1,3)} - L^{(2,2)}\right)\left(L^{(1,4)} - L^{(2,3)}\right)\left(L^{(1,5)} - L^{(2,4)}\right)\left(L^{(1,6)} - L^{(2,5)}\right),\\
F_6 &= \left(L^{(1,1)} - L^{(2,3)}\right)\left(L^{(1,2)} - L^{(2,4)}\right)\left(L^{(1,3)} - L^{(2,5)}\right)\left(L^{(1,4)} - L^{(2,6)}\right)\left(L^{(1,5)} - L^{(2,1)}\right)\left(L^{(1,6)} - L^{(2,2)}\right)\\
&\times \left(L^{(1,1)} - L^{(2,5)}\right)\left(L^{(1,2)} - L^{(2,6)}\right)\left(L^{(1,3)} - L^{(2,1)}\right)\left(L^{(1,4)} - L^{(2,2)}\right)\left(L^{(1,5)} - L^{(2,3)}\right)\left(L^{(1,6)} - L^{(2,4)}\right),\\
F_7 &= \left(L^{(1,1)} - L^{(2,4)}\right)\left(L^{(1,2)} - L^{(2,5)}\right)\left(L^{(1,3)} - L^{(2,6)}\right)\left(L^{(1,4)} - L^{(2,1)}\right)\left(L^{(1,5)} - L^{(2,2)}\right)\left(L^{(1,6)} - L^{(2,3)}\right).
\end{aligned}$$

We find that

$$F_i(X_2, \ldots, X_{12}) = f_i \cdot G_i(X_2, \ldots, X_{12}), \quad (i = 1, \ldots, 7),$$

where f_i are integers or square roots of integers depending on m with $(f_1 \ldots f_7)^2 = |D_K|$ and G_i are primitive polynomials with integer coefficients. Then the index form equation corresponding to the basis $\{1, b_2, \ldots, b_{12}\}$ is equivalent to

$$G_1(x_2, \ldots, x_{12}) \ldots G_7(x_2, \ldots, x_{12}) = \pm 1 \ \text{ in } \ x_2, \ldots, x_{12} \in \mathbb{Z},$$

that is

$$G_i(x_2,\dots,x_{12}) = \pm 1 \ \ (i=1,\dots,7),$$

or equivalently

$$F_i(x_2,\dots,x_{12}) = \pm f_i \ \ (i=1,\dots,7).$$

If a power integral basis in K exists then there exist $x_2,\dots,x_{12}\in\mathbb{Z}$ satisfying the above equations. Direct calculation shows that

$$m \mid F_4^2 - F_5,\ \ m \mid F_4^2 - F_6,\ \ m \mid F_4 - F_7,\ \ m \mid F_5 - F_6,$$

$$m^3 \mid 729F_1 - F_2,\ m^3 \mid 4096F_1 - F_3^2,\ \ m^3 \mid 4096F_2 - 729F_3^2,$$

(where the polynomials involved have integer coefficients) which imply the assertion of our Theorem 13.
□

8 Computational remarks

Calculating an integral basis by performing the reduction by 2-factors and 3-factors gave a huge number of possible cases and types of the integral basis which are shown in our tables.

We had to calculate the factors of the index form in all cases of the integral basis explicitly. In a number field of degree n the index form has $n-1$ variables and degree $n(n-1)/2$. For $n=6,8,12$ this degree is $15,28,66$, respectively. Therefore it was impossible to calculate the complete index form and then to factorize it. We had to combine suitable linear factors such that their product is invariant under some subgroup of the Galois group of $K/\mathbb{Q}$. Then we explicitely calculated the products of these linear factors. These products having integer coefficients are the factors of the index form. Finally we had to find explicitely the divisors of these factors.

This procedure was supported by our experience but it was still a huge computation, in a large number of cases. The calculation in one case of the integral basis took only some seconds except in the last section with fields of degree 12 when it took about 60 minutes per case.

All calculations were performed in Maple [2]. A very careful and efficient handling of the formulas was absolutely necessary to manage the calculations.

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