On the coefficients of power sums of arithmetic progressions

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Abstract

We investigate the coefficients of the polynomial

\[ S_{m,r}(\ell) = r^n + (m + r)^n + (2m + r)^n + \cdots + ((\ell - 1)m + r)^n. \]

We prove that these can be given in terms of Stirling numbers of the first kind and \( r \)-Whitney numbers of the second kind. Moreover, we prove a necessary and sufficient condition for the integrity of these coefficients.

Key words: arithmetic progressions, power sums, Stirling numbers, \( r \)-Whitney numbers, Bernoulli polynomials

1991 MSC: 11B25, 11B68, 11B73

1 Introduction

Let \( n \) be a positive integer, and let

\[ S_n(\ell) = 1^n + 2^n + \cdots + (\ell - 1)^n \]

be the power sum of the first \( \ell - 1 \) positive integers. It is well known that \( S_n(\ell) \) is strongly related to the Bernoulli polynomials \( B_n(x) \) in the following way:

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Preprint submitted to 10 February 2017
way
\[ S_n(\ell) = \frac{1}{n+1} (B_{n+1}(\ell) - B_{n+1}) \]
where the polynomials \( B_n(x) \) are defined by the generating series
\[ \frac{t e^{tx}}{e^t - 1} = \sum_{k=0}^\infty B_k(x) \frac{t^k}{k!} \]
and \( B_n = B_n(0) \) is the \( n \)th Bernoulli number.

It is possible to find the explicit coefficients of \( \ell \) in \( S_n(\ell) \) [9]:
\[ S_n(\ell) = \sum_{i=0}^{n+1} \ell^i \left( \sum_{k=0}^{n} S_2(n,k) S_1(k+1,i) \frac{1}{k+1} \right), \tag{1} \]
where \( S_1(n,k) \) and \( S_2(n,k) \) are the (signed) Stirling numbers of the first and second kind, respectively.

Recently, Bazsó et al. [1] considered the more general power sum
\[ S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n, \]
where \( m \neq 0, r \) are coprime integers. Obviously, \( S_{1,0}^n(\ell) = S_n(\ell) \). They, among other things, proved that \( S_{m,r}^n(\ell) \) is a polynomial of \( \ell \) with the explicit expression
\[ S_{m,r}^n(\ell) = \frac{m^n}{n+1} \left( B_{n+1} \left( \ell + \frac{r}{m} \right) - B_{n+1} \left( \frac{r}{m} \right) \right). \tag{2} \]
In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [8] deduced a longer expression which contains already just binomial coefficients and Bernoulli numbers.

For some related diophantine results on \( S_{m,r}^n(\ell) \) see [3,10,15,16,2] and the references given there.

Our goal is to give the explicit form of the coefficients of the polynomial \( S_{m,r}^n(\ell) \), thus generalizing (1). In this expression the Stirling numbers of the first kind also will appear, but, in place of the Stirling numbers of the second kind a more general class of numbers arises, the so-called \( r \)-Whitney numbers introduced by the second author [13].

The \( r \)-Whitney numbers \( W_{m,r}(n,k) \) of the second kind are generalizations of the usual Stirling numbers of the second kind with the exponential generating function
\[ \sum_{n=k}^\infty W_{m,r}(n,k) \frac{z^n}{n!} \frac{e^{rz}}{k!} \left( \frac{e^{mz} - 1}{m} \right)^k. \]
For algebraic, combinatoric and analytic properties of these numbers see [5,14]
and [6,7], respectively.

First, we prove the following.

**Theorem 1** For all parameters $\ell > 1, n, m > 0, r \geq 0$ we have

$$S_{m,r}^n(\ell) = \sum_{i=0}^{n+1} \ell^i \left( \sum_{k=0}^{n} \frac{m^k W_{m,r}(n,k)}{k+1} S_1(k+1,i) \right).$$

**Proof.** The formula which connects the power sums and the $r$-Whitney numbers is the next one from [13]:

$$(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)x^k.$$ 

Here $x^k = x(x - 1) \cdots (x - k + 1)$ is the falling factorial. We can see that it is enough to sum from $x = 0, 1, \ldots, \ell - 1$ to get back $S_{m,r}^n(\ell)$. Hence

$$S_{m,r}^n(\ell) = \sum_{k=0}^{n} m^k W_{m,r}(n,k) \sum_{x=0}^{\ell-1} x^k.$$ 

The inner sum can be determined easily (see [9]):

$$\sum_{x=0}^{\ell-1} x^k = \frac{\ell^{k+1}}{k+1} + \delta_{k,0}.$$ 

The Kronecker delta will never appear, because if $k = 0$ then the $r$-Whitney number is zero (unless the trivial case $n = 0$, which we excluded). Therefore, as an intermediate formula, we now have that

$$S_{m,r}^n(\ell) = \sum_{k=0}^{n} m^k W_{m,r}(n,k) \frac{\ell^{k+1}}{k+1}.$$ 

The falling factorial $\ell^{k+1}$ is a polynomial of $\ell$ with Stirling number coefficients:

$$\ell^{k+1} = \sum_{i=0}^{k+1} S_1(k + 1, i) \ell^i.$$ 

Substituting this to the formula above, we obtain:

$$S_{m,r}^n(\ell) = \sum_{k=0}^{n} m^k W_{m,r}(n,k) \frac{k+1}{k+1} \sum_{i=0}^{k+1} S_1(k+1,i) \ell^i.$$ 

Since $S_1(k+1,i)$ is zero if $i > k + 1$, we can run the inner summation up to $n + 1$ (this is taken when $k = n$) to make the inner sum independent of $k$. 

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Altogether, we have that

\[ S_{m,r}^n(\ell) = \sum_{i=0}^{n+1} \ell^i \sum_{k=0}^{n} \frac{m^k W_{m,r}(n, k)}{k + 1} S_1(k + 1, i). \]

This is exactly the formula that we wanted to prove. \(\square\)

Now we give some elementary consequences of the theorem. The proofs are trivial.

**Remark.** The next properties of the polynomial \(S_{m,r}^n(\ell)\) hold true for all parameters \(\ell > 1, n > 0, r, m \geq 0:\)

(i) The constant term of \(S_{m,r}^n(\ell)\) is 0,
(ii) The leading coefficient of \(S_{m,r}^n(\ell)\) is \(m^n/(n + 1)\),
(iii) \(S_{m,r}^n(\ell)\) is a polynomial of \(\ell\) of degree \(n + 1\) unless \(m = 0\); in this latter case the degree is \(n\).

The above statements also follow from (2).

2 The integer property of the coefficients in \(S_{m,r}^n(\ell)\)

The coefficients of the polynomial \(S_{m,r}^n(\ell)\) are not integer in the overwhelming majority of the cases:

\[ S_{1,0}^1(\ell) = \frac{\ell(\ell - 1)}{2}, \]
\[ S_{2,5}^2(\ell) = \frac{1}{3} \ell(47 + 24\ell + 4\ell^2), \]

etc.

However, we revealed that in special cases the polynomial \(S_{m,r}^n(\ell)\) has integer coefficients. Several parameters are in the next table.

<table>
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For example,
\[ S_{2,1}^3(\ell) = \ell^2(2\ell^2 - 1), \]
or
\[ S_{2,3}^3(\ell) = \ell(2 + \ell)(2\ell^2 + 4\ell + 3). \]

From the formula of Theorem 1 it can be seen that if
\[(k + 1) \mid m^k W_{m,r}(n, k) \quad (k = 1, 2, \ldots, n),\]
then we get integer coefficients.

To find another condition which is necessary and sufficient for the integrity of the coefficients in \( S_{m,r}^n(\ell) \), we recall the following well known properties of Bernoulli polynomials and Bernoulli numbers.

\[ B_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k(y)x^{n-k}; \quad (3) \]

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}; \quad (4) \]

\[ B_3 = B_5 = B_7 = \ldots = 0. \quad (5) \]

By the denominator of a rational number \( q \) we mean the smallest positive integer \( d \) such that \( dq \) is an integer. We recall also the von Staudt theorem

\[ \Lambda_{2n} = \prod_{p \mid (2n-1)} p, \quad (6) \]

where \( \Lambda_n \) is the denominator of \( B_n \). In particular, \( \Lambda_n \) is a square-free integer, divisible by 6. For the proofs of (3)-(5) see e.g. the work of Brillhart [4].

Let \( 2 \leq j \leq n \) be an even number and put

\[ f(n, j) := \text{lcm} \left( \frac{\Lambda_j}{\text{gcd} \left( \Lambda_j, \binom{n+1}{j} \right)}, \frac{\Lambda_j}{\text{gcd} \left( \Lambda_j, \binom{n+1}{j+1} \right)} \right) \cdots, \]

\[ \frac{\Lambda_j}{\text{gcd} \left( \Lambda_j, \binom{n}{j} \right)} \right). \quad (7) \]

Further, we define

\[ F(n) := \begin{cases} \text{lcm} \left( \text{rad}(n + 1), f(n, 2), f(n, 4), \ldots, f(n, n) \right) & \text{if } n \text{ is even}, \\ \text{lcm} \left( \text{rad}(n + 1), f(n, 2), f(n, 4), \ldots, f(n, n - 1) \right) & \text{if } n \text{ is odd}, \end{cases} \quad (8) \]

where

\[ \text{rad}(n) = \prod_{p \mid n} p. \]
**Theorem 2** The polynomial \( S_{m,r}^n(\ell) \) has integer coefficients if and only if \( F(n) \mid m \).

**Proof.** By relations (2), (3) and (4) we can rewrite \( S_{m,r}^n(\ell) \) as follows:

\[
S_{m,r}^n(\ell) = \frac{m^n}{n+1} \left( B_{n+1} \left( \ell + \frac{r}{m} \right) - B_{n+1} \left( \frac{r}{m} \right) \right) = \frac{m^n}{n+1} \left( \sum_{k=0}^{n+1} \binom{n+1}{k} B_k \left( \frac{r}{m} \right) \ell^{n+1-k} \right) - B_{n+1} \left( \frac{r}{m} \right) = \frac{m^n}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k \left( \frac{r}{m} \right) \ell^{n+1-k} = \frac{m^n}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \sum_{j=0}^{k} \binom{k}{j} B_j \left( \frac{r}{m} \right)^{k-j} \ell^{n+1-k}. \tag{12}
\]

We denote the common denominator of the coefficients of \( S_{m,r}^n(\ell) \) by \( Q \). One can see from (9) that the polynomial has integral coefficients if and only if \( m \) is divisible by \( Q \). Thus we have to determine \( Q \).

By (12) we observe that neither \( m \) nor \( r \) occurs in \( Q \). Moreover, the only algebraic expressions which may affect \( Q \) in (12) are on one hand \( n+1 \) and on the other hand, the denominators of the Bernoulli numbers involved, which are \( 2, \Lambda_j (2 \leq j \leq n \text{ even}) \) by (5) and the von Staudt theorem.

It can easily be seen that \( n+1 \mid m^n \) if \( \text{rad}(n+1) \mid m \). Indeed, supposing the contrary, i.e., that \( \text{rad}(n+1) \mid m \) and \( n+1 \nmid m^n \), it implies that there is a prime factor \( p \) of \( n+1 \) such that \( p^{n+1} \) divides \( n+1 \). Hence \( 2^{n+1} \leq p^{n+1} \leq n+1 \), which is a contradiction.

Let \( 2 \leq j \leq n \) be an even index. It follows from (12) that the contribution of \( \Lambda_j \) to the common denominator \( Q \) is precisely \( f(n,j) \) defined in (7). In other words, if \( f(n,j) \mid m \), then every term of (12) containing the factor \( B_j \) has integer coefficients.

In conclusion, we obtained that \( Q \) is the least common multiple of \( \text{rad}(n+1) \) and \( f(n,j) \) for all even \( j \in [2, n] \), which number we denoted in (8) by \( F(n) \). The theorem is proved. \( \square \)

**Remark.** An easy consequence of our Theorem 2 is that \( S_n(\ell) = S_{1,0}^n(\ell) \notin \mathbb{Z}[x] \) for any positive integer \( n \).

Some small values of \( F(n) \) are listed in the following table. These are results of an easy computation in MAPLE.
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Acknowledgements

This scientific work was financed by Proyecto Prometeo de la Secretaría Nacional de Ciencia, Tecnología e Innovación (Ecuador). The first author was supported by the Hungarian Academy of Sciences and by the OTKA grant NK104208.

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