Short thesis for the degree of doctor of philosophy (PhD)

Convergence of Cesàro means with variable parameters of Walsh-Fourier series

by

Anas Ahmad Mohammad Abu Joudeh

Supervisor: Prof. Dr. Gát György Tamás

UNIVERSITY OF DEBRECEN
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The results described in the dissertation and in this thesis have been published (or accepted for publication) in the following two papers: [6],[7].

Introduction

The present thesis talks about convergence of Cesàro means with variable parameters for Walsh-Fourier series. It consists of an introduction, four chapters, an abstract and a bibliography. In the introduction, we present some important and well-known notions and definitions related to the new results appearing in the thesis. Moreover, we present some historical background.

In 1800's Jean Baptiste Joseph Fourier began to work on the theory of heat. In 1822, he published book with tittle of Théorie Analytic de la Chaleur (The Analytic Theory of Heat).

A great deal of effort has been expended after this work in this research area. It became and called Fourier theory and field of harmonic analysis. Fourier theory gained exceptional importance in theoretical content and also enormous scope and great relevance everywhere in applications such as electrical engineering.

One of the greatest achievements of mathematics in the twentieth century is the result of Carleson. In 1966 he prved the almost everywhere convergence of the partial sums of the (trigonometric) Fourier series of a square integrable function. On the other, hand in 1926 Kolmogoroff [5] gave the construction of an integrable function with everywhere divergent trigonometric Fourier series. That is, if we want to have some pointwise convergence result for each function belonging to the Lebesgue space L^1 then it is needed to use some summation method. The invention of Fejér [11] was to use the arithmetical means of the partial sums. Among others, he proved for continuous functions that these means converge to the function in the supreumum norm. One year later, Lebesgue proved the almost everywhere convergence of these so-called Fejér means to the function for each integrable function. That is, the behavior of the Fejér (or also called $(C, 1)$) means is better than the behavior of the partial sums in this point of view. This fact also justifies the investigation of various summation methods of Fourier series. Later on, we write about the (C, α) summation - which is a generalization of the Fejér summation- of Fourier series. The result of Lebesgue above for the (C, α) case $(\alpha > 0)$ is due to M. Riesz [33].

Moreover, Fourier analyis has been developed on other structures too. For example, the dyadic group is the simplest but nontrivial model of the complete product of finite groups. Representing the characters of the dyadic group ordered in the Paley's sense, we obtain the Walsh system.

A relatively new thing of the generalizations on the Walsh-Paley system is the Vilenkin system introduced by Vilenkin [37] in 1947. He used the set of all characters of the complete product of arbitrary cyclic groups to obtain the commutative case.

In Hungary a dyadic analysis team works leaded by F. Schipp having many results in this theory. For instance, he proved that the partial sums of the Vilenkin-Fourier series (even in the unbounded case) of a function in $L^p(G)$ $(1 < p < 1)$ converge in the appropriate norm to the function (Schipp [29], Simon [34]).

And also Young [41] from Canada .

With respect to noncommutative Vilenkin groups (complete direct product of not necessarily Abelian groups) some studies were appeared in [14] by Gát and Toledo. They obtained not only negative results for this situation. They proved the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case.

Preliminaries

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [32]) and others. The notion of the Hardy space $H(I)$ is introduced in the following way [32]. Set the definition of the nth ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$. the Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels, respectively and so for the Fejér or $(C, 1)$ means of f. the kernel of the summability method (C, α_n) and call it the (C, α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$. Finally, an introduction to the two-dimensional Fourier coefficients, the rectangular partial sums of the two-dimensional Fourier series, the rectangular Dirichlet kernels and the (C, α_a) Cesàro-Marcinkiewicz means of integrable function f for two variables.

The Standard Notions Of Dyadic Analysis

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [32]) and others. Denote by $\mathbb{N} := \{0, 1, ...\}, \mathbb{P} := \mathbb{N} \setminus \{0\}$, the set of natural numbers, the set of positive integers and $I := [0,1)$ the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set $B (B \subset I)$.

Denote by $L^p(I)$ the usual Lebesgue spaces and $\|.\|_p$ the corresponding norms $(1 \leq p \leq \infty)$. Set

$$\mathcal{J} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\}$$

the set of dyadic intervals and for given $x \in I$ and let $I_n(x)$ denote the interval $I_n(x) \in \mathcal{J}$ of length 2^{-n} which contains $x \, (n \in \mathbb{N})$. Also use the notation $I_n := I_n(0) \, (n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number $(x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\})$ we choose the expansion which terminates in 0's.

The Notions Of the Hardy space

The notion of the Hardy space $H(I)$ is introduced in the following way [32]. A function $a \in L^\infty(I)$ is called an atom, if either $a = 1$ or a has the following properties: *supp* $a \subseteq I_a, \|a\|_\infty \leq |I_a|^{-1}, \int_I a = 0$, for some $I_a \in \mathcal{J}$. We say that the function f belongs to H, if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i's are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^\infty |\lambda_i| < \infty$ is true. It is known that H is a

Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \in H.$$

Definition. The nth ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$ is:

$$\omega_n(x) := \prod_{j=0}^{\infty} (-1)^{x_j n_j},$$

where $\mathbb{N} \ni n = \sum_{n=0}^{\infty} n_j 2^j$ ($n_j \in \{0, 1\}$ ($j \in \mathbb{N}$)). It is known (see [23] or [36]) that for the elements of the system $(\omega_n, n \in \mathbb{N})$ we have the almost everywhere equality

$$\omega_n(x + y) = \omega_n(x)\omega_n(y),$$

where the operation $+$ is the so-called logical addition on I. That is, for any $x, y \in I$

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$

Definition. The Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels Denote by

$$\hat{f}(n) := \int_I f\omega_n d\lambda, \quad D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n^1 := \frac{1}{n+1} \sum_{k=0}^{n} D_k.$$

Definition. The Fejér or $(C, 1)$ means of f

It is also known that the Fejér or $(C, 1)$ means of f is

$$\sigma_n^1 f(y) := \frac{1}{n+1}\sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n^1(y+x)\, d\lambda(x)$$
$$= \frac{1}{n+1}\sum_{k=0}^{n}\int_I f(x) D_k(y+x)\, d\lambda(x), \quad (n \in \mathbb{N}, y \in I).$$

It is known [32] that for $n \in \mathbb{N}, x \in I$ it holds

$$D_{2^n}(x) = \begin{cases} 2^n \text{ , if } x \in I_n \\ 0 \quad \text{, if } x \notin I_n \end{cases}$$

and also that

$$D_n(x) = \omega_n(x)\sum_{k=1}^{\infty} D_{2^k}(x) n_k (-1)^{x_k},$$

where $n = \sum_{i=1}^{\infty} n_i 2^i$, $n_i = \{0, 1\}$ $(i \in \mathbb{N})$.

Definition. The (C, α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C, α_n) and call it the (C, α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

$$K_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} D_k,$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n + 1)(\alpha_n + 2)...(\alpha_n + n)}{k!}.$$

It is known [44] that $A_n^{\alpha_n} = \sum_{k=0}^{n} A_k^{\alpha_n - 1}$, $A_k^{\alpha_n} - A_{k+1}^{\alpha_n} = -\frac{\alpha_n A_k^{\alpha_n}}{k+1}$.

Definition. Cesàro means of integrable function f

The (C, α_n) Cesàro means of integrable function f is

$$\sigma_n^{\alpha_n} f(y) := \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} S_k f(y) = \int_I f(x) K_n^{\alpha_n}(y + x) d\lambda(x).$$

Definition. The two-dimensional Fourier coefficients

Now, for the two variable case we have for $x = \left(x^1, x^2\right), y = \left(y^1, y^2\right) \in I^2$, $n = (n_1, n_2) \in \mathbb{N}^2$ the two-dimensional Fourier coefficients

$$\hat{f}(n_1, n_2) := \int\limits_{I \times I} f(x^1, x^2) \omega_{n_1}(x^1) \omega_{n_2}(x^2) d\lambda(x^1, x^2).$$

Definition. Rectangular partial sums of the two-dimensional Fourier series

The rectangular partial sums of the two-dimensional Fourier are

$$S_{n_1, n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1 - 1} \sum_{k_2=0}^{n_2 - 1} \hat{f}(k_1, k_2) \omega_{k_1}(y^1) \omega_{k_2}(y^2).$$

Definition. Rectangular Dirichlet kernels.

The rectangular Dirichlet kernels are

$$D_{n_1,n_2}(z) := D_{n_1}(z^1)D_{n_2}(z^2) = \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \omega_{k_1}(z^1)\omega_{k_2}(z^2),$$

where $(z = (z^1, z^2) \in I^2)$.

Definition. Marcinkiewicz mean and kernel.

We have the n^{th} Marcinkiewicz mean and kernel

$$\sigma_n^1 f(y) := \frac{1}{n+1}\sum_{k=0}^{n} S_{j,j}f(y), \quad K_n^1(z) = \frac{1}{n+1}\sum_{j=0}^{n} D_{j,j}(z).$$

Thus, we get

$$\sigma_n^1 f(y^1, y^2) = \int\limits_{I\times I} f(x^1, x^2)K_n^1(y^1 + x^1, y^2 + x^2)d\lambda(x^1, x^2).$$

Definition. The (C, α_n) kernel or the Cesàro-Marcinkiewicz kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C, α_n)-Marcinkiewicz and call it the (C, α_a) kernel or the Cesàro-Marcin-

kiewicz kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

$$K_n^{\alpha_n}(x_1, x_2) = \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n} A_{n-k}^{\alpha_n-1}D_{j,j}(x_1, x_2)$$

where
$$A_k^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)...(\alpha_n+k)}{k!}.$$

Definition. The (C, α_n) Cesàro-Marcinkiewicz means of integrable function f for two variables

The (C, α_n) Cesàro-Marcinkiewicz means of integrable function f for two variables are

$$\sigma_n^{\alpha_n} f(y^1, y^2) = \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} S_{k,k} f(y^1, y^2)(x)$$
$$= \int_{I\times I} f(x^1, x^2) K_n^{\alpha_n}(y^1 + x^1, y^2 + x^2) d\lambda(x^1, x^2).$$
$$= \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} \int_{I\times I} A_{n-k}^{\alpha_n - 1} f(x^1, x^2) D_k(y^1 + x^1) D_k(y^2 + x^2) d\lambda(x^1, x^2).$$

Over all of the chapter discussing the generalized Marcinkiewicz-Cesàro means we suppose that monotone decreasing sequences (α_n) and (β_n) satisfy

$$\beta_n = \alpha_{2^n}, \quad \frac{\alpha_N}{A_N^{\alpha_N}} \log^{\delta}\left(1 + \frac{N}{n}\right) \le C \frac{\alpha_n}{A_n^{\alpha_n}} \ (N \ge n, n, N \in \mathbb{P})(1)$$

for some $\delta > 1$ and for some positive constant C. We remark that from condition (1) it follows that sequence $(\frac{\alpha_n}{A_n^{\alpha_n}})$ is quasi monotone decreasing. That is, for some $C > 0$ we have $\frac{\alpha_N}{A_N^{\alpha_N}} \le C \frac{\alpha_n}{A_n^{\alpha_n}}$ $(N \ge n, n, N \in \mathbb{P})$.

Cesàro means of Fourier series with variable parameters (C, α_{2^n})

We introduced the notion of Cesàro means of Fourier series with variable parameters. We proved the almost everywhere convergence of a subsequnce of the Cesàro (C, α_n) means of integrable functions. That is, $\sigma_{2^n}^{\alpha_{2^n}} f \to f$ for $f \in L^1(I)$, where I is the unit interval (representing the dyadic, or Walsh group) for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$.

The main theorems of this chapter was proving:

Theorem 1. Suppose that $1 > \alpha_n > 0$. Let $f \in L^1(I)$. Then we have the a.e convergence $\sigma_{2^n}^{\alpha_{2^n}} f \longrightarrow f$.

The method we used to prove Theorem 1 is to investigated the maximal operator $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_{2^n}^{\alpha_{2^n}} f|$. We also proved that this operator is of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

Theorem 2. Suppose that $1 > \alpha_n > 0$. Let $f \in H(I)$. Then we have

$$\|\sigma_*^\alpha f\|_1 \leq C\|f\|_H.$$

Moreover, the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

$$\|\sigma_*^\alpha f\|_p \leq C_p\|f\|_p \text{ for all } 1 < p \leq \infty.$$

Basically, in order to proved Theorem 1 we verified that the maximal operator $\sigma_*^\alpha f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we got this, the investigation of kernel functions, and its maximal function on the unit interval I by making a hole around zero. To have the proof of Theorem 2 is the standard way after having the fact that $\sigma_*^{\alpha} f$ is of weak type (L^1, L^1).

By using next important Lemmas:
Lemma 1. Let $1 > \alpha_a > 0$, $f \in L^1(I)$ such that supp $f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$ $(a, k \in \mathbb{N}, u \in I)$. Then we have

$$\int_{I \setminus I_k(u)} \sup_{n, a \in \mathbb{N}} |t_n^{\alpha_a} f| d\lambda \leq C \|f\|_1.$$

.
Lemma 2. The operator σ_*^{α} is of type (L^{∞}, L^{∞}).
Lemma 3. Let $1 > \alpha_n > 0$. The operator σ_*^{α} is of weak type (L^1, L^1) $(\sigma_*^{\alpha} f := \sup_n |\sigma_{2^n}^{\alpha_{2^n}} f|\ )$.

Proof of Theorem 1. Let P be a Walsh polynomial, where $P(x) = \sum_{i=0}^{2^k - 1} c_i \omega_i$. Then for all natural number $n \geq 2^k$ we have that $S_n P \equiv P$. Consequently, the relation $\sigma_{2^n}^{\alpha_{2^n}} P \longrightarrow P$ holds everywhere.
Now Let $\epsilon, \delta > 0$, $f \in L^1$ Let P be a polynomial such that $\|f - P\|_{L^1} < \delta$ Then

$$\lambda(\overline{\lim_n} |\sigma_{2^n}^{\alpha_{2^n}} f - f| > \epsilon) \leq \lambda(\overline{\lim_n} |\sigma_{2^n}^{\alpha_{2^n}} (f - P)| > \frac{\epsilon}{3})$$

$$+\lambda(\overline{\lim_n} |\sigma_{2^n}^{\alpha_{2^n}} P - P| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_n} |P - f| > \frac{\epsilon}{3})$$

$$\leq \lambda(\sup_n |\sigma_{2^n}^{\alpha_{2^n}} (f - P)| > \frac{\epsilon}{3}) + 0 + \frac{3}{\epsilon} \|P - f\|_1 \leq C \|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C}{\epsilon} \delta.$$

Because σ_*^α is of weak type (L^1, L^1). So for all $\delta > 0$ and consequently for arbitrary $\epsilon > 0$ we have

$$\lambda(\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > \epsilon) = 0.$$

By the set inclusion

$$\{\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > 0\} \subset \bigcup_{k=1}^{\infty} \{\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > \frac{1}{k}\}$$

and by the fact that the union of each member on the right side is a 0 measure set we have that the left side is also a 0 measure set. Thus,

$$\mu\{\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > 0\} = 0$$

$$\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| = 0 \quad a.e$$

$$\lim_n |\sigma_{2^n}^{\alpha_{2^n}} f - f| = 0 \quad a.e$$

$$\lim_n(\sigma_{2^n}^{\alpha_{2^n}} f - f) = 0 \quad a.e$$

$$\sigma_{2^n}^{\alpha_{2^n}} f \longrightarrow f \quad a.e$$

That is, the proof of Theorem 1 is complete.

Proof of Theorem 2. Lemma 2 and Lemma 3 by the interpolation theorem of Marcinkiewicz [45] gives that the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. Let a be an atom ($a \neq 1$ can be supposed), $\operatorname{supp} a \subset I_k(x)$, $\int_I a d\lambda = 0$ and

$\|a\|_\infty \leq 2^k$ for some $k \in \mathbb{N}$ and $x \in I$. Then , $n < 2^k$ implies $S_n a = 0$ because $\int_{I_k(x)} a(t)d\lambda(t) = 0$ That is,

$$\sigma_*^\alpha a = \sup_{2^n \geq 2^k} |\sigma_{2^n}^{\alpha_{2^n}} f|.$$

By the help Lemma 1 it gives

$$\int_{I \setminus I_k(x)} \sigma_*^\alpha a \; d\lambda \leq \int_{I \setminus I_k(x)} \sup_{n \geq 2^k} \left| \int_{I_k(x)} a(y) T_n^{\alpha_n}(z+y) d\lambda(y) \right| d\lambda(z)$$

$$\leq C\|a\|_1 \leq C.$$

Since the operator σ_*^α is of type (L^2, L^2) (i.e $\|\sigma_*^\alpha f\|_2 \leq C\|f\|_2$ for all $f \in L^2(I)$), then we have

$$\|\sigma_*^\alpha a\|_1 = \int_{I \setminus I_a} \sigma_*^\alpha a + \int_{I_k(x)} \sigma_*^\alpha a$$

$$\leq C + |I_k(x)|^{\frac{1}{2}} \|\sigma_*^\alpha a\|_2 \leq C + C2^{\frac{-k}{2}} \|a\|_2 \leq C + C2^{\frac{-k}{2}} 2^{\frac{k}{2}} \leq C.$$

That is, $\|\sigma_*^\alpha a\|_1 \leq C$ and consequently the σ-sublinearity of σ_*^α gives

$$\|\sigma_*^\alpha f\|_1 \leq \sum_{i=0}^{\infty} |\lambda_i| \|\sigma_*^\alpha a_i\|_1 \leq C \sum_{i=0}^{\infty} |\lambda_i| \leq C\|f\|_H$$

for all $\sum_{i=0}^{\infty} \lambda_i a_i \in H$. That is , the operator σ_*^α is of type (H, L). That is, the proof of Theorem 2 is complete.

Cesàro means of Fourier series with variable parameters (C, α_n)

We introduced the notion of Cesàro means of Fourier series with variable parameters. We proved the almost everywhere convergence of the Cesàro (C, α_n) means of integrable functions $\sigma_n^{\alpha_n} f \to f$ for each $f \in L^1(I)$, where $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. Provided that for some restriction set (discussed below) $\mathbb{N}_{\alpha,K} \ni n \to \infty$, where K is any but fixed natural number.

Set two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$. For instance $P(n, 1) = n$. Also set for sequences $\alpha = (\alpha_n)$ and positive reals K the subset of natural numbers

$$\mathbb{N}_{\alpha,K} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.$$

We can easily remark that for a sequence α such that $1 > \alpha_n \geq \alpha_0 > 0$ we have $\mathbb{N}_{\alpha,K} = \mathbb{N}$ for some K depending only on α_0. We also remark that $2^n \in \mathbb{N}_{\alpha,K}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $K \geq 1$.

In this chapter C denotes an absolute constant and C_K another one which may depend only on K. The introduction of (C, α_n) means of Fourier series due to Akhobadze (although for numerical series Kaplan published a paper in [27] 1960) investigated [1] the behavior of the L^1-norm convergence of $\sigma_n^{\alpha_n} f \to f$ for the trigonometric system. In this chapter it is also supposed that $1 > \alpha_n > 0$ for all n.

The main theorems of this chapter was proving:
Theorem 3. Suppose that $1 > \alpha_n > 0$. Let $f \in L^1(I)$. Then

we have the almost everywhere convergence $\sigma_n^{\alpha_n} f \to f$ provided that $\mathbb{N}_{\alpha,K} \ni n \to \infty$.

The method we used to proved Theorem 3 is to investigate the maximal operator $\sigma_*^{\alpha} f := \sup_{n\in\mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f|$. We also proved that this operator is a kind of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

Theorem 4. Suppose that $1 > \alpha_n > 0$. Let $|f| \in H(I)$. Then we have

$$\|\sigma_*^{\alpha} f\|_1 \leq C_K \||f|\|_H.$$

Moreover, the operator σ_*^{α} is of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

$$\|\sigma_*^{\alpha} f\|_p \leq C_{K,p} \|f\|_p, \ \ for\ all\ 1 < p \leq \infty.$$

For the sequence $\alpha_n = 1$ Theorem 4 is due to Fujii [12]. For the more general but constant sequence $\alpha_n = \alpha_1$ see Weisz [38].

Basically, in order to prove Theorem 3 we verified that the maximal operator $\sigma_*^{\alpha} f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we get this is the investigation of kernel functions and their maximal function on the unit interval I by making a hole around zero. Besides, some quasi locality issue (for the notion of quasi-locality see [32]). To have the proof of Theorem 4 is the standard way.

By using next important Lemmas:

Lemma 4. Let $1 > \alpha_n > 0$, $f \in L^1(I)$ such that supp $f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$. Then we have

$$\int_{I\setminus I_k(u)} \tilde{\sigma}_*^{\alpha} f d\lambda \leq C_K \|f\|_1,$$

where constant C_K can depend only on K.
Lemma 5. The operator $\tilde{\sigma}_*^\alpha$ is of type (L^∞, L^∞) ($\tilde{\sigma}_*^\alpha f := \sup_{n\in\mathbb{N}_{\alpha,K}} |\tilde{\sigma}_n^{\alpha_n} f|$).
Lemma 6. The operators $\tilde{\sigma}_*^\alpha$ and σ_*^α are of weak type (L^1, L^1).

Proof of Theorem 3. The proof is quite similar to the proof of Theorem 1 and that is why afew steps are omitted. Let $P \in \mathbf{P}$ be a polynomial where $P(x) = \sum_{i=0}^{2^k-1} c_i\omega_i$. Then for all natural number $n \geq 2^k$, $n \in \mathbb{N}_{\alpha,K}$ we have that $S_nP \equiv P$. Consequently, the statement $\sigma_n^{\alpha_n} P \longrightarrow P$ holds everywhere (of course not only for restricted $n \in \mathbb{N}_{\alpha,K}$). Now, let $\epsilon, \delta > 0$, $f \in L^1$. Let $P \in \mathbf{P}$ be a polynomial such that $\|f - P\|_1 < \delta$. Then

$$\begin{aligned}
&\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|\sigma_n^{\alpha_n} f - f| > \epsilon) \\
&\leq \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|\sigma_n^{\alpha_n}(f - P)| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|\sigma_n^{\alpha_n} P - P| > \frac{\epsilon}{3}) \\
&+ \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|P - f| > \frac{\epsilon}{3}) \\
&\leq C_K\|P - f\|_1\frac{3}{\epsilon} \\
&\leq \frac{C_K}{\epsilon}\delta
\end{aligned}$$

because σ_*^α is of weak type (L^1, L^1) (with any fixed $K > 0$). This holds for all $\delta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|\sigma_n^{\alpha_n} f - f| > \epsilon) = 0$$

and consequently we also have

$$\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|\sigma_n^{\alpha_n}f-f|>0)=0.$$

This finally gives

$$\sigma_n^{\alpha_n}f\longrightarrow f \quad a.e. \quad (n\in\mathbb{N}_{\alpha,K}).$$

This completes the proof of Theorem 3.

Proof of Theorem 4. The proof of this theorem are similar to those in the proof of Theorem 2 and we skip some steps. Inequality $|K_n^{\alpha_n}|\le\tilde{K}_n^{\alpha_n}$, Lemma 5 and Lemma 6 by the interpolation theorem of Marcinkiewicz [32] give that the operator σ_*^α is of type (L^p,L^p) for all $1<p\le\infty$. In the sequel we prove that operator $\tilde{\sigma}_*^\alpha f=\sup_{n\in\mathbb{N}_{\alpha,K}}|f*\tilde{K}_n^\alpha|$ is of type (H,L).

Let a be an atom ($a\neq 1$ can be supposed), $a\subset I_k(x)$, $\|a\|_\infty\le 2^k$ for some $k\in N$ and $x\in I$. Then, $n<2^k$, $n\in\mathbb{N}_{\alpha,K}$ implies $a*\tilde{K}_n^\alpha=0$ because $\tilde{K}_n^\alpha$ is $\mathcal{A}_k$ measurable for $n<2^k$ and $\int_{I_k(x)}a(t)d\lambda(t)=0$. That is,

$$\tilde{\sigma}_*^\alpha a=\sup_{\mathbb{N}_{\alpha,K}\ni n\ge 2^k}|\tilde{\sigma}_n^{\alpha_n}f|.$$

By the help Lemma 4 we have

$$\int_{I\setminus I_k(x)}\tilde{\sigma}_*^\alpha a\,d\lambda=$$

$$\int_{I\setminus I_k(x)}\sup_{\mathbb{N}_{\alpha,K}\ni n\ge 2^k}\left|\int_{I_k(x)}a(y)\tilde{K}_n^{\alpha_n}(z+y)d\lambda(y)\right|d\lambda(z)$$

$$\le C_K\int_{I_k(x)}|a(y)|d\lambda(y)\le C_K\|a\|_1\le C_K.$$

Since the operator $\tilde{\sigma}_*^\alpha$ is of type (L^2, L^2) (i.e $\|\tilde{\sigma}_*^\alpha f\|_2 \le C_K \|f\|_2$ for all $f \in L^2(I)$), then we have

$$\|\tilde{\sigma}_*^\alpha a\|_1 = \int_{I \setminus I_k(x)} \tilde{\sigma}_*^\alpha a + \int_{I_k(x)} \tilde{\sigma}_*^\alpha a \le C_K.$$

That is $\|\tilde{\sigma}_*^\alpha a\|_1 \le C_K$ and consequently the σ-sublinearity of $\tilde{\sigma}_*^\alpha$ gives

$$\|\tilde{\sigma}_*^\alpha f\|_1 \le \sum_{i=0}^\infty |\lambda_i| \|\tilde{\sigma}_*^\alpha a_i\|_1 \le C_K \sum_{i=0}^\infty |\lambda_i| \le C_K \|f\|_H$$

for all $\sum_{i=0}^\infty \lambda_i a_i \in H$. That is, the operator $\tilde{\sigma}_*^\alpha$ is of type (H, L). This by inequality $|K_n^{\alpha_n}| \le \tilde{K}_n^{\alpha_n}$ and then by $\|\sigma_*^\alpha f\|_1 \le \|\tilde{\sigma}_*^\alpha |f|\|_1 \le C_K \||f|\|_H$ completes the proof of Theorem 4.

Cesàro means of two variable Walsh-Fourier series (C, β_n)

We formulated and proved that the maximal operator of some (C, β_n) means of cubical partial sums of two variable Walsh-Fourier series of integrable functions is of weak type (L^1, L^1). Moreover, the (C, β_n)-means $\sigma_{2^n}^{\beta_n} f$ of the function $f \in L^1$ converge a.e. to f for $f \in L^1(I^2)$, for some sequences $1 > \beta_n \searrow 0$.

We supposed that (α_n) and (β_n) sequences are monotone decreasing sequences and they satisfy:

$$\beta_n = \alpha_{2^n}, \frac{\alpha_N}{A_N^{\alpha_N}} \log^\delta \left(1 + \frac{N}{n}\right) \le C \frac{\alpha_n}{A_n^{\alpha_n}} (N \ge n \in \mathbb{P}) = \mathbb{N} \setminus \{0\}$$

for some $\delta > 1$ and for some positive constant C. We remark that from the condition above it follows that sequence $(\frac{\alpha_n}{A_n^{\alpha_n}})$ is quasi monotone decreasing. That is, for some $C > 0$ we have $\frac{\alpha_N}{A_N^{\alpha_N}} \le C\frac{\alpha_n}{A_n^{\alpha_n}}$ $(N \ge n, n, N \in \mathbb{P})$.

The main theorem of this chapter is:

Theorem 5. Suppose that monotone decreasing sequence $1 > \beta_n > 0$ satisfies the condition $\frac{A_{2^n}^{\beta_n}}{\beta_n}\frac{\beta_N}{A_{2^N}^{\beta_N}}(N+1-n)^\delta \le C$ for every $\mathbb{N} \ni N \ge n \ge 1$ and for some $\delta > 1$. Let $f \in L^1(I^2)$. Then we have the almost everywhere convergence

$$\sigma_{2^n}^{\beta_n} f \to f.$$

Remark. In the proof of Theorem 5 we defined the sequence (α_n) in a way that $\alpha_{2^k} = \beta_k$ and for any $2^k \le n < 2^{k+1}$ let $\alpha_n = \alpha_{2^k} = \beta_k$. Then the sequence (α_n) satisfies that it is decreasing and $\frac{A_n^{\alpha_n}}{\alpha_n}\frac{\alpha_N}{A_N^{\alpha_N}}\log^\delta\left(1+\frac{N}{n}\right) \le C$ for every $\mathbb{N} \ni N \ge n \ge 1$. That is, condition above is fulfilled.

- We give two examples for sequences (β_n) like above. Example one: $\beta_k = \alpha_{2^k} = \alpha_n = \alpha \in (0,1)$ for every natural number k, n.
- Example two: Let $\alpha_n = 1/n$. Then it is not difficult to have that $A_n^{\alpha_n} \to 1$ and for the sequence (α_n) $CN/n \ge \log^\delta(1+N/n)$ trivially holds with some $\delta > 1$.

Proof of Theorem 5. Let $P \in \mathbf{P}$ be a two-dimensional Walsh polynomial, that is, $P(x) = \sum_{i,j=0}^{2^k-1} c_{i,j}\omega_i(x^1)\omega_j(x^2)$.

Then for all natural number $m \geq 2^k$ we have that $S_{m,m}P \equiv P$. Consequently, the statement $\sigma_{2^n}^{\beta_n} P \to P$ holds everywhere. This follows from the fact that for any fixed j it holds $\frac{A_{2^n-j}^{\beta_n-1}}{A_{2^n}^{\beta_n}} \to 0$ since for instance for $j = 1$ we have $\frac{A_{2^n-1}^{\beta_n-1}}{A_{2^n}^{\beta_n}} = \frac{\beta_n 2^n}{(2^n-1+\beta_n)(2^n+\beta_n)} \to 0$.

Now, let $\eta, \epsilon > 0$, $f \in L^1(I^2)$. Let $P \in \mathbf{P}$ be a two-dimensional Walsh polynomial such that $\|f - P\|_1 < \eta$. Then by the already seen method we get

$$\begin{aligned}
&\lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} f - f| > \epsilon) \\
&\leq \lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n}(f - P)| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} P - P| > \frac{\epsilon}{3}) \\
&+ \lambda(\overline{\lim_{n\in\mathbb{N}}}|P - f| > \frac{\epsilon}{3}) \\
&\leq C\|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C}{\epsilon}\eta
\end{aligned}$$

because σ_*^β is of weak type (L^1, L^1). This holds for all $\eta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$\lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} f - f| > \epsilon) = 0$$

and consequently we also have

$$\lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} f - f| > 0) = 0.$$

This finally gives $\sigma_{2^n}^{\beta_n} f \longrightarrow f$ a.e. This completes the proof of Theorem 5.

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