SHORT THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (PHD)

TOPOLOGICAL LOOP WITH SOLVABLE MULTIPLICATION GROUP

by

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The results described in the dissertation and in this thesis have been published in the following four papers [13], [14], [15] and [16]. In the dissertation we use the following notations:

G	Lie group
$\mathbf{g}$	Lie algebra
$L_{\mathcal{F}}$	elementary filiform loop
$\mathcal{F}_n$	the n-dimensional elementary filiform Lie group
$\mathbf{f}_n$	the n-dimensional elementary filiform Lie algebra
$Mult(L)$	the multiplication group of loop L
$\mathbf{mult}(L)$	the Lie algebra of group $Mult(L)$
$\mathcal{L}_2$	the 2-dimensional non-abelian Lie group
$\mathbf{l}_2$	the 2-dimensional non-abelian Lie algebra
$Z(L)$	the centre of loop L
Z	the centre of group $Mult(L)$
$\mathbf{z}$	the centre of Lie algebra $\mathbf{g}$
$\mathbf{g}'$	the commutator subalgebra of Lie algebra $\mathbf{g}$
$Inn(L)$	the inner mapping group of loop L
$\mathbf{inn}(L)$	the Lie algebra of group $Inn(L)$
$\widetilde{PSL_2}(\mathbb{R})$	the universal covering group of $PSL_2(\mathbb{R})$
$\mathbb{R}^n$	n-dimensional abelian Lie algebra
$\mathbf{n}_{\mathrm{rad}}$	the nilradical of Lie algebra $\mathbf{g}$
e	the identity element of loop L
$\Lambda(L)$	the set of all left translations of loop L
$P(L)$	the set of all right translations of loop L
G_ℓ	the group generated by all left translations of loop L
G_r	the group generated by all right translations of loop L

Introduction

The dissertation is devoted to investigate the relations between non-associative binary systems loops L and the transformation groups $Mult(L)$ generated by all left and right translations of L. This group is called the multiplication group of L. The action of the group $Mult(L)$ on L is transitive and effective. The stabilizer of the identity element of L in the group $Mult(L)$ is the inner mapping group $Inn(L)$ of L. The initial steps to treat loops came from the study of coordinate systems of non-desarguesian planes and from the investigation of topological questions in differential geometry (cf. [3]). Firstly R. Baer considered loops in connection with the group G_ℓ or G_r generated by their left or right translations (cf. [2]). The studies of A. A. Albert ([1]) and R. H. Bruck ([5]) strengthened the algebraic features of loops. They proved that every normal subloop of L corresponds to a normal subgroup of the group $Mult(L)$ and the orbit of a normal subgroup of $Mult(L)$ with respect to the identity element $e \in L$ results a normal subloop of L (cf. Theorems 3, 4 and 5 in [1] and Lemma 1.3, IV.1, in [5]). Hence the group $Mult(L)$ and the subgroup $Inn(L)$ play an essential role for the investigation of the structure of the L (cf. [1], [5], [6], [22], [23], [32], [33], [37], [38]). In [4] it is proved that the nilpotency of the group $Mult(L)$ forces that the loop L is centrally nilpotent. In this case the group $Inn(L)$ is commutative. For finite loops A. Vesanen ([42]) proved that from the solvability of the group $Mult(L)$ follows the classical solvability of the loop L. Analogously as in the group case a loop L is classically solvable if there is a subnormal series of L such that every factor loop is commutative. Using congruences defining the decomposition of a loop L into its left cosets xN, $x \in L$, with respect to the normal subloop N of L, D. Stanovský and P. Vojtěchovský developed commutator theory for loops (cf. [37]). If there exists a normal series $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$ of L with the property that for all $i = 1, \cdots, n$, the factor loop L_i/L_{i-1} is abelian in L/L_{i-1}, then the loop L is congruence solvable. In contrast to the group case the class of congruence solvable loops is a proper subclass of the class of classical solvable loops (cf. Exercise 10 in [18] and Construction 9.1 and Example 9.3 in [37]). Moreover, the iterated abelian, respectively central extensions, yield congruence solvable, respectively centrally nilpotent loops (cf. Corollaries 5.1 and 5.2 in [38]).

In this dissertation we deal with connected topological loops L. We follow the approach of P. T. Nagy and K. Strambach who consistently studied topological and differentiable loops using the tools of Lie theory. In [29] topological and differentiable loops L are realized as sharply transitive sections in Lie groups G_ℓ generated by the left translations of L. The subject of our investigation is connected topological loops L having a solvable Lie group G as the group $Mult(L)$ generated by all left and right translations of L. The action of the group $Mult(L)$ on the topological space L is transitive and effective. Each 1-dimensional connected topological loop having a locally compact group as its multiplication group is associative (cf. Theorem 18.18 in [29]). In the class of Lie groups the elementary filiform groups $\mathcal{F}_n$ with dimension $n \geq 4$ are the multiplication groups of 2-dimensional connected topological proper loops. Moreover, these loops are central extensions of a 1-dimensional Lie group by the group $\mathbb{R}$ (cf. [9]). Chapter 2 deals with the investigation of the classical and congruence solvable properties for topological loops. Using the results of Lie on transitive actions of Lie groups on the plane $\mathbb{R}^2$ (cf. [21]) and those on the groups $Mult(L)$ of L, if $\dim(L) \leq 2$, we obtain that all 3-dimensional connected topological loops L having solvable Lie groups as their multiplication groups are classically solvable (cf. Theorem 11). Applying the relation between iterated abelian extensions and congruence solvability we formulated necessary and sufficient conditions for 3-dimensional topological loops L to be congruence solvable (cf. Theorem 12). A particular interesting example (Example 1) illustrates that also for the topological case the class of congruence solvable loops forms a proper subclass of the class of classical solvable loops.

In Chapters $3, 4, 5, 6$ we discuss the question what solvable Lie groups can be represented as the multiplication groups of connected topological loops having dimension 3. Many authors investigated the general problem, what group can be realized as the group $Mult(L)$ of a loop L, in particular if L is a finite loop ([7], [8], [22], [27], [34]). Firstly, T. Kepka and M. Niemenmaa considered the latter question and answered it using group theoretical tools (cf. [33]). The conditions for a group G to be the multiplication group $Mult(L)$ of a loop L request the existence of two special left transversals S, T with respect to a subgroup K of G. The group K results in being the inner mapping group of L. The transversals S and T can be taken as the set $\Lambda(L)$ of the left translations and the set $P(L)$ of the right translations of L, respectively. The transversals S, T are K-connected and the set $S \cup T$ generates the group G (see Lemma 7). These criterions can be fruitfully applied for the topological case too (cf. [9]-[17]). In [11] it is found the at most 5-dimensional solvable connected simply connected Lie groups which are not nilpotent and can be realized as the group $Mult(L)$ for a 3-dimensional topological loop L.

The isomorphism classes of solvable Lie algebras $\mathbf{g}$ are classified in [24], [26], [25], [36], [41], if $\dim(\mathbf{g}) \leq 6$. Hence we restrict our consideration for these classes of Lie algebras. The main result of Chapter 3 says that each at most 3-dimensional connected topological loop L, such that the group $Mult(L)$ of L is a solvable Lie group of dimension ≤ 6, has nilpotency class 2 (cf. Theorem 14). To prove this result in Chapter 3 we describe the structure of the 3-dimensional connected simply connected topological loops L and their multiplication groups $Mult(L)$ if $Mult(L)$ are solvable Lie groups. Theorem 15 deals with the case that $Mult(L)$ has discrete centre. Theorems 16 and 17 treat the case that $Mult(L)$ has 1-dimensional and 2-dimensional centre, respectively. In Chapter 3 we give the steps of the procedure for the classification of the 6-dimensional solvable Lie groups which are multiplication groups of 3-dimensional connected simply connected topological loops L having a solvable Lie group G of dimension 6 as their multiplication group. Based on the results of Theorems 15, 16, 17 we formulated Proposition 18, which is applied in Chapter 4 to exclude some classes of 6-dimensional Lie algebras which are not the Lie algebras of the groups $Mult(L)$ of L. These Lie algebras are characterized by one of the following properties:

- they have discrete centre (cf. Propositions 20, 21, 22),
- they are indecomposable and have 2-dimensional centre (cf. Theorem 19),
- they are indecomposable and have 4-dimensional non-abelian nilradicals (cf. Proposition 20),
- they are indecomposable and their nilradical is either $\mathbb{R}^5$ or a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$ (cf. Proposition 21).

In Chapters 5, 6 we find the 6-dimensional solvable Lie algebras and their 3-dimensional abelian subalgebras which are the Lie algebras of the multiplication groups and those of the inner mapping groups of 3-dimensional connected topological loops L. Chapters 5 and 6 consist of Lie algebras having 1-dimensional and 2-dimensional centre, respectively.

In Chapter 5 we find that there are seven classes of 6-dimensional solvable indecomposable Lie algebras $\mathbf{g}$ with 5-dimensional nilradical which are the Lie algebras of $Mult(L)$ (cf. Theorem 23). The nilradical of the Lie algebras $\mathbf{g}$ is isomorphic either to $\mathbf{f}_3 \oplus \mathbb{R}^2$ or to $\mathbf{f}_4 \oplus \mathbb{R}$ or to the 5-dimensional indecomposable nilpotent Lie algebra such that its 2-dimensional centre coincides with its commutator ideal. Among the 6-dimensional solvable indecomposable Lie algebras having 4-dimensional nilradical there are three classes which are Lie algebras of the multiplication groups of L. The nilradical of these Lie algebras is $\mathbb{R}^4$. The corresponding simply connected Lie groups G and their subgroups K, which are the inner mapping groups of 3-dimensional connected simply connected topological loops L, are listed in Theorem 24. In Theorem 25 we give the 18 families of decomposable solvable Lie algebras with 1-dimensional centre which are the Lie algebras of the group $Mult(L)$. In Theorems 23, 25 we determine also the abelian subalgebras $\mathbf{k}$ of the Lie algebras $\mathbf{g}$ which are the Lie algebras of the inner mapping group $Inn(L)$. In Chapter 5 the centre $Z(L)$ of all 3-dimensional connected simply connected topological loops L is the group $\mathbb{R}$. Moreover, the factor loop $L/Z(L)$ is the group $\mathbb{R}^2$. Hence these loops have nilpotency class 2.

In Chapter 6 all Lie algebras are decomposable solvable Lie algebras (see Theorem 19). Among the 6-dimensional Lie algebras there are 9 families which can be realized as the Lie algebra of the group $Mult(L)$ of a 3-dimensional connected topological proper loop L (cf. Theorems 26, 27). In this case the centre $Z(L)$ of the loop L is the group $\mathbb{R}^2$ and the factor loop $L/Z(L)$ is the group $\mathbb{R}$. Therefore L is centrally nilpotent of class 2.

Hence our main results in the dissertation are the following:

Theorem 1. *Let L be a proper connected simply connected topological loop of dimension 3 having a solvable Lie group as its multiplication group $Mult(L)$.*
(a) Then L is classically solvable. There is a normal subgroup $N \cong \mathbb{R}$ of L. Every normal subgroup $N \cong \mathbb{R}$ of L lies in a 2-dimensional normal subloop M of L. The factor loop L/M is isomorphic to $\mathbb{R}$, whereas the loops M and L/N are isomorphic either to a 2-dimensional simply connected Lie group or to an elementary filiform loop.
(b) The loop L is congruence solvable if and only if either L has a non-discrete centre or L has discrete centre and is an abelian extension of a 1-dimensional normal subgroup $N \cong \mathbb{R}$ by the factor loop L/N isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$.

If the multiplication group $Mult(L)$ of an at most 3-dimensional connected topological proper loop L is a solvable Lie group of dimension ≤ 6, then in Chapter 3 we show the following:

Theorem 2. *If L is a connected topological proper loop L of dimension ≤ 3 such that its multiplication group $Mult(L)$ is an at most 6-dimensional solvable Lie group, then L has nilpotency class 2.*

Chapters 4, 5 and 6 are devoted to classify the solvable Lie groups of dimension ≤ 6 which can be represented as the groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L. Our main results are summarized in the following Theorems. To formulate these results we use the notation of [24], [26], [36], [41].

Theorem 3. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional solvable Lie algebra $\mathbf{g}$ having 1-dimensional centre. Then L is centrally nilpotent of class 2 and for the Lie algebra $\mathbf{g}$ we obtain:*

- *If $\mathbf{g}$ is an indecomposable Lie algebra having 5-dimensional nilradical, then the Lie algebra $\mathbf{g}$ is one of the following:* $\mathbf{g}_1 = \mathbf{g}_{6,14}^{a=0=b}$, $\mathbf{g}_2 = \mathbf{g}_{6,22}^{a=0}$, $\mathbf{g}_3 = \mathbf{g}_{6,17}^{\delta=1,a=0=\varepsilon}$, $\mathbf{g}_4 = \mathbf{g}_{6,51}^{\varepsilon=\pm 1}$, $\mathbf{g}_5 = \mathbf{g}_{6,54}^{a=0=b}$, $\mathbf{g}_6 = \mathbf{g}_{6,63}^{a=0}$, $\mathbf{g}_7 = \mathbf{g}_{6,25}^{a=0=b}$.
- *If $\mathbf{g}$ is an indecomposable Lie algebra with 4-dimensional nilradical, then for the Lie algebra $\mathbf{g}$ we get one of the following:* $\mathbf{g}_1 = N_{6,23}^{a}$, $a \in \mathbb{R}$, $\mathbf{g}_2 = N_{6,22}^{a}$, $a \in \mathbb{R}\backslash\{0\}$, $\mathbf{g}_3 = N_{6,27}$.
- *If $\mathbf{g}$ is a decomposable Lie algebra, then for the Lie algebra $\mathbf{g}$ we have one of the following:* $\mathbf{g}_1 = \mathbb{R} \oplus \mathbf{g}_{5,19}^{\alpha=0,\beta\neq 0}$, $\mathbf{g}_2 = \mathbb{R} \oplus \mathbf{g}_{5,20}^{\alpha=0}$, $\mathbf{g}_3 = \mathbb{R} \oplus \mathbf{g}_{5,27}$, $\mathbf{g}_4 = \mathbb{R} \oplus \mathbf{g}_{5,28}^{\alpha=0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,32}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,33}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,34}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,35}$, $\mathbf{g}_9 = \mathbf{l}_2 \oplus \mathbf{g}_{4,1}$, $\mathbf{g}_{10} = \mathbf{l}_2 \oplus \mathbf{g}_{4,3}$, $\mathbf{g}_{11} = \mathbf{f}_3 \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{12} = \mathbf{f}_3 \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{13} = \mathbf{f}_3 \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{14} = \mathbf{f}_3 \oplus \mathbf{g}_{3,5}^{p>0}$, $\mathbf{g}_{15} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{16} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{17} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{18} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}^{p>0}$.

Theorem 4. *Let L be a 3-dimensional connected simply connected topological proper loop having an at most 6-dimensional solvable Lie algebra $\mathbf{g}$ with 2-dimensional centre as the Lie algebra of the multiplication group $Mult(L)$ of L. Then L is centrally nilpotent of class 2 and the Lie algebra $\mathbf{g}$ is one of the following possibilities:*

1 *The nilpotent Lie algebras:* $\mathbb{R} \oplus \mathbf{f}_4$, $\mathbb{R} \oplus \mathbf{f}_5$.

2 *The solvable, non-nilpotent Lie algebras:* $\mathbf{g}_1 = \mathbb{R}^2 \oplus \mathbf{g}_{4,2}^{\alpha \neq 0}$, $\mathbf{g}_2 = \mathbb{R}^2 \oplus \mathbf{g}_{4,4}$, $\mathbf{g}_3 = \mathbb{R}^2 \oplus \mathbf{g}_{4,5}^{-1 \leq \gamma \leq \beta \leq 1, \gamma\beta \neq 0}$, $\mathbf{g}_4 = \mathbb{R}^2 \oplus \mathbf{g}_{4,6}^{p \geq 0, \alpha \neq 0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,8}^{0<|\gamma| \leq 1}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,10}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,14}^{p \neq 0}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,15}^{\gamma = 0}$.

1 Preliminaries

In this Chapter we collect notions, tools and results, which we use in the later investigation.

A set L equipped with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if for all $x \in L$ the left translation map $\lambda_x : L \to L$, $\lambda_x(y) = x \cdot y$ as well as the right translation map $\rho_x : L \to L$, $\rho_x(y) = y \cdot x$ are bijections and there is an element $e \in L$ with the property $x = e \cdot x = x \cdot e$. A loop L is proper if it is not associative.

The relation between loops and sharply transitive sections in groups is described in Section 1.2. of [29] in the following way: Denote by G_ℓ the group generated by the left translations of a loop L and by H the stabilizer of $e \in L$ in G_ℓ. The set $\Lambda(L)$ of the left translations of L is a subset of G_ℓ and operates sharply transitively on the left cosets $xH; x \in G_\ell$. The latter property says that for any given left cosets aH, bH there is precisely one left translation λ_z with $\lambda_z aH = bH$.

The core $Co_{G_\ell}(H)$ of the subgroup H in the group G_ℓ is the largest normal subgroup of G_ℓ contained in H. If G_ℓ is a group, H is one of its subgroups with $Co_{G_\ell}(H) = \{1\}$ and $\sigma : G_\ell/H \to G_\ell$ is a section such that

1. the image $\sigma(G_\ell/H)$ is a subset of G_ℓ with $\sigma(H) = 1 \in G_\ell$,
2. the action of $\sigma(G_\ell/H)$ on the factor space G_ℓ/H is sharply transitive,
3. $\sigma(G_\ell/H)$ generates G_ℓ,

then the multiplication on G_ℓ/H given by $xH * yH = \sigma(xH)yH$ defines a loop $L(\sigma)$ having G_ℓ as the group generated by its left translations.

The left, respectively the right division map is defined by $L \times L \to L : (x, y) \mapsto x\backslash y = \lambda_x^{-1}(y)$, respectively $(x, y) \mapsto y/x = \rho_x^{-1}(y)$. Moreover, denote by $\mu_x : L \to L$ the map $\mu_x(y) = y\backslash x$. One has $\mu_x^{-1}(y) = x/y$. The groups $Mult(L) = \langle \lambda_x, \rho_x;\ x \in L\rangle$ and $TMult(L) = \langle \lambda_x, \rho_x, \mu_x;\ x \in L\rangle$ are called the multiplication group and the total multiplication group of L. We denote by $Inn(L)$ and $TInn(L)$ the stabilizer of the identity element $e \in L$ in $Mult(L)$ and in $TMult(L)$, respectively. These subgroups of $Mult(L)$ and $TMult(L)$ are called the inner mapping group and the total inner mapping group of L.

A normal subloop N of L is the kernel of a loop homomorphism $\alpha : (L, \cdot) \to (L', *)$. A word W is a formal product of letters $\lambda_{t(\bar{x})}$, $\rho_{t(\bar{x})}$ and their inverses, where $t(\bar{x}) = t(x_1, \cdots, x_n)$ is a loop term. If we substitute elements u_i of a particular loop L for x_i into a word W and interpret $\lambda_{t(\bar{x})}$, $\rho_{t(\bar{x})}$ as translations of L, then we get an element $W_{\bar{u}}$ of $Mult(L)$. The word W is inner if $W_{\bar{u}}(e) = e$ for each loop L with identity element e and each assignment of elements $u_i \in L$. The notion of tot-inner word is defined analogously allowing $\mu_{t(\bar{x})}$ as generating letters. Let $\mathcal{W}$ be a set of tot-inner words such that each loop L satisfies the property $TInn(L) = \langle W_{\bar{u}} : W \in \mathcal{W}, u_i \in L\rangle$. Let L be a loop and N_1, N_2 be normal subloops of L. The commutator $[N_1, N_2]_L$ is the smallest normal subloop of L containing the set $\{W_{\bar{u}}(a)/W_{\bar{v}}(a) : W \in \mathcal{W}, a \in N_1, u_i, v_i \in L, u_i/v_i \in N_2\}$. For the set $\mathcal{W}$ one can choose the set $\{T_x, U_x, L_{x,y}, R_{x,y}, M_{x,y}\}$ of the tot-inner words $T_x = \rho_x^{-1}\lambda_x$, $U_x = \rho_x^{-1}\mu_x$, $L_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y$, $R_{x,y} = \rho_{yx}^{-1}\rho_x\rho_y$, $M_{x,y} = \mu_{y\backslash x}^{-1}\mu_x\mu_y$ (cf. Theorem 2.1. in [38]).

A normal subloop N of L is said to be central in L, respectively abelian in L, if $[N, L]_L = \{e\}$, respectively $[N, N]_L = \{e\}$. The centre $Z(L)$ of a loop L is the normal subloop of L consisting of all elements $z \in L$ that satisfy the identities $zx = xz$, $zx \cdot y = z \cdot xy$, $x \cdot yz = xy \cdot z$, $xz \cdot y = x \cdot zy$ for all $x, y \in L$. A normal subloop N is central in L precisely if one has $N \leq Z(L)$. The centre $Z(L)$ of L is a commutative normal subgroup of L. A loop L is classically solvable if there is a series $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$

of subloops of L such that L_{i-1} is normal in L_i and the factor loop L_i/L_{i-1} is an abelian group for all $i = 1, 2, \cdots, n$. A loop L is called congruence solvable, respectively nilpotent, if there exists a chain $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$ of normal subloops of L such that every factor loop L_i/L_{i-1} is abelian in L/L_{i-1}, respectively central in L/L_{i-1}. Based on the above remark this definition of nilpotence is equivalent to the classical concept of central nilpotence in loop theory. If we put $Z_0 = \{e\}$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then we say that L is centrally nilpotent of class n. The centrally nilpotent loops are congruence solvable. If $(A, +, 0)$ is a commutative group, $(F, \cdot, e)$ is a loop and $\varphi, \phi : F \times F \to \mathrm{Aut}(A)$, $\theta : F \times F \to A$ are functions with $\varphi(y, e) = Id = \phi(e, y)$, $\theta(e, y) = 0 = \theta(y, e)$ for every $y \in F$, then on $F \times A$ a loop is defined by

$$(x, a) \oplus (y, b) = (x \cdot y, \varphi(x, y)(a) + \phi(x, y)(b) + \theta(x, y)).$$

This loop has identity element $(e, 0)$ and it is called the abelian extension of A by F determined by the factor system $\Gamma = (\varphi, \phi, \theta)$. We denote it by $L = F \oplus_\Gamma A$. An abelian extension is central if $\varphi(x, y) = \phi(x, y) = Id$ for all $x, y \in F$. A loop L is said to be an iterated abelian, respectively central extension, if it has the form

$$((((A_0 \oplus_{\Gamma_1} A_1) \oplus_{\Gamma_2} A_2) \oplus_{\Gamma_3} \cdots \oplus_{\Gamma_{k-2}} A_{k-2}) \oplus_{\Gamma_{k-1}} A_{k-1}) \oplus_{\Gamma_k} A_k,$$

where A_i, $i = 0, \cdots, k$, are abelian groups and all extensions are abelian, respectively central (cf. Section 5 in [38] and Definition in [23], p. 380).

Corollaries 5.1 and 5.2 in [38], p. 380, prove:

Lemma 5. *A loop L is congruence solvable, respectively centrally nilpotent, precisely if it is an iterated abelian, respectively an iterated central extension.*

We often use the following relations between normal subloop N, factor loop L/N of a loop L and their multiplication groups

$Mult(N)$, $Mult(L/N)$ in connection with the multiplication group $Mult(L)$ of L (see in [1], Theorems 3, 4 and 5, in [5], IV.1, Lemma 1.3 and in [17], Lemma 2.3).

Lemma 6. *Let L be a loop having $Mult(L)$ as its multiplication group and e as its identity element.*
(i) A homomorphism α of L onto the loop $\alpha(L)$ with kernel N induces a homomorphism of the multiplication group $Mult(L)$ onto the group $Mult(\alpha(L))$. The set $M(N) = \{m \in Mult(L); xN = m(x)N, \text{ for all } x \in L\}$ forms a normal subgroup of $Mult(L)$ containing the group $Mult(N)$ for the normal subloop N. The factor group $Mult(L)/M(N)$ is isomorphic to the group $Mult(L/N)$ of the factor loop L/N.
(ii) For each normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L. We have $\mathcal{N} \leq M(\mathcal{N}(e))$.

If G is a group, and K is a subgroup of G, then a system S of representatives for the left cosets $xK, x \in G$, is called a left transversal to K in G. If S, T are two left transversals to K in G, then we say that these are K-connected, if for all $s \in S$ and $t \in T$ the product $s^{-1}t^{-1}st$ lies in K. For a loop L the sets $\Lambda(L) = \{\lambda_a;\ a \in L\}$, $P(L) = \{\rho_a;\ a \in L\}$ are $Inn(L)$-connected left transversals in the group $Mult(L)$. In Theorem 4.1 of [33] the following necessary and sufficient conditions are given for a group G to be the group $Mult(L)$ of a loop L.

Lemma 7. *A group G is isomorphic to the multiplication group of a loop precisely if there is a subgroup K with $Co_G(K) = \{1\}$ and there exist K-connected left transversals S and T such that $G = \langle S, T\rangle$.*

In the later investigation we will often use the following assertion (cf. Proposition 2.7. in [33]).

Lemma 8. *Let L be a loop having $Mult(L)$ as its multiplication group and $Inn(L)$ as its inner mapping group. One has*

$$Co_{Mult(L)}(Inn(L)) = \{1\}$$

and the normalizer $N_{Mult(L)}(Inn(L))$ *equals to the direct product* $Inn(L) \times Z$*, where* Z *denotes the centre of* $Mult(L)$*.*

A topological loop is a topological space L such that the three binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y$, $(x, y) \mapsto y/x : L \times L \to L$ are continuous. In this case the multiplication group of L is a topological transformation group such that in general it has no natural (finite dimensional) differentiable structure. The condition that the group $Mult(L)$ is a Lie group restricts strongly the isomorphic classes of $Mult(L)$ as well as those of L. In the dissertation we suppose that the group $Mult(L)$ is a solvable Lie group. In the further considerations the following lemma is often applied.

Lemma 9. *Each connected topological loop has a universal covering loop, which is simply connected. If* L *is a 3-dimensional connected simply connected topological loop such that the group* $Mult(L)$ *is a solvable Lie group, then* L *is homeomorphic to* $\mathbb{R}^3$*.*

The first assertion is proved in [20], IX.1, whereas the second one is shown in Lemma 3.3 of [10], p. 390.

An elementary filiform Lie group $\mathcal{F}_n$ is a connected simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra $\mathbf{f}_n$ has a basis $\{e_1, \cdots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$. A 2-dimensional connected simply connected loop $L_{\mathcal{F}}$ is said to be elementary filiform, if its multiplication group is an elementary filiform group $\mathcal{F}_n$ with $n \geq 4$. A Lie algebra is called indecomposable, if it is not the direct sum of two proper ideals. Otherwise, the Lie algebra is decomposable.

Lemma 10. *Each elementary filiform loop* $L_{\mathcal{F}}$ *has nilpotency class 2.*

The proof of this Lemma can be found in [9], p. 420.

2 Classical solvable, congruence solvable topological loops

In this Chapter we prove the following theorems:

Theorem 11. *If L is a 3-dimensional connected simply connected topological loop such that its multiplication group is a solvable Lie group, then L is classically solvable. The loop L has a 1-dimensional normal subgroup N isomorphic to $\mathbb{R}$. For each 1-dimensional normal subgroup N there exists a normal series $\{e\} = L_0 \leq N = L_1 \leq M = L_2 \leq L = L_3$ of L such that every factor loop L_i/L_{i-1}, $i = 1, 2, 3$, is the group $\mathbb{R}$. Moreover, the loops M and L/N are isomorphic either to a 2-dimensional simply connected Lie group or to a loop $L_{\mathcal{F}}$.*

Theorem 12. *Let L be a 3-dimensional connected simply connected topological proper loop with a solvable Lie multiplication group. The loop L is congruence solvable if and only if L has one of the following properties:*

- *the centre of L has dimension 1 or 2,*
- *L has discrete centre and is an abelian extension of a normal subgroup $N \cong \mathbb{R}$ by the factor loop L/N isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$.*

The following construction shows that the class of congruence solvable loops is a proper subclass of the class of classical solvable loops also for the topological case.

Example 1. *Let $(Q, \cdot, 1)$ be a topological loop of dimension n having a normal subloop Q_1 such that the factor loop Q/Q_1 is isomorphic to the group $\mathbb{R}$. Let $\phi : (Q, \cdot) \to (\mathbb{R}, +)$ be a homomorphism. We consider a one-parameter family of loops $\Gamma_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(a, b) \mapsto \Gamma_t(a, b) = a *_t b$, $t \in \mathbb{R}$, such that $\Gamma_0(a, b) = a + b$ and Γ_t is not commutative for some $t \in \mathbb{R}$. Suppose that for all $t \in \mathbb{R}$ the loops Γ_t have the same identity element 0. We denote by $\Delta_t(a, b) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$*

the right division map $(a,b) \mapsto \Delta_t(a,b) = a/_t b$, $t \in \mathbb{R}$, *of the loop* Γ_t. *For the loops* Γ_t, $t \neq 0$, *we can take loops defined by the sharply transitive section* $\sigma_t : \widetilde{PSL_2}(\mathbb{R})/\mathcal{L}_2 \to \widetilde{PSL_2}(\mathbb{R})$ *determined by the functions* $f(u) = \exp[\frac{1}{6}\sin^2 t \cos u(\cos u - 1)]$ *and* $g(u) = (f(u)^{-1} - f(u)) \cot u$ *(see Proposition 18.15 and its proof in [29], pp. 244-245). All loops* Γ_t, $t \neq 0$, *are proper and hence they are not commutative (cf. Corollary 18.19. in [29], p. 248). The multiplication*

$$(x,a) \circ (y,b) = (x \cdot y, \Gamma_{\phi(x \cdot y)}(a,b))$$

on $Q \times \mathbb{R}$ *defines a loop* L_ϕ *which is an extension of the group* $\mathbb{R}$ *by the loop* Q. *The loop* L_ϕ *has the identity element* $(1,0)$ *since one has* $(1,0) \circ (y,b) = (y, \Gamma_{\phi(y)}(0,b)) = (y,b) = (y,b) \circ (1,0)$. *Hence the loop* L_ϕ *is an Albert extension of the group* $\mathbb{R}$ *by the loop* $(Q,\cdot)$ *given by the one-parameter family* Γ_t *of the loop multiplications on* $\mathbb{R}$ *(see [28], p. 4). Let* x *be an element of* Q *with* $\phi(x) \neq 0$. *We obtain* $T(x,a)(1,c) = ((x,a)\circ(1,c))/(x,a) = (x, \Gamma_{\phi(x)}(a,c))/(x,a) = (1, \Delta_{\phi(x)}(\Gamma_{\phi(x)}(a,c),a))$, *which is not independent of* $a \in \mathbb{R}$ *because the loop* $\Gamma_{\phi(x)}$ *is not commutative. Hence the normal subgroup* $\mathbb{R}$ *is not abelian in the loop* L_ϕ *(see Proof of Theorem 4.1 in [38], p. 377). In particular if the loop* $(Q,\cdot)$ *is the group* $\mathcal{L}_2$ *or a loop* $L_{\mathcal{F}}$, *then this construction yields a* 3*-dimensional connected topological loop, which is a non-abelian extension of the group* $\mathbb{R}$ *by the loop* $(Q,\cdot)$.

Note 13. *We are very thankful to Péter T. Nagy for the construction in Example 1.*

3 Topological loops with solvable Lie multiplication groups of dimension at most 6 are centrally nilpotent

From now on we restrict us for those solvable Lie groups which have dimension at most 6. The reason for this restriction is that the classification of the corresponding Lie algebras is complete (cf. [25], [36], [41]). Using this restriction we show:

Theorem 14. *If L is a connected topological proper loop of dimension ≤ 3 such that its multiplication group $Mult(L)$ is an at most 6-dimensional solvable Lie group, then L has nilpotency class 2.*

To show Theorem 14 we give the description of the structure of the 3-dimensional connected simply connected topological loops and their multiplication groups $Mult(L)$, if $Mult(L)$ is a solvable Lie group.

In Theorem 15 we deal with the case that $Mult(L)$ has discrete centre.

Theorem 15. *Let L be a proper connected simply connected topological loop of dimension 3 having a solvable Lie group with discrete centre as its multiplication group $Mult(L)$. The loop L is classically solvable. It has a connected normal subgroup N isomorphic to $\mathbb{R}$ and the factor loop L/N is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$. The dimension of the group $Mult(L)$ is at least 6 and the group $Mult(L)$ has a normal subgroup $\mathcal{S}$ containing $Mult(N) \cong \mathbb{R}$ such that the factor group $Mult(L)/\mathcal{S}$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, if $L/N \cong \mathcal{L}_2$, or to a group $\mathcal{F}_n$, $n \geq 4$, if $L/N \cong L_{\mathcal{F}}$. For each normal subgroup N of L the loop L has a normal subloop M isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$ such that $N < M$ and L/M is isomorphic to $\mathbb{R}$. The group $Mult(L)$ contains a normal subgroup V such that $Mult(L)/V \cong \mathbb{R}$ and the orbit $V(e)$ is the loop M. The inner mapping group $Inn(L)$ of L, the multiplication group $Mult(M)$ of M and the commutator subgroup of $Mult(L)$ are subgroups of V. The normalizer $N_{Mult(L)}(Inn(L))$ is $Inn(L)$.*

In Theorem 16 the group $Mult(L)$ has 1-dimensional centre.

Theorem 16. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is a solvable Lie group with 1-dimensional centre Z. Then the loop L is congruence solvable. The orbit $K(e)$, where K is a 1-dimensional*

connected normal subgroup of $Mult(L)$, is a normal subgroup of L isomorphic to $\mathbb{R}$. Moreover, one of the following possibilities holds:
(a) If the factor loop $L/K(e)$ is isomorphic to $\mathbb{R}^2$, then L has nilpotency class 2 and the orbit $K(e)$ coincides with the centre $Z(L)$ of L. The connected simply connected group $Mult(L)$ is a semidirect product of the abelian normal subgroup $P = Z \times Inn(L)$ by a group $Q \cong \mathbb{R}^2$ and the orbit $P(e)$ is $Z(L)$.
(b) If the factor loop $L/K(e)$ is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$, then $Mult(L)$ has a normal subgroup $\mathcal{S}$ containing K such that the orbits $\mathcal{S}(e)$ and $K(e)$ coincide. The factor group $Mult(L)/\mathcal{S}$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, if $L/K(e) \cong \mathcal{L}_2$, or to a Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/K(e) \cong L_{\mathcal{F}}$. In particular, if $K(e) = Z(L)$ and $L/Z(L)$ is isomorphic either to the group $\mathbb{R}^2$ or to a loop $L_{\mathcal{F}}$, then L is centrally nilpotent of class 3. The loop L contains a 2-dimensional normal subloop M with $K(e) < M$ and the group $Mult(L)$ has a normal subgroup V as in Theorem 15.

In Theorem 17 we consider the case that the centre of $Mult(L)$ has dimension 2.

Theorem 17. *If L is a proper connected simply connected topological loop of dimension 3 such that its multiplication group $Mult(L)$ is a solvable Lie group with 2-dimensional centre Z, then L has nilpotency class 2. The group $Mult(L)$ is a semidirect product of the normal subgroup $V = Z \times Inn(L) \cong \mathbb{R}^{m-1}$ by a group $Q \cong \mathbb{R}$, where $\mathbb{R}^2 = Z \cong Z(L)$ and $m = dim(Mult(L))$. For every 1-dimensional connected subgroup N of Z the orbit $N(e)$ is a connected central subgroup of L and the factor loop $L/N(e)$ is isomorphic either to $\mathbb{R}^2$ or to a loop $L_{\mathcal{F}}$. In particular, if the group $Mult(L)$ is indecomposable, then one has $L/N(e) \cong L_{\mathcal{F}}$. If $L/N(e) \cong \mathbb{R}^2$, then Theorem 16 (a) holds. If $L/N(e) \cong L_{\mathcal{F}}$, then the group $Mult(L)$ contains a normal subgroup $\mathcal{S}$ with $N < \mathcal{S}$. The factor group $Mult(L)/\mathcal{S}$ is isomorphic to a Lie group $\mathcal{F}_n$ with $n \geq 4$.*

Our next aim is to determine the 6-dimensional solvable Lie groups which are multiplication groups of 3-dimensional connected

simply connected topological loops.
Procedure of the classification:
1. step: For each 6-dimensional solvable Lie algebra $\mathbf{g}$ we have to find a suitable linear representation of the corresponding connected simply connected Lie group G.
2. step: As $dim(L) = 3$ we determine those 3-dimensional Lie subgroups K of G which have no non-trivial normal subgroup of G and satisfy the condition that the normalizer $N_G(K)$ is the direct product $K \times Z$, where Z is the centre of G (cf. Lemma 8).
3. step: We have to find left transversals S and T to K in G such that for all $s \in S$ and $t \in T$ one has $s^{-1}t^{-1}st \in K$ and G is generated by $S \cup T$ (cf. Lemma 7).
3.1. Since the transversals S and T are continuous, they are determined by 3 continuous real functions of 3 variables. The condition that the products $s^{-1}t^{-1}st$, $s \in S$ and $t \in T$, are in K is formulated by functional equations. Solving these functional equations we obtain the possible forms of the left transversals S and T. The left transversals S and T are the set $\Lambda(L)$ of all left translations and the set $P(L)$ of all right translations of L, respectively. These sets play an important role for the construction of the loop multiplication using the group G_ℓ, respectively G_r (cf. [29], p. 17-18).
3.2. We check whether the set $S \cup T$ generates the group G. If this is the case, then G is the multiplication group $Mult(L)$ of a loop L and K is the inner mapping group of L.

Proposition 18 is useful to exclude those 6-dimensional solvable Lie algebras which are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional connected topological loops L.

Proposition 18. *Suppose L is a proper connected simply connected topological loop of dimension 3 such that the Lie algebra of its multiplication group is a 6-dimensional solvable Lie algebra $\mathbf{g}$.*
a) For all 1-dimensional ideals $\mathbf{i}$ of $\mathbf{g}$ the orbits $I(e)$, where I is the simply connected Lie group of $\mathbf{i}$, are normal subgroups of L isomorphic to $\mathbb{R}$. We have one of the following possibilities:
(i) The factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$. Then $\mathbf{g}$ contains the

ideal $\mathbf{p} = \mathbf{c} \oplus \mathbf{inn}(\mathbf{L}) \cong \mathbb{R}^4$ *such that the commutator ideal* $\mathbf{g}'$ *of* $\mathbf{g}$ *lies in* $\mathbf{p}$ *and* $\mathbf{c}$ *is a* 1*-dimensional subalgebra of the centre* $\mathbf{z}$ *of* $\mathbf{g}$*.*
(ii) The factor loop $L/I(e)$ *is isomorphic either to the group* $\mathcal{L}_2$ *or to a loop* $L_{\mathcal{F}}$*. Then* $\mathbf{g}$ *has an ideal* $\mathbf{s}$ *such that* $\mathbf{i} \leq \mathbf{s}$ *and the factor Lie algebra* $\mathbf{g}/\mathbf{s}$ *is isomorphic either to* $\mathbf{l}_2 \oplus \mathbf{l}_2$ *or to a Lie algebra* $\mathbf{f}_n$*,* $n = 4, 5$*.*
b) If $\mathbf{a}$ *is an ideal of* $\mathbf{g}$ *such that dim*$(\mathbf{a}) = 2$*,* $\mathbf{a} \leq \mathbf{g}'$ *and the factor Lie algebra* $\mathbf{g}/\mathbf{a}$ *is isomorphic neither to* $\mathbf{l}_2 \oplus \mathbf{l}_2$ *nor to* $\mathbf{f}_4$*, then the orbit* $A(e)$*, where* A *is the simply connected Lie group of* $\mathbf{a}$*, is either a* 2*-dimensional connected normal subloop* M *of* L *or the factor loop* $L/A(e)$ *is isomorphic to* $\mathbb{R}^2$*.*
(iii) Assume $A(e) = M$*. Then there exists a* 5*-dimensional ideal* $\mathbf{v}$ *of* $\mathbf{g}$ *such that the Lie algebra* $\mathbf{inn}(\mathbf{L})$*, the Lie algebra* $\mathbf{mult}(M)$ *and the ideal* $\mathbf{g}'$ *are subalgebras of* $\mathbf{v}$*. Moreover, for all ideals* $\mathbf{b}$ *of* $\mathbf{g}$ *with dim*$(\mathbf{b}) \geq 3$ *and* $\mathbf{a} < \mathbf{b} \leq \mathbf{g}'$ *the orbit* $B(e)$*, where* B *is the simply connected Lie group of* $\mathbf{b}$*, coincides with* M*. One has* $\mathbf{a} \cap \mathbf{inn}(\mathbf{L}) = \{0\}$ *and the intersection* $\mathbf{b} \cap \mathbf{inn}(\mathbf{L})$ *has dimension dim*$(\mathbf{b}) - 2$*.*
(iv) If the factor loop $L/A(e)$ *is isomorphic to* $\mathbb{R}^2$*, then we have case (i).*
c) If the Lie algebra $\mathbf{g}$ *is indecomposable, then its centre* $\mathbf{z}$ *has dimension* ≤ 1*, the subalgebra* $\mathbf{c}$ *in case a) (i) coincides with* $\mathbf{z}$ *and the ideal* $\mathbf{p}$ *lies in the nilradical* $\mathbf{n}_{rad}$*.*
d) If dim$(\mathbf{n}_{rad}) = 4$*, then the ideal* $\mathbf{p}$ *equals to* $\mathbf{n}_{rad}$*. Moreover, if* $\mathbf{n}_{rad}$ *is not commutative or the centre* $\mathbf{z}$ *of* $\mathbf{g}$ *is trivial, then for each* 2*-dimensional abelian ideal* $\mathbf{a}$ *of* $\mathbf{g}$ *such that the factor Lie algebra* $\mathbf{g}/\mathbf{a}$ *is isomorphic neither to* $\mathbf{l}_2 \oplus \mathbf{l}_2$ *nor to* $\mathbf{f}_4$ *and for each nilpotent ideal* $\mathbf{s}$ *of* $\mathbf{g}$ *having dimension* > 2 *the orbits* $A(e)$*,* $\mathcal{S}(e)$*, where* A*,* $\mathcal{S}$ *are the simply connected Lie groups of* $\mathbf{a}$*,* $\mathbf{s}$*, respectively, are the same* 2*-dimensional normal subloop* M *of* L*. There is a* 5*-dimensional ideal* $\mathbf{v}$ *of* $\mathbf{g}$ *with the same properties as in case b) (iii). If* $\mathbf{g}$ *differs from the Lie algebra* $N_{6,28}$ *in Table III in [41], p. 1349, then the loop* M *is isomorphic to* $\mathbb{R}^2$*.*
e) If dim$(\mathbf{n}_{rad}) = 5$*, then the factor loop* $L/I(e)$ *in case a) is not isomorphic to the group* $\mathcal{L}_2$*.*

4 Solvable Lie groups which are not the multiplication groups of 3-dimensional topological loops

In this Chapter, we focus our attention to the classes of the following 6-dimensional solvable Lie groups:

- Indecomposable solvable Lie groups with 2-dimensional centre.
- Indecomposable solvable Lie groups such that their Lie algebras have one of the following nilradicals: a 4-dimensional non-abelian nilpotent Lie algebra, $\mathbb{R}^5$, a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$.
- Solvable Lie groups with discrete centre.

We prove that the Lie algebras of the above listed Lie groups are not the Lie algebras of the multiplication groups of 3-dimensional topological loops. Firstly, in Theorem 19 we state that the at most 6-dimensional indecomposable solvable Lie algebras with 2-dimensional centre are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional connected topological loops L.

Theorem 19. *There does not exist any* 3*-dimensional proper connected topological loop* L *having an at most* 6*-dimensional indecomposable solvable Lie group with* 2*-dimensional centre as the group* $Mult(L)$ *of* L*.*

Proposition 20 says that the 6-dimensional solvable indecomposable Lie algebras with 4-dimensional nilradical having trivial centre or non-abelian nilradical are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Proposition 20. *Let* $\mathbf{g}$ *be a* 6*-dimensional solvable indecomposable Lie algebra with* 4*-dimensional nilradical* $\mathbf{n}_{rad}$ *such that either* $\mathbf{n}_{rad}$ *is not commutative or the centre of* $\mathbf{g}$ *is trivial. There does not exist* *any* 3*-dimensional connected topological loop* L *having* $\mathbf{g}$ *as the Lie algebra of the multiplication group of* L*.*

In Proposition 21 we exclude the 6-dimensional solvable indecomposable Lie algebras having either a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$, or $\mathbb{R}^5$, as their nilradical.

Proposition 21. *There does not exist any* 3*-dimensional connected topological loop* L *such that the Lie algebra of the group* $Mult(L)$ *is a* 6*-dimensional indecomposable solvable Lie algebra having one of the following nilradicals: (a)* $[e_2, e_4] = e_3$, $[e_2, e_5] = e_1$, $[e_4, e_5] = e_2$*; (b)* $[e_2, e_4] = e_1$, $[e_3, e_5] = e_1$*; (c)* $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$*; (d)* $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$*; (e) the Lie algebra* $\mathbf{f}_5$*; (f) the Lie algebra* $\mathbb{R}^5$*.*

Proposition 22 shows that the 6-dimensional solvable decomposable Lie algebras with trivial centre are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Proposition 22. *The* 6*-dimensional decomposable solvable Lie algebras with trivial centre are not the Lie algebras of the multiplication groups of* 3*-dimensional topological loops.*

5 6-dimensional solvable Lie groups having 1-dimensional centre

In this Chapter we determine the 6-dimensional solvable Lie groups with 1-dimensional centre which are the multiplication groups of 3-dimensional topological loops L. In the class of the 6-dimensional indecomposable solvable Lie groups with 5-dimensional nilradical there are 7 families which are the groups $Mult(L)$ of L (cf. Theorem 23). We find that among the 6-dimensional indecomposable solvable Lie groups with 4-dimensional nilradical only three families can be represented as the group $Mult(L)$ of L (cf. Theorem 24). Finally, there are 18 families of 6-dimensional decomposable solvable Lie groups which are the group $Mult(L)$ of L (cf. Theorem 25). In all these cases we determine the inner mapping subgroups $Inn(L)$ of L. The corresponding loops L have 1-dimensional centre and nilpotency class 2. Hence Theorem 14 is valid.

In Theorem 23 we consider the case that the Lie algebra $\mathbf{mult}(\mathbf{L})$ of the multiplication group of L is a 6-dimensional solvable indecomposable Lie algebra with 5-dimensional nilradical.

Theorem 23. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional solvable indecomposable Lie algebra having 5-dimensional nilradical. Then L has nilpotency class 2 and the following pairs $(\mathbf{g}, \mathbf{k})$ of Lie algebras are the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the subgroup $Inn(L)$:*
$\mathbf{g}_1 := \mathbf{g}_{6,14}^{a=b=0}$: $[e_2, e_3] = e_1 = [e_5, e_6]$, $[e_4, e_6] = e_4$, $\mathbf{k}_{1,1} = \langle e_2, e_4 + e_1, e_5 \rangle$, $\mathbf{k}_{1,2} = \langle e_3, e_4 + e_1, e_5 \rangle$;
$\mathbf{g}_2 := \mathbf{g}_{6,22}^{a=0}$: $[e_2, e_3] = e_1 = [e_5, e_6]$, $[e_2, e_6] = e_3$, $[e_4, e_6] = e_4$, $\mathbf{k}_2 = \langle e_3, e_4 + e_1, e_5 \rangle$,
$\mathbf{g}_3 := \mathbf{g}_{6,17}^{\delta=1, a=\varepsilon=0}$: $[e_2, e_3] = e_1 = [e_4, e_6]$, $[e_3, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{3,1} = \langle e_3, e_4, e_5 + e_1 \rangle$, $\mathbf{k}_{3,2} = \langle e_2, e_4, e_5 + e_1 \rangle$;
$\mathbf{g}_4 := \mathbf{g}_{6,51}^{\varepsilon=\pm 1}$: $[e_1, e_5] = e_2$, $[e_4, e_5] = e_1$, $[e_3, e_6] = e_3$, $[e_4, e_6] = \varepsilon e_2$, $\mathbf{k}_4 = \langle e_1 + a_1 e_2, e_3 + e_2, e_4 \rangle$, $a_1 \in \mathbb{R}$;
$\mathbf{g}_5 := \mathbf{g}_{6,54}^{a=b=0}$: $[e_3, e_5] = e_1 = [e_1, e_6]$, $[e_4, e_5] = e_2$, $[e_3, e_6] = e_3$, $\mathbf{k}_5 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_6 := \mathbf{g}_{6,63}^{a=0}$: $[e_3, e_5] = e_1 = [e_1, e_6]$, $[e_3, e_6] = e_3$, $[e_4, e_5] = e_2 = [e_4, e_6]$, $\mathbf{k}_6 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_7 := \mathbf{g}_{6,25}^{a=b=0}$: $[e_2, e_3] = e_1 = [e_1, e_6]$, $[e_2, e_6] = e_2$, $[e_4, e_6] = e_5$, $\mathbf{k}_7 = \langle e_1 + e_5, e_2 + \varepsilon e_5, e_4 \rangle$, $\varepsilon = 0, 1$.
The multiplication group $Mult(L)$ and the inner mapping group $Inn(L)$ of L are isomorphic to the linear groups of matrices whose multiplications are given by:

$$Mult(L)_1 : g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1+x_2y_3-x_3y_2-x_6y_5, x_2+y_2,$$
$$x_3+y_3, x_4+y_4e^{-x_6}, x_5+y_5, x_6+y_6),$$
$$Inn(L)_{1,1}=\{g(u_1,u_3,0,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$Inn(L)_{1,2}=\{g(u_1,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$Mult(L)_2: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1+x_2y_3-x_3y_2-x_6(y_5+x_2y_2), x_2+y_2,$$
$$x_3+y_3-x_6y_2, x_4+y_4e^{-x_6}, x_5+y_5, x_6+y_6),$$
$$Inn(L)_2=\{g(u_1,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$Mult(L)_3: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1-x_6y_4+(\frac{1}{2}x_6^2+x_3)y_2, x_2+y_2,$$
$$x_3+y_3, x_4+y_4-x_6y_2, x_5+y_5e^{-x_6}, x_6+y_6),$$
$$Inn(L)_{3,1}=\{g(u_2,u_3,0,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$Inn(L)_{3,2}=\{g(u_2,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$Mult(L)_4: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1+x_5y_4, x_2+y_2+x_5y_1+\varepsilon x_4y_6+\frac{1}{2}x_5^2y_4,$$
$$x_3+y_3e^{-x_6}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$Inn(L)_4=\{g(u_1,a_1u_1+u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$a_1\in\mathbb{R}, \varepsilon=\pm1,$$
$$Mult(L)_5: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+x_5y_3)e^{-x_6}, x_2+y_2+x_5y_4,$$
$$x_3+y_3e^{-x_6}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$Inn(L)_5=\{g(u_1,u_1+a_2u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$
$$Mult(L)_6: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$

$$g(x_1+(y_1+y_3x_5)e^{-x_6}, x_2+y_2-(x_5+x_6)y_4,$$
$$x_3+y_3e^{-x_6}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$Inn(L)_6 = \{g(u_1, u_1+a_2u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i=1,2,3\}, a_2 \in \mathbb{R},$$
$$Mult(L)_7 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1+(y_1+y_2x_3)e^{-x_6}, x_2+y_2e^{-x_6},$$
$$x_3+y_3, x_4+y_4, x_5+y_5-x_4y_6, x_6+y_6),$$
$$Inn(L)_7 = \{g(u_1, u_2, 0, u_3, u_1+\varepsilon u_2, 0); u_i \in \mathbb{R}, i=1,2,3\},\ \varepsilon = 0,1,$$

In Theorem 24 we treat the case that the group $Mult(L)$ of a 3-dimensional connected simply connected topological proper loop L has 4-dimensional nilradical.

Theorem 24. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional solvable indecomposable Lie algebra having 4-dimensional nilradical. Then L has nilpotency class 2 and the following pairs $(\mathbf{g}, \mathbf{k})$ of Lie algebras are the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the subgroup $Inn(L)$:*

- $\mathbf{g_1} := N^a_{6,23}$: $[e_1,e_3]=[e_4,e_2]=e_3$, $[e_1,e_4]=[e_2,e_3]=e_4$, $[e_1,e_5]=e_6$, $[e_2,e_5]=ae_6$, $a\in\mathbb{R}$, $\mathbf{k}_1=\langle e_3+\varepsilon_1e_6, e_4+\varepsilon_2e_6, e_5+\varepsilon_3e_6\rangle$, $\varepsilon_i\in\{0,1\}$, $i=1,2,3$, *such that* $\varepsilon_1^2+\varepsilon_2^2\neq 0$.
- $\mathbf{g_2} := N^a_{6,22}$: $[e_1,e_3]=e_3$, $[e_1,e_5]=e_6$, $[e_2,e_6]=ae_3$, $[e_2,e_4]=e_4$, $a\in\mathbb{R}\setminus\{0\}$, $\mathbf{k}_2=\langle e_3+e_6, e_4+e_6, e_5+\varepsilon_1e_6\rangle$, $\varepsilon=0,1$.
- $\mathbf{g_3} := N_{6,27}$: $[e_1,e_3]=e_4$, $[e_1,e_5]=[e_2,e_6]=e_6$, $[e_1,e_6]=[e_5,e_2]=-e_5$, $\mathbf{k}_3=\langle e_3+\varepsilon_1e_4, e_5+\varepsilon_2e_4, e_4+\varepsilon_3e_4\rangle$, $\varepsilon_i\in\{0,1\}$, $i=1,2,3$, *such that* $\varepsilon_2^2+\varepsilon_3^2\neq 0$.

The multiplication group $Mult(L)$ *and the inner mapping group* $Inn(L)$ *of* L *are isomorphic to the linear groups of matrices whose multiplications are defined by:*

$$Mult(L)_1 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1e^{x_5}\cos(x_6)-y_2e^{x_5}\sin(x_6), x_2+y_2e^{x_5}\cos(x_6)+y_1e^{x_5}\sin(x_6),$$

$$x_3 + y_3, x_4 + y_4 + (ax_6 + x_5)y_3, x_5 + y_5, x_6 + y_6), a \in \mathbb{R},$$

$Inn(L)_1 = \{g(u_1, u_2, u_3, \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, 0, 0); u_1, u_2, u_3 \in \mathbb{R}\}$, $\varepsilon_k \in \{0, 1\}$, $k = 1, 2, 3$, *such that* $\varepsilon_1^2 + \varepsilon_2^2 \neq 0$.

$$Mult(L)_2 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1e^{x_5+ax_6}, x_2 + y_2e^{x_6}, x_3 + y_3,$$

$$x_4 + y_4 + x_5y_3, x_5 + y_5, x_6 + y_6), a \in \mathbb{R} \setminus \{0\},$$

$Inn(L)_2 = \{g(u_1, u_2, u_3, u_1+u_2+\varepsilon u_3, 0, 0);\ u_1, u_2, u_3 \in \mathbb{R}\}$, $\varepsilon = 0, 1$.

$$Mult(L)_3 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1, x_2 + y_2 + x_5y_1, x_3 + y_3e^{x_6}\cos(x_5) - y_4e^{x_6}\sin(x_5),$$

$$x_4 + y_4e^{x_6}\cos(x_5) + y_3e^{x_6}\sin(x_5), x_5 + y_5, x_6 + y_6),$$

$Inn(L)_3 = \{g(u_1, \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, u_2, u_3, 0, 0); u_1, u_2, u_3 \in \mathbb{R}\}$, $\varepsilon_k \in \{0, 1\}, k = 1, 2, 3$, *such that* $\varepsilon_2^2 + \varepsilon_3^2 \neq 0$.

In Theorem 25 we list the 6-dimensional decomposable solvable Lie groups with 1-dimensional centre which are the groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L.

Theorem 25. *Let* L *be a connected simply connected topological loop of dimension* 3 *such that its multiplication group* $Mult(L)$ *is a* 6*-dimensional decomposable solvable Lie group having* 1*-dimensional centre. Then* L *has nilpotency class* 2*. Moreover, the following Lie algebra pairs* $(\mathbf{g}, \mathbf{k})$ *are the Lie algebra* $\mathbf{g}$ *of the group* $Mult(L)$ *and the subalgebra* $\mathbf{k}$ *of the subgroup* $Inn(L)$*:*
If $\mathbf{g}$ *has the form* $\mathbf{g} = \mathbb{R} \oplus \mathbf{h} = \langle f_1 \rangle \oplus \langle e_1, e_2, e_3, e_4, e_5 \rangle$*, where* $\mathbf{h}$ *is a* 5*-dimensional solvable indecomposable Lie algebra with trivial centre, then one has:*

- $\mathbf{g}_1 = \mathbb{R} \oplus \mathbf{g}_{5,19}^{\alpha=0,\beta\neq 0}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = \beta e_4$, $\mathbf{k}_{1,\epsilon} = \langle e_1 + f_1, e_2 + \epsilon f_1, e_4 + f_1 \rangle$, $\epsilon = 0, 1$,

- $\mathbf{g}_2 = \mathbb{R} \oplus \mathbf{g}_{5,20}^{\alpha=0}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_1 + e_4$, $\mathbf{k}_{2,\epsilon} = \langle e_1 + f_1, e_2 + \epsilon f_1, e_4 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$, $\epsilon = 0, 1$,

- $\mathbf{g}_3 = \mathbb{R} \oplus \mathbf{g}_{5,27}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_3 + e_4$, $[e_4, e_5] = e_1 + e_4$, $\mathbf{k}_3 = \langle e_1 + f_1, e_3, e_4 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$,

- $\mathbf{g}_4 = \mathbb{R} \oplus \mathbf{g}_{5,28}^{\alpha=0}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_3 + e_4$, $[e_4, e_5] = e_4$, $\mathbf{k}_4 = \langle e_1 + a_1 f_1, e_3, e_4 + f_1 \rangle$, $a_1 \in \mathbb{R}\backslash\{0\}$,

- $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,32}$: $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = he_1 + e_3$, $\mathbf{k}_5 = \langle e_1 + f_1, e_2 + a_2 f_1, e_3 \rangle$, $h, a_2 \in \mathbb{R}$,

- $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,33}$: $[e_1, e_4] = e_1$, $[e_3, e_4] = \beta e_3$, $[e_2, e_5] = e_2$, $[e_3, e_5] = \gamma e_3$, $\beta^2 + \gamma^2 \neq 0$, $\mathbf{k}_6 = \langle e_1 + f_1, e_2 + f_1, e_3 + f_1 \rangle$,

- $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,34}$: $[e_1, e_4] = \alpha e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_2$, $\mathbf{k}_7 = \langle e_1 + f_1, e_2 + f_1, e_3 + a_3 f_1 \rangle$, $\alpha, a_3 \in \mathbb{R}$,

- $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,35}$: $[e_1, e_4] = he_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_1, e_5] = \alpha e_1$, $[e_2, e_5] = -e_3$, $[e_3, e_5] = e_2$, $h^2 + \alpha^2 \neq 0$, $\mathbf{k}_{8,1} = \langle e_1 + f_1, e_2 + f_1, e_3 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{8,2} = \langle e_1 + f_1, e_2, e_3 + f_1 \rangle$.

If $\mathbf{g}$ *is the Lie algebra* $\mathbf{l}_2 \oplus \mathbf{n} = \langle f_1, f_2 \rangle \oplus \langle e_1, e_2, e_3, e_4 \rangle$, *where* $\mathbf{n}$ *is a 4-dimensional solvable Lie algebra with 1-dimensional centre* $\langle e_1 \rangle$, *then we have:*

- $\mathbf{g}_9 = \mathbf{l}_2 \oplus \mathbf{g}_{4,1}$: $[f_1, f_2] = f_1$, $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $\mathbf{k}_9 = \langle f_1 + e_1, e_2 + a_2 e_1, e_3 \rangle$, $a_2 \in \mathbb{R}$,

- $\mathbf{g}_{10} = \mathbf{l}_2 \oplus \mathbf{g}_{4,3}$: $[f_1, f_2] = f_1$, $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$, $\mathbf{k}_{10} = \langle f_1 + e_2, e_1 + e_2, e_3 \rangle$.

If $\mathbf{g}$ *is one of the following Lie algebras* $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$ *and* $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$*,* $i = 2, 3, 4, 5$*, where the centre of* $\mathbf{f}_3 = \langle e_1, e_2, e_3 \rangle$ *is* $\langle e_1 \rangle$ *and* $\mathbf{g}_{3,i} = \langle e_4, e_5, e_6 \rangle$ *is a* 3*-dimensional solvable Lie algebra with trivial centre, then one has:*

- $\mathbf{g}_{11} = \mathbf{f}_3 \oplus \mathbf{g}_{3,2}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_4 + e_5$, $\mathbf{k}_{11,1} = \langle e_2, e_4 + e_1, e_5 \rangle$, $\mathbf{k}_{11,2} = \langle e_3, e_4 + e_1, e_5 \rangle$,

- $\mathbf{g}_{12} = \mathbf{f}_3 \oplus \mathbf{g}_{3,3}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{12,1} = \langle e_2, e_4 + e_1, e_5 + e_1 \rangle$, $\mathbf{k}_{12,2} = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle$,

- $\mathbf{g}_{13} = \mathbf{f}_3 \oplus \mathbf{g}_{3,4}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = he_5$, $-1 \leq h < 1$, $h \neq 0$, $\mathbf{k}_{13,1} = \langle e_2, e_4 + e_1, e_5 + e_1 \rangle$, $\mathbf{k}_{13,2} = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle$,

- $\mathbf{g}_{14} = \mathbf{f}_3 \oplus \mathbf{g}_{3,5}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = pe_4 - e_5$, $[e_5, e_6] = e_4 + pe_5$, $p > 0$, $\mathbf{k}_{14,1} = \langle e_2, e_4 + e_1, e_5 + a_3 e_1 \rangle$, $\mathbf{k}_{14,2} = \langle e_3, e_4 + e_1, e_5 + a_3 e_1 \rangle$, $a_3 \in \mathbb{R} \setminus \{0\}$, $\mathbf{k}_{14,3} = \langle e_2, e_4, e_5 + e_1 \rangle$, $\mathbf{k}_{14,4} = \langle e_3, e_4, e_5 + e_1 \rangle$,

- $\mathbf{g}_{15} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_4 + e_5$, $\mathbf{k}_{15} = \langle f_1 + e_3, e_4 + e_3, e_5 \rangle$,

- $\mathbf{g}_{16} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,3}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{16} = \langle f_1 + e_3, e_4 + e_3, e_5 + e_3 \rangle$,

- $\mathbf{g}_{17} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,4}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = he_5$, $-1 \leq h < 1$, $h \neq 0$, $\mathbf{k}_{17} = \langle f_1 + e_3, e_4 + e_3, e_5 + e_3 \rangle$,

- $\mathbf{g}_{18} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = pe_4 - e_5$, $[e_5, e_6] = e_4 + pe_5$, $p > 0$, $\mathbf{k}_{18,1} = \langle f_1 + e_3, e_4 + e_3, e_5 + a_3 e_3 \rangle$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{18,2} = \langle f_1 + e_3, e_4, e_5 + e_3 \rangle$.

The multiplication group $Mult(L)$ *and the inner mapping group* $Inn(L)$ *of* L *are isomorphic to the linear groups of matrices whose multiplications are given by:*

$$Mult(L)_1 : g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+(y_1-x_3y_2)e^{x_5}, x_2+y_2e^{x_5},$$
$$(x_3+y_3)e^{x_5+y_5}, x_4+y_4e^{bx_5}, x_5+y_5, x_6+y_6),$$
$$Inn(L)_{1,\epsilon}=\{g(u_1,u_2,0,u_3,0,u_1+\epsilon u_2+u_3); u_i\in\mathbb{R}, i=1,2,3\},$$
$$b\in\mathbb{R}\backslash\{0\}, \epsilon=0,1,$$
$$Mult(L)_2: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1-x_3y_2+x_5y_4)e^{x_5}, x_2+y_2e^{x_5},$$
$$(x_3+y_3)e^{x_5+y_5}, x_4+y_4e^{x_5}, x_5+y_5, x_6+y_6),$$
$$Inn(L)_{2,\epsilon}=\{g(u_1,u_2,0,u_3,0,u_1+\epsilon u_2+a_3u_3); u_i\in\mathbb{R}, i=1,2,3\},$$
$$\epsilon=0,1, a_3\in\mathbb{R},$$
$$Mult(L)_3: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+x_5y_4+\frac{1}{2}(2x_2+x_5^2)y_3)e^{x_5},$$
$$(x_2+y_2+x_5y_5+\frac{1}{2}y_5^2+\frac{1}{2}x_5^2)e^{x_5+y_5}, x_3+y_3e^{x_5},$$
$$x_4+(y_4+x_5y_3)e^{x_5}, x_5+y_5, x_6+y_6),$$
$$Inn(L)_3=\{g(u_1,0,u_2,u_3,0,u_1+a_3u_3); u_i\in\mathbb{R}, i=1,2,3\}, a_3\in\mathbb{R},$$
$$Mult(L)_4: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+x_2y_3)e^{x_5}, (x_2+y_2)e^{x_5+y_5},$$
$$x_3+y_3e^{x_5}, x_4+(y_4+x_5y_3)e^{x_5}, x_5+y_5, x_6+y_6),$$
$$Inn(L)_4=\{g(u_1,0,u_2,u_3,0,a_1u_1+u_3); u_i\in\mathbb{R}, i=1,2,3\}, a_1\neq 0,$$
$$Mult(L)_5: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+x_4y_2+ax_5y_3+\frac{1}{2}x_4^2y_3)e^{x_5}, x_2+(y_2+x_4y_3)e^{x_5},$$
$$x_3+y_3e^{x_5}, (x_4+y_4)e^{x_5+y_5}, x_5+y_5, x_6+y_6), a\in\mathbb{R},$$
$$Inn(L)_5=\{g(u_1,u_2,u_3,0,0,u_1+a_2u_2); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$

$$Mult(L)_6 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1e^{x_4}, x_2+y_2e^{x_5}, x_3+y_3e^{ax_5+bx_4}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$Inn(L)_6 = \{g(u_1,u_2,u_3,0,0,u_1+u_2+u_3); u_i \in \mathbb{R}, i=1,2,3\}, a^2+b^2 \neq 0,$$
$$Mult(L)_7 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1e^{ax_4+x_5}, x_2+(y_2+x_5y_3)e^{x_4}, x_3+y_3e^{x_4},$$
$$x_4+y_4e^{ax_4+x_5}, (x_5+y_5)e^{x_4+y_4}, x_6+y_6),$$
$$Inn(L)_7 = \{g(u_1,u_2,u_3,0,0,u_1+u_2+a_3u_3); u_i \in \mathbb{R}, i=1,2,3\}, a, a_3 \in \mathbb{R},$$
$$Mult(L)_8 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1e^{ax_5+bx_4}, x_2+(y_2cos(x_5)-y_3sin(x_5))e^{x_4},$$
$$x_3+(y_3cos(x_5)+y_2sin(x_5))e^{x_4}, x_4+y_4, x_5+y_5, x_6+y_6), a^2+b^2 \neq 0,$$
$$Inn(L)_{8,1} = \{g(u_1,u_2,u_3,0,0,u_1+u_2+a_3u_3); u_i \in \mathbb{R}, i=1,2,3\}, a_3 \in \mathbb{R},$$
$$Inn(L)_{8,2} = \{g(u_1,u_2,u_3,0,0,u_1+u_3); u_i \in \mathbb{R}, i=1,2,3\},$$
$$Mult(L)_9 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1+x_4y_2+\frac{1}{2}x_4^2y_3, x_2+y_2+x_4y_3,$$
$$x_3+y_3, x_4+y_4, x_5+y_5e^{x_6}, x_6+y_6),$$
$$Inn(L)_9 = \{g(u_1+a_2u_2,u_2,u_3,0,u_1,0); u_i \in \mathbb{R}, i=1,2,3\}, a_2 \in \mathbb{R},$$
$$Mult(L)_{10} : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1e^{x_4}, x_2+y_2+x_4y_3, x_3+y_3, x_4+y_4, x_5+y_5e^{x_6}, x_6+y_6),$$
$$Inn(L)_{10} = \{g(u_1,u_1+u_3,u_2,0,u_3,0); u_i \in \mathbb{R}, i=1,2,3\},$$
$$Mult(L)_{11} : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1+x_2y_3, x_2+y_2, x_3+y_3, x_4+(y_4+x_6y_5)e^{x_6}, x_5+y_5e^{x_6}, x_6+y_6),$$
$$Inn(L)_{11,1} = \{g(u_2,u_1,0,u_2,u_3,0); u_i \in \mathbb{R}, i=1,2,3\},$$
$$Inn(L)_{11,2} = \{g(u_2,0,u_1,u_2,u_3,0); u_i \in \mathbb{R}, i=1,2,3\},$$

$$Mult(L)_{12} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_2y_3, x_2 + y_2, x_3 + y_3, x_4 + y_4e^{x_6}, x_5 + y_5e^{x_6}, x_6 + y_6),$$
$$Inn(L)_{12,1} = \{g(u_2 + u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Inn(L)_{12,2} = \{g(u_2 + u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Mult(L)_{13} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_2y_3, x_2 + y_2, x_3 + y_3, x_4 + y_4e^{x_6},$$
$$x_5 + y_5e^{hx_6}, x_6 + y_6), -1 \le h < 1, h \ne 0,$$
$$Inn(L)_{13,1} = \{g(u_2 + u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Inn(L)_{13,2} = \{g(u_2 + u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Mult(L)_{14} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_2y_3, x_2 + y_2, x_3 + y_3, x_4 + (y_4cos(x_6) + y_5sin(x_6))e^{px_6},$$
$$x_5 + (y_5cos(x_6) - y_4sin(x_6))e^{px_6}, x_6 + y_6), p > 0,$$
$$Inn(L)_{14,1} = \{g(u_2+a_3u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3, \}, a_3 \ne 0,$$
$$Inn(L)_{14,2} = \{g(u_2+a_3u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3, \}, a_3 \ne 0,$$
$$Inn(L)_{14,3} = \{g(u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Inn(L)_{14,4} = \{g(u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Mult(L)_{15} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1+y_1e^{x_2}, x_2+y_2, x_3+y_3e^{x_2}, x_4+(y_4+x_6y_5)e^{x_6}, x_5+y_5e^{x_6}, x_6+y_6),$$
$$Inn(L)_{15} = \{g(u_1, 0, u_1 + u_2, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Mult(L)_{16} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1e^{x_2}, x_2 + y_2, x_3 + y_3e^{x_2}, x_4 + y_4e^{x_6}, x_5 + y_5e^{x_6}, x_6 + y_6),$$
$$Inn(L)_{16} = \{g(u_1, 0, u_1 + u_2 + u_3, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$Mult(L)_{17} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1e^{x_2}, x_2 + y_2, x_3 + y_3e^{x_2}, x_4 + y_4e^{x_6},$$

$$x_5 + y_5 e^{hx_6}, x_6 + y_6), -1 \le h < 1, h \neq 0,$$

$$Inn(L)_{17} = \{g(u_1, 0, u_1 + u_2 + u_3, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$Mult(L)_{18} : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1 e^{x_2}, x_2 + y_2, x_3 + y_3 e^{x_2}, x_4 + (y_4 cos(x_6) + y_5 sin(x_6))e^{px_6},$$

$$x_5 + (y_5 cos(x_6) - y_4 sin(x_6))e^{px_6}, x_6 + y_6), p > 0,$$

$$Inn(L)_{18,1} = \{g(u_1, 0, u_1+u_2+a_3u_3, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_3 \in \mathbb{R},$$

$$Inn(L)_{18,2} = \{g(u_1, 0, u_1 + u_3, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}.$$

6 6-dimensional solvable Lie group having 2-dimensional centre

In this Chapter we determine the at most 6-dimensional solvable Lie groups with 2-dimensional centre which can be represented as the multiplication groups $Mult(L)$ of 3-dimensional connected simply connected topological proper loops L. These Lie groups are decomposable (cf. Theorem 19). Moreover, the centre $Z(L)$ of the corresponding loops is isomorphic to $\mathbb{R}^2$ such that the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}$. These loops are centrally nilpotent of class 2.

Theorem 26. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group is an at most 6-dimensional decomposable nilpotent Lie group. Then the loops L have nilpotency class 2 and the multiplication groups $Mult(L)$ of L are the groups $\mathbb{R} \times \mathcal{F}_4$, $\mathbb{R} \times \mathcal{F}_5$.*

Theorem 27. *Let L be a 3-dimensional connected simply connected topological loop which has a 6-dimensional solvable non-nilpotent Lie algebra with 2-dimensional centre as the Lie algebra $\mathbf{g}$ of its multiplication group $Mult(L)$. Then L have nilpotency class 2 and the following Lie algebra pairs $(\mathbf{g}, \mathbf{k})$ are the Lie algebra $\mathbf{g}$ of the group*

Mult(L) *and the subalgebra* $\mathbf{k}$ *of the inner mapping group* *Inn*(L)*:* *If* $\mathbf{g}_i = \mathbb{R}^2 \oplus \mathbf{n}_i = \langle f_1, f_2 \rangle \oplus \langle e_1, \cdots, e_4 \rangle$, $i = 1, \cdots, 4$, *where* $\mathbf{n}$ *is a* 4*-dimensional solvable indecomposable Lie algebra with trivial centre, then one has*

- $\mathbf{n}_1 = \mathbf{g}_{4,2}^{\alpha \neq 0}$: $[e_1, e_4] = \alpha e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$, $\mathbf{k}_1 = \langle e_1 + f_1, e_2 + f_1, e_3 \rangle$,

- $\mathbf{n}_2 = \mathbf{g}_{4,4}$: $[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3$, $\mathbf{k}_2 = \langle e_1 + f_1, e_2 + a_2 f_1, e_3 + a_3 f_1 \rangle$, $a_2, a_3 \in \mathbb{R}$,

- $\mathbf{n}_3 = \mathbf{g}_{4,5}^{-1 \leq \gamma \leq \beta \leq 1, \gamma\beta \neq 0}$: $[e_1, e_4] = e_1, [e_2, e_4] = \beta e_2, [e_3, e_4] = \gamma e_3$, $\mathbf{k}_3 = \langle e_1 + f_1, e_2 + f_1, e_3 + f_1 \rangle$,

- $\mathbf{n}_4 = \mathbf{g}_{4,6}^{p \geq 0, \alpha \neq 0}$: $[e_1, e_4] = \alpha e_1, [e_2, e_4] = p e_2 - e_3, [e_3, e_4] = e_2 + p e_3$, $\mathbf{k}_{4,1} = \langle e_1 + f_1, e_2 + f_1, e_3 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{4,2} = \langle e_1 + f_1, e_2, e_3 + f_1 \rangle$.

If $\mathbf{g}_j = \mathbb{R} \oplus \mathbf{h}_j = \langle f_1 \rangle \oplus \langle e_1, e_2, e_3, e_4, e_5 \rangle$, *where* $\mathbf{h}$ *is a* 5*-dimensional solvable indecomposable Lie algebra with* 1*-dimensional centre, then we have*

- $\mathbf{h}_5 = \mathbf{g}_{5,8}^{0 < |\gamma| \leq 1}$: $[e_2, e_5] = e_1, [e_3, e_5] = e_3, [e_4, e_5] = \gamma e_4$, $\mathbf{k}_{5,\epsilon} = \langle e_2 + \epsilon f_1, e_3 + e_1, e_4 + e_1 \rangle$, $\epsilon = 0, 1$,

- $\mathbf{h}_6 = \mathbf{g}_{5,10}$: $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_4$, $\mathbf{k}_{6,\epsilon} = \langle e_2, e_3 + \epsilon f_1, e_4 + e_1 \rangle$, $\epsilon = 0, 1$,

- $\mathbf{h}_7 = \mathbf{g}_{5,14}^{p \neq 0}$: $[e_2, e_5] = e_1, [e_3, e_5] = p e_3 - e_4, [e_4, e_5] = e_3 + p e_4$, $\mathbf{k}_{7,\epsilon} = \langle e_2 + \epsilon f_1, e_3 + e_1, e_4 + a_3 e_1 \rangle$, $\epsilon = 0, 1$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{7,\delta} = \langle e_2 + \delta f_1, e_3, e_4 + e_1 \rangle$, $\delta = 0, 1$,

- $\mathbf{h}_8 = \mathbf{g}_{5,15}^{\gamma = 0}$: $[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_4, e_5] = e_3$, $\mathbf{k}_{8,\epsilon} = \langle e_1 + e_3, e_2, e_4 + \epsilon f_1 \rangle$, $\epsilon = 0, 1$.

The linear representations of the multiplication group *Mult*(L) *and the inner mapping group* *Inn*(L) *of* L *are given by:*

$$Mult(L)_1 : g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1e^{ax_4}, x_2+(y_2+x_4y_3)e^{x_4}, x_3+y_3e^{x_4},$$
$$x_4+y_4, x_5+y_5, x_6+y_6), a \neq 0,$$
$$Inn(L)_1 = \{g(u_1,u_2,u_3,0,u_1+u_2,0); u_i \in \mathbb{R}, i=1,2,3\},$$
$$Mult(L)_2 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+(y_1+x_4y_2+\frac{1}{2}x_4^2y_3)e^{x_4}, x_2+(y_2+x_4y_3)e^{x_4},$$
$$x_3+y_3e^{x_4}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$Inn(L)_2 = \{g(u_1,u_2,u_3,0,u_1+a_2u_2+a_3u_3,0); u_i \in \mathbb{R}, i=1,2,3\}, a_2,a_3 \in \mathbb{R},$$
$$Mult(L)_3 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1e^{x_4}, x_2+y_2e^{ax_4}, x_3+y_3e^{bx_4}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$Inn(L)_3 = \{g(u_1,u_2,u_3,0,u_1+u_2+u_3,0); u_i \in \mathbb{R}, i=1,2,3\},$$
$$-1 \leq a \leq b \leq 1, ab \neq 0$$

,

$$Mult(L)_4 : (x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1e^{ax_4}, x_2+(y_2cos(x_4)+y_3sin(x_4))e^{bx_4},$$
$$x_3+(y_3cos(x_4)-y_2sin(x_4))e^{bx_4}, x_4+y_4, x_5+y_5, x_6+y_6), a \neq 0, b \geq 0,$$
$$Inn(L)_{4,1} = \{g(u_1,u_2,u_3,0,u_1+u_2+a_3u_3,0); u_i \in \mathbb{R}, i=1,2,3\}, a_3 \in \mathbb{R},$$
$$Inn(L)_{4,2} = \{g(u_1,u_2,u_3,0,u_1+u_3,0); u_i \in \mathbb{R}, i=1,2,3\},$$
$$Mult(L)_5 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1+x_5y_2, x_2+y_2, x_3+y_3e^{x_5},$$
$$x_4+y_4e^{cx_5}, x_5+y_5, x_6+y_6), 0 < |c| \leq 1,$$
$$Inn(L)_{5,\epsilon} = \{g(u_2+u_3,u_1,u_2,u_3,0,\epsilon u_1); u_i \in \mathbb{R}, i=1,2,3\}, \epsilon = 0,1,$$
$$Mult(L)_6 : g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) =$$
$$g(x_1+y_1+x_5y_2+\frac{1}{2}x_5^2y_3, x_2+y_2+x_5y_3,$$

$$x_3 + y_3, x_4 + y_4 e^{x_5}, x_5 + y_5, x_6 + y_6),$$

$$Inn(L)_{6,\epsilon} = \{g(u_3, u_1, u_2, u_3, 0, \epsilon u_2); u_i \in \mathbb{R}, i = 1, 2, 3\}, \epsilon = 0, 1,$$

$$Mult(L)_7 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1 + x_2 y_5, x_2 + y_2, x_3 + (y_3 cos(x_5) - y_4 sin(x_5))e^{px_5},$$

$$x_4 + (y_4 cos(x_5) + y_3 sin(x_5))e^{px_5}, x_5 + y_5, x_6 + y_6), p \neq 0,$$

$$Inn(L)_{7,\epsilon} = \{g(u_2 + a_3 u_3, u_1, u_2, u_3, 0, \epsilon u_1); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$\epsilon = 0, 1, a_3 \in \mathbb{R},$$

$$Inn(L)_{7,\delta} = \{g(u_3, u_1, u_2, u_3, 0, \varepsilon u_1); u_i \in \mathbb{R}, i = 1, 2, 3\}, \delta = 0, 1,$$

$$Mult(L)_8 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+(y_1+y_2 x_5)e^{x_5}, x_2+y_2 e^{x_5}, x_3+y_3+x_5 y_4, x_4+y_4, x_5+y_5, x_6+y_6),$$

$$Inn(L)_{8,\epsilon} = \{g(u_1, u_2, u_1, u_3, 0, \epsilon u_3); u_i \in \mathbb{R}, i = 1, 2, 3\}, \epsilon = 0, 1,$$

Corollary 28. *All solvable decomposable Lie groups of dimension* 6 *which are the groups* $Mult(L)$ *of* 3*-dimensional connected topological loops* L *have* 1*- or* 2*-dimensional centre and* 3*-dimensional commutator subgroup.*

Corollary 29. *Each solvable Lie group of dimension* 6 *which is realized as the multiplication group* $Mult(L)$ *of a* 3*-dimensional connected topological proper loop* L *has* 1*- or* 2*-dimensional centre and* 2*- or* 3*-dimensional commutator subgroup.*

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Á. Figula and A. Al-Abayechi, *Topological loops having solvable indecomposable Lie groups as their multiplication groups*, Transformation Groups, **26**, no. 1 (2021), 279-303.

Á. Figula and A. Al-Abayechi, *Topological loops with solvable multiplication groups of dimension at most six are centrally nilpotent*, Int. J. Group Theory, **9**, no. 2 (2020), 81-94.

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List of conference talks of the author

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4. A. Al-Abayechi, *On the structure of topological loops with solvable multiplication groups*, 5th International Conference Riemannian Geometry and Applications, Faculty of Railways, Roads and Bridges, Technical University of Civil Engineering Bucharest and Faculty of Mathematics and Computer Science, University of Bucharest, Romania, January 15-17, 2021.

5. A. Al-Abayechi, *Decomposable solvable multiplication Lie groups and topological loops*, Ischia Group Theory Online Conference, 25-26 March 2021.

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List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (1)

1. Figula, Á., Ficzere, K., **Al-Abayechi, A.**: Topological loops with six-dimensional solvable multiplication groups having five-dimensional nilradical.
   Ann. Math. Inform. 50, 71-87, 2019. ISSN: 1787-5021.
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Foreign language scientific articles in international journals (3)

2. Figula, Á., **Al-Abayechi, A.**: Topological loops having solvable indecomposable Lie groups as their multiplication groups.
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   Int. J. Group Theory. 9 (2), 81-94, 2020. ISSN: 2251-7650.
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Foreign language conference proceedings (1)

5. **Al-Abayechi, A.**: Topological loops having solvable Lie groups as their multiplication groups. In: IX. Interdiszciplináris Doktorandusz Konferencia 2020 = 9th Interdisciplinary Doctoral Conference 2020 : Tanulmánykötet. Szerk.: Csiszár Beáta, Hankó Csilla, Kajos Luca Fanni, Mező Emerencia, Pécsi Tudományegyetem Doktorandusz Önkormányzat, Pécs, 6-17, 2020. ISBN: 9789634295839

Total IF of journals (all publications): 0,75
Total IF of journals (publications related to the dissertation): 0,75

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

02 June, 2021

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Nyilvántartási szám: DEENK/343/2021.PL
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Jelölt: Al-Abayechi, Ameer
Doktori Iskola: Matematika- és Számítástudományok Doktori Iskola
MTMT azonosító: 10072191

A PhD értekezés alapjául szolgáló közlemények

<u>Idegen nyelvű tudományos közlemények hazai folyóiratban</u> (1)

1. Figula, Á., Ficzere, K., **Al-Abayechi, A.**: Topological loops with six-dimensional solvable multiplication groups having five-dimensional nilradical.
   Ann. Math. Inform. 50, 71-87, 2019. ISSN: 1787-5021.
   DOI: http://dx.doi.org/10.33039/ami.2019.08.001

<u>Idegen nyelvű tudományos közlemények külföldi folyóiratban</u> (3)

2. Figula, Á., **Al-Abayechi, A.**: Topological loops having solvable indecomposable Lie groups as their multiplication groups.
   Transform. Groups. 26 (1), 279-303, 2021. ISSN: 1083-4362.
   DOI: http://dx.doi.org/10.1007/s00031-020-09604-1
   IF: 0.75 (2019)

3. **Al-Abayechi, A.**, Figula, Á.: Geodesic vectors and flat totally geodesic subalgebras in nilpotent metric Lie algebras.
   Itogi Nauki i Techniki. Ser. Sovrem. Mat. Pril. Temat. Obz. 177, 10-23, 2020. ISSN: 0233-6723.
   DOI: http://dx.doi.org/10.36535/0233-6723-2020-177-10-23

4. Figula, Á., **Al-Abayechi, A.**: Topological loops with solvable multiplication groups of dimension at most six are centrally nilpotent.
   Int. J. Group Theory. 9 (2), 81-94, 2020. ISSN: 2251-7650.
   DOI: http://dx.doi.org/10.22108/IJGT.2019.114770.1522

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Idegen nyelvű konferencia közlemények (1)

5. **Al-Abayechi, A.**: Topological loops having solvable Lie groups as their multiplication groups. In: IX. Interdiszciplináris Doktorandusz Konferencia 2020 = 9th Interdisciplinary Doctoral Conference 2020 : Tanulmánykötet. Szerk.: Csiszár Beáta, Hankó Csilla, Kajos Luca Fanni, Mező Emerencia, Pécsi Tudományegyetem Doktorandusz Önkormányzat, Pécs, 6-17, 2020. ISBN: 9789634295839

A közlő folyóiratok összesített impakt faktora: 0,75
A közlő folyóiratok összesített impakt faktora (az értekezés alapjául szolgáló közleményekre): 0,75

A DEENK a Jelölt által az iDEa Tudóstérbe feltöltött adatok bibliográfiai és tudománymetriai ellenőrzését a tudományos adatbázisok és a Journal Citation Reports Impact Factor lista alapján elvégezte.

Debrecen, 2021.06.02.

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