Inequalities on two variable Gini and Stolarsky means

doktori (Ph.D.) értekezés

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Contents

CHAPTER 1

Introduction

There is an extended literature concerning the so-called Gini and Stolarsky means. These two variable homogenous means play important roles both in the theory of means and in the application of inequalities in various branches of mathematics.

We recall now the definition of these means. If x, y are positive numbers, then their Gini mean is defined by:

$$G_{a,b}(x,y) = \begin{cases} \left(\frac{x^a+y^a}{x^b+y^b}\right)^{\frac{1}{a-b}}, & \text{if } a \neq b, \\ \exp\left(\frac{x^a \log x + y^a \log y}{x^a+y^a}\right), & \text{if } a = b, \end{cases}$$

while their Stolarsky mean is the following:

$$S_{a,b}(x,y) = \begin{cases} \left(\frac{b(x^a-y^a)}{a(x^b-y^b)}\right)^{\frac{1}{a-b}}, & \text{if } (a-b)ab \neq 0, x \neq y, \\ \exp\left(-\frac{1}{a} + \frac{x^a \log x - y^a \log y}{x^a-y^a}\right), & \text{if } a = b \neq 0, x \neq y, \\ \left(\frac{x^a-y^a}{a(\log x - \log y)}\right)^{\frac{1}{a}}, & \text{if } a \neq 0,\ b = 0, x \neq y, \\ \sqrt{xy}, & \text{if } a = b = 0, \\ x, & \text{if } x = y. \end{cases}$$

This setting is a special case of a more general one. Namely, the concept of the Gini mean can be defined for any number of positive variables in the following way:

$$G_{a,b;n}(\mathbf{x}) = G_{a,b;n}(x_1, x_2, \ldots, x_n) = \begin{cases} \left(\frac{\sum_{i=1}^n x_i^a}{\sum_{i=1}^n x_i^b}\right)^{\frac{1}{a-b}}, & \text{if } a \neq b, \\ \exp\left(\frac{\sum_{i=1}^n x_i^a \log x_i}{\sum_{i=1}^n x_i^a}\right), & \text{if } a = b. \end{cases}$$

Similarly, the Stolarsky – or, as it is also known in the literature: the difference mean – of n ($n \in \mathbb{N}$) positive numbers can be given by divided differences.

The power mean with exponent p of the positive numbers x and y can be obtained both as $G_{0,p}(x,y)$ and $S_{2p,p}(x,y)$. That is, the power means are included in both classes of means. More surprisingly, as it has recently been proved by Alzer and Ruscheweyh [**4**], the class of power means forms exactly the intersection of the classes of Gini and Stolarsky means.

In particular, $S_{2,1}$, $S_{0,0}$, and $S_{-2,-1}$ are the *arithmetic*, *geometric*, and *harmonic* means, respectively. The special settings $S_{1,0}$ and $S_{1,1}$ are called *logarithmic* and *identric* means.

These definitions create a continuous, moreover, infinitely many times differentiable function

$$(a, b, x, y) \mapsto M_{a,b}(x, y)$$

on the domain $\mathbb{R}^2 \times \mathbb{R}_+^2$, where $M_{a,b}(x, y)$ can stand for either $G_{a,b}(x, y)$ or $S_{a,b}(x, y)$.

The Gini and Stolarsky means provide us a large field for research. Extended surveys have been done towards the comparison of them. Others aim at Hölder or Minkowski-type relations – and so on.

In the recent years, under the supervision of Professor Zsolt Páles, my studies were directed first of all to these topics. This activity can be followed by [**16**], [**17**], [**15**], [**19**], [**18**], [**14**], [**13**] (the first five are joint works with my supervisor). This thesis covers the main part of the results.

In the meantime some of our theorems, concepts have been completed or reformulated. Additionally, after publishing the above papers some new connections were found. Therefore, I changed the original structures of the articles and tried to reorganize them in the way as it seemed to be the most logical.

My thesis consists of three main parts. Chapters 2, 3 and 4 serve as preliminaries for the rest of my work. In Chapters 5, 6 and 7 I present the results, obtained in the field of comparison of our means. Finally, Chapters 8, 9 and 10 deal with the generalized Minkowski-type inequalities, concerning the two variable Gini and Stolarsky means.

The results are builded on known statements. To distinguish them, the next convention was followed. The new results, published first in [**16**], [**17**], [**15**], [**19**], [**18**], [**14**] and [**13**] were numbered numerically, like, for example, THEOREM 1.1, while the known results were labelled by letters: LEMMA 1.A. In the first case the reference number is also indicated.

Finally, in the cases when a property holds both for the Gini and the Stolarsky means, the notation $M_{a,b}$ (or sometimes simply: M) will be applied. If an additional specification is needed, $M = G$ stands, for instance, for the fact that the statement concerns only the Gini means.

CHAPTER 2

Preliminary results

2.1. Introduction

In this chapter we collect the most important elementary properties of the Gini and Stolarsky means. Section 2 consists of a (well known) basic identity that will help us several times to avoid the unnecessary duplications when proving certain statements for the parameter pairs (a, b) and $(-a, -b)$.

Nevertheless the cases in the definitions of Gini and Stolarsky means seem quite different, we will see that they all can be derived from the case of equal parameters, which – in a sense – plays a central role in our treatment. The following statement can be found in the literature (see e.g. [**53**]), whose proof will also be presented for sake of completeness. Lemma 2.B turns out also to be very useful, since it makes possible to apply the Hermite-Hadamard-type theorems. The details can be read in Section 3.

Lemma 2.1, presented in Section 4 is the first result that can directly be converted to inequalities for Gini and Stolarsky means.

In Section 5 we get familiar with three elementary functions, playing important roles at the comparison theorems. Their fundamental and useful properties will also be presented.

Finally, in the last section we describe the first (partial) derivatives of the Gini and Stolarsky means since we will apply them many times. (Due to the symmetry, it is enough to calculate them by their first variable.)

2.2. An elementary identity

LEMMA 2.A. *For $a, b \in \mathbb{R}$, we have the identity*

$$M_{a,b}(x, y) = \left[M_{-a,-b}(x^{-1}, y^{-1})\right]^{-1} \qquad (x, y \in \mathbb{R}_+). \tag{2.1}$$

2.3. Representations by integral averages of the means with equal parameters

LEMMA 2.B. *Let the positive numbers x and y be fixed. Then for any real numbers a, b $(a \neq b)$ the following formula holds:*

$$\log M_{a,b}(x, y) = \frac{1}{a - b} \int_b^a \log M_{t,t}(x, y)dt. \tag{2.2}$$

PROOF. We may assume that $x \neq y$, since the case $x = y$ is trivial. For Gini means, we have

$$\frac{1}{a-b}\int_b^a \log G_{t,t}(x,y)dt = \frac{1}{a-b}\int_b^a \frac{x^t\log x + y^t \log y}{x^t+y^t}dt$$
$$= \frac{1}{a-b}\Big[\log\left(x^t+y^t\right)\Big]_b^a = \frac{1}{a-b}\log\frac{x^a+y^a}{x^b+y^b} = \log G_{a,b}(x,y).$$

In the Stolarsky case we will assume that $x > y$ and $a > b$. If $0 < b < a$ or $b < a < 0$ then

$$\frac{1}{a-b}\int_b^a \log S_{t,t}(x,y)dt = \frac{1}{a-b}\int_b^a \left(-\frac{1}{t} + \frac{x^t\log x - y^t\log y}{x^t-y^t}\right)dt$$
$$= \frac{1}{a-b}\left[\log\left(\frac{x^t-y^t}{t}\right)\right]_b^a = \frac{1}{a-b}\log\frac{\frac{x^a-y^a}{a}}{\frac{x^b-y^b}{b}} = \log S_{a,b}(x,y).$$

If $0 = b < a$ or $b < a = 0$ then we can apply the continuity of the integral as the function of its bounds. For example,

$$\frac{1}{a}\int_0^a \log S_{t,t}(x,y)dt = \lim_{b\to 0+}\left(\frac{1}{a-b}\int_b^a\left(-\frac{1}{t}+\frac{x^t\log x - y^t\log y}{x^t-y^t}\right)\right)dt$$
$$= \frac{1}{a}\lim_{b\to 0+}\left[\log\left(\frac{x^t-y^t}{t}\right)\right]_b^a = \frac{1}{a}\left(\log\frac{x^a-y^a}{a} - \lim_{b\to 0+}\log\frac{x^b-y^b}{b}\right)$$
$$= \frac{1}{a}\left(\log\frac{x^a-y^a}{a} - \log(\log x - \log y)\right) = \log S_{a,0}(x,y).$$

Finally, in the case $b < 0 < a$

$$\frac{1}{a-b}\int_b^a \log S_{t,t}(x,y)dt = \frac{1}{a-b}\Big(\int_b^0 \log S_{t,t}(x,y)dt + \int_0^a \log S_{t,t}(x,y)dt\Big)$$
$$= \frac{1}{a-b}\left(a\frac{1}{a}\Big(\log\frac{x^a-y^a}{a} - \log(\log x - \log y)\Big)\right.$$
$$\left. - b\frac{1}{b}\left(\log\frac{x^b-y^b}{b} - \log(\log x-\log y)\right)\right) = \log S_{a,b}(x,y).$$

$\square$

2.4. A consequence of Karamata's inequality

LEMMA 2.1.([**18**]) *For any positive* $x \neq 1$,

$$\frac{x(x+1)}{2} < \left(\frac{x-1}{\log x}\right)^3. \tag{2.3}$$

PROOF. By Karamata's classical inequality (see [**40**, p. 272]), we have that

$$\frac{x+x^{1/3}}{1+x^{1/3}} < \frac{x-1}{\log x}. \tag{2.4}$$

Thus, it suffices to show that

$$\frac{x(x+1)}{2} < \left(\frac{x+x^{1/3}}{1+x^{1/3}}\right)^3. \tag{2.5}$$

Dividing both sides by x, then multiplying them by $2(1+x^{1/3})^3$, finally, collecting the terms on the right side, one can easily check that (2.5) turns to

$$0 < (x^{2/3}+x^{1/3}+1)(x^{1/3}-1)^4,$$

which is obviously true for all positive $x \neq 1$. □

The inequality stated in the above lemma can be translated to an inequality concerning the geometric, arithmetic and logarithmic means.

COROLLARY 2.2.([**18**]) *For all* $x, y > 0$,

$$S_{0,0}^2(x,y) \cdot S_{2,1}(x,y) \leq S_{1,0}^3(x,y). \tag{2.6}$$

PROOF. If $x = y$, then (2.6) is obvious. If $x \neq 1$ and $y = 1$, then (2.6) is literally the same as (2.3), hence (2.6) holds in this case, too. Now replacing x by x/y in (2.3), and using the homogeneity of the Stolarsky means, we get that (2.6) is valid for all positive $x \neq y$. □

REMARK 2.3. Arguing in the same way as in the proof of Corollary 2.2, one can deduce that the inequalities (2.4) and (2.5) are equivalent to

$$S_{0,0}^2(x,y) \cdot G_{\frac{2}{3},\frac{1}{3}}(x,y) \leq S_{1,0}^3(x,y)$$

and

$$S_{2,1}(x,y) = G_{0,1}(x,y) \leq G_{\frac{2}{3},\frac{1}{3}}(x,y),$$

respectively. The latter inequality can also be derived from the comparison theorem of two variable Gini means (cf. [**48**], [**49**], [**16**] and [**19**]).

2.5. Auxiliary functions and their properties

DEFINITION 2.4. Define the functions $\mathcal{E}, \mathcal{M}, \mathcal{L} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathcal{E}(a,b) := \begin{cases} \dfrac{|a|-|b|}{a-b}, & \text{if } a \neq b, \\ \operatorname{sign}(a), & \text{if } a = b, \end{cases} \tag{2.7}$$

$$\mathcal{M}(a,b) := \begin{cases} \min\{a,b\}, & \text{if } a,b \geq 0, \\ 0, & \text{if } ab < 0, \\ \max\{a,b\}, & \text{if } a,b \leq 0, \end{cases} \tag{2.8}$$

$$\mathcal{L}(a,b) := \begin{cases} \dfrac{a-b}{\log(a/b)}, & \text{if } 0 < ab \text{ and } a \neq b, \\ a, & \text{if } 0 < ab \text{ and } a = b, \\ 0, & \text{if } ab \leq 0. \end{cases} \tag{2.9}$$

LEMMA 2.5.([**16**]) *For the function $\mathcal{E}$, the following statements hold:*

(i) $\mathcal{E}$ *is symmetric with respect to its variables and it is also odd, that is* $\mathcal{E}(-a,-b) = -\mathcal{E}(a,b)$ *for all* $a, b \in \mathbb{R}$*;*
(ii) $\mathcal{E}$ *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *and* $\mathcal{E}(a,-a) = 0$ *for all* $a \in \mathbb{R}$*;*
(iii) For fixed $a \in \mathbb{R}$*, the function* $b \mapsto \mathcal{E}(a,b)$ *is increasing over* $\mathbb{R}$*. If* $a < 0$*, then this function is strictly increasing on* $\mathbb{R}_+$*, furthermore,* $-1 \leq \mathcal{E}(a,b) \leq 1$ *for all* $a, b \in \mathbb{R}$*.*

PROOF. These results can immediately be obtained from the definitions of the function $\mathcal{E}$. □

The function $\mathcal{L}$ can be considered as the extension of the logarithmic mean $L = S_{1,0}$. (Clearly, if $a, b > 0$ then $\mathcal{L}(a,b) = S_{0,1}(a,b)$, that is, $\mathcal{L}$ coincides with the logarithmic mean on $\mathbb{R}^2_+$.)

LEMMA 2.6.([**17**]) *For the function $\mathcal{L}$, the following statements hold:*

(i) $\mathcal{L}$ *is symmetric with respect to its variables and it is also odd, that is,* $\mathcal{L}(-a,-b) = -\mathcal{L}(a,b)$ *for all* $a, b \in \mathbb{R}$*;*
(ii) $\mathcal{L}$ *is continuous on* $\mathbb{R}^2$*;*
(iii) For fixed $a \in \mathbb{R}$*, the function* $b \mapsto \mathcal{L}(a,b)$ *is increasing over* $\mathbb{R}$*. If* $a > 0$*, then this function is strictly increasing on* $\mathbb{R}_+$ *and* $\lim_{b\to\infty} \mathcal{L}(a,b) = \infty$*;*
(iv) $\mathcal{L}$ *is concave over the region* $[0,\infty)^2$*.*

PROOF. The first three properties of $\mathcal{L}$ can be obtained easily. We note that (iii) is the consequence of the strict concavity of the $\log$ function.

For (iv), we have to prove that the matrix

$$\begin{pmatrix} \partial_{11}\mathcal{L} & \partial_{12}\mathcal{L} \\ \partial_{21}\mathcal{L} & \partial_{22}\mathcal{L} \end{pmatrix}$$

is nonpositive definite. Really, due to the symmetry of $\mathcal{L}$,

$$\begin{vmatrix} \partial_{11}\mathcal{L} & \partial_{12}\mathcal{L} \\ \partial_{21}\mathcal{L} & \partial_{22}\mathcal{L} \end{vmatrix} = 0,$$

and an elementary calculation shows that $\partial_{11}\mathcal{L} \leq 0$ on the domain indicated. □

REMARK 2.7. For (iv) we may also refer to a statement, detailed later in Chapter 10. Namely, by Theorem 10.A, $S_{1,0}$ satisfies the reversed Minkowski inequality. Thus, using also the homogeneity,

$$\begin{aligned} &\mathcal{L}\big(ta_1 + (1-t)b_1, ta_2 + (1-t)b_2\big) \\ &\qquad \geq \mathcal{L}(ta_1, ta_2) + \mathcal{L}\big((1-t)b_1, (1-t)b_2\big) = t\mathcal{L}(a_1,a_2) + (1-t)\mathcal{L}(b_1,b_2) \end{aligned}$$

for $a_1, a_2, b_1, b_2 > 0$, $t \in [0,1]$. By continuity, this inequality extends for a_1, a_2, $b_1, b_2 \geq 0$, hence $\mathcal{L}$ is concave on $[0,\infty)^2$.

The next lemma will also be used several times.

LEMMA 2.8.([**17**]) *Suppose that* $2 \leq a+b$. *Then*

$$\mathcal{L}(a,b) \underset{(\geq)}{\leq} \mathcal{L}(1,a+b-1), \qquad \textit{if} \qquad \min\{a,b\} \underset{(\geq)}{\leq} 1.$$

PROOF. Due to the symmetry, we can assume that $a \leq b$.
First, let us assume that $a \leq 1$. Then $b \geq 2-a \geq 1$.
In the case $a \leq 0$,

$$\mathcal{L}(a,b) = 0 < \mathcal{L}(1,a+b-1).$$

In the case $0 < a \leq 1$ the only way to provide $a = b$ is $a = b = 1$. In this case our statement is trivial. Consequently, we may suppose that $a \neq b$.

Define $t := \dfrac{b-1}{b-a}$. With this choice of t, we have that $t \in [0,1)$, furthermore

$$ta + (1-t)b = 1 \qquad \text{and} \qquad tb + (1-t)a = a+b-1.$$

Due to this reason and the concavity of $\mathcal{L}$ (see Lemma 2.6 (iv)), we get that

$$\begin{aligned}\mathcal{L}(a,b) &= t\mathcal{L}(a,b) + (1-t)\mathcal{L}(b,a)\\ &\leq \mathcal{L}\big(ta+(1-t)b, tb+(1-t)a\big)\\ &= \mathcal{L}(1,a+b-1).\end{aligned}$$

Suppose now that $a \geq 1$. Define $t := \dfrac{b-1}{a+b-2}$. (For the above mentioned reason, we will enclose the case $a = b$.)

Again, $t \in [0,1)$,

$$t \cdot 1 + (1-t)(a+b-1) = a \qquad \text{and} \qquad t(a+b-1) + (1-t)\cdot 1 = b.$$

Consequently, applying Lemma 2.6 (i) and (iv),

$$\begin{aligned}\mathcal{L}(1,a+b-1) &= t\mathcal{L}(1,a+b-1) + (1-t)\mathcal{L}(a+b-1,1)\\ &\leq \mathcal{L}\big(t+(1-t)(a+b-1), t(a+b-1)+(1-t)\big)\\ &= \mathcal{L}(a,b).\end{aligned}$$

Thus, the proof of the lemma is complete. □

2.6. On the partial derivatives of the Gini and Stolarsky means

An elementary computation yields the following formulae for the partial derivatives of the Gini and Stolarsky means.

REMARK 2.9. Let a, b be any real numbers. Then

$$\frac{\partial}{\partial x}G_{a,b}(x,y) = \begin{cases} \frac{1}{a-b}G_{a,b}(x,y)\left(a\frac{x^{a-1}}{x^a+y^a} - b\frac{x^{b-1}}{x^b+y^b}\right), & \text{if } a \neq b, \\ G_{a,a}(x,y)\left(a\frac{x^{a-1}y^a(\log x - \log y)}{(x^a+y^a)^2} + \frac{x^{a-1}}{x^a+y^a}\right), & \text{if } a = b, \end{cases} \tag{2.10}$$

$$\frac{\partial}{\partial x}S_{a,b}(x,y) = \begin{cases} \frac{1}{a-b}S_{a,b}(x,y)\left(a\frac{x^{a-1}}{x^a-y^a} - b\frac{x^{b-1}}{x^b-y^b}\right), & \text{if } (a-b)ab \neq 0, x \neq y, \\ S_{a,a}(x,y)\left(-a\frac{x^{a-1}y^a(\log x - \log y)}{(x^a-y^a)^2} + \frac{x^{a-1}}{x^a-y^a}\right), & \text{if } a = b \neq 0, x \neq y, \\ S_{a,0}(x,y)\left(\frac{x^{a-1}}{x^a-y^a} - \frac{1}{ax(\log x - \log y)}\right), & \text{if } a \neq 0,\ b = 0, x \neq y, \\ \frac{1}{2}\sqrt{\frac{y}{x}}, & \text{if } a = b = 0, \\ \frac{1}{2}, & \text{if } x = y. \end{cases} \tag{2.11}$$

It is also easy to see that all these functions are continuous on the domain $(a, b, x, y) \in \mathbb{R}^2 \times \mathbb{R}_+^2$.

CHAPTER 3

Asymptotic properties

3.1. Introduction

In this chapter we list a number of asymptotic properties of Gini and/or Stolarsky means. In the first section we will perform elementary transformations on the means, while in the last one the limits of some composed expressions will be calculated.

These results will be extremely useful when proving the comparison theorems.

We will try to preserve the symmetry of the treatment concerning the two families of our means, that is, if possible, we will present the theorems simultaneously.

3.2. Elementary asymptotics

LEMMA 3.1.([**16**]) *Assume that* $a, b \in \mathbb{R}$. *Then*

$$\lim_{z\to\infty}\big(M_{a,b}(x+z, y+z) - z\big) = \frac{x+y}{2} \qquad (x, y \in \mathbb{R}_+).$$

PROOF. Using the homogeneity of $M_{a,b}$ and replacing y by $1/t$, we get

$$\begin{aligned}\lim_{z\to\infty}\big(M_{a,b}(x+z, y+z) - z\big) &= \lim_{t\to 0+}\frac{M_{a,b}(tx+1, ty+1) - 1}{t}\\ &= \frac{\partial}{\partial t}M_{a,b}(tx+1, ty+1)\Big|_{t=0}.\end{aligned}$$

We can apply the chain rule of the differentiation of composed functions. Applying Remark 2.9, one can directly check that in all cases

$$\begin{aligned}&\frac{\partial}{\partial t}M_{a,b}(tx+1, ty+1)\Big|_{t=0}\\ &\qquad = x\cdot\frac{\partial}{\partial x}M_{a,b}(x,y)\Big|_{(x,y)=(1,1)} + y\cdot\frac{\partial}{\partial y}M_{a,b}(x,y)\Big|_{(x,y)=(1,1)} = \frac{x+y}{2},\end{aligned}$$

which completes the proof. □

REMARK 3.2. This result is known also for any homogeneous, symmetric means (cf. [**1**], [**2**], [**9**] or the proof of Theorem 8.3).

LEMMA 3.3.([**16**]) *Suppose that a, b are real numbers. Then*
(3.1)

$$\lim_{x\to 0+} G_{a,b}(x,y) = \begin{cases} y, & \text{if } \min\{a,b\} > 0, \\ y \cdot 2^{-\frac{1}{\max\{a,b\}}}, & \text{if } \min\{a,b\} = 0 \text{ and } \max\{a,b\} > 0, \\ 0, & \text{if } \min\{a,b\} < 0 \text{ or } (a,b) = (0,0), \end{cases}$$

while

$$\lim_{x\to 0+} S_{a,b}(x,y) = \begin{cases} y \cdot e^{-\frac{1}{\mathcal{L}(a,b)}}, & \text{if } \min\{a,b\} > 0, \\ 0, & \text{if } \min\{a,b\} \le 0. \end{cases} \tag{3.2}$$

PROOF. The statement easily follows from the definition of our means. For (3.1) suppose first that $a, b > 0$, $a \neq b$. Then

$$\lim_{x\to 0+} G_{a,b}(x,y) = \lim_{x\to 0+} \Big(\frac{x^a + y^a}{x^b + y^b}\Big)^{\frac{1}{a-b}} = \big(y^{a-b}\big)^{\frac{1}{a-b}} = y.$$

For the case $a = b > 0$ we note first that $\lim_{x\to 0+} x^a \log x = 0$. Thus,

$$\begin{aligned} \lim_{x\to 0+} G_{a,a}(x,y) &= \lim_{x\to 0+} \exp\Big(\frac{x^a \log x + y^a \log y}{x^a + y^a}\Big) \\ &= \lim_{x\to 0+} \exp\Big(\frac{y^a \log y}{x^a + y^a}\Big) = \exp\Big(\frac{y^a \log y}{y^a}\Big) = y. \end{aligned}$$

Suppose now that $\min\{a,b\} = 0$ and $\max\{a,b\} > 0$, for example, $a > 0$ and $b = 0$. Then

$$\lim_{x\to 0+} G_{a,0}(x,y) = \lim_{x\to 0+} \Big(\frac{x^a + y^a}{2}\Big)^{\frac{1}{a}} = y \cdot 2^{-\frac{1}{a}},$$

as we stated. Finally, assume that $\min\{a,b\} < 0$, for example, $a < 0$ and $a < b$. Observe first that

$$\lim_{t\to 0+} \frac{t^a + 1}{t^b + 1} = \infty.$$

(If $b \geq 0$ then the numerator tends to ∞ and the denominator tends to 1, thus their ratio also tends to ∞. If $b < 0$ then we can apply L'Hospital's rule to obtain our result.) Consequently,

$$\lim_{x\to 0+} \frac{x^a + y^a}{x^b + y^b} = y^{a-b} \lim_{x\to 0+} \frac{(x/y)^a + 1}{(x/y)^b + 1} = y^{a-b} \lim_{t\to 0+} \frac{t^a + 1}{t^b + 1} = \infty.$$

Since the exponent $1/(a-b)$ is negative, we get that $\lim_{x\to 0+} G_{a,b}(x,y) = 0$, as we stated.

If $a = b < 0$ then $\lim_{x\to 0+} x^a \log x = -\infty$, therefore,

$$\lim_{x\to 0+} G_{a,a}(x,y) = \lim_{x\to 0+} \exp\Big(\frac{x^a \log x + y^a \log y}{x^a + y^a}\Big) = 0,$$

Since the case $(a,b) = (0,0)$ is trivial, the proof of (3.1) is complete.

To prove (3.2), suppose first that $a \neq b$. Then

$$\lim_{x\to 0+} S_{a,b}(x,y) = \lim_{x\to 0+}\Big(\frac{x^a - y^a}{a}\frac{b}{x^b - y^b}\Big)^{\frac{1}{a-b}} = \big(\frac{-y^a}{a}\frac{b}{-y^b}\big)^{\frac{1}{a-b}}$$
$$= y\Big(\frac{b}{a}\Big)^{\frac{1}{a-b}} = y\cdot e^{-\frac{1}{\mathcal{L}(a,b)}}.$$

Moreover,

$$\lim_{x\to 0+} S_{a,a}(x,y) = \lim_{x\to 0+}\exp\Big(-\frac{1}{a} + \frac{x^a\log x + y^a\log y}{x^a + y^a}\Big)$$
$$= \exp\Big(-\frac{1}{a} + \frac{-y^a\log y}{-y^a}\Big)$$
$$= \exp\Big(-\frac{1}{a} + \log y\Big) = y\cdot e^{-\frac{1}{a}} = y\cdot e^{-\frac{1}{\mathcal{L}(a,a)}}.$$

The proof of the case $\min\{a,b\} \le 0$ is similar to the one applied for the Gini setting, therefore, it is omitted. □

REMARK 3.4. Applying the homogeneity of our means, the previous lemma can be reformulated in the following way:

(3.3)
$$\lim_{x\to\infty}\frac{G_{a,b}(x,1)}{x} = \begin{cases} 1, & \text{if } \min\{a,b\} > 0,\\ 2^{-\frac{1}{\max\{a,b\}}}, & \text{if } \min\{a,b\} = 0 \text{ and } \max\{a,b\} > 0,\\ 0, & \text{if } \min\{a,b\} < 0 \text{ or } (a,b) = (0,0),\end{cases}$$

while

$$\lim_{x\to\infty}\frac{S_{a,b}(x,1)}{x} = \begin{cases} e^{-\frac{1}{\mathcal{L}(a,b)}}, & \text{if } \min\{a,b\} > 0,\\ 0, & \text{if } \min\{a,b\} \le 0.\end{cases} \tag{3.4}$$

REMARK 3.5. Due to the continuity of $M_{a,b}(x,y)$ in its variables, Lemma 3.3 can be extended in the following way: Suppose that a,b are real numbers. Then

(3.5)
$$\lim_{\substack{x\to 0+\\ y\to z}} G_{a,b}(x,y) = \begin{cases} z, & \text{if } \min\{a,b\} > 0,\\ z\cdot 2^{-\frac{1}{\max\{a,b\}}}, & \text{if } \min\{a,b\} = 0 \text{ and } \max\{a,b\} > 0,\\ 0, & \text{if } \min\{a,b\} < 0 \text{ or } (a,b) = (0,0),\end{cases}$$

while

$$\lim_{\substack{x\to 0+\\ y\to z}} S_{a,b}(x,y) = \begin{cases} z\cdot e^{-\frac{1}{\mathcal{L}(a,b)}}, & \text{if } \min\{a,b\} > 0,\\ 0, & \text{if } \min\{a,b\} \le 0.\end{cases} \tag{3.6}$$

LEMMA 3.6.([**16**]) *Assume that* $a,b > 1$. *Then*

$$\lim_{z\to\infty}\big(G_{a,b}(x,y+z) - z\big) = y \qquad (x,y \in \mathbb{R}_+).$$

PROOF. Using the homogeneity of $G_{a,b}$ and replacing z by $1/t$, we get

$$\lim_{z\to\infty}\big(G_{a,b}(x,y+z)-z\big)=\lim_{t\to 0}\frac{G_{a,b}(tx,ty+1)-1}{t}.$$

Due to Remark 3.5, the numerator on the right hand side goes to 0, hence we can apply L'Hospital's rule again to obtain the statement. By the chain rule, again,

$$\frac{\partial}{\partial t}G_{a,b}(tx,ty+1)\Big|_{t=0} = x\cdot\lim_{(x,y)\to(0,1)}\frac{\partial}{\partial x}G_{a,b}(x,y)+y\cdot\lim_{(x,y)\to(0,1)}\frac{\partial}{\partial y}G_{a,b}(x,y)=y.$$

□

3.3. Composed asymptotics

THEOREM 3.7.([**19**]) *Let $a,b\in\mathbb{R}$ be arbitrary. Then*

$$\lim_{t\to 1}\frac{M_{a,b}(t,1)-\frac{t+1}{2}}{(t-1)^2}=\begin{cases}\frac{a+b-1}{8}, & \textit{if } M=G,\\ \frac{a+b-3}{24}, & \textit{if } M=S.\end{cases}\tag{3.7}$$

PROOF. Let t be an arbitrary positive number. By Remark 2.9,

$$\frac{\partial}{\partial t}G_{a,b}(t,1)=\begin{cases}\frac{1}{a-b}\Big(a\frac{t^{a-1}}{t^a+1}-b\frac{t^{b-1}}{t^b+1}\Big)G_{a,b}(t,1), & \text{if } a\neq b,\\ \frac{t^{2a-1}+(a\log t+1)t^{a-1}}{(t^a+1)^2}G_{a,b}(t,1), & \text{if } a=b,\end{cases}$$

and a direct calculation shows that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}G_{a,b}(t,1)&=\frac{1}{a-b}\frac{\partial}{\partial t}G_{a,b}(t,1)\Big(a\frac{t^{a-1}}{t^a+1}-b\frac{t^{b-1}}{t^b+1}\Big)\\&\quad+\frac{1}{a-b}G_{a,b}(t,1)\Big(a\frac{(a-1)t^{a-2}-t^{2a-2}}{(t^a+1)^2}-b\frac{(b-1)t^{b-2}-t^{2b-2}}{(t^b+1)^2}\Big),\end{aligned}$$

while

$$\begin{aligned}\frac{\partial^2}{\partial t^2}G_{a,a}(t,1)&=\frac{\partial}{\partial t}G_{a,a}(t,1)\frac{(a\log t+1)t^{a-1}+t^{2a-1}}{(t^a+1)^2}\\&\quad+G_{a,a}(t,1)\Big(\frac{at^{a-2}+(a-1)(a\log t+1)t^{a-2}}{(t^a+1)^2}\\&\qquad-\frac{2at^{a-1}(a\log t\cdot t^{a-1}+t^{2a-1}+t^{a-1})}{(t^a+1)^3}\Big).\end{aligned}$$

It means that

$$\lim_{t\to 1}\frac{\partial}{\partial t}G_{a,b}(t,1)=\frac{1}{2}$$

and

$$\lim_{t\to 1}\frac{\partial^2}{\partial t^2}G_{a,b}(t,1)=\frac{a+b-1}{4}.$$

Consequently, applying L'Hospital's rule twice,

$$\begin{aligned}\lim_{t\to1}\frac{G_{a,b}(t,1)-\frac{t+1}{2}}{(t-1)^2} &= \lim_{t\to1}\frac{\frac{\partial}{\partial t}G_{a,b}(t,1)-\frac12}{2(t-1)}=\lim_{t\to1}\frac{\frac{\partial^2}{\partial t^2}G_{a,b}(t,1)}{2}\\ &= \frac{\frac{\partial^2}{\partial t^2}G_{a,b}(1,1)}{2}=\frac{a+b-1}{8},\end{aligned}$$

that is, we are ready with the proof for Gini means.

For Stolarsky means, we will follow the same method.

Let t be a positive number, different from 1. After some calculations, we get (from Remark 2.9) that

$$\frac{\partial}{\partial t}S_{a,b}(t,1)=\begin{cases}\frac{1}{a-b}\frac{(a-b)t^{a+b-1}-at^{a-1}+bt^{b-1}}{t^{a+b}-t^a-t^b+1}S_{a,b}(t,1), & \text{if } (a-b)ab\neq0,\\ \frac{t^{2a-1}-(a\log t+1)t^{a-1}}{(t^a-1)^2}S_{a,b}(t,1), & \text{if } a=b\neq0,\\ \frac{at^a\log t-t^a+1}{at\log t(t^a-1)}S_{a,b}(t,1), & \text{if } a\neq0,\ b=0.\end{cases}$$

(The case $a=b=0$ was covered by the first part of the proof.)

Applying L'Hospital's law twice for the fractions, one can easily check that

$$\lim_{t\to1}\frac{\partial}{\partial t}S_{a,b}(t,1)=\frac12.$$

We will also need the limit of the second derivative. Elementary calculations show that in the case $(a-b)ab\neq0$

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}S_{a,b}(t,1)\\ &=\frac{1}{a-b}\Big(-a\frac{t^{2a-2}+(a-1)t^{a-2}}{(t^a-1)^2}+b\frac{t^{2b-2}+(b-1)t^{b-2}}{(t^b-1)^2}\Big)S_{a,b}(t,1)\\ &\quad+\frac{1}{a-b}\frac{(a-b)t^{a+b-1}-at^{a-1}+bt^{b-1}}{t^{a+b}-t^a-t^b+1}\frac{\partial}{\partial t}S_{a,b}(t,1).\end{aligned}$$

Using L'Hospital's law twice for the fractions, again, and our previous results, we get that

$$\lim_{t\to1}\frac{\partial^2}{\partial t^2}S_{a,b}(t,1)=\frac{a+b-3}{12}.$$

After similar calculations one can get the same result in the other two cases as well.

The rest of the proof can be treated as it happened when proving the theorem for Gini means. □

For the next theorem we will need a simple statement.

LEMMA 3.8.([**19**]) *For any real number a*

$$\lim_{t\to\infty}\frac{\log(e^{at}+e^{-at})}{t}=|a|, \tag{3.8}$$

$$\lim_{t\to\infty}\frac{\log\left(\frac{e^{at}-e^{-at}}{a}\right)}{t}=|a|. \tag{3.9}$$

PROOF. After some transformations and applying L'Hospital's law,

$$\begin{aligned}\lim_{t\to\infty}\frac{\log(e^{at}+e^{-at})}{t}&=\lim_{t\to\infty}\frac{\log(2\frac{e^{at}+e^{-at}}{2})}{t}=\lim_{t\to\infty}\frac{\log\cosh at+\log 2}{t}\\&=\lim_{t\to\infty}a\frac{\sinh at}{\cosh at}=\lim_{t\to\infty}a\tanh at=a\operatorname{sign}a=|a|.\end{aligned}$$

The proof of the other statement is completely similar. □

THEOREM 3.9.([**19**]) *Let* $a,b\in\mathbb{R}$ *be arbitrary. Then*

$$\lim_{t\to\infty}\frac{\log M_{a,b}(e^t,e^{-t})}{t}=\mathcal{E}(a,b). \tag{3.10}$$

PROOF. Suppose first that $M=G$.
If $a\neq b$, then – applying Lemma 3.8 –

$$\lim_{t\to\infty}\frac{\log G_{a,b}(e^t,e^{-t})}{t}=\lim_{t\to\infty}\frac{\log\left(e^{at}+e^{-at}\right)^{\frac{1}{t}}-\log\left(e^{bt}+e^{-bt}\right)^{\frac{1}{t}}}{a-b}=\frac{|a|-|b|}{a-b}.$$

On the other hand, if $a=b$, then

$$\begin{aligned}\lim_{t\to\infty}\frac{\log G_{a,a}(e^t,e^{-t})}{t}&=\lim_{t\to\infty}\frac{e^{at}\log e^t+e^{-at}\log e^{-t}}{t(e^{at}+e^{-at})}=\lim_{t\to\infty}\tanh(at)\\&=\operatorname{sign}(a).\end{aligned}$$

Consider now the case $M=S$.
In the case $(a-b)ab\neq 0$ we can apply Lemma 3.8, again:

$$\lim_{t\to\infty}\frac{\log S_{a,b}(e^t,e^{-t})}{t}=\lim_{t\to\infty}\frac{\log\left(\frac{e^{at}-e^{-at}}{a}\right)^{\frac{1}{t}}-\log\left(\frac{e^{bt}-e^{-bt}}{b}\right)^{\frac{1}{t}}}{a-b}=\frac{|a|-|b|}{a-b}.$$

Suppose now that $a=b\neq 0$. Then

$$\lim_{t\to\infty}\frac{\log S_{a,a}(e^t,e^{-t})}{t}=\lim_{t\to\infty}\frac{-\frac{1}{a}+\frac{e^{at}t+e^{-at}t}{e^{at}-e^{-at}}}{t}=\lim_{t\to\infty}\coth at=\operatorname{sign}a.$$

Finally, if $a\neq 0$ and $b=0$, then we use Lemma 3.8:

$$\begin{aligned}\lim_{t\to\infty}\frac{\log S_{a,0}(e^t,e^{-t})}{t}&=\lim_{t\to\infty}\frac{\log\frac{e^{at}-e^{-at}}{2at}}{at}=\lim_{t\to\infty}\frac{\log\frac{e^{at}-e^{-at}}{a}-\log 2t}{at}\\&=\frac{\lim_{t\to\infty}\frac{\log\frac{e^{at}-e^{-at}}{a}}{t}}{a}=\frac{|a|}{a}=\frac{|a|-|0|}{a-0}.\end{aligned}$$

The case $a=b=0$ has been covered while handling the Gini case, so the proof is complete. □

THEOREM 3.10.([**19**]) *Let* $a, b \in \mathbb{R}$ *be arbitrary. Then*

$$\lim_{t\to\infty} \frac{\log \dfrac{t + \log G_{a,b}(e^t, e^{-t})}{t - \log G_{a,b}(e^t, e^{-t})}}{2t} = \mathcal{M}(a, b). \tag{3.11}$$

PROOF. For (3.11) it is enough to show that

$$\lim_{t\to\infty} \frac{\log\big(t + \log G_{a,b}(e^t, e^{-t})\big)}{2t} = \begin{cases} 0, & \text{if } a, b \geq 0, \\ 0, & \text{if } ab < 0, \\ \max\{a, b\}, & \text{if } a, b \leq 0, \end{cases} \tag{3.12}$$

and

$$\lim_{t\to\infty} \frac{\log\big(t - \log G_{a,b}(e^t, e^{-t})\big)}{2t} = \begin{cases} -\min\{a, b\}, & \text{if } a, b \geq 0, \\ 0, & \text{if } ab < 0, \\ 0, & \text{if } a, b \leq 0, \end{cases} \tag{3.13}$$

because (3.11) is just the difference of (3.12) and (3.13). In the proof of (3.12) we distinguish two cases. If either $ab < 0$ or $a, b \geq 0$, then $-1 < \mathcal{E}(a, b) \leq 1$. Thus, in view of (3.10), we get

$$\begin{aligned} \lim_{t\to\infty} \frac{\log\big(t + \log G_{a,b}(e^t, e^{-t})\big)}{2t} &= \lim_{t\to\infty} \frac{\log t + \log\big(1 + \log G_{a,b}(e^t, e^{-t})/t\big)}{2t} \\ &= \lim_{t\to\infty} \frac{\log t}{2t} + \lim_{t\to\infty} \frac{\log\big(1 + \mathcal{E}(a, b)\big)}{2t} = 0. \end{aligned}$$

Finally, we consider the case $a, b \leq 0$. We may assume that $b \leq a \leq 0$; we shall prove that

$$\lim_{t\to\infty} \frac{\log\big(t + \log G_{a,b}(e^t, e^{-t})\big)}{2t} = a. \tag{3.14}$$

If $b < a$, then a short calculation shows that

$$\frac{\log\big(t + \log G_{a,b}(e^t, e^{-t})\big)}{2t} = \frac{\log\left[\dfrac{1}{a-b} \log\left(\dfrac{1 + e^{2at}}{1 + e^{2bt}}\right)\right]}{2t}.$$

To determine the limit of the last expression, we can apply L'Hospital's rule. Computing the derivatives of the numerator and the denominator of the right hand side term, after some transformations, it turns out that their ratio is equal to

$$\frac{a + (a-b)e^{2bt} - be^{-2(a-b)t}}{(1 + e^{2at})(1 + e^{2bt})\left(\dfrac{\log(1 + e^{2at})}{e^{2at}} - \dfrac{\log(1 + e^{2bt})}{e^{2at}}\right)}. \tag{3.15}$$

In (3.15) the limit of the numerator equals a. With L'Hospital's rule it can immediately be checked that

$$\lim_{t\to\infty}\frac{\log(1+e^{2pt})}{e^{2qt}}=\begin{cases}1, & \text{if } p=q<0,\\ \log 2, & \text{if } p=q=0,\\ 0, & \text{if } p<q\leq 0,\end{cases}\tag{3.16}$$

which means that the the limit of (3.15) in each cases equals a, as we stated.

The case of the equal parameters in (3.12) can be treated similarly. The proof of (3.13) is completely analogous, therefore, it is omitted. $\square$

REMARK 3.11. Using the homogeneity of the Gini means, one can obtain the following more general statements

$$\lim_{(x,y)\to(t,t)}\frac{G_{a,b}(x,y)-\frac{x+y}{2}}{(x-y)^2}=\frac{a+b-1}{8t}\qquad(t>0),\tag{3.17}$$

$$\lim_{t\to\infty}\big(G_{a,b}(x^t,y^t)\big)^{\frac{1}{t}}=\sqrt{xy}\Big[\max\big\{\sqrt{x/y},\sqrt{y/x}\big\}\Big]^{\mathcal{E}(a,b)}\qquad(x,y>0),\tag{3.18}$$

and

$$\lim_{t\to\infty}\left(\frac{\log y^t-\log G_{a,b}(x^t,y^t)}{\log G_{a,b}(x^t,y^t)-\log x^t}\right)^{\frac{1}{t}}=(x/y)^{\mathcal{M}(a,b)}\qquad(x,y>0).\tag{3.19}$$

Observe that with $x=e$, $y=1/e$ and after taking the logarithm of both sides, the relations (3.18) and (3.19) reduce to (3.10) and (3.11), respectively. Conversely, using the homogeneity of Gini means, (3.17), (3.18), and (3.19) can easily be deduced from (3.7), (3.10), and (3.11).

CHAPTER 4

Extensions and applications of the Hermite-Hadamard inequality

4.1. Introduction

The so-called Hermite-Hadamard inequality [**28**] is one of the most investigated classical inequalities concerning convex functions. It reads as follows:

Theorem 4.A. *Let* $\mathcal{I} \subset \mathbb{R}$ *be an interval and* $f : \mathcal{I} \to \mathbb{R}$ *be a concave (convex) function. Then, for all subinterval* $[a, b] \subset \mathcal{I}$ *with non-empty interior,*

$$f\left(\frac{a+b}{2}\right) \underset{(\leq)}{\geq} \frac{1}{b-a}\int_a^b f(x)dx \underset{(\leq)}{\geq} \frac{f(a)+f(b)}{2} \tag{4.1}$$

holds.

An account on the history of this inequality can be found in [**41**]. Surveys on various generalizations and developments can be found in [**44**] and [**25**]. The description of best possible inequalities of Hadamard-Hermite type are due to Fink [**26**]. A generalization to higher-order convex function can be found in [**6**], while [**7**] offers a generalization for functions that are Beckanbach-convex with respect to a two dimensional linear space of continuous functions.

In this form (4.1) is valid only for functions that are purely convex or concave on their whole domain. We will see that under appropriate conditions the same inequalities can be stated for a much larger family of functions. It will turn out that the results, obtained for this situation, can be applied for the Gini and Stolarsky means. In this way, we will get new inequalities for these classes of two variable homogeneous means.

4.2. The weighted Hermite-Hadamard inequality for convex *or* concave functions

In this section we will extend Theorem 4.A, replacing the arithmetic mean by more general means, applying weight functions.

Given a positive, locally integrable weight function $\varrho : \mathcal{I} \to \mathbb{R}_+$, define the ϱ-mean of a and b by

$$M_\varrho(a, b) := \frac{\int_a^b x\varrho(x)dx}{\int_a^b \varrho(x)dx}.$$

Then the following statement holds:

LEMMA 4.1.([**14**]) *Let* $\mathfrak{I} \subset \mathbb{R}$ *be an interval,* $f : \mathfrak{I} \to \mathbb{R}$ *be a concave (convex) function and* $\varrho : \mathfrak{I} \to \mathbb{R}$ *a positive, locally integrable weight function. Then, for all subintervals* $[a, b] \subset \mathfrak{I}$ *with non-empty interior,*

$$
\begin{aligned}
f\left(M_\varrho(a,b)\right) \underset{(\leq)}{\geq} & \frac{1}{\int_a^b \varrho(x)dx}\int_a^b f(x)\varrho(x)dx \\
\underset{(\leq)}{\geq} & \frac{b-M_\varrho(a,b)}{b-a}f(a) + \frac{M_\varrho(a,b)-a}{b-a}f(b).
\end{aligned} \tag{4.2}
$$

PROOF. Suppose that f is concave over $\mathfrak{I}$ and let the linear function $e(x) := cx + d$ be a support line of the function f at the point $M_\varrho(a, b)$. Let

$$g(x) = \frac{f(b) - f(a)}{b-a} \cdot x + \frac{bf(a) - af(b)}{b-a}.$$

be the chord of f from $(a, f(a))$ to $(b, f(b))$. Then, applying the concavity,

$$e(x) \geq f(x) \geq g(x) \quad (x \in \mathfrak{I}),$$

that is,

$$\frac{\int_a^b e(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} \geq \frac{\int_a^b f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} \geq \frac{\int_a^b g(x)\varrho(x)dx}{\int_a^b \varrho(x)dx}. \tag{4.3}$$

After a calculation, we obtain that

$$
\begin{aligned}
\frac{\int_a^b e(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} &= \frac{\int_a^b (cx+d)\varrho(x)dx}{\int_a^b \varrho(x)dx} = c\frac{\int_a^b x\varrho(x)dx}{\int_a^b \varrho(x)dx} + d \\
&= cM_\varrho(a,b) + d = f(M_\varrho(a,b))
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\int_a^b g(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} &= \frac{\int_a^b \left(\frac{f(b)-f(a)}{b-a}x + \frac{bf(a)-af(b)}{b-a}\right)\varrho(x)dx}{\int_a^b \varrho(x)dx} \\
&= \frac{f(b)-f(a)}{b-a}M_\varrho(a,b) + \frac{bf(a)-af(b)}{b-a} \\
&= \frac{b-M_\varrho(a,b)}{b-a}f(a) + \frac{M_\varrho(a,b)-a}{b-a}f(b),
\end{aligned}
$$

which proves (4.2).
For convex functions the proof is similar. □

(It can immediately be seen that Theorem 4.A is a special case of Lemma 4.1 with $\varrho(x) \equiv 1$.)

REMARK. The primary motivation for the various extension of the Hermite-Hadamard inequality, such as those obtained by Zsolt Páles and the author [**15**] is to provide inequalities for the Gini and Stolarsky means.

Lemma 4.1 can also be applied, for instance, to give an upper and a lower bound for the Stolarsky mean $S_{r,s}(\xi,\eta)$. For, suppose that $f(x) = x^{r-s}$, $\varrho(x) := x^{s-1}$. Then $M_\varrho(\xi,\eta) = S_{s,s+1}(\xi,\eta)$, while $\frac{\int_\xi^\eta f(x)\varrho(x)dx}{\int_\xi^\eta \varrho(x)dx} = (S_{r,s}(\xi,\eta))^{r-s}$. In this way we can give bounds for the general Stolarsky mean in terms of a more special instance of it, namely, by the one where the difference of the parameters equals 1.

4.3. Odd and even functions with respect to a point

In the following we will encounter with functions showing two kinds of symmetry.

DEFINITION 4.2. Let $\mathfrak{I}$ be a real interval, $m \in \mathfrak{I}$. We say that the function $f : \mathfrak{I} \to \mathbb{R}$ is *odd with respect to the point* m, if $t \mapsto f(m+t) - f(m)$ is odd, that is,

$$f(m-t) + f(m+t) = 2f(m) \qquad (t \in (\mathfrak{I} - m) \cap (m - \mathfrak{I})), \tag{4.4}$$

while it is said to be *even with respect to the point* m, if $t \mapsto f(m+t)$ is even, that is,

$$f(m-t) = f(m+t) \qquad (t \in (\mathfrak{I} - m) \cap (m - \mathfrak{I})). \tag{4.5}$$

REMARK 4.3. Observe that when $\mathfrak{I}$ is closed and m is one its endpoints then $(\mathfrak{I} - m) \cap (m - \mathfrak{I})$ is either empty or the singleton $\{m\}$, therefore the condition $t \in (\mathfrak{I} - m) \cap (m - \mathfrak{I})$ does not mean any restriction on f.

For the integral of the product of odd and even functions with respect to the midpoint of the same interval, the following statement is true:

LEMMA 4.4.([**14**]) *Let* $g, h : [\alpha,\beta] \to \mathbb{R}$ *be integrable functions over* $[\alpha,\beta]$, g *be odd and* h *be even with respect to the point* $(\alpha+\beta)/2$. *Then*

$$\int_\alpha^\beta g(x)h(x)dx = g\left(\frac{\alpha+\beta}{2}\right)\int_\alpha^\beta h(x)dx.$$

PROOF. Let m denote the midpoint of $[\alpha,\beta]$. By splitting the integral at the point m and applying (4.4) and (4.5) for g and h, respectively, we get that

$$\begin{aligned}
&\int_\alpha^\beta g(x)h(x)dx\\
&\qquad = \int_\alpha^m g(x)h(x)dx + \int_m^\beta \big((2g(m) - g(2m-x)\big)h(2m-x)dx\\
&\qquad = \int_\alpha^m g(x)h(x)dx - \int_m^\alpha \big((2g(m) - g(y)\big)h(y)dy\\
&\qquad = \int_\alpha^m \big(g(x)h(x) + 2g(m)h(x) - g(x)h(x)\big)dx\\
&\qquad = 2g(m)\int_\alpha^m h(x)dx = g(m)\int_\alpha^\beta h(x)dx.
\end{aligned}$$

□

REMARK 4.5. As a special case of Lemma 4.4 with $h(x) \equiv 1$ we get the following statement:

$$\int_{m-\alpha}^{m+\alpha} g(x)dx = 2\alpha g(m)$$

for any positive α in $(\mathfrak{I} - m) \cap (m - \mathfrak{I})$.

4.4. The weighted Hermite-Hadamard inequality for convex-concave functions

THEOREM 4.6.([**14**]) *Let the function $f : \mathfrak{I} \to \mathbb{R}$ be odd with respect to the element $m \in \mathfrak{I}$, $\varrho : \mathfrak{I} \to \mathbb{R}$ a positive, locally integrable weight function, which is even with respect to m, and let $[a,b]$ be a subinterval of $\mathfrak{I}$ with non-empty interior. Then the following statement is valid:*
If f is convex over the interval $\mathfrak{I} \cap (-\infty, m]$ and concave over $\mathfrak{I} \cap [m,\infty)$, then

$$\begin{aligned}
f\left(M_\varrho(a,b)\right) &\underset{(\le)}{\ge} \frac{1}{\int_a^b \varrho(x)dx}\int_a^b f(x)\varrho(x)dx\\
&\underset{(\le)}{\ge} \frac{b - M_\varrho(a,b)}{b-a} f(a) + \frac{M_\varrho(a,b) - a}{b-a} f(b),
\end{aligned} \tag{4.6}$$

if $\dfrac{a+b}{2} \underset{(\le)}{\ge} m$. In (4.6) *the reversed inequalities are valid if f is concave over the interval $\mathfrak{I} \cap (-\infty, m]$ and convex over $\mathfrak{I} \cap [m,\infty)$.)*

PROOF. First we shall prove the left hand side inequality.
Suppose that $m \le (a+b)/2$, f is convex over the interval $\mathfrak{I}\cap(-\infty, m]$ and concave over $\mathfrak{I} \cap [m,\infty)$. (The other cases can be derived from this situation, applying one

of the transformations $f(m-x)$, $-f(x)$ and $-f(m-x)$.) Moreover, the case $m \leq a$ has no interest, so we may assume that $m > a$. Then, applying Lemma 4.4,

$$\begin{aligned} f\left(M_\varrho(a,b)\right) &= f\left(\frac{\int_a^{2m-a} x\varrho(x)dx + \int_{2m-a}^b x\varrho(x)dx}{\int_a^b \varrho(x)dx}\right) \\ &= f\left(\frac{m\int_a^{2m-a} \varrho(x)dx + \int_{2m-a}^b x\varrho(x)dx}{\int_a^b \varrho(x)dx}\right) \\ &= f\left(\frac{\int_a^{2m-a} \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot m + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot\frac{\int_{2m-a}^b x\varrho(x)dx}{\int_{2m-a}^b \varrho(x)dx}\right). \end{aligned}$$

Since $\frac{\int_{2m-a}^b x\varrho(x)dx}{\int_{2m-a}^b \varrho(x)dx} = M_\varrho(2m-a,b)$ – that is, a mean of $2m-a$ and b –, we get that

$$b \geq \frac{\int_{2m-a}^b x\varrho(x)dx}{\int_{2m-a}^b \varrho(x)dx} \geq 2m-a > m.$$

Therefore, both m and $M_\varrho(2m-a,b)$ belong to the concavity domain of f. Applying the concavity of f, we conclude that the previous expression is greater than or equal to

$$\begin{aligned} &\frac{\int_a^{2m-a} \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f(m) + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f\left(M_\varrho(2m-a,b)\right) \\ &\quad = \frac{f(m)\int_a^{2m-a} \varrho(x)dx}{\int_a^b \varrho(x)dx} + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f\left(M_\varrho(2m-a,b)\right). \end{aligned}$$

Using Lemma 4.4, again, one can substitute the first numerator of the right hand side phrase by $\int_a^{2m-a} f(x)\varrho(x)dx$. Summarizing the above calculations, we obtain

$$f\left(M_\varrho(a,b)\right) \geq \frac{\int_a^{2m-a} f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx}\cdot f\left(M_\varrho(2m-a,b)\right). \tag{4.7}$$

Since f is concave over the interval $[2m-a,b]$, we can apply the left hand side inequality of Lemma 4.1 and get that

$$f\left(M_\varrho(2m-a,b)\right) \geq \frac{1}{\int_{2m-a}^b \varrho(x)dx}\int_{2m-a}^b f(x)\varrho(x)dx.$$

Substituting this in (4.7) we obtain that

$$\begin{aligned} f\left(M_\varrho(a,b)\right) \geq & \frac{\int_a^{2m-a} f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} \\ & + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^b \varrho(x)dx} \cdot \frac{1}{\int_{2m-a}^b \varrho(x)dx} \int_{2m-a}^b f(x)\varrho(x)dx \\ = & \frac{\int_a^{2m-a} f(x)\varrho(x)dx + \int_{2m-a}^b f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx} = \frac{\int_a^b f(x)\varrho(x)dx}{\int_a^b \varrho(x)dx}, \end{aligned}$$

that is, the proof of the first inequality is complete.

To prove the second inequality in (4.6), it is enough to prove that

$$\int_a^b f(x)\varrho(x)dx \geq \frac{\int_a^b (b-x)\varrho(x)dx}{b-a} f(a) + \frac{\int_a^b (x-a)\varrho(x)dx}{b-a} f(b), \tag{4.8}$$

which is apparently equivalent to the second inequality in (4.6). We need the following simple statements:

$$\text{(A)} \quad f(m) \geq \frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b),$$
$$\text{(B)} \quad f(2m-a) \geq \frac{b-2m+a}{b-a} f(a) + \frac{2(m-a)}{b-a} f(b).$$

For (A), observe that f is concave over the interval $[m,b]$, containing the point $2m-a$. Thus,

$$f(2m-a) \geq \frac{b-2m+a}{b-m} f(m) + \frac{m-a}{b-m} f(b). \tag{4.9}$$

Substituting $2f(m) - f(a)$ for $f(2m-a)$ in (4.9), we obtain – after some transformations – (A).
Moreover, if we put in (4.9) $\frac{f(a)+f(2m-a)}{2}$ in place of $f(m)$, after rearranging the inequality, we get (B).
After these preparations, we are ready to prove (4.8). First, applying Lemma 4.4,

$$\begin{aligned} \int_a^b f(x)\varrho(x)dx &= \int_a^{2m-a} f(x)\varrho(x)dx + \int_{2m-a}^b f(x)\varrho(x)dx \\ &= f(m) \int_a^{2m-a} \varrho(x)dx + \int_{2m-a}^b f(x)\varrho(x)dx. \end{aligned}$$

In the first term on the right hand side, we may apply (A) for $f(m)$. Moreover, we can apply the right hand side inequality of Lemma 4.1 to the second term of the last expression, since f is concave in the interval $[2m-a, b]$:

$$\int_{2m-a}^{b} f(x)\varrho(x)dx \geq \frac{\int_{2m-a}^{b}(b-x)\varrho(x)dx}{b-2m+a} f(2m-a) + \frac{\int_{2m-a}^{b}(x-2m+a)\varrho(x)dx}{b-2m+a} f(b).$$

Applying (A) and (B) for $f(m)$ and $f(2m-a)$, we get that

$$\begin{aligned}
&\int_{a}^{b} f(x)\varrho(x)dx \\
&\quad\geq \left(\frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b)\right) \int_{a}^{2m-a} \varrho(x)dx \\
&\quad + \frac{\int_{2m-a}^{b}(b-x)\varrho(x)dx}{b-2m+a} \left(\frac{b-2m+a}{b-a} f(a) + \frac{2(m-a)}{b-a} f(b)\right) \\
&\quad + \frac{\int_{2m-a}^{b}(x-2m+a)\varrho(x)dx}{b-2m+a} f(b) \\
&\quad= \left[\frac{b-m}{b-a}\int_{a}^{2m-a} \varrho(x)dx + \frac{\int_{2m-a}^{b}(b-x)\varrho(x)dx}{b-a}\right] f(a) \\
&\quad + \left[\frac{m-a}{b-a}\int_{a}^{2m-a} \varrho(x)dx + \frac{2(m-a)}{b-a}\frac{\int_{2m-a}^{b}(b-x)\varrho(x)dx}{b-2m+a}\right. \\
&\quad \left. + \frac{\int_{2m-a}^{b}(x-2m+a)\varrho(x)dx}{b-2m+a}\right] f(b).
\end{aligned}$$

Finally, we will check that the coefficients of $f(a)$ and $f(b)$ are the desired ones. First, from Lemma 4.4 we get that $\int_{a}^{2m-a}(m-x)\varrho(x)dx = 0$. Thus,

$$\begin{aligned}
&\frac{b-m}{b-a}\int_{a}^{2m-a} \varrho(x)dx + \frac{\int_{2m-a}^{b}(b-x)\varrho(x)dx}{b-a} \\
&\quad= \frac{1}{b-a}\left(\int_{a}^{2m-a}(b-m)\varrho(x)dx + \int_{2m-a}^{b}(b-x)\varrho(x)dx\right) \\
&\quad= \frac{1}{b-a}\left(\int_{a}^{2m-a}(b-x)\varrho(x)dx + \int_{2m-a}^{b}(b-x)\varrho(x)dx\right) \\
&\quad= \frac{1}{b-a}\int_{a}^{b}(b-x)\varrho(x)dx.
\end{aligned}$$

This accounts for the coefficient of $f(a)$. Moreover,

$$\begin{aligned}\frac{2(m-a)}{b-a}\frac{\int_{2m-a}^{b}(b-x)\varrho(x)dx}{b-2m+a}&+\frac{\int_{2m-a}^{b}(x-2m+a)\varrho(x)dx}{b-2m+a}\\&=\int_{2m-a}^{b}\frac{2(b-x)(m-a)+(x-2m+a)(b-a)}{(b-a)(b-2m+a)}\varrho(x)dx\\&=\int_{2m-a}^{b}\frac{x-a}{b-a}\varrho(x)dx,\end{aligned}$$

while, with Lemma 4.4, again,

$$\frac{m-a}{b-a}\int_{a}^{2m-a}\varrho(x)dx=\int_{a}^{2m-a}\frac{m-a}{b-a}\varrho(x)dx=\int_{a}^{2m-a}\frac{x-a}{b-a}\varrho(x)dx.$$

Therefore, the coefficient of $f(b)$ equals

$$\int_{a}^{2m-a}\frac{x-a}{b-a}\varrho(x)dx+\int_{2m-a}^{b}\frac{x-a}{b-a}\varrho(x)dx=\frac{1}{b-a}\int_{a}^{b}(x-a)\varrho(x)dx,$$

as required. $\square$

In the special case when $\varrho(x)\equiv 1$, our statement can be read as follows:

COROLLARY 4.7.([**14**]) *Let the function $f:\mathfrak{I}\to\mathbb{R}$ be odd with respect to the element $m\in\mathfrak{I}$, and let $[a,b]$ be a subinterval of $\mathfrak{I}$ with non-empty interior. Then the following statement is valid:*
If f is convex over the interval $\mathfrak{I}\cap(-\infty,m]$ and concave over $\mathfrak{I}\cap[m,\infty)$, then

$$f\left(\frac{a+b}{2}\right)\underset{(\leq)}{\geq}\frac{1}{b-a}\int_{a}^{b}f(x)dx\underset{(\leq)}{\geq}\frac{f(a)+f(b)}{2}, \tag{4.10}$$

if $\frac{a+b}{2}\underset{(\leq)}{\geq}m$. In (4.6) *the reversed inequalities are valid if f is concave over the interval $\mathfrak{I}\cap(-\infty,m]$ and convex over $\mathfrak{I}\cap[m,\infty)$.)*

That is, in this case our inequalities are literally the same as those in Theorem 4.A.

4.5. An application for Gini and Stolarsky means

Our aim is to apply the results in Corollary 4.7 for Gini and Stolarsky means. For this purpose we will show that, for fixed positive x,y, the function

$$\mu_{x,y}:\mathbb{R}\to\mathbb{R},\quad t\mapsto\log M_{t,t}(x,y) \tag{4.11}$$

satisfies the assumptions of Corollary 4.7.

LEMMA 4.8.([**18**]) *Let x,y be arbitrary positive numbers. Then the function $\mu_{x,y}$ defined in* (4.11) *has the following properties:*

(i)

$$\mu_{x,y}(t) + \mu_{x,y}(-t) = 2\mu_{x,y}(0) \quad (t \in \mathbb{R}),$$

(ii) $\mu_{x,y}$ *is convex over* $\mathbb{R}_-$ *and concave over* $\mathbb{R}_+$.

PROOF. (i) For Gini means:

$$\begin{aligned}\mu_{x,y}(t) + \mu_{x,y}(-t) &= \frac{x^t \log x + y^t \log y}{x^t + y^t} + \frac{x^{-t} \log x + y^{-t} \log y}{x^{-t} + y^{-t}} \\ &= \frac{x^t \log x + y^t \log y}{x^t + y^t} + \frac{y^t \log x + x^t \log y}{y^t + x^t} \\ &= \frac{x^t \log(xy) + y^t \log(xy)}{x^t + y^t} = \log(xy) = 2\mu_{x,y}(0),\end{aligned}$$

while for Stolarsky means – assuming that $t \neq 0$ –

$$\begin{aligned}\mu_{x,y}(t) + \mu_{x,y}(-t) &= -\frac{1}{t} + \frac{x^t \log x - y^t \log y}{x^t - y^t} + \frac{1}{t} + \frac{x^{-t} \log x - y^{-t} \log y}{x^{-t} - y^{-t}} \\ &= \frac{x^t \log x - y^t \log y}{x^t - y^t} + \frac{y^t \log x - x^t \log y}{y^t - x^t} \\ &= \frac{x^t \log(xy) - y^t \log(xy)}{x^t - y^t} = \log(xy) = 2\mu_{x,y}(0).\end{aligned}$$

(ii) If $x = y$, then $\mu_{x,y}(t) = x$ for all $t \in \mathbb{R}$, hence $\mu_{x,y}$ is convex-concave everywhere. Therefore, we may assume that $x \neq y$.

In the case of Gini means,

$$t^3 \mu''_{x,y}(t) = -\frac{x^t y^t (\log x^t - \log y^t)^3 (x^t - y^t)}{(x^t + y^t)^3}.$$

The sign of $x^t - y^t$ is the same as that of $\log x^t - \log y^t$, therefore, $t^3 \mu''_{x,y}(t) \geq 0$ for all $t \in \mathbb{R}$. Thus, $\mu_{x,y}$ is convex over $\mathbb{R}_-$ and concave over $\mathbb{R}_+$.

In the setting of Stolarsky means, we have that

$$\begin{aligned}t^3 \mu''_{x,y}(t) &= -2 + \frac{x^t y^t (\log x^t - \log y^t)^3 (x^t + y^t)}{(x^t - y^t)^3} \\ &= -2 \left(1 - \frac{S^2_{0,0}(x^t, y^t) S_{2,1}(x^t, y^t)}{S^3_{1,0}(x^t, y^t)} \right).\end{aligned}$$

In view of Corollary 2.2, it follows that $t^3 \mu''_{x,y}(t) \geq 0$ for all $t \in \mathbb{R}$. Therefore, $\mu_{x,y}$ is convex over $\mathbb{R}_-$ and concave over $\mathbb{R}_+$ in this case, too. $\square$

As a consequence of Lemma 4.8 and Corollary 4.7, we can provide a lower and an upper estimate for $M_{a,b}$ in terms of the means $M_{\frac{a+b}{2},\frac{a+b}{2}}$ and $\sqrt{M_{a,a} \cdot M_{b,b}}$.

THEOREM 4.9.([**18**]) *Let a, b be real numbers so that $a+b \underset{(\leq)}{\geq} 0$. Then*

$$M_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} M_{a,b}(x,y) \underset{(\leq)}{\geq} \sqrt{M_{a,a}(x,y)M_{b,b}(x,y)}$$

holds for any positive numbers x, y.

PROOF. Let x, y be fixed positive numbers. By Lemma 4.8, the function $\mu_{x,y}$ is odd with respect to $m = 0$ and is convex (concave) on $\mathbb{R}_-$ (on $\mathbb{R}_+$). Therefore, Corollary 4.7 can be applied to $f := \mu_{x,y}$. Then

$$\mu_{x,y}\left(\frac{a+b}{2}\right) \underset{(\leq)}{\geq} \frac{1}{a-b}\int_b^a \mu_{x,y}(t)dt \underset{(\leq)}{\geq} \frac{\mu_{x,y}(a)+\mu_{x,y}(b)}{2}$$

if $\frac{a+b}{2} \underset{(\leq)}{\geq} 0$. Thus, by the definition of $\mu_{x,y}$ and in view of Lemma 2.B, the following inequality holds:

$$\log M_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} \log M_{a,b}(x,y) \underset{(\leq)}{\geq} \frac{\log M_{a,a}(x,y)+\log M_{b,b}(x,y)}{2}$$

if $a+b \underset{(\leq)}{\geq} 0$. Applying the exponential function to this inequality, we get that

$$M_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} M_{a,b}(x,y) \underset{(\leq)}{\geq} \sqrt{M_{a,a}(x,y)M_{b,b}(x,y)}$$

if $a+b \underset{(\leq)}{\geq} 0$. Hence the stated inequalities follow in the Gini and Stolarsky means setting, respectively. □

4.6. A variant of the Hermite-Hadamard inequality

In the sequel, we shall need a new variant of (4.1), where the left hand side is replaced by a certain weighted arithmetic mean of $f(a)$ and $f(b)$. To state this result, we recall the notions of the positive and negative part functions defined, for $x \in \mathbb{R}$, by

$$x^+ := \max\{x, 0\} = \frac{|x|+x}{2} \qquad \text{and} \qquad x^- := \max\{-x, 0\} = \frac{|x|-x}{2}.$$

LEMMA 4.10.([**19**]) *Let $f : \mathcal{I} \to \mathbb{R}$ be odd with respect to an element $m \in \mathcal{I}$, furthermore, suppose that f is increasing. Then, for any interval $[a, b] \subset \mathcal{I}$,*

(4.12)

$$\big((b-m)^+ - (a-m)^+\big)f(b) + \big((a-m)^- - (b-m)^-\big)f(a) \underset{(\leq)}{\geq} \int_a^b f(x)dx$$

holds if $\frac{a+b}{2} \underset{(\leq)}{\geq} m$.

PROOF. We consider first the case $\frac{a+b}{2} \geq m$ and assume that $a \leq m$. Then $m \leq 2m-a \leq b$. Applying that f is odd with respect to the point m, Remark 4.5 and using also the monotonicity, we have

$$\begin{aligned}
\int_a^b f(t)dt &= \int_a^m f(t)dt + \int_m^{2m-a} f(t)dt + \int_{2m-a}^b f(t)dt \\
&= 2(m-a)f(m) + \int_{2m-a}^b f(t)dt \\
&= \big((m-a)f(a) + (m-a)f(2m-a)\big) + \int_{2m-a}^b f(t)dt \\
&\leq (m-a)f(a) + (m-a)f(b) + (a+b-2m)f(b) \\
&= (b-m)f(b) + (m-a)f(a) \\
&= \big((b-m)^+ - (a-m)^+\big)f(b) + \big((a-m)^- - (b-m)^-\big)f(a).
\end{aligned}$$

If $m \leq a$, then, using only the monotonicity of f, we get

$$\begin{aligned}
\int_a^b f(t)dt \leq (b-a)f(b) = \big((b-m)^+ - (a-m)^+\big)&f(b) \\
+ \big((a-m)^- - (b-m)^-\big)&f(a).
\end{aligned}$$

In the case $\frac{a+b}{2} \leq m$, a similar argument completes the proof. $\square$

REMARK 4.11. Using the monotonicity of f, it is elementary to see that the left hand side of (4.12) can equivalently be written also in the form

$$\min\big\{(b-m)f(b) + (m-a)f(a)\ ,\ (b-a)f(b)\big\} \tag{4.13}$$

if $\frac{a+b}{2} \geq m$. Indeed, if $a \leq m$, then

$$\begin{aligned}
&\big((b-m)^+ - (a-m)^+\big)f(b) + \big((a-m)^- - (b-m)^-\big)f(a) \\
&\qquad = (b-m)f(b) + (m-a)f(a)
\end{aligned}$$

and $(b-m)f(b) + (m-a)f(a) \leq (b-a)f(b)$ which results (4.13). In the case $m \leq a$, we have that

$$\begin{aligned}
&\big((b-m)^+ - (a-m)^+\big)f(b) + \big((a-m)^- - (b-m)^-\big)f(a) \\
&\qquad = (b-a)f(b)
\end{aligned}$$

and $(b-m)f(b) + (m-a)f(a) \geq (b-a)f(b)$ which also leads to (4.13). Analogously, if $\frac{a+b}{2} \geq m$, then the left hand side of (4.12) can be replaced by

$$\max\big\{(b-m)f(b) + (m-a)f(a)\ ,\ (b-a)f(a)\big\}.$$

Now we apply the the result of Lemma 4.10 for Gini and Stolarsky means to replace the left hand side in Theorem 4.9 by a weighted geometric mean of $M_{a,a}$ and $M_{b,b}$. Due to the symmetry of $M_{a,b}(x,y)$ in its parameters, it suffices to handle the situation $b \le a$ in the sequel.

THEOREM 4.12.([**19**]) *If* $b < a$ *and* $a+b \underset{(\le)}{\ge} 0$, *then, for all positive* x, y,

$$\left(M_{a,a}(x,y)\right)^{\frac{a^+-b^+}{a-b}} \left(M_{b,b}(x,y)\right)^{\frac{b^--a^-}{a-b}} \underset{(\le)}{\ge} M_{a,b}(x,y). \tag{4.14}$$

PROOF. Using the result of F. Qi [**52**], we get that the function $g_{x,y}$ defined in

$$m_{x,y} : \mathbb{R} \to \mathbb{R}_+, \quad t \mapsto \log M_{t,t}(x,y) \tag{4.15}$$

is strictly increasing for all fixed $x, y > 0$ with $x \neq y$. (This statement also follows from the comparison theorems of M – see later.) Applying Lemma 2.B, we obtain that

$$\log M_{a,b}(x,y) = \frac{1}{a-b}\int_b^a m_{x,y}(t)dt \qquad (x, y \in \mathbb{R}_+). \tag{4.16}$$

To prove (4.14), we can restrict ourselves to the upper direction in the inequalities; we could use an analogous argument for the reversed signs. Then, in view of Lemma 4.10,

$$\begin{aligned}
\log M_{a,b}(x,y) &= \frac{1}{a-b}\int_b^a m_{x,y}(t)dt \\
&\le \frac{a^+-b^+}{a-b} m_{x,y}(a) + \frac{b^--a^-}{a-b} m_{x,y}(b) \\
&= \frac{a^+-b^+}{a-b} \log M_{a,a}(x,y) + \frac{b^--a^-}{a-b} \log M_{b,b}(x,y),
\end{aligned}$$

which is equivalent to the desired inequality (4.14). □

Combining the results of Theorem 4.9 and Theorem 4.12, we get the following lower and upper estimates for the Gini/Stolarsky mean $M_{a,b}$ in terms of a weighted geometric mean of $M_{a,a}$ and $M_{b,b}$:

COROLLARY 4.13.([**19**]) *For all real* a, b *with* $b \le a$, $(a,b) \neq (0,0)$ *and for all positive* x, y,

(i) $\left(M_{a,a}(x,y)\right)^{\frac{1}{2}}\left(M_{b,b}(x,y)\right)^{\frac{1}{2}} \le M_{a,b}(x,y) \le M_{a,a}(x,y)$, *if* $0 \le b \le a$,

(ii) $\left(M_{a,a}(x,y)\right)^{\frac{1}{2}}\left(M_{b,b}(x,y)\right)^{\frac{1}{2}} \le M_{a,b}(x,y)$
$\le \left(M_{a,a}(x,y)\right)^{\frac{a}{a-b}}\left(M_{b,b}(x,y)\right)^{\frac{-b}{a-b}}$, *if* $0 \le -b \le a$,

(iii) $\left(M_{a,a}(x,y)\right)^{\frac{1}{2}}\left(M_{b,b}(x,y)\right)^{\frac{1}{2}} \ge M_{a,b}(x,y)$
$\ge \left(M_{a,a}(x,y)\right)^{\frac{a}{a-b}}\left(M_{b,b}(x,y)\right)^{\frac{-b}{a-b}}$, *if* $0 \le a \le -b$,

(iv) $\left(M_{a,a}(x,y)\right)^{\frac{1}{2}}\left(M_{b,b}(x,y)\right)^{\frac{1}{2}} \ge M_{a,b}(x,y) \ge M_{b,b}(x,y)$, *if* $b \le a \le 0$.

PROOF. All the left hand side inequalities follow from the right hand side inequality of Theorem 4.9. The right hand side inequalities in (i), (ii), (iii), and (iv) are consequences of the inequality (4.14). $\square$

CHAPTER 5

Comparison of Gini means

5.1. Introduction

The comparison problem of two variable Gini means on $\mathbb{R}_+$ was solved by Páles [**48**]. The main result of that paper reads as follows.

Theorem 5.A. *Suppose that* $a, b, c, d \in \mathbb{R}$, $(a-b)(c-d) \neq 0$. *Then*

$$G_{a,b}(x,y) \le G_{c,d}(x,y) \qquad (x, y \in \mathbb{R}_+) \tag{5.1}$$

holds if and only if

$$\text{(i)} \qquad a+b \le c+d,$$

$$\text{(ii)} \quad \begin{cases} \min\{a,b\} \le \min\{c,d\}, & \text{if } \min\{a,b,c,d\} \ge 0, \\ \max\{a,b\} \le \max\{c,d\}, & \text{if } \max\{a,b,c,d\} \le 0, \\ \dfrac{|a|-|b|}{a-b} \le \dfrac{|c|-|d|}{c-d}, & \text{if } \min\{a,b,c,d\} < 0 \\ & \quad < \max\{a,b,c,d\}. \end{cases} \tag{5.2}$$

This theorem does not offer conditions when $(a-b)(c-d) = 0$. In order to cover this case as well, in [**16**] we extended Theorem 5.A. In the meantime, the theorem obtained a new appearance, due to the auxiliary functions $\mathcal{E}$ and $\mathcal{M}$, introduced in Definition 2.4. Our theorem has the following form:

THEOREM 5.1.([**15**]) *Let* $a, b, c, d \in \mathbb{R}$ *be arbitrary parameters. Then*

$$G_{a,b}(x,y) \le G_{c,d}(x,y) \tag{5.3}$$

holds for all positive x *and* y *if and only if* a, b, c, d *satisfy the following three conditions:*

$$a+b \le c+d, \tag{5.4}$$

$$\mathcal{E}(a,b) \le \mathcal{E}(c,d), \tag{5.5}$$

$$\mathcal{M}(a,b) \le \mathcal{M}(c,d). \tag{5.6}$$

In the sequel we will present a new, stand-alone proof for Theorem 5.1, therefore, the original method will not be detailed – we note here only that the main idea in [**16**] was that the (equal) parameters could be approached by appropriate sequences of different parameters. The process is analogous to that in the next chapter, applied for the extension of the comparison theorem of Stolarsky means.

5.2. On certain directional derivatives

DEFINITION 5.2. Let the positive numbers x and y be fixed. Then define the function $\mathcal{G}_{x,y}$ as follows:

$$\mathcal{G}_{x,y} : \mathbb{R}^2 \to \mathbb{R}, \quad (a,b) \mapsto \mathcal{G}_{x,y}(a,b) := G_{a,b}(x,y).$$

The importance of the inequalities established in Corollary 4.13 will be clear when we apply them to investigate the directional derivatives of the function $\mathcal{G}_{x,y}$.

LEMMA 5.3.([**19**]) *Suppose that $b \le a$ and let the positive numbers x and y be fixed. The directional derivative of the function $\mathcal{G}_{x,y}$ in direction $d = (d_1, d_2)$ is nonnegative at the point (a,b) if and only if*

$$\text{(5.7)} \quad \begin{cases} \big(G_{a,a}(x,y)\big)^{d_1}\big(G_{b,b}(x,y)\big)^{-d_2} \ge \big(G_{a,b}(x,y)\big)^{d_1-d_2}, & \textit{if} \quad a \ne b, \\ d_1 + d_2 \ge 0, & \textit{if} \quad a = b. \end{cases}$$

PROOF. The directional derivative of $\mathcal{G}_{x,y}$ at (a,b) in direction d can be calculated as follows:

$$\partial_d \mathcal{G}_{x,y}(a,b) = d_1 \cdot \partial_1 \mathcal{G}_{x,y}(a,b) + d_2 \cdot \partial_2 \mathcal{G}_{x,y}(a,b).$$

(Here ∂_i stands for the partial derivative with respect to the ith variable.) An easy calculation shows that, in the case $a \ne b$,

$$\partial_1 \mathcal{G}_{x,y}(a,b) = \frac{G_{a,b}(x,y)}{a-b} \log \frac{G_{a,a}(x,y)}{G_{a,b}(x,y)}$$

and

$$\partial_2 \mathcal{G}_{x,y}(a,b) = \frac{G_{a,b}(x,y)}{a-b} \log \frac{G_{a,b}(x,y)}{G_{b,b}(x,y)}.$$

Therefore,

$$\partial_d \mathcal{G}_{x,y}(a,b) = \frac{G_{a,b}(x,y)}{a-b} \log \frac{\big(G_{a,a}(x,y)\big)^{d_1}\big(G_{b,b}(x,y)\big)^{-d_2}}{\big(G_{a,b}(x,y)\big)^{d_1-d_2}}.$$

By the assumption $a > b$, this expression is nonnegative if and only if

$$\big(G_{a,a}(x,y)\big)^{d_1}\big(G_{b,b}(x,y)\big)^{-d_2} \ge \big(G_{a,b}(x,y)\big)^{d_1-d_2},$$

that is, the first inequality in (5.7) holds. In the case $a = b$, we have that

$$\partial_1 \mathcal{G}_{x,y}(a,a) = \partial_2 \mathcal{G}_{x,y}(a,a) = \Big(\frac{\log x - \log y}{x^a + y^a}\Big)^2 (xy)^a G_{a,a}(x,y).$$

Thus, it is easily seen that $\partial_d \mathcal{G}_{x,y}(a,a) \ge 0$ if and only if $d_1 + d_2 \ge 0$. $\square$

Combining Lemma 5.3 and Corollary 4.13, we get the following result.

COROLLARY 5.4.([**19**]) *For $b \le a$ with $(a,b) \neq (0,0)$, define the vectors $u_{a,b}$ and $v_{a,b}$ in the following way:*

(5.8)
$$u_{a,b} = \begin{cases} (1,0), & \text{if } 0 \le b \le a, \\ \left(1, \frac{b}{a}\right), & \text{if } 0 < -b < a, \\ (1,-1), & \text{if } 0 < a \le -b, \\ (1,-1), & \text{if } b \le a \le 0, \end{cases} \qquad v_{a,b} = \begin{cases} (-1,1), & \text{if } 0 \le b \le a, \\ (-1,1), & \text{if } 0 < -b \le a, \\ \left(\frac{a}{b}, 1\right), & \text{if } 0 < a < -b, \\ (0,1), & \text{if } b \le a \le 0. \end{cases}$$

Then the maps $(a,b) \mapsto u_{a,b}$ and $(a,b) \mapsto v_{a,b}$ are continuous on the domain indicated, furthermore, the directional derivative of the function $\mathcal{G}_{x,y}$, at the point (a,b) is nonnegative in the directions $u_{a,b}$ and $v_{a,b}$ for all fixed positive numbers x, y.

PROOF. In view of Lemma 5.3, we have to check (5.7) for the vectors $d = u_{a,b}$ and $d = v_{a,b}$. By substituting these values of d into (5.7), it is easily seen that the conditions obtained are equivalent to the inequalities established in Corollary 4.13. □

The following lemma offers a useful sufficient condition for the comparison of two Gini means.

LEMMA 5.5.([**19**]) *Let the positive numbers x and y be fixed and let (a,b), (c,d) be two arbitrary points in $\mathbb{R}^2$. Suppose that the directional derivative of $\mathcal{G}_{x,y}$ is nonnegative in the direction $(c-a, d-b)$ at any point of the segment*

$$[(a,b),(c,d)] = \big\{ \big(a + t(c-a),\ b + t(d-b)\big) \mid t \in [0,1] \big\}.$$

Then

$$G_{a,b}(x,y) \le G_{c,d}(x,y).$$

PROOF. Define

$$\begin{aligned} \varphi_{x,y}(t) &:= G_{a+t(c-a),b+t(d-b)}(x,y) \\ &= \mathcal{G}_{x,y}\big(a+t(c-a), b+t(d-b)\big) \qquad (t \in [0,1]). \end{aligned}$$

Then φ is differentiable on $\mathbb{R}$, and

$$\begin{aligned} \varphi'_{x,y}(t) &= \partial_1 \mathcal{G}_{x,y}\big(a+t(c-a), b+t(d-b)\big) \cdot (c-a) \\ &\qquad + \partial_2 \mathcal{G}_{x,y}\big(a+t(c-a), b+t(d-b)\big) \cdot (d-b) \\ &= \partial_{(c-a,d-b)} \mathcal{G}_{x,y}\big(a+t(c-a), b+t(d-b)\big) \ge 0 \end{aligned}$$

by our assumption. Thus $\varphi_{x,y}$ is nondecreasing on $[0,1]$, whence

$$G_{a,b}(x,y) = \varphi_{x,y}(0) \le \varphi_{x,y}(1) = G_{c,d}(x,y)$$

follows. □

5.3. A new proof of the comparison theorem for Gini means

(Necessity.) Assume that (5.3) holds for all $x, y > 0$. Then, substituting $x = t$, $y = 1$, we get that

$$\frac{G_{a,b}(t,1) - \frac{t+1}{2}}{(t-1)^2} \leq \frac{G_{c,d}(t,1) - \frac{t+1}{2}}{(t-1)^2} \qquad (t > 0, t \neq 1).$$

Now, taking the limit $t \to 1$ and using (3.7) of Theorem 3.7, we obtain

$$\frac{a+b-1}{8} \leq \frac{c+d-1}{8},$$

which is equivalent to (5.4).

The substitution $x = e^t$, $y = e^{-t}$ results from (5.3) that

$$\frac{\log G_{a,b}(e^t, e^{-t})}{t} \leq \frac{\log G_{c,d}(e^t, e^{-t})}{t} \qquad (t > 0).$$

Therefore, taking the limit $t \to \infty$ and using Theorem 3.9, we get the necessity of (5.5).

Finally, again with the substitution $x = e^t$, $y = e^{-t}$ we have (by the strict mean value property of the Gini means) that

$$-t < \log G_{a,b}(e^t, e^{-t}) \leq \log G_{c,d}(e^t, e^{-t}) < t \qquad (t > 0),$$

therefore,

$$0 < \frac{t + \log G_{a,b}(e^t, e^{-t})}{t - \log G_{a,b}(e^t, e^{-t})} \leq \frac{t + \log G_{c,d}(e^t, e^{-t})}{t - \log G_{c,d}(e^t, e^{-t})} \qquad (t > 0).$$

Hence, in view of Theorem 3.10,

$$\begin{aligned} \mathcal{M}(a,b) &= \lim_{t\to\infty} \frac{\log \dfrac{t + \log G_{a,b}(e^t, e^{-t})}{t - \log G_{a,b}(e^t, e^{-t})}}{2t} \leq \lim_{t\to\infty} \frac{\log \dfrac{t + \log G_{c,d}(e^t, e^{-t})}{t - \log G_{c,d}(e^t, e^{-t})}}{2t} \\ &= \mathcal{M}(c,d) \end{aligned}$$

and thus the necessity of (5.6) is also proved.

(Sufficiency.) Assume that conditions (5.4), (5.5), and (5.6) hold and let x and y be two arbitrary positive numbers throughout the proof.

In order to distinguish various cases according to the positions of the points (a, b) and (c, d), consider the following five subsets of the half-plain $H := \{(s,t) \mid t \leq s\}$:

$$\begin{aligned} &H_1 := \{(s,t) \mid t \leq s \leq 0,\ (s,t) \neq (0,0)\}, \quad &&H_2 := \{(s,t) \mid 0 < s < -t\}, \\ &H_3 := \{(s,t) \in H \mid s + t = 0\}, && \\ &H_4 := \{(s,t) \mid 0 \leq -t < s\}, &&H_5 := \{(s,t) \mid 0 < t \leq s\}. \end{aligned}$$

Evidently $H = H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$, furthermore, $H_1 \cup H_2 = \{(s,t) \in H \mid s + t < 0\}$, $H_4 \cup H_5 = \{(s,t) \in H \mid s + t > 0\}$, and $H_i \cap H_j = \emptyset$ for all $1 \leq i < j \leq 5$.

By the symmetry of the parameters, we can assume that $a \geq b$ and $c \geq d$, that is, $(a,b),(c,d) \in H$. Due to the pairwise disjointness of the sets H_1, H_2, H_3, H_4, and H_5, there exist unique indices i,j such that $(a,b) \in H_i$ and $(c,d) \in H_j$. Applying conditions (5.4), (5.5), and (5.6), we prove first that $i \leq j$ holds. If $(a,b) \in H_1$, then there is nothing to prove. If $(a,b) \in H_2$, then $a > 0$, hence $\mathcal{E}(a,b) > -1$. Thus, condition (5.5) yields that $\mathcal{E}(c,d) > -1$, hence, (c,d) cannot be in H_1, i.e., it belongs to $H_2 \cup H_3 \cup H_4 \cup H_5$. In the case when $(a,b) \in H_3$, we have that $a+b=0$. Hence, by (5.4), it follows that $c+d \geq 0$, i.e., $(c,d) \in H_3 \cup H_4 \cup H_5$. If $(a,b) \in H_4$, then $a+b>0$. Thus (5.4) yields that $c+d>0$, which is equivalent to $(c,d) \in H_4 \cup H_5$. Finally, if (a,b) is in H_5, then $0 < b$. Therefore, $\mathcal{M}(a,b) = \min\{a,b\} > 0$, whence, in view of condition (5.6), we get that $\mathcal{M}(c,d) > 0$. Thus (c,d) has to be an element of H_5, too.

In the rest of the proof, we may assume that $(a,b) \in H_i$ and $(c,d) \in H_j$ for some indices $1 \leq i \leq j \leq 5$. Our aim is to prove that

$$G_{a,b}(x,y) \leq G_{p,q}(x,y) \leq G_{c,d}(x,y), \tag{5.9}$$

where (p,q) is defined by distinguishing five cases *(i)*-*(v)* according to the possible positions of the points (a,b) and (c,d) in H as follows:

$$(p,q) := \begin{cases} \textit{(i)} \quad (0,0), & \text{if} \quad (a,b) \in H_1 \cup H_2 \cup H_3, \\ & \qquad (c,d) \in H_3 \cup H_4 \cup H_5; \\ \textit{(ii)} \quad (c, a+b-c), & \text{if} \quad (a,b) \in H_1, \\ & \qquad (c,d) \in H_1; \\ \textit{(iii)} \quad \left(\frac{a+b}{c+d}c, \frac{a+b}{c+d}d\right), & \text{if} \quad (a,b) \in H_1 \cup H_2, \\ & \qquad (c,d) \in H_2; \\ \textit{(iv)} \quad \left(\frac{c+d}{a+b}a, \frac{c+d}{a+b}b\right), & \text{if} \quad (a,b) \in H_4, \\ & \qquad (c,d) \in H_4 \cup H_5; \\ \textit{(v)} \quad (c+d-b, b), & \text{if} \quad (a,b) \in H_5, \\ & \qquad (c,d) \in H_5. \end{cases}$$

For, we will show that at any point of the two segments $[(a,b),(p,q)]$ and $[(p,q),(c,d)]$ the directional derivatives of $\mathcal{G}_{x,y}$, defined by the vectors $(p-a, q-b)$ and $(c-p, d-q)$, respectively, are nonnegative. Thus, in view of Lemma 5.5, the desired inequality (5.9) results, which demonstrates the statement.

Case (i). We have to show that $G_{a,b}(x,y) \leq G_{0,0}(x,y) \leq G_{c,d}(x,y)$ if $a+b \leq 0 \leq c+d$. Due to the symmetry, we may deal only with the first inequality.

If $(a,b) \in H_1$, then the direction $(p-a, q-b) = (-a,-b) = (-a-b)(0,1) + (-a)(1,-1)$ is a cone combination of the vectors $u = (1,-1)$ and $v = (0,1)$. Since the segment $[(a,b),(0,0)]$ is contained in $H_1 \cup \{(0,0)\}$, hence, by Corollary 5.4, the directional derivative of $\mathcal{G}_{x,y}$ in the direction $(-a,-b)$ is nonnegative at any point of $[(a,b),(0,0)]$. Thus, Lemma 5.5 results $G_{a,b}(x,y) \leq G_{0,0}(x,y)$.

Similarly, if $(a,b) \in H_2$, then $(p-a, q-b) = (-a,-b) = (-b)\left(\frac{a}{b}, 1\right)$. Therefore, by Corollary 5.4, the directional derivative of $\mathcal{G}_{x,y}$ in the direction $(-a,-b)$ is nonnegative at any point of the segment $[(a,b),(0,0)] \subset H_2 \cup \{(0,0)\}$. Applying Lemma 5.5, $G_{a,b}(x,y) \le G_{0,0}(x,y)$ follows again.

Finally, if $(a,b) \in H_3$, then an elementary computation shows that the identity $G_{a,b}(x,y) = G_{0,0}(x,y)$ holds.

Case (ii). Since $(c,d) \in H_1$, we know that $c \le 0$, and due to condition (5.4), we get that $a+b-c \le d \le 0$. Thus, $(p,q) \in H_1$, whence, by the convexity of H_1, it follows that $[(a,b),(p,q)] \subset H_1$. By condition (5.6), we have that $a \le c$, hence the vector $(p-a, q-b) = (c-a, a-c)$ is obtained from $u = (1,-1)$ by multiplication with a nonnegative scalar. Therefore, by Corollary 5.4, the directional derivative of $\mathcal{G}_{x,y}$ in the direction u is nonnegative at any point of the segment $[(a,b),(p,q)]$. Applying Lemma 5.5, we obtain that $G_{a,b}(x,y) \le G_{p,q}(x,y)$.

We have a similar situation for the segment $[(p,q),(c,d)] \subset H_1$. The vector $(c-p, d-q) = (0, c+d-a-b)$ is co-directional with $v = (0,1)$ (by condition (5.4)). Using Corollary 5.4 again, the directional derivative of $\mathcal{G}_{x,y}$ in direction v is nonnegative at any point of the segment $[(p,q),(c,d)]$. Therefore, by Lemma 5.5, we get that $G_{p,q}(x,y) \le G_{c,d}(x,y)$.

Case (iii). Then $(c,d) \in H_2$ yields that $c > 0$ and $d < 0$. The inequality $ad - bc \ge 0$ is obviously valid if $(a,b) \in H_1$, because then a and b are nonpositive. In the case $(a,b) \in H_2$, we have that $b < 0 < a$, therefore, the condition (5.5) ensures that $ad - bc \ge 0$ also holds. Thus the direction $(p-a, q-b) = \frac{bc-ad}{c+d}(1,-1)$ is co-directional with the vector $u = (1,-1)$. Using Corollary 5.4, we can see that the directional derivative of $\mathcal{G}_{x,y}$, in the direction u is nonnegative at any point of the segment $[(a,b),(p,q)] \subset H_1 \cup H_2$. Hence, by Lemma 5.5, it follows that $G_{a,b}(x,y) \le G_{p,q}(x,y)$.

Observe that $(p,q) \in H_2$, hence we have that segment $[(p,q),(c,d)]$ is also contained in H_2. On the other hand, $(c-p, d-q) = \frac{(c+d-a-b)d}{c+d}\left(\frac{c}{d}, 1\right)$, hence $(c-p, d-q)$ is co-directional with the vector $v = \left(\frac{c}{d}, 1\right)$. In view of Corollary 5.4 again, we obtain that the directional derivative of $\mathcal{G}_{x,y}$ in direction v is nonnegative at any point of the segment $[(p,q),(c,d)] \subset H_2$. Therefore, by Lemma 5.5, we get that $G_{p,q}(x,y) \le G_{c,d}(x,y)$.

The Cases *(iv), (v)* are completely analogous to Cases *(iii), (ii),* respectively. Therefore, the details are left to the reader.

CHAPTER 6

Comparison of Stolarsky means

6.1. Introduction

The comparison problem

$$S_{a,b}(x,y) \le S_{c,d}(x,y) \tag{6.1}$$

on $\mathbb{R}_+$ (i.e., if $x, y \in \mathbb{R}_+$ in (6.1)) was solved by Leach and Sholander [**31**]. Páles [**47**] gave a new proof for this result. In [**49**] Páles solved the comparison problem (6.1) on any subinterval (α, β) of $\mathbb{R}_+$. Several particular inequalities involving $S_{a,b}$ and their special cases were dealt with by Alzer [**3**], Brenner [**8**], Brenner and Carlson, [**9**] Burk [**10**], Carlson [**11**], Dodd [**24**], Leach and Sholander [**30**], Lin [**33**], Pittinger [**50**], [**51**], Sándor [**53**], Seiffert [**54**], [**55**], Stolarsky [**56**], [**57**], Székely [**58**]. Neuman [**42**] studied multivariable weighted logarithmic means, Leach and Sholander [**32**] dealt also with difference means of several variables.

In [**47**] the following result can be read:

Theorem 6.A. *Let $a, c, b, d \in \mathbb{R}$ and assume that $a \neq b$, $c \neq d$. Then the comparison inequality* (6.1) *holds for all $x, y \in \mathbb{R}_+$ if and only if the conditions*

$$a + b \le c + d \tag{6.2}$$

and

$$\begin{cases} \mathcal{L}(a,b) \le \mathcal{L}(c,d), & \text{if } 0 \le \min\{a,b,c,d\} \\ & \quad \text{or } \max\{a,b,c,d\} \le 0, \\ \mathcal{E}(a,b) \le \mathcal{E}(c,d), & \text{if } \min\{a,b,c,d\} < 0 < \max\{a,b,c,d\} \end{cases} \tag{6.3}$$

are satisfied.

6.2. Comparison theorem for Stolarsky means revisited

In the following result, we restate the necessary and sufficient condition for the comparison of Stolarsky means found by Páles [**47**]. The original result in [**47**] (see Theorem 6.A above) does not cover the case of equal parameters, that is, the conditions $a \neq b$, $c \neq d$ were also assumed. The conditions (6.5) and (6.6) also differs from the analogous condition (6.3) of Theorem 6.A, though they turn out to be equivalent to each other.

THEOREM 6.1.([**17**]) *Let* $a, b, c, d \in \mathbb{R}$ *be arbitrary parameters. Then the comparison inequality* (6.1) *holds for all positive* x *and* y *if and only if* a, b, c, d *satisfy the following three conditions:*

$$a + b \leq c + d, \tag{6.4}$$

$$\mathcal{E}(a, b) \leq \mathcal{E}(c, d), \tag{6.5}$$

$$\mathcal{L}(a, b) \leq \mathcal{L}(c, d). \tag{6.6}$$

First we show that if (6.2) holds, then the conditions (6.3) and (6.5)-(6.6) are equivalent to each other. Obviously, if (6.5) and (6.6) hold, then (6.3) is also valid. Assume now that (6.3) is satisfied. We distinguish three cases.

Case 1: $0 \leq \min\{a, b, c, d\}$.

Then $\mathcal{L}(a, b) \leq \mathcal{L}(c, d)$ by (6.3), that is, (6.6) holds. If $(c, d) \neq (0, 0)$, then $\mathcal{E}(c, d) = 1$, hence $\mathcal{E}(a, b) \leq 1 = \mathcal{E}(c, d)$. If $c = d = 0$, then, by (6.2), $a + b \leq 0$, hence $a = b = 0$ and $\mathcal{E}(a, b) = 0 = \mathcal{E}(c, d)$. Thus, (6.5) is also satisfied.

Case 2: $\min\{a, b, c, d\} < 0 < \max\{a, b, c, d\}$.

Then we have $\mathcal{E}(a, b) \leq \mathcal{E}(c, d)$ by (6.3), that is, (6.5) holds automatically. Now we show that $\min\{a, b\} \leq 0$ and $\max\{c, d\} \geq 0$. If, on the contrary, $\min\{a, b\} > 0$, then $1 = \mathcal{E}(a, b)$, hence $\mathcal{E}(a, b) \leq \mathcal{E}(c, d)$ yields that $\min\{c, d\} \geq 0$. Thus we get the contradiction $\min\{a, b, c, d\} \geq 0$. A similar argument validates $\max\{c, d\} \geq 0$. Due to these inequalities (and the definition of $\mathcal{L}$), we have that $\mathcal{L}(a, b) \leq 0 \leq \mathcal{L}(c, d)$, i.e., (6.6) is also satisfied.

Case 3: $\max\{a, b, c, d\} \leq 0$.

The argument is completely analogous to that followed in Case 1.

Summarizing what we have proved, we can see that Theorem 6.A remains valid if condition (6.3) is replaced by (6.5)-(6.6). Now we show that the conditions (6.2) and (6.3) are necessary and sufficient in order that (6.1) be valid for all positive x, y.

(Necessity.) Assume that (6.1) is satisfied for all positive x, y. Then, substituting $x = t$, $y = 1$, we get that

$$\frac{S_{a,b}(t, 1) - \frac{t+1}{2}}{(t-1)^2} \leq \frac{S_{c,d}(t, 1) - \frac{t+1}{2}}{(t-1)^2} \qquad (t > 0, t \neq 1).$$

Now, taking the limit $t \to 1$ and using (3.7) of Theorem 3.7, we obtain

$$\frac{a + b - 3}{24} \leq \frac{c + d - 3}{24},$$

which is equivalent to (6.2).

The proof of the condition $\mathcal{E}(a, b) \leq \mathcal{E}(c, d)$ is literally the same as we presented when proving (5.5), based on Theorem 3.9.

To prove $\mathcal{L}(a, b) \leq \mathcal{L}(c, d)$, suppose first that $0 \leq \min\{a, b, c, d\}$. The cases $ab = cd = 0$ and $ab = 0 < cd$ are trivial, while – applying (3.2) in Lemma 3.3 the case $ab > 0 = cd$ is impossible. That is, we may assume that $a, b, c, d > 0$. But, by Lemma 3.3, again, we have that

$$y \cdot e^{-\frac{1}{\mathcal{L}(a,b)}} \leq y \cdot e^{-\frac{1}{\mathcal{L}(a,b)}},$$

which is equivalent to our statement. Finally, the case $\max\{a, b, c, d\} \leq 0$ can be deduced from the previous one, due to Lemma 2.A and Lemma 2.6(i).

(Sufficiency.) Assume that (6.2), (6.5) and (6.6) – that is, (6.3) – are valid. Define the following sequences:

$$a^{(n)} := a, \quad b^{(n)} := b - \frac{1}{n}, \quad c^{(n)} := c, \quad d^{(n)} := d + \frac{1}{n} \qquad (n \in \mathbb{N}).$$

Then, for n large enough, $a^{(n)} \neq b^{(n)}$, $c^{(n)} \neq d^{(n)}$. Obviously,

$$a^{(n)} + b^{(n)} = a + b - \frac{1}{n} < a + b \leq c + d \leq c + d + \frac{1}{n} = c^{(n)} + d^{(n)},$$

furthermore, by (6.2), (6.5), (6.6) and the monotonicity properties of $\mathcal{L}$ and $\mathcal{E}$ (see Lemma 2.5(iii) and Lemma 2.8(iii)),

$$\begin{aligned}\mathcal{L}\big(a^{(n)}, b^{(n)}\big) &\leq \mathcal{L}\Big(a, b - \frac{1}{n}\Big) \leq \mathcal{L}(a, b)\\ &\leq \mathcal{L}(c, d) \leq \mathcal{L}\Big(c, d + \frac{1}{n}\Big) \leq \mathcal{L}\big(c^{(n)}, d^{(n)}\big),\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}\big(a^{(n)}, b^{(n)}\big) &\leq \mathcal{E}\Big(a, b - \frac{1}{n}\Big) \leq \mathcal{E}(a, b)\\ &\leq \mathcal{E}(c, d) \leq \mathcal{E}\Big(c, d + \frac{1}{n}\Big) \leq \mathcal{E}\big(c^{(n)}, d^{(n)}\big).\end{aligned}$$

Therefore, by Theorem 6.A, we get

$$S_{a^{(n)},b^{(n)}}(x, y) \leq S_{c^{(n)},d^{(n)}}(x, y) \qquad (x, y \in \mathbb{R}_+). \tag{6.7}$$

Taking the limit $n \to \infty$ and using the continuity of $S_{a,b}(x, y)$ with respect to the parameters a, b, we get that (6.1) holds for all positive x, y.

REMARK 6.2. In case of sufficiency, our proof followed the method we used in [**17**]. The necessity, however, was proved here with a new method: we applied the asymptotic properties presented in Chapter 3. The original proof, built on sequences, can be found in [**17**].

CHAPTER 7

Comparison of Gini and Stolarsky means

7.1. Introduction

The comparison problem for the means of the same kind, but with different parameters has covered by the previous two chapters. These results, however, describe only the cases where at the two sides of the comparison inequality there stand *means of the same kind*. Our present aim is to state necessary/sufficient conditions for the comparison of Gini *and* Stolarsky means. The first results in this direction are due to Neuman and Páles [**43**] who investigated the comparison of Gini and Stolarsky means of equal parameters and proved that, for given real numbers a, b, the comparison inequality

$$G_{a,b}(x,y) \underset{(\geq)}{\leq} S_{a,b}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if $a+b \underset{(\geq)}{\leq} 0$. Another problem, the so-called strong comparison problem (which is equivalent to the monotonicity property of the ratios of the means in question), has recently been investigated by Hästö [**29**].

In the next section some necessary conditions are obtained. After it, we formulate two propositions that offer necessary and sufficient conditions for the comparison problem in a particular setting. These results will play an important role in the last section when stating the sufficient conditions for the comparison of Gini and Stolarsky means.

In the sequel, we restrict our attention to the inequality $G_{a,b} \leq S_{c,d}$ only because the analogous inequality $S_{a,b} \leq G_{c,d}$ is equivalent to $G_{-c,-d} \leq S_{-a,-b}$, therefore the results for the second type of the comparison inequality can easily be derived from what we will be obtain.

7.2. Necessary conditions

In this section we derive conditions that are necessary for the mixed comparison inequality of Gini and Stolarsky means.

THEOREM 7.1.([**15**]) *Suppose that the inequality*

$$G_{a,b}(x,y) \leq S_{c,d}(x,y) \tag{7.1}$$

holds for any positive x, y. Then

$$3(a+b) \leq c+d, \tag{7.2}$$

$$\mathcal{E}(a,b) \leq \mathcal{E}(c,d), \tag{7.3}$$

$$\min\{a,b\} \leq \min\{c,d\}, \tag{7.4}$$

$$\text{if } \min\{a,b\} = 0 < \max\{a,b\} \text{ then } \max\{a,b\} \leq \log 2 \cdot \mathcal{L}(c,d). \tag{7.5}$$

PROOF. Substituting $(t,1)$ in place of (x,y) in (7.1), after some transformations we get that

$$\frac{G_{a,b}(t,1) - \frac{t+1}{2}}{(t-1)^2} \leq \frac{S_{c,d}(t,1) - \frac{t+1}{2}}{(t-1)^2} \qquad (t \in \mathbb{R}_+ \setminus \{1\}).$$

Performing the limit $t \to 1$ and applying Theorem 3.7, we get (7.2).

For (7.5), after the same substitution

$$\frac{G_{a,b}(t,1)}{t} \leq \frac{S_{c,d}(t,1)}{t} \qquad (t \in \mathbb{R}_+).$$

Taking the limit of both sides as $t \to \infty$ we can apply the result in Remark 3.5. Since the limit of the right hand side is less than 1, the inequality $\min\{a,b\} > 0$ would yield a contradiction – that is, we are ready with (7.5). Moreover, (7.4) is also a direct consequence of Remark 3.5.

Finally, applying Theorem 3.9, we obtain (7.3), that is, the proof is complete. □

7.3. Particular comparison inequalities

In this section we examine three particular cases of the comparison of Gini and Stolarsky means. These statements will turn out to be useful tools in formulating sufficient conditions for the general comparison problem. In these cases the parameters a, b and c, d of the Gini and Stolarsky means are chosen so that the necessary condition (7.2) of Theorem 7.1 hold with equality.

The third inequality was presented by Zsolt Páles and the author as an open problem at the 3-rd Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in 2003 [**12**].)

THEOREM 7.2.([**15**]) *The inequality*

$$G_{a,b}(x,y) \leq S_{3a,3b}(x,y) \tag{7.6}$$

holds for all positive x, y if and only if $a+b \leq 0$, while the reversed inequality holds if and only if $a+b \geq 0$.

PROOF. We deal only with the characterization of the inequality (7.6), the investigation of the reversed inequality is completely analogous.

Suppose first that $a \neq b$, for example, $a > b$ and $ab \neq 0$. Using the symmetry and homogeneity of the means, setting $t = \log\sqrt{x/y}$, (7.6) can be rewritten in the equivalent form

$$\left(\frac{\cosh(at)}{\cosh(bt)}\right)^{\frac{1}{a-b}} \le \left(\frac{\frac{\sinh(3at)}{3a}}{\frac{\sinh(3bt)}{3b}}\right)^{\frac{1}{3a-3b}} \qquad (t \in \mathbb{R}_+),$$

which is also equivalent to

$$\frac{\sinh(3bt)}{bt\cosh^3(bt)} \le \frac{\sinh(3at)}{at\cosh^3(at)} \qquad (t \in \mathbb{R}_+). \tag{7.7}$$

To investigate this inequality, introduce the function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \frac{\sinh 3x}{x(\cosh x)^3}.$$

It can immediately be seen that f is even. We also claim that it is decreasing on $\mathbb{R}_+$. We can easily obtain that

$$f'(x) = \frac{6x\cosh 2x - \sinh 4x - \sinh 2x}{2x^2(\cosh x)^4} \qquad (x \in \mathbb{R} \setminus \{0\}).$$

Thus, it suffices to show that $6x\cosh 2x - \sinh 4x - \sinh 2x := h(x)$ is negative for any positive x. (Therefore, f' is negative on $\mathbb{R}_+$.)

Expanding the function h into McLaurin series, we get that

$$\begin{aligned} h(x) &= \sum_{i=0}^{\infty}\left(\frac{6\cdot 2^{2i}}{(2i)!} - \frac{4^{2i+1}}{(2i+1)!} - \frac{2^{2i+1}}{(2i+1)!}\right)\cdot x^{2i+1} \\ &= \sum_{i=0}^{\infty}(3i+1-2^{2i})\frac{2^{2i+2}}{(2i+1)!}\cdot x^{2i+1}. \end{aligned}$$

Here the coefficient $3i + 1 - 2^{2i}$ vanishes for $i = 0$ and $i = 1$ and is negative if $i \ge 2$. Therefore, $h(x)$ and also $f'(x)$ is negative for all positive x. Thus, the even f is decreasing on the positive half line and increasing on negative reals. It readily follows from this that (7.7), i.e., $f(bt) \le f(at)$ is valid if and only if $|a| \le |b|$. One can easily observe that this inequality together with $b < a$ holds if and only if $a + b \le 0$.

In the cases $a = b$ or $ab = 0$, the necessity and sufficiency of the condition $a + b \le 0$ can similarly be verified. □

REMARK 7.3. Theorem 7.2 implies the result of Neuman and Páles [**43**], since (by Theorem 6.1) $S_{3a,3b}(x, y) \le S_{a,b}(x, y)$ holds for all positive x, y if and only if $a + b \le 0$.

THEOREM 7.4.([**15**]) *The inequality*

$$G_{a,b}(x, y) \le S_{2a+b,a+2b}(x, y) \tag{7.8}$$

holds for all positive x, y *if and only if* $ab(a+b) \le 0$, *while the reversed inequality holds if and only if* $ab(a + b) \ge 0$.

PROOF. The case $a = b$ is covered by Theorem 7.2. Moreover, if $ab = 0$, then the inequality turns to an identity, since $G_{a,0}(x, y) = S_{2a,a}(x, y)$ for all positive x, y.

In the case $2a + b = 0$, $ab \neq 0$, with the notation $t = \log \sqrt{x/y}$, (7.8) is equivalent to the inequality

$$\left(\frac{\cosh(at)}{\cosh(-2at)}\right)^{\frac{1}{3a}} \leq \left(\frac{\sinh(-3at)}{-3at}\right)^{\frac{1}{-3a}} \qquad (t \in \mathbb{R}_+). \tag{7.9}$$

Thus, we have to show that (7.9) holds if and only if $a < 0$.

For, we will prove that the inequality

$$\frac{\cosh(x)}{\cosh(-2x)} \geq \left(\frac{\sinh(-3x)}{-3x}\right)^{-1} \tag{7.10}$$

holds for all $x \neq 0$. The functions on the two sides of this inequality are even, so we may assume that $x > 0$. Then, inequality (7.10) can be rewritten into the form

$$\sinh(4x) + \sinh(2x) - 6x\cosh(2x) \geq 0.$$

As we have seen it in the proof of Theorem 7.2, the left hand side of this inequality is nonnegative. Thus (7.10) follows for all $x \neq 0$. In view of (7.10), the inequality (7.9) holds for all $t > 0$ if and only if $a < 0$, which completes the proof in this case.

The case $a + 2b = 0$, $ab \neq 0$ can be treated similarly.

We may assume now that $a \neq b$, e.g. $a > b$, $ab \neq 0$, and $(a+2b)(2a+b) \neq 0$. Now (7.8) can be rewritten in the equivalent form

$$\frac{\cosh(at)}{\cosh(bt)} \leq \frac{\frac{\sinh(2a+b)t}{2a+b}}{\frac{\sinh(a+2b)t}{a+2b}} \qquad (t \in \mathbb{R}_+), \tag{7.11}$$

or, rearranging this and dividing both sides by the positive t,

$$\frac{\cosh(at)\sinh(a+2b)t}{(a+2b)t} \leq \frac{\cosh(bt)\sinh(2a+b)t}{(2a+b)t} \qquad (t \in \mathbb{R}_+).$$

Finally, applying the product-to-sum formulas and denoting $2t$ by s, our statement is equivalent to the following:

$$\frac{\sinh(a+b)s + \sinh(bs)}{(a+2b)s} \leq \frac{\sinh(a+b)s + \sinh(as)}{(2a+b)s} \qquad (s \in \mathbb{R}_+). \tag{7.12}$$

Introduce the function $f(x) = 1 + \frac{x}{3!} + \frac{x^2}{5!} + \frac{x^3}{7!} + \dots$ on $\mathbb{R}_+$. Clearly, f is strictly convex on $\mathbb{R}_+$ and $\sinh x = x \cdot f(x^2)$ holds for any real x. Using this notation, (7.12) transforms to

$$\frac{(a+b)f\big((a+b)^2s^2\big) + bf(b^2s^2)}{a+2b} - \frac{(a+b)f\big((a+b)^2s^2\big) + af(a^2s^2)}{2a+b} \leq 0. \tag{7.13}$$

The left hand side of (7.13) is, however, the product of $ab(a-b)(a+b)\cdot s^4$ and the 2nd-order divided difference $[(a+b)^2 s^2, a^2 s^2, b^2 s^2; f]$. The function f being strictly convex on $\mathbb{R}_+$, we have that this divided difference is positive if $|a| \neq |b|$ and $|a|, |b| \neq 0$. Consequently, (7.13) holds if and only if $ab(a-b)(a+b)$ is nonpositive. With the assumption $a > b$ this yields that (7.13), that is, (7.11) is valid if and only if $ab(a+b) \leq 0$. $\square$

THEOREM 7.5.([**13**]) *For any positive numbers x and y*

$$G_{-1+\frac{2}{\sqrt{5}},-1-\frac{2}{\sqrt{5}}}(x,y) \leq S_{-3,-3}(x,y). \tag{7.14}$$

To prove this, we have to study some simple statements.

PROPOSITION 7.6. ([**13**]) *Theorem 7.5 is equivalent to the following statement:*
For any nonnegative t

$$G_{-1+\frac{2}{\sqrt{5}},-1-\frac{2}{\sqrt{5}}}(e^t, e^{-t}) \leq S_{-3,-3}(e^t, e^{-t}) \tag{7.15}$$

holds.

PROOF. Proposition 7.6 is a special case of Theorem 7.5.
Inversely, let x and y be two arbitrary positive numbers. Due to the symmetry, we may assume that $x \geq y$. Define $t := \log\sqrt{x/y}$. Then, applying Proposition 7.6 for this nonnegative t, we get that

$$G_{-1+\frac{2}{\sqrt{5}},-1-\frac{2}{\sqrt{5}}}\left(\sqrt{\frac{x}{y}}, \sqrt{\frac{y}{x}}\right) \leq S_{-3,-3}\left(\sqrt{\frac{x}{y}}, \sqrt{\frac{y}{x}}\right).$$

Multiplying both sides with the (positive) $\sqrt{xy}$ and applying the homogeneity of the means, we obtain (7.14). $\square$

DEFINITION 7.7. Define the following functions:

$$g_c(t) := \frac{1}{2c}\left(\log\cosh(1-c)t - \log\cosh(1+c)t\right) \quad (t \in \mathbb{R}),$$

$$s(t) := \begin{cases} 1/3 - t\coth(3t), & \text{if } t \neq 0; \\ 0, & \text{if } t = 0, \end{cases}$$

where c stands for an arbitrary real number.

REMARK 7.8. Since $M_{p,q}(x,y)$ is infinitely many times differentiable for any choice of M, p and q, the same holds for their logarithm, that is, for any real number c, the functions g_c and s are infinitely many time differentiable over $\mathbb{R}$.

PROPOSITION 7.9. ([**13**]) *Theorem 7.5 is equivalent to the following statement:*
For any nonnegative t

$$g_\varepsilon(t) \leq s(t) \tag{7.16}$$

holds. (Here and in the following ε stands for the real number $2/\sqrt{5}$.)

PROOF. It is enough to show the equivalence of Proposition 7.9 and Proposition 7.6.
Since both (7.16) and (7.15) turns to identity for $t = 0$, we will assume that $t > 0$. Applying the definitions of the Gini and Stolarsky means in this case,

$$\begin{aligned} g_c(t) &= \frac{1}{2c} \log \frac{e^{(-1+c)t} + e^{(-1+c)(-t)}}{e^{(-1-c)t} + e^{(-1-c)(-t)}} \\ &= \frac{1}{2c} \log \frac{\cosh(-1+c)t}{\cosh(-1-c)t} = \frac{1}{2c}(\log\cosh(1-c)t - \log\cosh(1+c)t) \end{aligned}$$

and

$$s(t) = \frac{1}{3} + \frac{e^{-3t}\cdot t - e^{3t}\cdot(-t)}{e^{-3t} - e^{3t}}\frac{1}{3} + t\frac{\cosh(-3t)}{\sinh(-3t)} = \frac{1}{3} - t\coth(3t).$$

Consequently, (7.15) can be written as

$$\frac{1}{2\varepsilon}(\log\cosh(1-\varepsilon)t - \log\cosh(1+\varepsilon)t) \le \frac{1}{3} - t\coth(3t),$$

as we stated. □

We will show now that the function $s - g_\varepsilon$ is increasing over $[0, \infty)$. Since its value equals 0 at the point 0, it will follow from this increase that $s - g_\varepsilon$ is nonnegative over the nonnegative half-line. By Proposition 7.9, this result will complete the proof of Theorem 7.5.

PROPOSITION 7.10. ([**13**]) *$(s - g_\varepsilon)'(t) > 0$, if $t > 0$ and $(s - g_\varepsilon)'(0) = 0$.*

PROOF. After some calculations, we obtain that for any positive number t

$$(s - g_c)'(t) = \frac{n_c(t)}{d_c(t)}, \tag{7.17}$$

where

$$\begin{aligned} n_c(t) =& (1-c)\sinh(6+2c)t - (1+c)\sinh(6-2c)t - 2c\sinh 4t \\ &+ 12ct\cosh 2ct + 12ct\cosh 2t - 2\sinh 2ct - 2c\sinh 2t \end{aligned} \tag{7.18}$$

and

$$d_c(t) = 8c\cosh(1+c)t\cosh(1-c)t\cdot\sinh^2 3t. \tag{7.19}$$

We have to show that the numerator of $(s - g_c)'(t)$, that is, $n_c(t)$ is positive for $c = \varepsilon$. For brevity, let $2t$ be denoted by y. After this substitution, introduce the function f_c for the numerator, that is,

$$\begin{aligned} f_c(y) :=& (1-c)\sinh(3+c)y - (1+c)\sinh(3-c)y - 2c\sinh 2y \\ &+ 6cy\cosh cy + 6cy\cosh y - 2\sinh cy - 2c\sinh y \qquad (y \in \mathbb{R}). \end{aligned}$$

We have to show that f_c is positive for any positive real number y.

The function f_c can be expanded into McLaurin series. Since

$$\sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} \qquad (x \in \mathbb{R})$$

and

$$x \cosh ax = \sum_{k=0}^{\infty} \frac{1}{(2k)!} a^{2k} x^{2k+1} \qquad (x \in \mathbb{R}),$$

we get that

$$f_c(y) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \cdot \lambda_{c,k} \cdot y^{2k+1} \qquad (y \in \mathbb{R}), \tag{7.20}$$

where, for any nonnegative integer k,

$$\begin{aligned}\lambda_{c,k} = {} & (1-c)(3+c)^{2k+1} - (1+c)(3-c)^{2k+1} \\ & - 2c2^{2k+1} - 2c^{2k+1} - 2c + (6c^{2k+1} + 6c)(2k+1).\end{aligned}$$

For $c = \varepsilon$ we have that for any nonnegative integer k

$$\begin{aligned}\lambda_{\varepsilon,k} = {} & \left(1 - \frac{2}{\sqrt{5}}\right)\left(3 + \frac{2}{\sqrt{5}}\right)^{2k+1} - \left(1 + \frac{2}{\sqrt{5}}\right)\left(3 - \frac{2}{\sqrt{5}}\right)^{2k+1} - \frac{8}{\sqrt{5}} \cdot 2^{2k} \\ & - 2 \cdot \left(\frac{2}{\sqrt{5}}\right)^{2k+1} - \frac{4}{\sqrt{5}} + \left[6 \cdot \left(\frac{2}{\sqrt{5}}\right)^{2k+1} + \frac{12}{\sqrt{5}}\right](2k+1).\end{aligned}$$

We will prove that $\lambda_{\varepsilon,0} = \lambda_{\varepsilon,1} = \lambda_{\varepsilon,2} = 0$ and $\lambda_{\varepsilon,k} > 0$, if $k \geq 3$.
The first statement can directly be checked. For the second one, introduce the sequence

$$a_k = \left(1 - \frac{2}{\sqrt{5}}\right)\left(3 + \frac{2}{\sqrt{5}}\right)^{2k+1} - \left(1 + \frac{2}{\sqrt{5}}\right)\left(3 - \frac{2}{\sqrt{5}}\right)^{2k+1} - \frac{8}{\sqrt{5}} \cdot 2^{2k}$$

and

$$b_k = -2 \cdot \left(\frac{2}{\sqrt{5}}\right)^{2k+1} - \frac{4}{\sqrt{5}} + \left[6 \cdot \left(\frac{2}{\sqrt{5}}\right)^{2k+1} + \frac{12}{\sqrt{5}}\right](2k+1),$$

where $k = 0, 1, 2, \ldots$. As $\lambda_{\varepsilon,k} = a_k + b_k$ $(k = 0, 1, 2, \ldots)$, it is enough to prove that
(A) $a_k > 0$, if $k \geq 3$ and (B) $b_k > 0$, if $k \geq 3$.

For (A), we will show that $d_k := a_k / 2^{2k+1} > 0$ for $k \geq 3$. Since $d_3 = \frac{2997}{1000}\sqrt{5}$, it suffices to verify that d_k is increasing. Really, for any nonnegative integer m,

$$d_{m+1} - d_m = \left(\frac{1}{4} + \frac{\sqrt{5}}{50}\right)\left(\frac{3}{2} + \frac{1}{\sqrt{5}}\right)^{2m+1} - \left(\frac{1}{4} - \frac{\sqrt{5}}{50}\right)\left(\frac{3}{2} - \frac{1}{\sqrt{5}}\right)^{2m+1} > 0,$$

since this inequality is equivalent to

$$\left(\frac{\frac{3}{2} + \frac{1}{\sqrt{5}}}{\frac{3}{2} - \frac{1}{\sqrt{5}}}\right)^{2m+1} > \frac{\frac{1}{4} - \frac{\sqrt{5}}{50}}{\frac{1}{4} + \frac{\sqrt{5}}{50}},$$

which is obviously true (the two sides are separated by the number 1).

For (B), observe first that $b_0 = \frac{16}{\sqrt{5}}$. We show, again, that b_k is also increasing. For, let m be an arbitrary nonnegative integer. We have that

$$\frac{5}{4}(b_{m+1} - b_m) = (11 - 3m) \cdot \left(\frac{2}{\sqrt{5}}\right)^{2m+1} + 6\sqrt{5}.$$

The right hand side of this inequality is positive, because the inequality

$$(11 - 3m) \cdot \left(\frac{2}{\sqrt{5}}\right)^{2m+1} + 6\sqrt{5} > 0$$

is equivalent to

$$\left(\frac{\sqrt{5}}{2}\right)^{2m+1} > \frac{1}{2\sqrt{5}} \cdot m - \frac{11}{6\sqrt{5}}.$$

For $m = 0$ the difference of the left and the right side is positive. This difference, as the function of m, can be extended to be a continuous function which has a positive derivative, therefore, is increasing. In particular, this difference is positive for any positive integer.

Consequently, we have proven that $\lambda_{\varepsilon,k} = a_k + b_k \geq 0$ for all nonnegative integers k – in particular, $\lambda_{\varepsilon,k} > 0$ for $k \geq 3$ – that is, $f_\varepsilon(y) > 0$. It means that $(s - g_\varepsilon)'(t) > 0$, if $t > 0$. The statement $(s - g_\varepsilon)'(0) = 0$ can immediately be checked, that is, the proof is complete. □

7.4. Sufficient conditions

In this section, according to the position of the pair $(a, b) \in \mathbb{R}^2$, we give sufficient conditions for the Gini-Stolarsky comparison inequality. These conditions are sometimes (unfortunately not always) also necessary.

THEOREM 7.11.([**15**]) *Let a, b be positive numbers. Then there are no parameters c, d so that the inequality*

$$G_{a,b}(x, y) \leq S_{c,d}(x, y)$$

be valid for all positive numbers x, y.

PROOF. This is a direct consequence of (7.5) in Theorem 7.1. $\square$

THEOREM 7.12.([**15**]) *Let* a, b *be real numbers so that* $\min\{a, b\} = 0 < \max\{a, b\}$. *Then*

$$G_{a,b}(x, y) \le S_{c,d}(x, y)$$

is valid for all positive numbers x, y *if and only if*

(a) $3a \le c + d$,
(b) $a \le \log 2 \cdot \mathcal{L}(c, d)$.

PROOF. The necessity of the condition follows form Theorem 7.1. For the sufficiency, assume that $0 = b < a$. Then $G_{a,b}(x, y) = G_{a,0}(x, y) = S_{a,2a}(x, y)$, and we can apply Theorem 6.1 to the inequality $S_{a,2a}(x, y) \le S_{c,d}(x, y)$. Now (a) is equivalent to $a + 2a \le c + d$ and (b) yields $\mathcal{L}(a, 2a) \le \mathcal{L}(c, d)$. Thus c, d must be positive, whence $\mathcal{E}(a, 0) \le \mathcal{E}(c, d)$ also follows. Therefore, in view of Theorem 6.1, (a) and (b) yield that $G_{a,0}(x, y) = S_{a,2a}(x, y) \le S_{c,d}(x, y)$ holds for all positive x, y. $\square$

THEOREM 7.13.([**15**]) *Let* a, b *be real numbers so that* $ab < 0$ *and* $a + b \ge 0$. *If*

(a) $3(a + b) \le c + d$,
(b) $\mathcal{L}\{a + 2b, 2a + b\} \le \mathcal{L}(c, d)$,
(c) $\mathcal{E}(2a + b, a + 2b) \le \mathcal{E}(c, d)$,

then

$$G_{a,b}(x, y) \le S_{c,d}(x, y)$$

is valid for all positive numbers x, y.

PROOF. By Theorem 7.4, the condition $ab(a + b) \le 0$ yields that, for any positive x, y,

$$G_{a,b}(x, y) \le S_{a+2b,2a+b}(x, y).$$

Moreover, the conditions guarantee that Theorem 6.1 can be applied to obtain

$$S_{a+2b,2a+b}(x, y) \le S_{c,d}(x, y).$$

Combining these two inequalities, the desired inequality follows.. $\square$

THEOREM 7.14.([**15**]) *Let* a, b *be real numbers so that* $ab < 0$ *and* $a + b \le 0$. *Then*

$$G_{a,b}(x, y) \le S_{c,d}(x, y)$$

is valid for all positive numbers x, y *if and only if*

(a) $3(a + b) \le c + d$,
(b) $\mathcal{E}(a, b) \le \mathcal{E}(c, d)$.

PROOF. The necessity of conditions (a) and (b) is the consequence of Theorem 7.1.

By Theorem 7.4, the condition $a+b \le 0$ yields that, for all positive x, y,

$$G_{a,b}(x,y) \le S_{3a,3b}(x,y).$$

To complete the proof of the sufficiency, we will also prove that

$$S_{3a,3b}(x,y) \le S_{c,d}(x,y).$$

For, by Theorem 6.1, we have to ensure that, in addition to (a), $\mathcal{E}(3a,3b) \le \mathcal{E}(c,d)$ and $\mathcal{L}(3a,3b) \le \mathcal{L}(c,d)$ hold. The first inequality trivially follows from (b) since $\mathcal{E}(3a,3b) \le \mathcal{E}(a,b)$. On the other hand, $ab < 0$, consequently, $\mathcal{L}(3a,3b) = 0$. Thus it suffices to show that $\mathcal{L}(c,d)$ is nonnegative. Indeed, in the opposite case we have that $c, d < 0$, thus $\mathcal{E}(c,d)$ is equal to -1, while $\mathcal{E}(a,b) > -1$ — in contradiction with (b). □

REMARK 7.15. It is clear that the conditions of Theorem 7.13 and Theorem 7.14 coincide in the case $ab < 0$, $a+b=0$ – that is, $b=-a$, $b \ne 0$.

THEOREM 7.16.([**15**]) *Let a, b be real numbers, $a, b \le 0$. If*

(a) $3(a+b) \le c+d$,

(b) $\mathcal{L}(2a+b, a+2b) \le \mathcal{L}(c,d)$,

then

$$G_{a,b}(x,y) \le S_{c,d}(x,y)$$

is valid for all positive numbers x, y.

PROOF. First we check our statement when $(a,b) = (0,0)$. In this case $G_{0,0}(x,y) = \sqrt{xy} = S_{0,0}(x,y)$, so we can use Theorem 6.1 again. Then (a) and (b) are equivalent to the inequality $0 \le c+d$ which, by Theorem 6.1, results that $S_{0,0} \le S_{c,d}$ is valid.

In the rest of the proof, we assume that $(a,b) \ne (0,0)$.

In view of Theorem 7.4, it is clear that, for all positive x, y,

$$G_{a,b}(x,y) \le S_{2a+b,a+2b}(x,y).$$

We have that $a, b \le 0$ and $a+b < 0$, consequently, $\mu(2a+b, a+2b) = -1$, whence $\mu(2a+b, a+2b) \le \mu(c,d)$ follows. Therefore, using Theorem 6.1, the conditions (a) and (b) imply that

$$S_{2a+b,a+2b}(x,y) \le S_{c,d}(x,y),$$

which combined with the previous inequality results our statement. □

REMARK 7.17. In the domain $a, b \le 0$ we also have the inequality $G_{a,b} \le S_{3a,3b}$ which could be used to obtain that the inequality in (a) and $\mathcal{L}(3a,3b) \le \mathcal{L}(c,d)$ form also a system of sufficient conditions. However, applying Theorem 6.1, it easily follows that $S_{2a+b,a+2b} \le S_{3a,3b}$ holds, too. Thus, the sufficient condition obtained this way is essentially weaker than that of Theorem 7.16.

LEMMA 7.18.([**13**])

$$(s-g_c)'(0)=(s-g_c)''(0)=0, \tag{7.21}$$

$$(s-g_c)^{(iii)}(0)=0, \tag{7.22}$$

while

$$(s-g_c)^{(iv)}(0)=\frac{32}{5}-8c^2. \tag{7.23}$$

PROOF. With L'Hospital's law, one can directly check that

$$s'(0)=g_c'(0)=0$$

and

$$s''(0)=g_c''(0)=-2,$$

that is, (7.21) holds.

We will need the values of the derivatives of n_c and d_c (defined in (7.18) and (7.19)) at 0. For, we determine first the ones of f_c and h_c, where $h_c(2t)=d_c(t)$, and – as before – $f_c(2t)=n_c(t)$.

By means of (7.20) we get that

$$f_c'(0)=\lambda_{c,0}=0, \quad f_c''(0)=0, \quad f_c^{(iii)}(0)=\lambda_{c,1}=0,$$
$$f_c^{(iv)}(0)=0, \quad f_c^{(v)}(0)=\lambda_{c,2}=288c-360c^3.$$

Moreover, following the method, applied in the previous proof, we can obtain the McLaurin expansion of $h_c(y)$. Namely,

$$\begin{aligned} &8c\cosh(1+c)t\cosh(1-c)t\cdot\sinh^2 3t \\ &\quad=-2c\cosh cy-2c\cosh y+c\cosh(3+c)y \\ &\quad+c\cosh(3-c)y+c\cosh 4y+c\cosh 2y \\ &\quad=\sum_{k=0}^{\infty}\frac{1}{(2k)!}\mu_{c,k}\cdot y^{2k} \qquad (y\in\mathbb{R}), \end{aligned}$$

where

$$\mu_{c,k}=-2c\cdot c^{2k}-2c+c\cdot(3+c)^{2k}+c\cdot(3-c)^{2k}+c\cdot 4^{2k}+c\cdot 2^{2k} \qquad (k=0,1,2,\dots).$$

In particular,

$$h_c(0)=\mu_{c,0}=0, \quad h_c'(0)=0, \quad h_c''(0)=\mu_{c,1}=36c, \quad h_c^{(iii)}(0)=0.$$

That is, applying that $n_c(t)=f_c(2t)$ and $d_c(t)=h_c(2t)$,

$$\begin{aligned} &n_c(0)=0, \quad n_c'(0)=0, \quad n_c''(0)=0, \quad n_c^{(iii)}(0)=0, \\ &n_c^{(iv)}(0)=0, \quad n_c^{(v)}(0)=32(288c-360c^3) \end{aligned} \tag{7.24}$$

and

$$d_c(0)=0, \quad d_c'(0)=0, \quad d_c''(0)=4\cdot 36c, \quad d_c^{(iii)}(0)=0. \tag{7.25}$$

From (7.17) we obtain that $n_c(t) = (s-g)'(t)d_c(t)$.

Performing the forth derivative of both sides, we have that

$$n_c^{(iv)}(t) = (s-g)'(t)d_c^{(iv)}(t) + 4(s-g)''(t)d_c^{(iii)}(t) + 6(s-g)^{(iii)}(t)d_c''(t) \\ + 4(s-g)^{(iv)}(t)d_c'(t) + (s-g)^{(v)}(t)d_c(t).$$

Putting $t = 0$ in this equation and using (7.21), (7.24) and (7.25), we have that $(s-g)^{(iii)}(0)$ must be equal to 0, that is, we have obtained (7.22).

Calculating the fifth derivative of this identity, we get the following equation:

$$n_c^{(v)}(t) = (s-g)'(t)d_c^{(v)}(t) + 5(s-g)''(t)d_c^{(iv)}(t) + 10(s-g)^{(iii)}(t)d_c^{(iii)}(t) \\ + 10(s-g)^{(iv)}(t)d_c''(t) + 5(s-g)^{(v)}(t)d_c'(t) + (s-g)^{(vi)}(t)d_c(t).$$

Substituting $t = 0$, and applying (7.21), (7.22), (7.24) and (7.25), we have that

$$32(288c - 360c^3) = 10(s-g)^{(iv)}(0) \cdot 144c,$$

that is,

$$(s-g)^{(iv)}(0) = \frac{32}{5} - 8c^2,$$

as we stated in (7.23). □

PROPOSITION 7.19. ([**13**]) *Let c be a real number. Then*

$$G_{-1+c,-1-c}(x,y) \le S_{-3,-3}(x,y) \tag{7.26}$$

holds for any positive x, y if and only if $|c| \le \varepsilon$.

PROOF. Suppose first that (7.26) is valid for any positive x, y.

Putting e^t and e^{-t} into the place of x and y, then taking the logarithm of the sides of (7.26), it turns to the inequality

$$s(t) - g_c(t) \ge 0 \qquad (t \in \mathbb{R}). \tag{7.27}$$

Assume that $|c| > \varepsilon$. As we know from Lemma 7.18, in this case $(s-g_c)'(0) = (s-g_c)''(0) = (s-g_c)^{(iii)}(0) = 0$, and $(s-g_c)^{(iv)}(0) < 0$, that is, the function $s - g_c$ has a strict local maximum at 0. Since its value equals 0 at 0, we have that in an appropriate place τ of the neighborhood of 0, $s(\tau) - g_c(\tau) < 0$, which contradicts to (7.27).

Suppose now that $|c| \le \varepsilon$. Then, applying the comparison theorem for Gini means (Theorem 5.1) and Theorem 7.5, we have that

$$G_{-1+c,-1-c}(x,y) \le G_{-1+\varepsilon,-1-\varepsilon}(x,y) \le S_{-3,-3}(x,y) \quad (x, y \in \mathbb{R}_+),$$

that is, we obtained (7.26). □

PROPOSITION 7.20. ([**13**]) *Let $a, b < 0$. Then*

$$G_{-\frac{2a}{a+b},-\frac{2b}{a+b}}(x,y) \le S_{-3,-3}(x,y) \tag{7.28}$$

holds for any positive x, y if and only if

$$\frac{a}{b} \in [9 - 4\sqrt{5}, 9 + 4\sqrt{5}]. \tag{7.29}$$

PROOF. Denote the number $\frac{a-b}{a+b}$ by c. In the sense of Proposition 7.19, the only thing to prove is that the inequality $\left|\frac{a-b}{a+b}\right| \le \varepsilon$ is equivalent to (7.29).

Indeend, the inequality

$$-\varepsilon \le \frac{a-b}{a+b} \le \varepsilon$$

is equivalent to

$$\frac{1-\varepsilon}{1+\varepsilon} \le \frac{a}{b} \le \frac{1+\varepsilon}{1-\varepsilon},$$

that is, to

$$9-4\sqrt{5} \le \frac{a}{b} \le 9+4\sqrt{5}.$$

□

PROPOSITION 7.21. ([**13**]) *Let* $a, b < 0$. *Then*

$$G_{a,b}(x,y) \le S_{\frac{3}{2}(a+b),\frac{3}{2}(a+b)}(x,y) \tag{7.30}$$

holds for any positive x, y *if and only if* (7.29) *is valid.*

PROOF. Since

$$M_{\lambda\alpha,\lambda\beta}(x,y) = \Big(M_{\alpha,\beta}(x^\lambda, y^\lambda)\Big)^{\frac{1}{\lambda}}$$

both for $M = G$ and $M = S$, we have that the statements

$$G_{\alpha,\beta}(x,y) \le S_{\gamma,\delta}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

and

$$G_{\lambda\alpha,\lambda\beta}(x,y) \le S_{\lambda\gamma,\lambda\delta}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

are equivalent to each other.

Applying this with $\lambda = -\frac{a+b}{2}$, $\alpha = -\frac{2a}{a+b}$, $\beta = -\frac{2b}{a+b}$, (7.28) turns to the (equivalent) (7.30), that is, both of them are equivalent to the condition (7.29). □

THEOREM 7.22.([**13**]) *Let* $a, b < 0$ *and suppose that* (7.29) *holds. Then*

$$G_{a,b}(x,y) \le S_{c,d}(x,y) \tag{7.31}$$

holds for any positive x, y *if and only if*

$$3(a+b) \le c+d \tag{7.32}$$

is valid.

PROOF. We have seen in Theorem 7.1 that (7.32) is necessary for (7.31) to hold.

For the sufficiency, suppose now that (7.32) holds. As one can immediately check, the conditions of the comparison theorem for Stolarsky means are satisfied (see Theorem 6.1) for the comparison

$$S_{\frac{3}{2}(a+b),\frac{3}{2}(a+b)}(x,y) \le S_{c,d}(x,y) \qquad (x, y \in \mathbb{R}_+).$$

Combining this with (7.30), we obtain our statement. □

REMARK 7.23. Summarizing our results: we can characterize the general Gini-Stolarsky comparison in the cases $a, b > 0$, $\min\{a, b\} = 0 < \max\{a, b\}$, $ab < 0, \quad a + b \leq 0$, finally, $a, b < 0, \quad a/b \in [9 - 4\sqrt{5}, 9 + 4\sqrt{5}]$. However, if $ab < 0$ and $a + b \geq 0$, $a, b \leq 0$, or $a, b < 0, \quad a/b \notin [9 - 4\sqrt{5}, 9 + 4\sqrt{5}]$, then we have only necessary, but not sufficient and sufficient, but not necessary conditions – so, we have these cases as open problems.

CHAPTER 8

Minkowski inequality for general two variable means

8.1. Introduction

The survey concerning the Minkowski-type (or reversed-Minkowski-type) behavior of our means has an extended literature. (In the next chapters we will refer to some of the most important papers on this topic.) In general, the question for the two variable setting is the following: under what conditions will the inequality

$$M_{a,b}(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M_{a,b}(x_1, y_1) + M_{a,b}(x_2, y_2) \tag{8.1}$$

be valid for all positive x_1, x_2, y_1, y_2? The direction "$\leq$" is called Minkowski, while the opposite is the reversed-Minkowski inequality.

A possibility to generalize this problem is that each appearance of $M_{a,b}$ is replaced by a mean of the same kind but having different parameters, that is, we ask for necessary and sufficient conditions such that

$$M_{a_0,b_0}(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M_{a_1,b_1}(x_1, y_1) + M_{a_2,b_2}(x_2, y_2) \tag{8.2}$$

be valid for all positive x_1, x_2, y_1, y_2.

In the next two chapters we will study the question (8.2) first for the Gini, then for the Stolarsky means. The following general results will prove to be useful during that process.

8.2. Minkowski inequality and convexity

In this section we show that the Minkowski inequality (8.2) can be reduced to convexity type inequalities with respect to certain one variable functions derived from the means M_{a_i,b_i}. Our considerations will be even more general here, we shall deal with Minkowski type inequalities for arbitrary two variable homogeneous means.

THEOREM 8.1.([**17**]) *Let $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be two variable homogeneous means. Then*

$$M_0(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M_1(x_1, y_1) + M_2(x_2, y_2) \tag{8.3}$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ *if and only if*

$$m_0\big(t\xi + (1-t)\eta\big) \underset{(\geq)}{\leq} tm_1(\xi) + (1-t)m_2(\eta) \tag{8.4}$$

is satisfied for all $t \in (0,1)$, $\xi, \eta \in \mathbb{R}_+$, *where*

$$m_i(t) = M_i(t,1) \qquad (t \in \mathbb{R}_+,\ i = 0,1,2).$$

PROOF. Due to the homogeneity of the means, (8.3) can equivalently be written as

$$(y_1+y_2)M_0\left(\frac{x_1+x_2}{y_1+y_2}, 1\right) \underset{(\geq)}{\leq} y_1M_1\left(\frac{x_1}{y_1}, 1\right) + y_2M_2\left(\frac{x_2}{y_2}, 1\right),$$

that is, (8.3) is equivalent to the inequality

$$m_0\left(\frac{y_1}{y_1+y_2}\frac{x_1}{y_1} + \frac{y_2}{y_1+y_2}\frac{x_2}{y_2}\right) \underset{(\geq)}{\leq} \frac{y_1}{y_1+y_2}m_1\left(\frac{x_1}{y_1}\right) + \frac{y_2}{y_1+y_2}m_2\left(\frac{x_2}{y_2}\right). \tag{8.5}$$

After defining

$$t := \frac{y_1}{y_1+y_2}, \qquad \xi := \frac{x_1}{y_1}, \qquad \eta := \frac{x_2}{y_2},$$

we can see that (8.5) is valid for all positive x_1, x_2, y_1, y_2 if and only if (8.4) holds on the domain indicated. □

If all the means in (8.3) are equal, then, as a corollary of the above result, we obtain the following characterization of homogeneous two variable means that satisfy the Minkowski or the reversed Minkowski inequality. In the special cases when the mean is a Gini or a Stolarsky mean, this result appeared in [**38**], [**39**].

COROLLARY 8.2.([**17**]) *Let* $M : \mathbb{R}_+^2 \to \mathbb{R}$ *be a two variable homogeneous mean. Then*

$$M(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M(x_1,y_1) + M(x_2,y_2) \tag{8.6}$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ *if and only if the function*

$$m(t) = M(t,1) \qquad (t \in \mathbb{R}_+,\ i = 0,1,2).$$

is convex (resp. concave) on $\mathbb{R}_+$.

PROOF. Observe that, in the case $M_0 = M_1 = M_2 = M$, condition (8.4) reduces to the convexity of the function $m_0 = m_1 = m_2 = m$. □

Using Theorem 8.1, now we derive two necessary conditions for (8.3).

THEOREM 8.3.([**17**]) *Let* $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ *be two variable homogeneous means such that* M_0 *is symmetric on* $\mathbb{R}_+^2$, *and differentiable at the point* $(1,1)$. *Assume that* (8.3) *holds for all* $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$. *Then*

$$M_0 \underset{(\geq)}{\leq} M_i \qquad \text{and} \qquad A \underset{(\geq)}{\leq} M_i \qquad (i = 1,2), \tag{8.7}$$

where A denotes the arithmetic mean.

PROOF. By Theorem 8.1, we have that (8.4) is valid for all $t \in (0,1)$, $\xi, \eta \in \mathbb{R}_+$. By substituting $u = \xi = \eta$ and taking the limits $t \to 0$ and $t \to 1$, we get that

$$m_0(u) \underset{(\geq)}{\leq} m_i(u) \qquad (u \in \mathbb{R}_+,\ i = 1, 2).$$

Using the homogeneity of the means, it follows directly that the first inequality of (8.7) is satisfied.

Substituting $\eta = 1$, using that $m_0(1) = m_2(1) = 1$, (8.4) implies

$$\frac{m_0[t(\xi-1)+1] - m_0(1)}{t} \underset{(\geq)}{\leq} m_1(\xi) - 1 \qquad (0 < t < 1,\ \xi \in \mathbb{R}_+).$$

Computing the limit as $t \to 0$ and using the differentiability of M_0 at $(1,1)$, we get that

$$\partial_1 M_0(1,1)(\xi - 1) \underset{(\geq)}{\leq} m_1(\xi) - 1 \qquad (\xi \in \mathbb{R}_+). \tag{8.8}$$

The function M_0 being a mean, we have $M_0(x,x) = x$ for all $x > 0$. Thus, differentiating this identity at $x = 1$, we derive

$$\partial_1 M_0(1,1) + \partial_2 M_0(1,1) = 1.$$

Due to the symmetry of M, we also have $\partial_1 M_0(1,1) = \partial_2 M_0(1,1)$. Hence $\partial_1 M_0(1,1) = 1/2$. Thus (8.8) reduces to

$$\frac{\xi + 1}{2} \underset{(\geq)}{\leq} m_1(\xi) \qquad (\xi \in \mathbb{R}_+),$$

which results the second inequality in (8.7) for $i = 1$. The proof in the case $i = 2$ is analogous. □

In order to derive also sufficient conditions, we need the following definition.

DEFINITION 8.4. Let $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be two variable means. A mean $M : \mathbb{R}_+^2 \to \mathbb{R}$ is called a *Minkowski-separator* (resp. *reversed-Minkowski-separator*) for (M_0, M_1, M_2) if M satisfies the (reversed) Minkowski inequality (8.6), furthermore,

$$M_0 \underset{(\geq)}{\leq} M \qquad \text{and} \qquad M \underset{(\geq)}{\leq} M_i \quad (i = 1, 2).$$

The existence of the (reversed-)Minkowski-separator yields a trivial but useful sufficient condition for the (reversed) Minkowski inequality (8.3).

THEOREM 8.5.([**17**]) *Let $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be two variable means. Suppose that there exists a (reversed-)Minkowski-separator for (M_0, M_1, M_2). Then the (reversed) Minkowski inequality* (8.3) *holds for all positive x_1, x_2, y_1, y_2.*

PROOF. Assume that M is a (reversed-)Minkowski-separator for (M_0, M_1, M_2). Then,

$$\begin{aligned} M_0(x_1+x_2, y_1+y_2) &\underset{(\geq)}{\leq} M(x_1+x_2, y_1+y_2) \\ &\underset{(\geq)}{\leq} M(x_1, y_1) + M(x_2, y_2) \\ &\underset{(\geq)}{\leq} M_1(x_1, y_1) + M_2(x_2, y_2) \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$, that is, (8.3) holds. $\square$

CHAPTER 9

Generalized Minkowski inequality for Gini means

9.1. Introduction

Minkowski's inequality for the special Gini mean with $a-b=1$ was treated by Beckenbach [**5**]. Concerning the general case

(9.1)
$$G_{a,b;n}(\mathbf{x_1}+\mathbf{x_2}) \le G_{a,b;n}(\mathbf{x_1})+G_{a,b;n}(\mathbf{x_2}) \qquad (\mathbf{x_1},\mathbf{x_2}\in\mathbb{R}_+^n,\ n=2,3,\dots),$$

Dresher and also Danskin proved that the conditions

$$0 \le \min\{a,b\} \le 1 \le \max\{a,b\} \tag{9.2}$$

are sufficient for (9.1) to hold. Losonczi [**35**] showed that the inequality (9.1) is not only sufficient but it is also necessary for (9.2) to hold. He also proved that the reverse inequality

(9.3)
$$G_{a,b;n}(\mathbf{x_1}+\mathbf{x_2}) \ge G_{a,b;n}(\mathbf{x_1})+G_{a,b;n}(\mathbf{x_2}) \qquad (\mathbf{x_1},\mathbf{x_2}\in\mathbb{R}_+^n,\ n=2,3,\dots)$$

holds if and only if

$$\min\{a,b\} \le 0 \le \max\{a,b\} \le 1 \tag{9.4}$$

is satisfied. In [**37**], the inequalities (9.1), (9.3) were characterized in the case, where the coordinates of the variables $\mathbf{x_1},\mathbf{x_2}$ vary only in a subinterval (α,β) of $\mathbb{R}_+$.

Another possibility to generalize (9.1) is that each appearance of $G_{a,b}$ is replaced by a possibly different Gini mean, that is we ask for necessary and sufficient conditions such that

(9.5)
$$G_{a_0,b_0;n}(\mathbf{x_1}+\mathbf{x_2}) \le G_{a_1,b_1;n}(\mathbf{x_1})+G_{a_2,b_2;n}(\mathbf{x_2}) \quad (\mathbf{x_1},\mathbf{x_2}\in\mathbb{R}_+^n,\ n=2,3,\dots)$$

be valid. The result obtained by Páles [**45**] states that (9.5) is valid on the domain indicated if and only if

$$\begin{aligned} &\text{(i)} && a_1,a_2,b_1,b_2\ge 0,\\ &\text{(ii)} && \max\{1,a_0,b_0\}\le\max\{a_i,b_i\}, \qquad (i=1,2),\\ &\text{(iii)} && \min\{a_0,b_0\}\le\min\{1,a_1,b_1,a_2,b_2\}. \end{aligned} \tag{9.6}$$

The reversed inequality

(9.7)
$$G_{a_0,b_0;n}(\mathbf{x_1}+\mathbf{x_2}) \geq G_{a_1,b_1;n}(\mathbf{x_1})+G_{a_2,b_2;n}(\mathbf{x_2}) \quad (\mathbf{x_1},\mathbf{x_2}\in\mathbb{R}_+^n,\ n=2,3,\dots)$$

was also characterized in [**45**]. It holds if and only if

$$\text{(9.8)}\qquad \begin{aligned} &\text{(i)} && 1\geq a_1,a_2,b_1,b_2,\\ &\text{(ii)} && \min\{0,a_0,b_0\}\geq\min\{a_i,b_i\}, \qquad (i=1,2),\\ &\text{(iii)} && \max\{a_0,b_0\}\geq\max\{0,a_1,b_1,a_2,b_2\}. \end{aligned}$$

Further methods and results were obtained by Daróczy and Losonczi [**22**], Losonczi [**34**], [**35**], Páles [**46**] for characterizing inequalities (of quite general form) involving quasiarithmetic means weighted by weightfunctions and by Daróczy [**20**] [**21**], Losonczi [**36**], Daróczy and Páles [**23**] and Páles [**47**] for more general means (deviation and quasideviation means).

In these general results, however, one has to suppose that *the inequalities hold for all* $n = 2, 3, \dots$. Fixing the number of variables n in (9.1), (9.3), (9.5), and (9.7), we obtain new problems to investigate. The first step in this direction is of course studying the case $n = 2$ and inequalities (9.1) and (9.3). This was done in the paper of Losonczi and Páles [**38**]. The main result of [**38**] can be formulated as follows.

Theorem 9.A. (Losonczi–Páles [**38**]) *Let* $a, b \in \mathbb{R}$. *Then the inequality*

$$\text{(9.9)}\qquad G_{a,b}(x_1+x_2,y_1+y_2)\leq G_{a,b}(x_1,y_1)+G_{a,b}(x_2,y_2)$$

holds if and only if

$$\text{(9.10)}\qquad 0\leq\min\{a,b\}\leq 1\leq a+b.$$

Our aim is to characterize the situation when the more general inequality

$$\text{(9.11)}\qquad G_{a_0,b_0}(x_1+x_2,y_1+y_2)\leq G_{a_1,b_1}(x_1,y_1)+G_{a_2,b_2}(x_2,y_2)$$

holds for all positive x_1, y_1, x_2, y_2. Our main result is contained in the following theorem.

THEOREM 9.1.([**16**]) *Let* $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$. *Then* (9.11) *holds if and only if*

$$\text{(9.12)}\qquad \begin{aligned} &\text{(i)} && a_1,a_2,b_1,b_2\geq 0,\\ &\text{(ii)} && \max\{1,a_0+b_0\}\leq\min\{a_1+b_1,a_2+b_2\},\\ &\text{(iii)} && \min\{a_0,b_0\}\leq\min\{1,a_1,b_1,a_2,b_2\}. \end{aligned}$$

The proof of the necessity of conditions (i)–(iii) of this result will be obtained with the help of a sequence of lemmas. The proof of the sufficiency is based on Theorem 9.A, since, as it will turn out, conditions (i)–(iii) of Theorem 9.1 are necessary and sufficient for the existence of some parameters $a, b \in \mathbb{R}$ such that (9.9) is valid and

$$G_{a_0,b_0} \le G_{a,b}, \qquad G_{a,b} \le G_{a_1,b_1}, \qquad G_{a,b} \le G_{a_2,b_2}$$

hold. Thus any inequality of the form (9.11) is a *weakening* of inequality (9.9) for some $a, b \in \mathbb{R}$.

Concerning the inequality

$$G_{a,b}(x_1 + x_2, y_1 + y_2) \ge G_{a,b}(x_1, y_1) + G_{a,b}(x_2, y_2), \tag{9.13}$$

which is reversed to (9.9), there are only necessary (but not sufficient) and sufficient (but not necessary) conditions presented in [**38**]. Therefore, the investigation of the inequality reversed to (9.11) is left as an open problem.

It is interesting to note that the analogous problems, that is, the Minkowski and reversed Minkowski inequalities for the so called Stolarsky means can be characterized completely (see Losonczi–Páles [**39**]).

9.2. Proof of the generalized Minkowski inequality for Gini means

Because of the symmetry, we can assume that

$$a_0 \le b_0, \qquad a_1 \le b_1, \qquad a_2 \le b_2. \tag{9.14}$$

(Necessity.)

Assume now that the Minkowski inequality (9.11) holds.

Theorem 8.3 yields that for $i = 1, 2$

$$G_{0,1}(x_1, y_1) \le G_{a_i,b_i}(x_1, y_1) \qquad (x_1, y_1 \in \mathbb{R}_+). \tag{9.15}$$

Consequently, by Theorem 5.1, $0 + 1 \le a_i + b_i$, therefore,

$$1 \le \min\{a_1 + b_1, a_2 + b_2\}. \tag{9.16}$$

Using (9.14), (9.16) implies that that $b_1, b_2 > 0$. If a_1, for example, were negative, then (9.15) would also yield that $\mathcal{E}(1, 0) \le \mathcal{E}(a_1, b_1)$, that is, $1 \le \frac{|a_1|-|b_1|}{a_1-b_1}$. This inequality, however, implies $a_1 \ge 0$. The contradiction obtained shows that $a_1 \ge 0$ and similarly $a_2 \ge 0$. Thus (9.12)(i) is proved.

In order to complete (9.12)(ii), one may apply Theorem 8.3, again: for $i = 1, 2$

$$G_{a_0,b_0}(x_1, y_1) \le G_{a_i,b_i}(x_1, y_1) \qquad (x_1, y_1 \in \mathbb{R}_+). \tag{9.17}$$

Thus, by Theorem 5.1, the inequality $a_0 + b_0 \le a_i + b_i$ holds. These inequalities, together with (9.16), yield (9.12)(ii).

To obtain (9.12)(iii), we show first that

$$\min\{a_0, b_0\} \le \min\{a_1, b_1, a_2, b_2\}. \tag{9.18}$$

In the case $\min\{a_0, b_0\} < 0$, (9.18) is obvious because the right hand side is nonnegative. In case of $\min\{a_0, b_0\} \geq 0$, (9.17) and Theorem 5.1 yield

$$\min\{a_0, b_0\} \leq \min\{a_1, b_1\}.$$

Similarly

$$\min\{a_0, b_0\} \leq \min\{a_2, b_2\}.$$

Hence (9.18) is valid.

To complete the proof, we have only to show that $\min\{a_0, b_0\} \leq 1$. On the contrary, suppose that $a_0, b_0 > 1$.

We know, by (9.18), that in this case $a_1, b_1, a_2, b_2 > 1$ (and consequently $a_1, b_1, a_2, b_2 > 0$). Taking the limit $y_1 \to 0$ in (9.11) and applying Lemma 3.3, we obtain

$$G_{a_0,b_0}(x_1, y_1 + y_2) \leq G_{a_1,b_1}(x_1, y_1) + y_2 \qquad (x_1, y_1, y_2 \in \mathbb{R}_+).$$

Thus

$$\lim_{y_2 \to \infty} \big(G_{a_0,b_0}(x_1, y_1 + y_2) - y_2\big) \leq G_{a_1,b_1}(x_1, y_1) \qquad (x_1, y_1 \in \mathbb{R}_+). \tag{9.19}$$

By Lemma 3.6, the limit of the left hand side is y_1, that is,

$$y_1 \leq G_{a_1,b_1}(x_1, y_1),$$

for all positive x_1 and y_1. If $x_1 < y_1$, then the inequality obtained contradicts the mean value property of $G_{a,b}$.

Thus the proof of Theorem 9.1 is completed.

(Sufficiency.)

Define

$$a := \min\{a_1, a_2, 1\}, \qquad b := \min\{a_1 + b_1, a_2 + b_2\} - a.$$

We are going to prove that $G_{a,b}$ is a Minkowski-separator for G_{a_0,b_0}, G_{a_1,b_1} and G_{a_2,b_2}. In the sense of Theorem 8.5, this will guarantee our statement. We have to check the following three statements.

(I) $G_{a,b}$ satisfies the Minkowski inequality (9.9),

(II) $G_{a_0,b_0} \leq G_{a,b}$,

(III) $G_{a,b} \leq G_{a_i,b_i}$ $(i = 1, 2)$.

In order to prove (I), we have to verify that (9.10) holds. By the definition of a and (9.12)(i), we get that $0 \leq a \leq 1$. By (9.12)(ii), we also have

$$\min\{a_1 + b_1, a_2 + b_2\} \geq 1 \geq a.$$

Thus

$$b = \min\{a_1 + b_1, a_2 + b_2\} - a \geq 0,$$

whence $0 \leq \min\{a, b\} \leq 1$. Using again the definition of a, b, and (9.12)(ii), we obtain

$$a + b = \min\{a_1 + b_1, a_2 + b_2\} \geq 1. \tag{9.20}$$

Therefore $G_{a,b}$ satisfies the Minkowski inequality (9.9).

In order to prove (II), we distinguish two cases.

If $a_0 < 0$, then $G_{a_0,b_0} \le G_{a,b}$ holds if and only if

$$a_0 + b_0 \le a + b, \qquad \mathcal{E}(a_0, b_0) \le \mathcal{E}(a, b). \tag{9.21}$$

The first inequality follows from (9.12)(ii):

$$a_0 + b_0 \le \max\{1, a_0 + b_0\} \le \min\{a_1 + b_1, a_2 + b_2\} = a + b.$$

Due to (9.20), $\max\{a, b\} > 0$. Therefore, we have that $\mathcal{E}(a, b) = 1$. Thus, by Lemma 2.5, the second inequality in (9.21) is obvious.

If $a_0 \ge 0$ then $G_{a_0,b_0} \le G_{a,b}$ holds if and only if

$$a_0 + b_0 \le a + b, \qquad \min\{a_0, b_0\} \le \min\{a, b\}. \tag{9.22}$$

The proof of the first inequality coincides with that of the previous case. To obtain the second inequality, we show that $a_0 \le a$ and $a_0 \le b$. Since then

$$\min\{a_0, b_0\} = a_0 \le \min\{a, b\}.$$

By (9.12)(iii),

$$a_0 = \min\{a_0, b_0\} \le \min\{1, a_1, b_1, a_2, b_2\} = \min\{1, a_1, a_2\} = a.$$

In order to obtain $a_0 \le b$ we need to show that

$$a_0 \le \min\{a_1 + b_1, a_2 + b_2\} - \min\{a_1, a_2, 1\},$$

which is equivalent to the inequalities

$$a_0 + \min\{a_1, a_2, 1\} \le a_i + b_i \qquad (i = 1, 2).$$

On the other hand, by (9.12)(iii) and (9.14)

$$a_0 + \min\{a_1, a_2, 1\} \le a_0 + a_i \le a_i + a_i \le a_i + b_i \qquad (i = 1, 2).$$

Thus we have proved (II).

To obtain (III), we have to show that

$$a + b \le a_i + b_i, \qquad \min\{a, b\} \le \min\{a_i, b_i\} \qquad (i = 1, 2). \tag{9.23}$$

The first inequality obviously follows from the definition of b. On the other hand,

$$\min\{a, b\} \le a = \min\{a_1, a_2, 1\} \le a_i = \min\{a_i, b_i\} \qquad (i = 1, 2),$$

therefore the second inequality of (9.23) is also valid. Thus the proof of (III) is also complete.

Using the above method, one can get the following generalization of Theorem 9.1.

THEOREM 9.2.([**16**]) *Let* $k \geq 2$ *and* $a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_k \in \mathbb{R}$. *Then*

$$G_{a_0,b_0}((x_1, y_1) + \cdots + (x_k, y_k)) \leq G_{a_1,b_1}(x_1, y_1) + \cdots + G_{a_k,b_k}(x_k, y_k)$$

holds for all positive $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ *if and only if*

(i) $a_1, \ldots, a_k, b_1, \ldots, b_k \geq 0,$

(ii) $\max\{1, a_0 + b_0\} \leq \min\{a_1 + b_1, \ldots, a_k + b_k\},$

(iii) $\min\{a_0, b_0\} \leq \min\{1, a_1, b_1, \ldots, a_k, b_k\}.$

CHAPTER 10

Generalized Minkowski inequality for Stolarsky means

10.1. Introduction

In a paper [**39**] by Losonczi and Páles, necessary and sufficient conditions for the Minkowski and the reversed Minkowski inequality have been found. The results contained in Theorems 1 and 2 of this paper can be formulated in the following united form.

Theorem 10.A. *Let $a, b \in \mathbb{R}$. Then the Minkowski inequality (or the reversed Minkowski inequality)*

$$S_{a,b}(x_1 + x_2, y_1 + y_2) \underset{(\geq)}{\leq} S_{a,b}(x_1, y_1) + S_{a,b}(x_2, y_2) \tag{10.1}$$

holds for all positive x_1, x_2, y_1, y_2 if and only if the conditions

$$3 \underset{(\geq)}{\leq} a + b \qquad \text{and} \qquad 1 \underset{(\geq)}{\leq} \min\{a, b\}. \tag{10.2}$$

are satisfied.

A possibility to generalize (10.1) is that each appearance of $S_{a,b}$ is replaced by a possibly different Stolarsky mean, that is, we ask for necessary and sufficient conditions such that

$$S_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \underset{(\geq)}{\leq} S_{a_1,b_1}(x_1, y_1) + S_{a_2,b_2}(x_2, y_2) \tag{10.3}$$

be valid for all positive x_1, x_2, y_1, y_2.

10.2. Necessary conditions

The main result of this section offers a necessary condition for the Minkowski inequality (10.3). As we shall see below, the condition is not sufficient in the general case.

THEOREM 10.1.([**17**]) *Let $a_0, b_0, a_1, b_1, a_2, b_2 \in \mathbb{R}$. If the Minkowski inequality*

$$S_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \leq S_{a_1,b_1}(x_1, y_1) + S_{a_2,b_2}(x_2, y_2) \tag{10.4}$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$*, then*

$$\left\{\begin{array}{rcl} \max\{a_0+b_0,3\} & \le & \min\{a_1+b_1, a_2+b_2\} \\ \max\big\{\mathcal{L}(a_0,b_0),\mathcal{L}(1,2)\big\} & \le & \min\big\{\mathcal{L}(a_1,b_1),\mathcal{L}(a_2,b_2)\big\}. \end{array}\right. \tag{10.5}$$

If the reversed Minkowski inequality

$$S_{a_0,b_0}(x_1+x_2, y_1+y_2) \ge S_{a_1,b_1}(x_1,y_1) + S_{a_2,b_2}(x_2,y_2) \tag{10.6}$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$*, then*

$$\left\{\begin{array}{rcl} \min\{a_0+b_0,3\} & \ge & \max\{a_1+b_1, a_2+b_2\} \\ \min\big\{\mathcal{L}(a_0,b_0),\mathcal{L}(1,2)\big\} & \ge & \max\big\{\mathcal{L}(a_1,b_1),\mathcal{L}(a_2,b_2)\big\} \\ \mathcal{E}(a_0,b_0) & \ge & \max\big\{\mathcal{E}(a_1,b_1),\mathcal{E}(a_2,b_2)\big\}. \end{array}\right. \tag{10.7}$$

PROOF. Assume that (10.4) holds on the domain indicated. Then, by Theorem 8.3,

$$S_{a_0,b_0} \le S_{a_i,b_i} \qquad S_{1,2} = A \le S_{a_i,b_i} \qquad (i = 1,2).$$

Due to our Theorem 8.1, these inequalities yield that, for $i = 1, 2$,

$$a_0 + b_0 \le a_i + b_i, \qquad \mathcal{L}(a_0,b_0) \le \mathcal{L}(a_i,b_i), \qquad \mathcal{E}(a_0,b_0) \le \mathcal{E}(a_i,b_i), \tag{10.8}$$

and

$$1 + 2 \le a_i + b_i, \qquad \mathcal{L}(1,2) \le \mathcal{L}(a_i,b_i), \qquad \mathcal{E}(1,2) \le \mathcal{E}(a_i,b_i). \tag{10.9}$$

The value of $\mathcal{L}(a,b)$ is positive if and only if $0 < \min\{a,b\}$, therefore $\mathcal{L}(1,2) \le \mathcal{L}(a_i,b_i)$ yields that $0 < \min\{a_i,b_i\}$. Thus $\mathcal{E}(a_i,b_i) = 1$ and the conditions

$$\mathcal{E}(a_0,b_0) \le \mathcal{E}(a_i,b_i), \qquad \mathcal{E}(1,2) \le \mathcal{E}(a_i,b_i)$$

hold automatically if the second inequality of (10.9) is valid. Combining the first two inequalities of (10.8) and (10.9), the condition (10.5) of the theorem follows.

Now assume that the reversed Minkowski inequality (10.6) holds. Then Theorem 8.3 yields

$$S_{a_0,b_0} \ge S_{a_i,b_i} \qquad S_{1,2} = A \ge S_{a_i,b_i} \qquad (i = 1,2).$$

By Theorem 8.1, it follows from these inequalities that, for $i = 1, 2$,

$$a_0 + b_0 \ge a_i + b_i, \qquad \mathcal{L}(a_0,b_0) \ge \mathcal{L}(a_i,b_i), \qquad \mathcal{E}(a_0,b_0) \ge \mathcal{E}(a_i,b_i), \tag{10.10}$$

and

$$3 \ge a_i + b_i, \qquad \mathcal{L}(1,2) \ge \mathcal{L}(a_i,b_i), \qquad \mathcal{E}(1,2) \ge \mathcal{E}(a_i,b_i). \tag{10.11}$$

The last inequality in (10.11) is always valid since $\mathcal{E}(1,2) = 1 \ge \mathcal{E}(a_i,b_i)$. Thus, it can be omitted. Combining the rest of the inequalities from (10.10) and (10.11), we can deduce the condition (10.7). $\square$

REMARK 10.2. The conditions (10.5) and (10.7) of the theorem are not sufficient for (10.5) and (10.6) to hold, since even in the case $a_0 = a_1 = a_2 = a$, $b_0 = b_1 = b_2 = b$ they are different from the corresponding condition of (10.2) which is necessary and sufficient for (10.4) and (10.6), that is, for (10.1) in this setting.

10.3. Minkowski-separators and sufficient conditions

Introduce the following notations

$$\begin{aligned} \mathcal{M} &= \big\{(a,b) \in \mathbb{R}^2 \,:\, 3 \le a+b,\ 1 \le \min\{a,b\}\big\}, \\ \mathcal{M}^* &= \big\{(a,b) \in \mathbb{R}^2 \,:\, a+b \le 3,\ \min\{a,b\} \le 1\big\}. \end{aligned}$$

By Theorem 10.A, a pair (a,b) belongs to $\mathcal{M}$ (resp. $\mathcal{M}^*$) if and only if the Minkowski (resp. reversed Minkowski) inequality (10.1) holds.

DEFINITION 10.3. We say that (a,b) is a *Minkowski-separator* for $(a_0,b_0,a_1,b_1,a_2,b_2) \in \mathbb{R}^6$ if

$$(a,b) \in \mathcal{M}, \qquad S_{a_0,b_0} \le S_{a,b} \qquad \text{and} \qquad S_{a,b} \le S_{a_i,b_i} \quad (i=1,2).$$

We say that (a,b) is a *reversed-Minkowski-separator* for $(a_0,b_0,a_1,b_1,a_2,b_2) \in \mathbb{R}^6$ if

$$(a,b) \in \mathcal{M}^*, \qquad S_{a,b} \le S_{a_0,b_0} \qquad \text{and} \qquad S_{a_i,b_i} \le S_{a,b} \quad (i=1,2).$$

Clearly, (a,b) is a (reversed-)Minkowski-separator for $(a_0,b_0,a_1,b_1,a_2,b_2) \in \mathbb{R}^6$ if and only if the mean $S_{a,b}$ is a (reversed-) Minkowski-separator for the means $(S_{a_0,b_0}, S_{a_1,b_1}, S_{a_2,b_2})$. Due to Theorem 8.5, we have the following sufficient condition for the inequalities (10.4) and (10.6).

COROLLARY 10.4.([**17**]) *Let $a_0,b_0,a_1,b_1,a_2,b_2 \in \mathbb{R}$. Suppose that there exists a Minkowski-separator (resp. reversed-Minkowski-separator) for $(a_0,b_0,a_1,b_1,a_2,b_2)$. Then the Minkowski inequality* (10.4) *(resp. the reversed Minkowski inequality* (10.6)*) holds for all positive x_1,x_2,y_1,y_2.*

In the following theorems we characterize the existence of a (reversed-)Minkowski-separator, therefore, in view of the previous theorem, we obtain sufficient conditions for the (reversed) Minkowski inequality.

THEOREM 10.5.([**17**]) *For $(a_0,b_0,a_1,b_1,a_2,b_2) \in \mathbb{R}^6$ there exists a Minkowski-separator if and only if*

$$\left\{ \begin{aligned} \max\{a_0+b_0, 3\} &\le \min\{a_1+b_1, a_2+b_2\}, \\ \max\big\{\mathcal{L}(a_0,b_0), \mathcal{L}(1,2),\ &\mathcal{L}(1, a_0+b_0-1)\big\} \\ &\le \min\big\{\mathcal{L}(a_1,b_1), \mathcal{L}(a_2,b_2)\big\} \end{aligned} \right. \tag{10.12}$$

THEOREM 10.6.([**17**]) *For* $(a_0, b_0, a_1, b_1, a_2, b_2) \in \mathbb{R}^6$ *there exists a reversed-Minkowski-separator if and only if*

$$(10.13)\quad \begin{cases} \min\{a_0 + b_0, 3\} & \geq & \max\{a_1 + b_1, a_2 + b_2\}, \\ \min\big\{\mathcal{L}(a_0, b_0), \mathcal{L}(1, 2)\big\} & \geq & \max\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\}, \\ \mathcal{E}(a_0, b_0) & \geq & \max\big\{\mathcal{E}(a_1, b_1), \mathcal{E}(a_2, b_2)\big\}, \\ \text{finally, if } \min\{a_0, b_0\} \geq 1 \text{ then} & & \\ \mathcal{L}(1, a_0 + b_0 - 1) & \geq & \max\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\}. \end{cases}$$

REMARK 10.7. In view of Corollary 10.4, we can see that (10.12) and (10.13) are sufficient conditions for the Minkowski inequality (10.4) and the reversed Minkowski inequality (10.6), respectively.
One can also see that in the case $a_0 = a_1 = a_2 = a$, $b_0 = b_1 = b_2 = b$ these conditions are also necessary and reduce to the necessary and sufficient condition (10.2) of Theorem 10.A.

PROOF OF THEOREM 10.5. *(Necessity.)* Suppose that there exists a Minkowski-separator $(a, b) \in \mathcal{M}$ for $(a_0, b_0, a_1, b_1, a_2, b_2)$. Then, by Corollary 10.4, the Minkowski inequality (10.4) holds. Thus, by Theorem 10.1, we have that condition (10.5) is satisfied. In order to prove that (10.12) is valid, we have to show that

$$\mathcal{L}(1, a_0 + b_0 - 1) \leq \min\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\}$$

holds, too.

Then, $\min(a, b) \geq 1$, hence, by Lemma 2.8, we get that $\mathcal{L}(1, a + b - 1) \leq \mathcal{L}(a, b)$. Due to the inequalities $S_{a_0,b_0} \leq S_{a,b} \leq S_{a_i,b_i}$ (and Theorem 9.A), we also have $a_0 + b_0 \leq a + b$ and $\mathcal{L}(a, b) \leq \mathcal{L}(a_i, b_i)$ ($i = 1, 2$). Therefore, using the monotonicity property of $\mathcal{L}$, it follows that

$$\mathcal{L}(1, a_0 + b_0 - 1) \leq \mathcal{L}(1, a + b - 1) \leq \mathcal{L}(a, b) \leq \min\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\},$$

that is, we obtain our statement.

(Sufficiency.) Suppose now that the condition (10.12) holds. We show that in this case there exists an appropriate Minkowski-separator.

Case 1: $3 \leq a_0 + b_0$ and $\min\{a_0, b_0\} < 1$.

Then $(a, b) = (1, a_0 + b_0 - 1)$ is Minkowski-separator. Indeed, it is clear that $(a, b) \in \mathcal{M}$. Moreover, $a + b = a_0 + b_0$, therefore, by the first inequality of (10.12),

$$a_0 + b_0 \leq a + b \leq a_i + b_i \qquad (i = 1, 2).$$

The inequalities

$$\mathcal{L}(a_0, b_0) \leq \mathcal{L}(a, b), \qquad \mathcal{E}(a_0, b_0) \leq \mathcal{E}(a, b),$$

trivially hold since $\mathcal{E}(a, b) = 1$ and Lemma 2.8 yields

$$\mathcal{L}(a_0, b_0) \leq \mathcal{L}(1, a_0 + b_0 - 1) = \mathcal{L}(a, b).$$

Thus, by Theorem 9.A, we get that $S_{a_0,b_0} \leq S_{a,b}$.

Finally, by the last inequality of (10.12),

$$\mathcal{L}(a,b) = \mathcal{L}(1, a_0 + b_0 - 1) \leq \mathcal{L}(a_i, b_i) \qquad (i = 1,2).$$

Since $\mathcal{L}(1,2) \leq \mathcal{L}(a_i, b_i)$, hence $\min\{a_i, b_i\} > 0$. Therefore,

$$\mathcal{E}(a,b) = \mathcal{E}(1, a_0 + b_0 - 1) \leq 1 = \mathcal{E}(a_i, b_i) \qquad (i = 1,2).$$

Thus we have also proved that $S_{a,b} \leq S_{a_i,b_i}$. Consequently, (a,b) is a Minkowski-separator.

Case 2: $3 \leq a_0 + b_0$ and $1 \leq \min\{a_0, b_0\}$.

Using Theorem 9.A, it can immediately be checked that (a_0, b_0) is an appropriate Minkowski-separator.

Case 3: $a_0 + b_0 < 3$ and $\mathcal{L}(a_0, b_0) \leq \mathcal{L}(1,2)$.

Again, by Theorem 9.A, it can easily be seen that $(1,2)$ is suitable for being a Minkowski-separator.

Case 4: $a_0 + b_0 < 3$ and $\mathcal{L}(1,2) < \mathcal{L}(a_0, b_0)$.

By the symmetry, we may assume that $a_0 \leq b_0$. We show first that $1 \leq a_0$. In the opposite case, $a_0 \leq b_0 < 3 - a_0$ implies that $\min\{a_0, 3 - a_0\} = a_0 < 1$, therefore by Lemma 2.8 we would get

$$\mathcal{L}(a_0, b_0) < \mathcal{L}(a_0, 3 - a_0) \leq \mathcal{L}(1, a_0 + 3 - a_0 - 1) = \mathcal{L}(1,2),$$

contradicting the conditions of Case 4.

Now, let us choose such a real number $c \in [1, 3/2]$ such that

$$\mathcal{L}(a_0, b_0) = \mathcal{L}(c, 3 - c)$$

be valid. In order to see that such a value c exists, define the continuous function $\varphi(t) = \mathcal{L}(t, 3 - t)$. Applying the conditions of this case, and the monotonicity properties of $\mathcal{L}$, we find that

$$\varphi(1) = \mathcal{L}(1,2) < \mathcal{L}(a_0, b_0) < \mathcal{L}(a_0, 3 - a_0) = \varphi(a_0),$$

$\mathcal{L}(c, 3-c)\varphi(c) = \mathcal{L}(a_0, b_0)$ holds. Define $(a,b) = (c, 3-c)$. We verify that this is a Minkowski-separator.

As $1 \leq c \leq \frac{3}{2}$, we get that $(a,b) \in \mathcal{M}$. Applying the conditions of this case, it is clear that

$$a_0 + b_0 \leq a + b = c + (3 - c) = 3$$

and

$$\mathcal{L}(a_0, b_0) = \mathcal{L}(a,b) = \mathcal{L}(c, 3-c), \qquad \mathcal{E}(a_0, b_0) \leq \mathcal{E}(a,b) = \mathcal{E}(c, 3-c) = 1,$$

furthermore, by (10.12),

$$a + b = 3 \leq a_i + b_i \quad \text{and} \quad \mathcal{L}(a,b) \leq \mathcal{L}(a_i, b_i), \quad \mathcal{E}(a,b) \leq \mathcal{E}(a_i, b_i) = 1$$

are valid ($i = 1,2$). Thus, by Theorem 9.A, we get that $S_{a_0,b_0} \leq S_{a,b} \leq S_{a_i,b_i}$ is valid, i.e., $(c, 3-c)$ is the desired Minkowski separator. □

PROOF OF THEOREM 10.6. *(Necessity.)* Suppose now that (a,b) is a reversed Minkowski-separator for $(a_0,b_0,a_1,b_1,a_2,b_2)$. Then, by Corollary 10.4, the reversed Minkowski inequality (10.6) holds. Thus, by Theorem 8.5, condition (10.7) is also satisfied. In order to verify (10.13), we need to show that if $\min\{a_0,b_0\} \geq 1$ then

$$\mathcal{L}(1, a_0+b_0-1) \geq \min\big\{\mathcal{L}(a_1,b_1), \mathcal{L}(a_2,b_2)\big\}. \tag{10.14}$$

We distinguish two cases.

Case 1: $a+b \leq 2$.

First we show that, in this case, $\mathcal{L}(a,b) \leq 1$. This inequality is trivial if $\min\{a,b\} \leq 0$ since then $\mathcal{L}(a,b) \leq 0$. If $\min\{a,b\} > 0$, then, by the concavity of $\mathcal{L}$, we get

$$\mathcal{L}(a,b) = \frac{\mathcal{L}(a,b)+\mathcal{L}(b,a)}{2} \leq \mathcal{L}\Big(\frac{a+b}{2}, \frac{b+a}{2}\Big) = \frac{a+b}{2} \leq 1.$$

Thus, the inequalities $S_{a_i,b_i} \leq S_{a,b}$ and $\min\{a_0,b_0\} \geq 1$ yield

$$\max\big\{\mathcal{L}(a_1,b_1), \mathcal{L}(a_2,b_2)\big\} \leq \mathcal{L}(a,b) \leq 1 = \mathcal{L}(1,1) \leq \mathcal{L}(1,a_0+b_0-1).$$

Case 2: $a+b > 2$.

Then, by Lemma 2.8, the inequality $\min\{a,b\} \leq 1$, yields that $\mathcal{L}(1,a+b-1) \geq \mathcal{L}(a,b)$. Due to the inequalities $S_{a_0,b_0} \geq S_{a,b} \geq S_{a_i,b_i}$ (and Theorem 9.A), we also have $a_0+b_0 \geq a+b$ and $\mathcal{L}(a,b) \geq \mathcal{L}(a_i,b_i)$ ($i=1,2$). Therefore, using the monotonicity properties of $\mathcal{L}$, it follows that

$$\mathcal{L}(1,a_0+b_0-1) \geq \mathcal{L}(1,a+b-1) \geq \mathcal{L}(a,b) \geq \max\big\{\mathcal{L}(a_1,b_1), \mathcal{L}(a_2,b_2)\big\}.$$

(Sufficiency.) Suppose now that the condition (10.13) holds. We are going to prove that there exists an appropriate reversed-Minkowski-separator.

Case 1: $a_0+b_0 \leq 3$ and $1 < \min\{a_0,b_0\}$.

We show that $(a,b) = (1, a_0+b_0-1)$ is a reversed-Minkowski-separator. Clearly, $(a,b) \in \mathcal{M}^*$. Moreover, $a+b = a_0+b_0$, hence, by the first inequality of (10.13),

$$a_0+b_0 \leq a+b \leq a_i+b_i \qquad (i=1,2).$$

The inequality $\mathcal{E}(a,b) \leq \mathcal{E}(a_0,b_0)$ trivially holds since $\mathcal{E}(a_0,b_0) = 1$.
By Lemma 2.8, we also have

$$\mathcal{L}(a,b) = \mathcal{L}(1,a_0+b_0-1) \leq \mathcal{L}(a_0,b_0).$$

Thus, due to Theorem 9.A, we deduce that $S_{a,b} \leq S_{a_0,b_0}$.

By the conditions of this case and the third inequality of (10.13),

$$\mathcal{L}(a_i,b_i) \leq \mathcal{L}(1,a_0+b_0-1) = \mathcal{L}(a,b) \quad \text{and} \quad \mathcal{E}(a_i,b_i) \leq \mathcal{E}(1,a_0+b_0-1) = 1$$

are valid ($i=1,2$). Thus we have also proved that $S_{a,b} \leq S_{a_i,b_i}$. Therefore, (a,b) is indeed a reversed-Minkowski-separator.

Case 2: $a_0+b_0 \leq 3$ and $\min\{a_0,b_0\} \leq 1$.

Due to Theorem 9.A, one can see that (a_0, b_0) is an appropriate reversed-Minkowski-separator.

Case 3: $3 < a_0 + b_0$ and $\mathcal{L}(1,2) \le \mathcal{L}(a_0, b_0)$.

Again, by Theorem 9.A, it can be obtained that $(1,2)$ is a reversed-Minkowski-separator.

Case 4: $3 < a_0 + b_0$ and $\mathcal{L}(a_0, b_0) < \mathcal{L}(1,2)$.

We consider two subcases. If $\min\{a_0, b_0\} > 0$ then define the function $\varphi(t) = \mathcal{L}(t, 3-t)$. By the assumptions of the case, $0 = \varphi(0) < \mathcal{L}(a_0, b_0) < \mathcal{L}(1,2) = \varphi(1)$. Thus, by the continuity of φ, there exists a number $0 < c < 1$ such that

$$\mathcal{L}(a_0, b_0) = \mathcal{L}(c, 3-c). \tag{10.15}$$

Clearly, we also have

$$\mathcal{E}(a_0, b_0) = \mathcal{E}(c, 3-c) \tag{10.16}$$

(because both sides are equal to 1).

In the case $\min\{a_0, b_0\} \le 0$ define the function $\psi(t) = \mathcal{E}(t, 3-t)$. Then $\psi(0) = 1$ and $\lim_{t\to-\infty} \psi(t) = 0$. On the other hand, due to the conditions of this case, we have $0 < \mathcal{E}(a_0, b_0) \le 1$. Thus, there exists a number $c \le 0$ such that (10.16) holds. Then, trivially, (10.15) is also valid (because both sides are equal to 0).

Define now $(a, b) = (c, 3-c)$. We are going to show that (a, b) is a reversed-Minkowski-separator.

Since $c < 1$, we get that $(a, b) \in \mathcal{M}^*$. Applying the conditions of this case, it is clear that

$$a + b = c + (3 - c) = 3 \le a_0 + b_0.$$

This inequality, together with (10.15) and (10.16) implies that $S_{a,b} \le S_{a_0,b_0}$.

Furthermore, by the first condition of (10.13) and (10.15), (10.16),

$$a_i + b_i \le 3 = a + b,$$

and

$$\mathcal{L}(a_i, b_i) \le \mathcal{L}(a_0, b_0) = \mathcal{L}(a, b), \quad \mathcal{E}(a_i, b_i) \le \mathcal{E}(a_0, b_0) = \mathcal{E}(a, b)$$

for $i = 1, 2$. Therefore, we get that the inequality $S_{a_i,b_i} \le S_{a,b}$ is also valid, i.e., $(c, 3-c)$ is the desired reversed Minkowski separator. $\square$

Summary

The Gini and Stolarsky means have an extended literature. These means are defined for two variables, in the most general case, by

$$G_{a,b}(x,y) = \left(\frac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}}$$

and

$$S_{a,b}(x,y) = \left(\frac{b(x^a - y^a)}{a(x^b - y^b)}\right)^{\frac{1}{a-b}},$$

respectively. (The complete definitions, covering all cases can be read in Chapter 1.)

In this thesis I tried to collect and systematize the results of the previous years, obtained under the supervision of Professor Zsolt Páles. (The order of the chapters does not correspond to the chronological order of the results involved – looking backwards, hidden interconnections became known.)

I put in the center of my studies the various comparison theorems for the Gini and/or Stolarsky means, and the generalized Minkowski inequalities regarding them. To handle (state and prove) these propositions, some preparations were needed. In Chapter 2 I collected some well-known, but important identities. Moreover, for sake of the organic treatment, I introduced the functions $\mathcal{E}$, $\mathcal{M}$ and $\mathcal{L}$, which play the key roles in the comparison theorems, defined by

$$\mathcal{E}(a,b) := \begin{cases} \dfrac{|a|-|b|}{a-b}, & \text{if } a \neq b, \\ \operatorname{sign}(a), & \text{if } a = b, \end{cases} \qquad \mathcal{M}(a,b) := \begin{cases} \min\{a,b\}, & \text{if } a,b \geq 0, \\ 0, & \text{if } ab < 0, \\ \max\{a,b\}, & \text{if } a,b \leq 0 \end{cases}$$

and

$$\mathcal{L}(a,b) := \begin{cases} \dfrac{a-b}{\log(a/b)}, & \text{if } 0 < ab \text{ and } a \neq b, \\ a, & \text{if } 0 < ab \text{ and } a = b, \\ 0, & \text{if } ab \leq 0. \end{cases}$$

Their basic properties (for example, continuity, convexity, monotonicity) were also examined here.

Many of the inequalities for the Gini and Stolarsky means can be deduced from certain asymptotic properties. For, in Chapter 3 I listed some of the more

important asymptotic properties. A few of them could be formulated in a uniform way. (Throughout this thesis, I used the notation M for the means in the formulae, valid both for the Gini and the Stolarsky means.) In this field we obtained that for any real numbers a and b the following statements hold:

THEOREM.

$$\lim_{x\to 0+} G_{a,b}(x,y) = \begin{cases} y, & \textit{if} \min\{a,b\}>0, \\ y\cdot 2^{-\frac{1}{\max\{a,b\}}}, & \textit{if} \min\{a,b\}=0 \,\&\, \max\{a,b\}>0, \\ 0, & \textit{if} \min\{a,b\}<0 \textit{ or } (a,b)=(0,0), \end{cases}$$

$$\lim_{x\to 0+} S_{a,b}(x,y) = \begin{cases} y\cdot e^{-\frac{1}{\mathcal{L}(a,b)}}, & \textit{if} \min\{a,b\}>0, \\ 0, & \textit{if} \min\{a,b\}\le 0, \end{cases}$$

$$\begin{aligned} \lim_{z\to\infty}\big(M_{a,b}(x+z,y+z)-z\big) &= \frac{x+y}{2} \qquad (x,y\in\mathbb{R}_+), \\ \lim_{z\to\infty}\big(G_{a,b}(x,y+z)-z\big) &= y, \qquad (x,y\in\mathbb{R}_+), \\ \lim_{t\to 1}\frac{M_{a,b}(t,1)-\frac{t+1}{2}}{(t-1)^2} &= \begin{cases} \frac{a+b-1}{8}, & \textit{if } M=G, \\ \frac{a+b-3}{24}, & \textit{if } M=S, \end{cases} \\ \lim_{t\to\infty}\frac{\log M_{a,b}(e^t,e^{-t})}{t} &= \mathcal{E}(a,b), \\ \lim_{t\to\infty}\frac{\log\dfrac{t+\log G_{a,b}(e^t,e^{-t})}{t-\log G_{a,b}(e^t,e^{-t})}}{2t} &= \mathcal{M}(a,b), \end{aligned}$$

moreover, if $a,b>1$, then

$$\lim_{z\to\infty}\big(G_{a,b}(x,y+z)-z\big)=y. \qquad (x,y\in\mathbb{R}_+).$$

As an other big unit of the preliminary results, in Chapter 4 some versions of the Hermite-Hadamard inequality were presented. We introduced the concept of the odd and even functions with respect to a point. Namely, we say a function $f:\mathcal{I}\to\mathbb{R}$ to be *odd with respect to the point* m, if $t\mapsto f(m+t)-f(m)$ is odd, that is,

$$f(m-t)+f(m+t)=2f(m) \qquad (t\in(\mathcal{I}-m)\cap(m-\mathcal{I})),$$

while it is said to be *even with respect to the point* m, if $t\mapsto f(m+t)$ is even, that is,

$$f(m-t)=f(m+t) \qquad (t\in(\mathcal{I}-m)\cap(m-\mathcal{I})).$$

The main ”Hermite-Hadamard-type” result is involved in the following theorem:

THEOREM. *Let the function $f:\mathcal{I}\to\mathbb{R}$ be odd with respect to the element $m\in\mathcal{I}$, $\varrho:\mathcal{I}\to\mathbb{R}$ a positive, locally integrable weight function, which is even with respect to m, and let $[a,b]$ be a subinterval of $\mathcal{I}$ with non-empty interior. Then the*

following statement is valid:
If f is convex over the interval $\mathfrak{I} \cap (-\infty, m]$ and concave over $\mathfrak{I} \cap [m, \infty)$, then

$$
\begin{aligned}
f\left(M_\varrho(a,b)\right) &\underset{(\le)}{\ge} \frac{1}{\int_a^b \varrho(x)dx} \int_a^b f(x)\varrho(x)dx \\
&\underset{(\le)}{\ge} \frac{b - M_\varrho(a,b)}{b-a} f(a) + \frac{M_\varrho(a,b) - a}{b-a} f(b),
\end{aligned}
$$

if $\frac{a+b}{2} \underset{(\le)}{\ge} m$. The reversed inequalities are valid if f is concave over the interval $\mathfrak{I} \cap (-\infty, m]$ and convex over $\mathfrak{I} \cap [m, \infty)$.)

If, in particular, $\varrho(x) \equiv 1$, we obtain the "un-weighted" version of the previous theorem. As it turns out, this statement can perfectly be applied for the function

$$\mu_{x,y} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \log M_{t,t}(x,y).$$

In this way, we can provide the following estimate for $M_{a,b}$:

THEOREM. *Let a, b be real numbers so that $a + b \underset{(\le)}{\ge} 0$. Then*

$$M_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\le)}{\ge} M_{a,b}(x,y) \underset{(\le)}{\ge} \sqrt{M_{a,a}(x,y) M_{b,b}(x,y)}$$

holds for any positive numbers x, y.

This inequality could be improved by means of the following version of the Hermite-Hadamard inequality:

THEOREM. *Let $f : \mathfrak{I} \to \mathbb{R}$ be symmetric with respect to an element $m \in \mathfrak{I}$, furthermore, suppose that f is increasing. Then, for any interval $[a, b] \subset \mathfrak{I}$,*

$$\left((b-m)^+ - (a-m)^+\right) f(b) + \left((a-m)^- - (b-m)^-\right) f(a) \underset{(\le)}{\ge} \int_a^b f(x)dx$$

holds if $\frac{a+b}{2} \underset{(\le)}{\ge} m$.

Consequently,

THEOREM. *If $b < a$ and $a + b \underset{(\le)}{\ge} 0$, then, for all positive x, y,*

$$\left(M_{a,a}(x,y)\right)^{\frac{a^+ - b^+}{a-b}} \left(M_{b,b}(x,y)\right)^{\frac{b^- - a^-}{a-b}} \underset{(\le)}{\ge} M_{a,b}(x,y),$$

therefore, we get the following estimates:

THEOREM. *For all real a, b with $b \le a$, $(a, b) \ne (0, 0)$ and for all positive x, y,* *(i)* $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \le M_{a,b}(x,y) \le M_{a,a}(x,y)$, *if* $0 \le b \le a$,

(ii) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \le M_{a,b}(x,y)$
$\le \big(M_{a,a}(x,y)\big)^{\frac{a}{a-b}}\big(M_{b,b}(x,y)\big)^{\frac{-b}{a-b}}$, *if* $0 \le -b \le a$,

(iii) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \ge M_{a,b}(x,y)$
$\ge \big(M_{a,a}(x,y)\big)^{\frac{a}{a-b}}\big(M_{b,b}(x,y)\big)^{\frac{-b}{a-b}}$, *if* $0 \le a \le -b$,

(iv) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \ge M_{a,b}(x,y) \ge M_{b,b}(x,y)$, *if* $b \le a \le 0$.

After these preparations, we were able to reformulate the known comparison theorems, both for the Gini and the Stolarsky means.

For the Gini means, the main result is the following:

THEOREM. *Let $a, b, c, d \in \mathbb{R}$ be arbitrary parameters. Then the comparison inequality*

$$G_{a,b}(x,y) \le G_{c,d}(x,y)$$

holds for all positive x and y if and only if a, b, c, d satisfy the following three conditions:

$$a+b \le c+d, \quad \mathcal{E}(a,b) \le \mathcal{E}(c,d), \quad \mathcal{M}(a,b) \le \mathcal{M}(c,d).$$

Nevertheless originally we proved this theorem in a completely different way, in my thesis I presented a method, builded on the above-mentioned preliminaries. (In the case of the Stolarsky means, however, the "original" way was followed.) Having introduced the function

$$\mathcal{G}_{x,y} : \mathbb{R}^2 \to \mathbb{R}, \quad (a,b) \mapsto \mathcal{G}_{x,y}(a,b) := G_{a,b}(x,y),$$

we had the following

THEOREM. *Suppose that $b \le a$ and let the positive numbers x and y be fixed. The directional derivative of the function $\mathcal{G}_{x,y}$ in direction $d = (d_1, d_2)$ is nonnegative at the point (a,b) if and only if*

$$\begin{cases} \big(G_{a,a}(x,y)\big)^{d_1}\big(G_{b,b}(x,y)\big)^{-d_2} \ge \big(G_{a,b}(x,y)\big)^{d_1-d_2}, & \textit{if} \quad a \ne b, \\ d_1 + d_2 \ge 0, & \textit{if} \quad a = b. \end{cases}$$

As a consequence of the previous statement, we obtain the following corollary:

THEOREM. *For $b \le a$ with $(a,b) \ne (0,0)$, define the vectors $u_{a,b}$ and $v_{a,b}$ in the following way:*

$$u_{a,b} = \begin{cases} (1,0), & \textit{if } 0 \le b \le a, \\ \big(1, \frac{b}{a}\big), & \textit{if } 0 < -b < a, \\ (1,-1), & \textit{if } 0 < a \le -b, \\ (1,-1), & \textit{if } b \le a \le 0, \end{cases} \qquad v_{a,b} = \begin{cases} (-1,1), & \textit{if } 0 \le b \le a, \\ (-1,1), & \textit{if } 0 < -b \le a, \\ \big(\frac{a}{b}, 1\big), & \textit{if } 0 < a < -b, \\ (0,1), & \textit{if } b \le a \le 0. \end{cases}$$

Then the maps $(a,b) \mapsto u_{a,b}$ *and* $(a,b) \mapsto v_{a,b}$ *are continuous on the domain indicated, furthermore, the directional derivative of the function* $\mathcal{G}_{x,y}$*, at the point* (a,b) *is nonnegative in the directions* $u_{a,b}$ *and* $v_{a,b}$ *for all fixed positive numbers* x, y*.*

After this, the sufficiency of the conditions of the comparison theorem for Gini means can quickly be completed by means of the following statement:

THEOREM. *Let the positive numbers* x *and* y *be fixed and let* (a,b)*,* (c,d) *be two arbitrary points in* $\mathbb{R}^2$*. Suppose that the directional derivative of* $\mathcal{G}_{x,y}$ *is nonnegative in the direction* $(c-a, d-b)$ *at any point of the segment*

$$[(a,b),(c,d)] = \big\{\big(a+t(c-a),\ b+t(d-b)\big) \mid t \in [0,1]\big\}.$$

Then

$$G_{a,b}(x,y) \le G_{c,d}(x,y).$$

For sake of completeness, we presented a new proof also for the necessity of the conditions. This part of Chapter 5 can easily be deduced from the asymptotic properties, listed above.

Unfortunately, this method, builded on the behavior of the directional derivatives, could not be followed in the case of Stolarsky means. We had to apply appropriate sequences instead to obtain the reformulated comparison theorem:

THEOREM. *Let* $a, b, c, d \in \mathbb{R}$ *be arbitrary parameters. Then the comparison inequality*

$$S_{a,b}(x,y) \le S_{c,d}(x,y)$$

holds for all positive x *and* y *if and only if* a, b, c, d *satisfy the following three conditions:*

$$a+b \le c+d, \quad \mathcal{E}(a,b) \le \mathcal{E}(c,d), \quad \mathcal{L}(a,b) \le \mathcal{L}(c,d).$$

(The necessity, however, could similarly be proven as that of the theorem, concerning the Gini means.)

The third – and last – chapter on the comparison problem is Chapter 7. Applying the appropriate asymptotic properties, we can easily obtain the following necessary conditions:

THEOREM. *Suppose that the inequality*

$$G_{a,b}(x,y) \le S_{c,d}(x,y)$$

holds for any positive x, y*. Then*

$$\begin{aligned} 3(a+b) &\le c+d, \\ \mathcal{E}(a,b) &\le \mathcal{E}(c,d), \\ \min\{a,b\} &\le \min\{c,d\}, \\ \textit{if } \min\{a,b\} = 0 < \max\{a,b\} \textit{ then } \max\{a,b\} &\le \log 2 \cdot \mathcal{L}(c,d). \end{aligned}$$

To obtain sufficient conditions, we apply three propositions:

THEOREM. *The inequality*

$$G_{a,b}(x,y) \le S_{3a,3b}(x,y)$$

holds for all positive x, y *if and only if* $a+b \le 0$*, while the reversed inequality holds if and only if* $a+b \ge 0$*. Moreover, the inequality*

$$G_{a,b}(x,y) \le S_{2a+b,a+2b}(x,y)$$

holds for all positive x, y *if and only if* $ab(a+b) \le 0$*, while the reversed inequality holds if and only if* $ab(a+b) \ge 0$*. Finally, for any positive variables* x*,* y *the following holds:*

$$G_{-1+\frac{2}{\sqrt{5}},-1-\frac{2}{\sqrt{5}}}(x,y) \le S_{-3,-3}(x,y).$$

Applying these statements, we obtain that

THEOREM. *(i) Let* a, b *be positive numbers. Then there are no parameters* c, d *so that the Gini-Stolarsky comparison inequality be valid for all positive* x*,* y*.*

(ii) Let a, b *be real numbers so that* $\min\{a,b\} = 0 < \max\{a,b\}$*. Then the Gini-Stolarsky comparison inequality is valid for all positive* x*,* y *if and only if*
- *(a)* $3a \le c+d$*,*
- *(b)* $a \le \log 2 \cdot \mathcal{L}(c,d)$*.*

(iii) Let a, b *be real numbers so that* $ab < 0$ *and* $a+b \ge 0$*. If*
- *(a)* $3(a+b) \le c+d$*,*
- *(b)* $\mathcal{L}\{a+2b, 2a+b\} \le \mathcal{L}(c,d)$*,*
- *(c)* $\mathcal{E}(2a+b, a+2b) \le \mathcal{E}(c,d)$*,*

then the Gini-Stolarsky comparison inequality is valid for all positive x*,* y*.*

(iv) Let a, b *be real numbers so that* $ab < 0$ *and* $a+b \le 0$*. Then the Gini-Stolarsky comparison inequality is valid for all positive* x*,* y *if and only if*
- *(a)* $3(a+b) \le c+d$*,*
- *(b)* $\mathcal{E}(a,b) \le \mathcal{E}(c,d)$*.*

(v) Let a, b *be real numbers,* $a, b \le 0$*. If*
- *(a)* $3(a+b) \le c+d$*,*
- *(b)* $\mathcal{L}(2a+b, a+2b) \le \mathcal{L}(c,d)$*,*

then the Gini-Stolarsky comparison inequality is valid for all positive x*,* y

(vi) Let a, b *be real numbers,* $a, b \le 0$*,* $a/b \in [9-4\sqrt{5}, 9+4\sqrt{5}]$*. Then the Gini-Stolarsky comparison inequality is valid for all positive* x*,* y *if and only if*

$$3(a+b) \le c+d.$$

In the last three chapters we dealt with the generalized Minkowski inequality, both for the Gini and the Stolarsky means. First, in Chapter 8, we give a general approach to the problem: under what conditions will the inequality

$$M_{a_0,b_0}(x_1+x_2,y_1+y_2) \underset{(\geq)}{\leq} M_{a_1,b_1}(x_1,y_1)+M_{a_2,b_2}(x_2,y_2)$$

be valid for all positive x_1, x_2, y_1, y_2?

The first step towards the answer was the observation that the ”Minkowski-property” can be translated into the language of convexity:

THEOREM. *Let $M : \mathbb{R}_+^2 \to \mathbb{R}$ be a two variable homogeneous mean. Then*

$$M(x_1+x_2,y_1+y_2) \underset{(\geq)}{\leq} M(x_1,y_1)+M(x_2,y_2)$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ if and only if the function

$$m(t) = M(t,1) \qquad (t \in \mathbb{R}_+,\ i = 0,1,2).$$

is convex (resp. concave) on $\mathbb{R}_+$.

Moreover, we have the following necessary conditions for the generalized Minkowski inequality:

THEOREM. *Let $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be two variable homogeneous means such that M_0 is symmetric on $\mathbb{R}_+^2$, and differentiable at the point $(1,1)$. Assume that*

$$M_0(x_1+x_2,y_1+y_2) \underset{(\geq)}{\leq} M_1(x_1,y_1)+M_2(x_2,y_2)$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$. Then

$$M_0 \underset{(\geq)}{\leq} M_i \qquad \textit{and} \qquad A \underset{(\geq)}{\leq} M_i \qquad (i = 1,2),$$

where A denotes the arithmetic mean.

The crucial point of Chapter 8 is the concept of the Minkowski-separator. This is defined as follows:

Let $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be two variable means. A mean $M : \mathbb{R}_+^2 \to \mathbb{R}$ is called a *Minkowski-separator* (resp. *reversed-Minkowski-separator*) for (M_0, M_1, M_2) if M satisfies the (reversed) Minkowski inequality, furthermore,

$$M_0 \underset{(\geq)}{\leq} M \qquad \text{and} \qquad M \underset{(\geq)}{\leq} M_i \quad (i = 1,2).$$

Namely, the existence of the separator guarantees the validity of the generalized Minkowski inequality, as it can be read in the next statement:

THEOREM. *Let $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be two variable means. Suppose that there exists a (reversed-)Minkowski-separator for (M_0, M_1, M_2). Then the (reversed) Minkowski inequality holds for all positive x_1, x_2, y_1, y_2.*

In the rest of the thesis we study this generalized Minkowski inequality for the Gini and Stolarsky means. The two cases show a relevant difference: it turns out that the existence of the Minkowski-separator – that is, the sufficiency of the generalized Minkowski-inequality – coincides the necessity. For Stolarsky means, however, this equivalence does not hold.

For the Gini means, in Chapter 9 we can state the following:

THEOREM. *Let* $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$. *Then*

$$G_{a_0,b_0}((x_1, y_1) + (x_2, y_2)) \le G_{a_1,b_1}(x_1, y_1) + G_{a_2,b_2}(x_2, y_2)$$

holds if and only if

$$\begin{array}{rl} \text{(i)} & a_1, a_2, b_1, b_2 \ge 0, \\ \text{(ii)} & \max\{1, a_0 + b_0\} \le \min\{a_1 + b_1, a_2 + b_2\}, \\ \text{(iii)} & \min\{a_0, b_0\} \le \min\{1, a_1, b_1, a_2, b_2\}. \end{array}$$

In Chapter 10 we collect the necessary conditions both for the Minkowski and the reversed-Minkowski inequality to hold:

THEOREM. *Let* $a_0, b_0, a_1, b_1, a_2, b_2 \in \mathbb{R}$. *If the Minkowski inequality*

$$S_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \le S_{a_1,b_1}(x_1, y_1) + S_{a_2,b_2}(x_2, y_2)$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$, *then*

$$\begin{cases} \max\{a_0 + b_0, 3\} & \le & \min\{a_1 + b_1, a_2 + b_2\} \\ \max\big\{\mathcal{L}(a_0, b_0), \mathcal{L}(1, 2)\big\} & \le & \min\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\}. \end{cases}$$

If the reversed Minkowski inequality

$$S_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \ge S_{a_1,b_1}(x_1, y_1) + S_{a_2,b_2}(x_2, y_2)$$

holds for all $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$, *then*

$$\begin{cases} \min\{a_0 + b_0, 3\} & \ge & \max\{a_1 + b_1, a_2 + b_2\} \\ \min\big\{\mathcal{L}(a_0, b_0), \mathcal{L}(1, 2)\big\} & \ge & \max\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\} \\ \mathcal{E}(a_0, b_0) & \ge & \max\big\{\mathcal{E}(a_1, b_1), \mathcal{E}(a_2, b_2)\big\}. \end{cases}$$

The existence of the Minkowski and the reversed Minkowski separator, however, holds under different conditions:

THEOREM. *For* $(a_0, b_0, a_1, b_1, a_2, b_2) \in \mathbb{R}^6$ *there exists a Minkowski-separator if and only if*

$$\begin{cases} \max\{a_0 + b_0, 3\} & \le \min\{a_1 + b_1, a_2 + b_2\}, \\ \max\big\{\mathcal{L}(a_0, b_0), \mathcal{L}(1, 2), & \mathcal{L}(1, a_0 + b_0 - 1)\big\} \\ & \le \min\big\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\big\} \end{cases}$$

For $(a_0, b_0, a_1, b_1, a_2, b_2) \in \mathbb{R}^6$ *there exists a reversed-Minkowski-separator if and only if*

$$\begin{cases} \min\{a_0 + b_0, 3\} & \geq & \max\{a_1 + b_1, a_2 + b_2\}, \\ \min\left\{\mathcal{L}(a_0, b_0), \mathcal{L}(1, 2)\right\} & \geq & \max\left\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\right\}, \\ \mathcal{E}(a_0, b_0) & \geq & \max\left\{\mathcal{E}(a_1, b_1), \mathcal{E}(a_2, b_2)\right\}, \\ \textit{finally, if } \min\{a_0, b_0\} \geq 1 \textit{ then} & & \\ \mathcal{L}(1, a_0 + b_0 - 1) & \geq & \max\left\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\right\}. \end{cases}$$

These conditions are sufficient for the Minkowski inequality and the reversed Minkowski inequality, respectively.

Összefoglalás

A Gini- és Stolarsky-közepekről számtalan értekezés készült. Ezeket a középértékeket két változó esetén – a legáltalánosabb esetben – a

$$G_{a,b}(x,y) = \left(\frac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}}$$

és

$$S_{a,b}(x,y) = \left(\frac{b(x^a - y^a)}{a(x^b - y^b)}\right)^{\frac{1}{a-b}},$$

formulákkal definiáljuk. (Az összes esetet lefedő teljes definíció az 1. fejezetben található.)

Dolgozatomban megkíséreltem összegyűjteni és rendszerezni az elmúlt években, Páles Zsolt professzor úr irányításával kapott eredményeket. (A fejezetek sorrendje nem felel meg a bennük foglalt eredmények kronológiai sorrendjének, mivel utólag visszatekintve rejtett kapcsolatokra derült fény.)

Vizsgálataim középpontjába a Gini és/vagy Stolarsky-közepekről szóló összehasonlítási tételeket és a velük kapcsolatos általánosított Minkowski-egyenlőtlenségeket helyeztem. Ahhoz, hogy ezen állításokat formába önteni és bizonyítani lehessen, némi előkészületekre volt szükség. A 2. fejezetben összegyűjtöttem néhány jól ismert, de fontos összefüggést. Továbbá, a következetes tárgyalásmód érdekében bevezettem az

$$\mathcal{E}(a,b) := \begin{cases} \dfrac{|a|-|b|}{a-b}, & \text{ha } a \neq b, \\ \operatorname{sign}(a), & \text{ha } a = b, \end{cases} \qquad \mathcal{M}(a,b) := \begin{cases} \min\{a,b\}, & \text{ha } a,b \geq 0, \\ 0, & \text{ha } ab < 0, \\ \max\{a,b\}, & \text{ha } a,b \leq 0 \end{cases}$$

és

$$\mathcal{L}(a,b) := \begin{cases} \dfrac{a-b}{\log(a/b)}, & \text{ha } 0 < ab \text{ és } a \neq b, \\ a, & \text{ha } 0 < ab \text{ és } a = b, \\ 0, & \text{ha } ab \leq 0. \end{cases}$$

összefüggések által definiált $\mathcal{E}$, $\mathcal{M}$ és $\mathcal{L}$ függvényeket, amelyek az összehasonlítási tételekben kulcsszerepet játszanak. Itt vizsgáltam ezek alapvető tulajdonságait is (folytonosság, konvexitás, monotonitás).

Több, Gini- és Stolarsky-közepekkel kapcsolatos egyenlőtlenség származtatható bizonyos aszimptotikus tulajdonságokból. Ezért a 3. fejezetben összegyűjtöttem ezek közül a legfontosabbakat. Némelyik állítást egységes szerkezetben lehetett tálalni, oly módon, hogy azok mind a Gini-, mind a Stolarsky-közepekre érvényesek legyenek. (Dolgozatomban M áll minden olyan helyen, ahol valamely összefüggés mindkét középérték-típusra fennáll.) A következőkre jutottam:

TÉTEL. *Bármely a, b valós paraméter esetén*

$$\lim_{x\to 0+} G_{a,b}(x,y) = \begin{cases} y, & \textit{ha} \min\{a,b\} > 0, \\ y\cdot 2^{-\frac{1}{\max\{a,b\}}}, & \textit{ha} \min\{a,b\} = 0 \,\&\, \max\{a,b\} > 0, \\ 0, & \textit{ha} \min\{a,b\} < 0 \textit{vagy} (a,b) = (0,0), \end{cases}$$

$$\lim_{x\to 0+} S_{a,b}(x,y) = \begin{cases} y\cdot e^{-\frac{1}{\mathcal{L}(a,b)}}, & \textit{ha} \min\{a,b\} > 0, \\ 0, & \textit{ha} \min\{a,b\} \le 0, \end{cases}$$

$$\lim_{z\to\infty} \big(M_{a,b}(x+z,y+z) - z\big) = \frac{x+y}{2} \qquad (x,y\in\mathbb{R}_+),$$

$$\lim_{z\to\infty} \big(G_{a,b}(x,y+z) - z\big) = y, \qquad (x,y\in\mathbb{R}_+),$$

$$\lim_{t\to 1} \frac{M_{a,b}(t,1) - \frac{t+1}{2}}{(t-1)^2} = \begin{cases} \frac{a+b-1}{8}, & \textit{ha } M = G, \\ \frac{a+b-3}{24}, & \textit{ha } M = S, \end{cases}$$

$$\lim_{t\to\infty} \frac{\log M_{a,b}(e^t,e^{-t})}{t} = \mathcal{E}(a,b),$$

$$\lim_{t\to\infty} \frac{\log\dfrac{t+\log G_{a,b}(e^t,e^{-t})}{t-\log G_{a,b}(e^t,e^{-t})}}{2t} = \mathcal{M}(a,b),$$

továbbá, ha $a,b>1$, úgy

$$\lim_{z\to\infty} \big(G_{a,b}(x,y+z) - z\big) = y. \qquad (x,y\in\mathbb{R}_+).$$

Az előkészületek egy másik nagy egységét a Hermite-Hadamard egyenlőtlenség különböző változatai alkotják. Ezekről a 4. fejezetben esik szó. Megjelenik a pontra nézve páratlan, illetve páros függvény fogalma. Nevezetesen, akkor mondjuk, hogy az $f : \mathcal{I} \to \mathbb{R}$ függvény *páratlan az m pontra nézve*, ha a $t \mapsto f(m+t) - f(m)$ függvény páratlan, azaz,

$$f(m-t) + f(m+t) = 2f(m) \qquad (t \in (\mathcal{I}-m)\cap(m-\mathcal{I})),$$

míg f-et *m-re nézve párosnak* hívjuk, ha $t\mapsto f(m+t)$ is páros, vagyis

$$f(m-t) = f(m+t) \qquad (t \in (\mathcal{I}-m)\cap(m-\mathcal{I})).$$

A legfontosabb ”Hermite-Hadamard-típusú” eredményt a következő tétel tartalmazza:

TÉTEL. *Legyen az $f : \mathcal{I} \to \mathbb{R}$ függvény páratlan az $m \in \mathcal{I}$ pontra nézve, $\varrho : \mathcal{I} \to \mathbb{R}$ pedig egy pozitív, lokálisan integrálható súlyfüggvény, amely páros m-re nézve, és legyen $[a, b]$ egy nemüres belsejű részintervalluma $\mathcal{I}$-nek. Ekkor a következő állítás érvényes:*
Ha f konvex az $\mathcal{I} \cap (-\infty, m]$ intervallumon és konkáv $\mathcal{I} \cap [m, \infty)$ fölött, akkor

$$f\left(M_\varrho(a, b)\right) \underset{(\leq)}{\geq} \frac{1}{\int_a^b \varrho(x)dx} \int_a^b f(x)\varrho(x)dx$$
$$\underset{(\leq)}{\geq} \frac{b - M_\varrho(a, b)}{b - a} f(a) + \frac{M_\varrho(a, b) - a}{b - a} f(b),$$

ha $\dfrac{a + b}{2} \underset{(\leq)}{\geq} m$. A fordított irányú egyenlőtlenségek igazak, ha f konkáv $\mathcal{I} \cap (-\infty, m]$-n és konvex $\mathcal{I} \cap [m, \infty)$-n.)

Speciálisan, ha $\varrho(x) \equiv 1$, úgy az előző tétel súlyozatlan verzióját kapjuk. Mint kiderül, ez az állítás kitűnően alkalmazható a

$$\mu_{x,y} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \log M_{t,t}(x, y).$$

függvényre. Ily módon a következő becslést kapjuk $M_{a,b}$-re:

TÉTEL. *Legyenek az a, b valós számok olyanok, melyekre $a + b \underset{(\leq)}{\geq} 0$. Ekkor*

$$M_{\frac{a+b}{2}, \frac{a+b}{2}}(x, y) \underset{(\leq)}{\geq} M_{a,b}(x, y) \underset{(\leq)}{\geq} \sqrt{M_{a,a}(x, y) M_{b,b}(x, y)}$$

érvényes minden pozitív x, y számra.

Ezt az egyenlőtlenséget a Hermite-Hadamard egyenlőtlenség következő változatával javítani lehet:

TÉTEL. *Legyen $f : \mathcal{I} \to \mathbb{R}$ páratlan az $m \in \mathcal{I}$ pontra nézve, továbbá, tegyük fel, hogy f növekszik. Ekkor bármely $[a, b] \subset \mathcal{I}$ intervallum esetén*

$$\big((b - m)^+ - (a - m)^+\big) f(b) + \big((a - m)^- - (b - m)^-\big) f(a) \underset{(\leq)}{\geq} \int_a^b f(x)dx$$

érvényes, ha $\dfrac{a + b}{2} \underset{(\leq)}{\geq} m$.

Következésképpen,

TÉTEL. *Ha $b < a$ és $a + b \underset{(\leq)}{\geq} 0$, akkor minden pozitív x, y-ra*

$$\big(M_{a,a}(x, y)\big)^{\frac{a^+ - b^+}{a - b}} \big(M_{b,b}(x, y)\big)^{\frac{b^- - a^-}{a - b}} \underset{(\leq)}{\geq} M_{a,b}(x, y),$$

így a következő becslésekhez jutunk:

TÉTEL. *Minden* a, b *valós számra, melyre* $b \le a$, $(a,b) \neq (0,0)$, *és minden pozitív* x, y *esetén*

(i) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \le M_{a,b}(x,y) \le M_{a,a}(x,y)$, *ha* $0 \le b \le a$,

(ii) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \le M_{a,b}(x,y)$
$\le \big(M_{a,a}(x,y)\big)^{\frac{a}{a-b}}\big(M_{b,b}(x,y)\big)^{\frac{-b}{a-b}}$, *ha* $0 \le -b \le a$,

(iii) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \ge M_{a,b}(x,y)$
$\ge \big(M_{a,a}(x,y)\big)^{\frac{a}{a-b}}\big(M_{b,b}(x,y)\big)^{\frac{-b}{a-b}}$, *ha* $0 \le a \le -b$,

(iv) $\big(M_{a,a}(x,y)\big)^{\frac{1}{2}}\big(M_{b,b}(x,y)\big)^{\frac{1}{2}} \ge M_{a,b}(x,y) \ge M_{b,b}(x,y)$, *ha* $b \le a \le 0$.

Ezek után az előkészületek után átfogalmazhatók a Gini- és Stolarsky-közepekre érvényes összehasonlítási tételek. Gini-közepekre a fő eredményünk a következő:

TÉTEL. *Legyenek* $a, b, c, d \in \mathbb{R}$ *tetszőleges paraméterek. A*

$$G_{a,b}(x,y) \le G_{c,d}(x,y)$$

összehasonlítási egyenlőtlenség pontosan akkor áll fenn minden x *és* y *esetén, ha* a, b, c, d*-re fennáll a következő három feltétel:*

$$a+b \le c+d, \quad \mathcal{E}(a,b) \le \mathcal{E}(c,d), \quad \mathcal{M}(a,b) \le \mathcal{M}(c,d).$$

Bár ezt a tételt eredetileg teljesen más módszerrel is bizonyítottuk, dolgozatomban a bizonyításra egy, a fentiekre épülő módszert használtam. (A Stolarsky-közepek esetén viszont az „eredeti” eljárást a alkalmaztam.) Bevezetve a

$$\mathcal{G}_{x,y} : \mathbb{R}^2 \to \mathbb{R}, \quad (a,b) \mapsto \mathcal{G}_{x,y}(a,b) := G_{a,b}(x,y),$$

függvényt, a következőt kapjuk:

TÉTEL. *Tegyük fel, hogy* $b \le a$ *és rögzítsük az* x *és* y *be pozitív számokat. A* $\mathcal{G}_{x,y}$ *függvény irány menti deriváltja a* $d = (d_1, d_2)$ *irányban pontosan akkor nemnegatív az* (a,b) *pontban, ha*

$$\begin{cases} \big(G_{a,a}(x,y)\big)^{d_1}\big(G_{b,b}(x,y)\big)^{-d_2} \ge \big(G_{a,b}(x,y)\big)^{d_1-d_2}, & \text{ha} \quad a \neq b, \\ d_1 + d_2 \ge 0, & \text{ha} \quad a = b. \end{cases}$$

A tétel következményeként adódik az alábbi állítás:

TÉTEL. *Az* $b \le a$, $(a,b) \neq (0,0)$ *számokra definiáljuk az* $u_{a,b}$ *és* $v_{a,b}$ *vektorokat a következőképpen*

$$u_{a,b} = \begin{cases} (1,0), & \text{ha } 0 \le b \le a, \\ \big(1, \frac{b}{a}\big), & \text{ha } 0 < -b < a, \\ (1,-1), & \text{ha } 0 < a \le -b, \\ (1,-1), & \text{ha } b \le a \le 0, \end{cases} \qquad v_{a,b} = \begin{cases} (-1,1), & \text{ha } 0 \le b \le a, \\ (-1,1), & \text{ha } 0 < -b \le a, \\ \big(\frac{a}{b}, 1\big), & \text{ha } 0 < a < -b, \\ (0,1), & \text{ha } b \le a \le 0. \end{cases}$$

Ekkor az $(a, b) \mapsto u_{a,b}$ *és* $(a, b) \mapsto v_{a,b}$ *leképezések folytonosak értelmezési tartományukon, továbbá a* $\mathcal{G}_{x,y}$ *függvény irány menti deriváltja az* (a, b) *pontban nemnegatív az* $u_{a,b}$ *és* $v_{a,b}$ *irányokban bármely* x, y *pozitív számok rögzítése esetén.*

Ezek után a Gini-közepek összehasonlítási tételében a feltételek elégségessége hamar megkapható a következő állítás segítségével:

TÉTEL. *Rögzítsük az* x *és* y *pozitív számokat és legyen* (a, b), (c, d) $\mathbb{R}^2$ *két tetszőleges pontja. Tegyük fel, hogy a* $\mathcal{G}_{x,y}$ *függvény irány menti deriváltja nemnegatív az* $(c - a, d - b)$ *irányban az*

$$[(a, b), (c, d)] = \big\{\big(a + t(c - a),\ b + t(d - b)\big) \mid t \in [0, 1]\big\}.$$

szakasz minden pontjában. Ekkor

$$G_{a,b}(x, y) \leq G_{c,d}(x, y).$$

A teljesség kedvéért a feltételek szükségességét is új módszerrel bizonyítottam. Az 5. fejezetnek ez a része könnyen levezethető a fent összegyűjtött aszimptotikus tulajdonságokból.

Sajnos ez a módszer, amelyet az irány menti deriváltak viselkedésére építettünk, a Stolarsky-közepekre nem bizonyult alkalmazhatónak. Ehelyett alkalmas sorozatokat lehetett felhasználni a következő összehasonlítási tétel bizonyításához:

TÉTEL. *Legyenek* $a, b, c, d \in \mathbb{R}$ *tetszőleges paraméterek. Ekkor az*

$$S_{a,b}(x, y) \leq S_{c,d}(x, y)$$

összehasonlítási egyenlőtlenség pontosan akkor teljesül minden pozitív x *és* y *számra, ha* a, b, c, d *eleget tesz a következő három feltételnek:*

$$a + b \leq c + d, \quad \mathcal{E}(a, b) \leq \mathcal{E}(c, d), \quad \mathcal{L}(a, b) \leq \mathcal{L}(c, d).$$

(A szükségességet viszont a Gini-közepekre vonatkozó analóg tételé mintájára lehet bizonyítani.)

Az összehasonlítási tétellel foglalkozó harmadik és egyben utolsó fejezet a 7. fejezet. Alkalmazva a megfelelő aszimptotikus tulajdonságokat, könnyen megkapjuk az alábbi szükséges feltételeket:

TÉTEL. *Tegyük fel, hogy a*

$$G_{a,b}(x, y) \leq S_{c,d}(x, y)$$

egyenlőtlenség fennáll minden x, y *pozitív számra. Ekkor*

$$\begin{aligned} 3(a + b) &\leq c + d, \\ \mathcal{E}(a, b) &\leq \mathcal{E}(c, d), \\ \min\{a, b\} &\leq \min\{c, d\}, \\ \text{\textit{és ha} } \min\{a, b\} = 0 < \max\{a, b\} \text{ \textit{akkor} } \max\{a, b\} &\leq \log 2 \cdot \mathcal{L}(c, d). \end{aligned}$$

Elégséges feltételek megalkotásához a következő három állítást használhatjuk fel:

TÉTEL. *A*

$$G_{a,b}(x,y) \le S_{3a,3b}(x,y)$$

egyenlőtlenség pontosan akkor érvényes minden pozitív x, y esetén, ha $a+b \le 0$, míg a fordított egyenlőtlenség pontosan akkor igaz, ha $a+b \ge 0$. Továbbá, a

$$G_{a,b}(x,y) \le S_{2a+b,a+2b}(x,y)$$

egyenlőtlenség pontosan akkor érvényes minden pozitív x, y esetén, ha $ab(a+b) \le 0$, míg a fordított egyenlőtlenség pontosan akkor igaz, ha $ab(a+b) \ge 0$. Végül, minden pozitív x, y számra fennáll, hogy

$$G_{-1+\frac{2}{\sqrt{5}},-1-\frac{2}{\sqrt{5}}}(x,y) \le S_{-3,-3}(x,y).$$

Ezen állítások felhasználásával kapjuk, hogy

TÉTEL. *(i) Legyenek a, b pozitív számok. Ekkor nem léteznek olyan c, d paraméterek,melyekre a Gini-Stolarsky összehasonlítási egyenlőtlenség minden pozitív x, y-ra teljesülne.*

(ii) Legyenek a, b olyan valós számok, melyekre $\min\{a,b\} = 0 < \max\{a,b\}$. Ekkor a Gini-Stolarsky összehasonlítási egyenlőtlenség pontosan akkor teljesül minden pozitív x, y-ra, ha
(a) $3a \le c+d$,
(b) $a \le \log 2 \cdot \mathcal{L}(c,d)$.

(iii) Legyenek a, b olyan valós számok, melyekre $ab < 0$ és $a+b \ge 0$. Ha
(a) $3(a+b) \le c+d$,
(b) $\mathcal{L}\{a+2b, 2a+b\} \le \mathcal{L}(c,d)$,
(c) $\mathcal{E}(2a+b, a+2b) \le \mathcal{E}(c,d)$,
akkor a Gini-Stolarsky összehasonlítási egyenlőtlenség minden pozitív x, y estén érvényes.

(iv) Legyenek a, b olyan valós számok, melyekre $ab < 0$ és $a+b \le 0$. Ekkor a Gini-Stolarsky összehasonlítási egyenlőtlenség pontosan akkor teljesül minden pozitív x, y-ra, ha
(a) $3(a+b) \le c+d$,
(b) $\mathcal{E}(a,b) \le \mathcal{E}(c,d)$.

(v) Legyenek a, b valós számok, $a, b \le 0$. Ha
(a) $3(a+b) \le c+d$,
(b) $\mathcal{L}(2a+b, a+2b) \le \mathcal{L}(c,d)$,
akkor a Gini-Stolarsky összehasonlítási egyenlőtlenség minden pozitív x, y estén érvényes.

(vi) *Legyenek a, b valós számok, $a, b \leq 0$, $a/b \in [9-4\sqrt{5}, 9+4\sqrt{5}]$. Ekkor a Gini-Stolarsky összehasonlítási egyenlőtlenség pontosan akkor teljesül minden pozitív x, y-ra, ha*

$$3(a+b) \leq c+d.$$

Az utolsó három fejezet a Gini-, illetve Stolarsky-közepekre felírható Minkowski-típusú egyenlőtlenséggel foglalkozik. A 8. fejezetben általánosan közelítjük meg a kérdést: milyen feltételek mellett teljesül az

$$M_{a_0,b_0}(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M_{a_1,b_1}(x_1, y_1) + M_{a_2,b_2}(x_2, y_2)$$

egyenlőtlenség minden pozitív x_1, x_2, y_1, y_2 esetén?

Az első lépést az az észrevétel jelenti, mely szerint a "Minkowski-tulajdonság" a konvexitás nyelvén a következőképpen fogalmazható meg:

TÉTEL. *Legyen $M : \mathbb{R}_+^2 \to \mathbb{R}$ egy kétváltozós, homogén közép. Ekkor*

$$M(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M(x_1, y_1) + M(x_2, y_2)$$

pontosan akkor áll fenn minden pozitív $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ esetén, ha az

$$m(t) = M(t, 1) \qquad (t \in \mathbb{R}_+,\ i = 0, 1, 2).$$

függvény konvex (konkáv) $\mathbb{R}_+$-en.

Továbbá, az általánosított Minkowski-egyenlőtlenséghez az alábbi szükséges feltételek teljesülnek:

TÉTEL. *Legyenek $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ olyan kétváltozós homogén közepek, melyekre M_0 szimmetrikus $\mathbb{R}_+^2$-n és differenciálható az $(1, 1)$ pontban. Tegyük fel, hogy*

$$M_0(x_1+x_2, y_1+y_2) \underset{(\geq)}{\leq} M_1(x_1, y_1) + M_2(x_2, y_2)$$

fennáll minden $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ esetén. Ekkor

$$M_0 \underset{(\geq)}{\leq} M_i \qquad \text{és} \qquad A \underset{(\geq)}{\leq} M_i \qquad (i = 1, 2),$$

ahol A a számtani közepet jelöli.

A 8. fejezet középponti fogalma a Minkowski-szeparátor fogalma. Ezt a következőképpen definiáljuk:

Legyenek $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ kétváltozós közepek. Egy $M : \mathbb{R}_+^2 \to \mathbb{R}$ közepet *Minkowski-szeparátornak* (illetve *fordított Minkowski-szeparátornak*) hívunk

(M_0, M_1, M_2)-re nézve, ha M eleget tesz a (fordított) Minkowski-egyenlőtlenségnek, valamint

$$M_0 \underset{(\geq)}{\leq} M \qquad \text{és} \qquad M \underset{(\geq)}{\leq} M_i \quad (i = 1, 2).$$

A szeparátor létezése az általánosított Minkowski-egyenlőtlenség teljesülését vonja maga után, amint az a következő állításból kiderül:

TÉTEL. *Legyenek $M_0, M_1, M_2 : \mathbb{R}_+^2 \to \mathbb{R}$ kétváltozós közepek. Tegyük fel, hogy létezik (fordított)Minkowski-szeparátor az (M_0, M_1, M_2) hármasra nézve. Ekkor a (fordított) Minkowski-egyenlőtlenség minden pozitív x_1, x_2, y_1, y_2 esetén fennáll.*

Dolgozatom hátralevő részében az általánosított Minkowski-egyenlőtlenséget a Gini- és a Stolarsky-közepekre vizsgálom. E két eset között lényeges eltérés mutatkozott: kiderült, hogy a Gini-közepek esetén a Minkowski-szeparátor létezése – vagyis az általánosított Minkowski-egyenlőtlenség elégségességi feltétele – egybeesik a szükségességgel. A Stolarsky-közepek esetén azonban ez az ekvivalencia nem teljesül.

Gini-közepekre a 9. fejezetben a következőt állítjuk:

TÉTEL. *Legyen $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$. Ekkor*

$$G_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \leq G_{a_1,b_1}(x_1, y_1) + G_{a_2,b_2}(x_2, y_2)$$

akkor és csak akkor teljesül, ha

$$\begin{aligned} &\text{(i)} && a_1, a_2, b_1, b_2 \geq 0, \\ &\text{(ii)} && \max\{1, a_0 + b_0\} \leq \min\{a_1 + b_1, a_2 + b_2\}, \\ &\text{(iii)} && \min\{a_0, b_0\} \leq \min\{1, a_1, b_1, a_2, b_2\}. \end{aligned}$$

A 10. fejezetben mind a Minkowski-, mind a fordított Minkowski-egyenlőtlenség szükséges feltételeit összegyűjtöttük:

TÉTEL. *Legyen $a_0, b_0, a_1, b_1, a_2, b_2 \in \mathbb{R}$. Ha a*

$$S_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \leq S_{a_1,b_1}(x_1, y_1) + S_{a_2,b_2}(x_2, y_2)$$

Minkowski-egyenlőtlenség fennáll minden $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ esetén, akkor

$$\begin{cases} \max\{a_0 + b_0, 3\} & \leq & \min\{a_1 + b_1, a_2 + b_2\} \\ \max\left\{\mathcal{L}(a_0, b_0), \mathcal{L}(1, 2)\right\} & \leq & \min\left\{\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2)\right\}. \end{cases}$$

Ha a

$$S_{a_0,b_0}(x_1 + x_2, y_1 + y_2) \geq S_{a_1,b_1}(x_1, y_1) + S_{a_2,b_2}(x_2, y_2)$$

fordított Minkowski-egyenlőtlenség áll fenn minden $x_1, y_1, x_2, y_2 \in \mathbb{R}_+$ *esetén, akkor*

$$\begin{cases} \min\{a_0+b_0,3\} & \geq & \max\{a_1+b_1,a_2+b_2\} \\ \min\big\{\mathcal{L}(a_0,b_0),\mathcal{L}(1,2)\big\} & \geq & \max\big\{\mathcal{L}(a_1,b_1),\mathcal{L}(a_2,b_2)\big\} \\ \mathcal{E}(a_0,b_0) & \geq & \max\big\{\mathcal{E}(a_1,b_1),\mathcal{E}(a_2,b_2)\big\}. \end{cases}$$

A Minkowski, illetve fordított Minkowski-szeparátor létezése azonban más feltételek mellett garantálható:

TÉTEL. *Az* $(a_0,b_0,a_1,b_1,a_2,b_2) \in \mathbb{R}^6$ *szám-hatoshoz pontosan akkor létezik Minkowski-szeparátor, ha*

$$\begin{cases} \max\{a_0+b_0,3\} & \leq \min\{a_1+b_1,a_2+b_2\}, \\ \max\big\{\mathcal{L}(a_0,b_0),\mathcal{L}(1,2), & \mathcal{L}(1,a_0+b_0-1)\big\} \\ & \leq \min\big\{\mathcal{L}(a_1,b_1),\mathcal{L}(a_2,b_2)\big\} \end{cases}$$

Az $(a_0,b_0,a_1,b_1,a_2,b_2) \in \mathbb{R}^6$ *szám-hatoshoz pedig pontosan akkor létezik fordított Minkowski-szeparátor, ha*

$$\begin{cases} \min\{a_0+b_0,3\} & \geq & \max\{a_1+b_1,a_2+b_2\}, \\ \min\big\{\mathcal{L}(a_0,b_0),\mathcal{L}(1,2)\big\} & \geq & \max\big\{\mathcal{L}(a_1,b_1),\mathcal{L}(a_2,b_2)\big\}, \\ \mathcal{E}(a_0,b_0) & \geq & \max\big\{\mathcal{E}(a_1,b_1),\mathcal{E}(a_2,b_2)\big\}, \\ \textit{végül, ha } \min\{a_0,b_0\} \geq 1\textit{, akkor} & & \\ \mathcal{L}(1,a_0+b_0-1) & \geq & \max\big\{\mathcal{L}(a_1,b_1),\mathcal{L}(a_2,b_2)\big\}. \end{cases}$$

Ezek a feltételek elégségesek a Minkowski- (illetve fordított Minkowski-) egyenlőtlenség teljesüléséhez.

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Inequalities on two variable
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