THE SMALLEST UNIVOQUE NUMBER IS NOT ISOLATED

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Dedicated to the 80th birthday of Professor Lajos Tândoróy.

Abstract. Komôrk and Loreti [9] showed that there exists a smallest univoque number $q \approx 1.787$. Later Allouche and Cassaigne [1] proved that this number is transcendental. The aim of this note is to construct a (decreasing) sequence of algebraic numbers converging to $q$.

1. Introduction

Given a real number $1 \leq q \leq 2$, there exists at least one sequence $(c_i)$ of zeroes and ones satisfying the equality

$$1 = c_1 \frac{1}{q} + c_2 \frac{1}{q^2} + c_3 \frac{1}{q^3} + \ldots$$

One such sequence, denoted by $(\gamma_i)$, can be obtained by the so-called greedy algorithm of Rényi [13]: proceeding by induction, we choose $c_i = 1$ whenever possible. Among all expansions for a given $q$, this is lexicographically the largest.

If $q = 2$, then this is the unique possible expansion: $c_i = 1$ for all $i$. Erdős, Horváth and Joo [5] discovered that there exist also smaller numbers $q$ having this curious uniqueness property; following Daróczy and Kátai [3] we call them univoque numbers. Subsequently, they were characterized algebraically in [6] (see also [10] for an extension of this result):

Theorem 1. A number $1 \leq q \leq 2$ is univoque if and only if there exists an expansion $(\gamma_i)$ of 1 satisfying the following two conditions (in the lexicographic sense):

$$\gamma_{i+1}\gamma_{i+2} \ldots < \gamma_1\gamma_2 \ldots \quad \text{whenever} \quad \gamma_i = 0$$

and

$$\gamma_{i+1}\gamma_{i+2} \ldots \gamma_i \gamma_{i+1} \ldots < \gamma_1\gamma_2 \ldots \quad \text{whenever} \quad \gamma_i = 1.$$

Here and in the sequel we use the notation $\bar{c} := 1 - c$.

Among several interesting properties of the set $U$ of univoque numbers, for which we refer to the papers [1], [2], [3], [4], [5], [8] and [9], we recall from [9] that there exists a smallest univoque number $q \approx 1.787$, and the corresponding expansion is given by the truncated Thue–Morse sequence

$$(\bar{c})_{i=1}^\infty = 1101001101\ldots$$

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The purpose of this note is to investigate the following two questions:

- One may wonder whether $q'$ is an isolated univoque number or not. In the first case one could look for the second smallest univoque number, and so on.
- Allouche and Cassaigne proved in [1] that $q'$ is transcendental. It is then natural to look for the smallest algebraic univoque number if it exists.

Both problems are solved by the following

**Theorem 2.** There exists a (decreasing) sequence of algebraic univoque numbers converging to $q'$. In particular, $q'$ is not an isolated point of $U$.

2. Proof of Theorem 2

For the purpose of the present paper, it is advantageous to adopt the following definition of the Thue-Morse sequence $(\tau_i)$: if

$$t = \varepsilon_k 2^k + \cdots + \varepsilon_0$$

is the dyadic expansion of some nonnegative integer $i$, then we define

$$\tau_i := \begin{cases} 1 & \text{if } \varepsilon_k + \cdots + \varepsilon_0 \text{ is odd}, \\ 0 & \text{if } \varepsilon_k + \cdots + \varepsilon_0 \text{ is even}. \end{cases}$$

In particular, $\tau_0 = 0$. See [9] for its equivalence with another usual definition.

Our main tool is the following strengthening of a property of the Thue-Morse sequence $\tau_1, \tau_2, \ldots$, established in [6].

**Lemma 3.** Let $1 \leq i < 2^{N+1}$ for some nonnegative integer $N$.

(a) If $\tau_i = 0$, then $\tau_{i+1} \ldots \tau_{i+2^N} < \tau_1 \ldots \tau_{2^N}$ in the lexicographic sense.

(b) If $\tau_i = 1$, then $\tau_{i+1} \ldots \tau_{i+2^N} < \tau_1 \ldots \tau_{2^N}$ in the lexicographic sense.

**Remark.** In fact, part (a) remains valid even if $\tau_i = 1$, except the case where $N = 0$ and $i = 1$, while part (b) remains always valid even if $\tau_i = 0$. An analogous property was established recently by Gledhill and Sidorenko [7].

**Proof.** Consider first the case $\tau_i = 0$. Then $\varepsilon_k + \cdots + \varepsilon_0$ is even and therefore $\varepsilon_k + \cdots + \varepsilon_0 \geq 2$ because $i \geq 1$ by assumption. Hence we may write $i = 2^n + 2^m + j$ with $2^n > 2^m > j \geq 0$. We claim that

$$\tau_{i+1} \ldots \tau_{i+2^N} < \tau_{j+1} \ldots \tau_{j+2^N}.$$

We distinguish two cases. If $n \geq m + 2$, then using (4) we have

$$\tau_{i+k} = \tau_{j+k} \quad \text{for} \quad 1 < k < 2^m - j$$

but

$$\tau_{i+2^m - j} = \tau_{j+2^m + j} = 0 < 1 = \tau_{2^m} = \tau_{j+2^m - j}.$$

Since

$$2^m \cdot j \leq 2^m \leq 2^N < 2^{N+1},$$

this proves (5).

If $n = m + 1$, then using (4) we obtain by a similar reasoning that

$$\tau_{i+k} = \tau_{j+k} \quad \text{for} \quad 1 \leq k < 2^{m+1} - j$$

but

$$\tau_{i+2^{m+1} - j} = \tau_{j+2^{m+1} - j} = 0 < 1 = \tau_{2^{m+1}} = \tau_{j+2^{m+1} - j}.$$
Since
\[ 2^{m+1} - j < 2^{m+1} = 2^n \leq 2^N, \]
(5) follows again.

Since \( \tau_j = \tau_i = 0 \), we may iterate (5) until we obtain \( j = 0 \), thereby proving the desired inequality.

Now consider the case \( \tau_j = -1 \) and write \( i = 2^m + j \) with \( 2^m > j > 0 \). We claim that
\[ (6) \quad \tau_{i+1} \cdots \tau_{i+2^N} < \tau_{j+1} \cdots \tau_{j+2^N}. \]
Indeed, using (4) we have
\[ \tau_{i+1} = \tau_{j+1} \quad \text{for} \quad 1 \leq k < 2^m - j \]
but
\[ \tau_{i+2^m-j} = \tau_{2^m} = 0 < 1 = \tau_{2^m} = \tau_{j+2^m-j}. \]
Since
\[ 2^m - j \leq 2^m \leq 2^N, \]
this proves (6).

If \( j = 0 \), then we are done. If \( j > 0 \), then we complete the proof by combining (5) and (6). \( \square \)

Now fix a nonnegative integer \( N \) and introduce the following sequence:

\[ (7) \quad c_i := \begin{cases} \tau_i & \text{if } 1 \leq i < 2^{N+1}, \\ 2^N & \text{if } i \geq 2^{N+1}. \end{cases} \]

This sequence was used for different purposes in a recent work of Glendinning and Sidorov [7]. Observe that the sequence \( (c_n) \) is periodic with period \( 2^N \) beginning with \( c_0 \). Let us write down the first 16 elements of the Thue-Morse sequence and of the sequences \( (c_n) \) for \( N = 0, 1, 2 \):

\[
\begin{align*}
(c_i) & : & \quad 1101001100101101\ldots \\
N = 0 & : & \quad 11111111111111\ldots \\
N = 1 & : & \quad 11010101010101\ldots \\
N = 2 & : & \quad 1101001100110011\ldots 
\end{align*}
\]

Let us note for further reference that

\[ (8) \quad \tau_i = \tau_{i-2^N} \quad \text{for} \quad 2^{N+1} \leq i < 2^{N+1} + 2^N. \]

Indeed, this follows easily from (4).

It is clear that the equation
\[ (9) \quad 1 = \frac{c_1}{\sqrt{3}} + \frac{c_2}{\sqrt{2}} + \frac{c_4}{\sqrt{2}} + \ldots \]
defines an algebraic number \( 1 < q_N \leq 2 \) satisfying \( q_N \to q^2 \) as \( N \to \infty \).
Proof of Theorem 2. Thanks to Theorem 1, it suffices to verify that the sequence \((c_n)\) is admissible in the following sense:

\[
(10) \quad c_{i+1} \cdots c_{i+2^n} < c_1 \cdots c_{2^n} \quad \text{whenever} \quad c_i = 0
\]

and

\[
(11) \quad c_{i+1} \cdots c_{i+2^n} < c_1 \cdots c_{2^n} \quad \text{whenever} \quad c_i = 1.
\]

For \(1 \leq i < 2^{N+1}\) both relations follow from the similar properties of the Thue–Morse sequence established in the preceding lemma because the first \(2^{N+1} + 2^N - 1\) of the two sequences coincide by equation (8).

For \(i \geq 2^{N+1}\) the relations (10) and (11) now follow by induction because the sequences \(c_{i+1} \cdots c_{i+2^n}\) and \(c_{i+1} \cdots c_{i+2^n}\) coincide, and also \(c_i = c_{i-2^n}\), so that \(c_i = 0\) implies \(c_{i-2^n} = 0\) and \(c_i = 1\) implies \(c_{i-2^n} = 1\). \(\square\)

REFERENCES


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