Graph labelings with restrictive conditions

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Graph labelings with restrictive conditions

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Contents

Part I

Introduction

My work during my PhD studies is made up of several parts: The main topic investigated is the distance-constrained labeling. This problem is motivated by radio communication, the goal is to assign the radio channels to the transmitters in a way that none of the channels interfere with each other and the frequency band is the lowest possible. I will give a brief summary of the theory, the research guidelines and the main results in Chapter 1.

During my theoretical work I dealt with a special vertex labeling of two graph classes, namely the $L(j, j-1, ..., 2, 1)-$labeling of trees and unit interval graphs. The studied values were known for $L(2, 1)-$ and $L(3, 2, 1)$–labelings, I generalized them for arbitrary natural number j. I present my results so far in detail in Chapters 2 and 3, respectively.

In addition to the theoretical studies I examined the problem in practical terms. I chose the linear programming for these studies. I formalized the distance-constrained labeling problem in terms of integer programming. This allowed me to make some comparisons between the graph model and a bit more precise model of the frequency assignment problem that was introduced by me. Chapter 4 describes the formalization, the new model and the computational results of the comparisons.

The last section (Chapter 6) of this thesis is about another kind of partitioning. In this case, not the vertices have to be partitioned according to a specific rule, but the edges of the graph. The main result here is the solution of a quite recently defined problem.

Part II

Theory of distance-constrained labeling

Before presenting my work I would like to make them more transparent and easier to understand. In this section I summarize the theory and the basic results of the topic. Since the most studied theoretical question in the area is how good the distance-constrained labeling numbers of graphs can be estimated, I devote subsections 1.1 and 1.2 to a small collection of such results. Subsection 1.3 describes some other interesting questions that have been raised so far by scientists.

1 Earlier results and background

Problems involving distance 2 In the highly influential paper [1], Hale introduced graph coloring problems with motivation from frequency assignment. This is the following: The radio channels are represented by nonnegative integers and have to be assigned to transmitters efficiently, such that the channels would not interfere with each other. It means that close transmitters receive different channels, and very close transmitters receive channels greater apart.

Since then, several coloring models have been proposed for this kind of problems, perhaps the most intensively studied one is $L(2,1)$–labeling. First investigated by Griggs and Yeh in [2], it requires to assign nonnegative integer labels to the vertices of a given graph G in such a way that vertices having a common neighbor must get distinct labels, and the labels of adjacent vertices must differ at least by 2. The graph invariant $\lambda_{2,1}(G)$ is defined as the smallest possible value of the largest label in such a labeling of G.

Analogously, an $L(j_1, j_2)$–labeling of G is an assignment of nonnegative integer labels to the vertices of G in such a way that the labels of adjacent vertices differ by j_1 or more, and those of vertices at distance 2 apart differ at least by j_2. A comprehensive survey of $L(j_1, j_2)-$labeling and bounds on the minimum $\lambda_{j_1,j_2}(G)$ of the largest label, with nearly 200 references, was given by Calamoneri in [3].

Higher levels of separation As regards vertices at larger distances apart, $L(3, 2, 1)-$labeling (first studied by Shao and Liu in [4]) and more generally $L(j_1, j_2, j_3)$–labeling (introduced by Shao in [5]) put three conditions, depending on the distances between vertices. Here the parameter j_i describes that the difference between the labels of vertices at distance i apart must be at least j_i.

In connection with our results on trees, an interesting unpublished theorem of Clipperton, Gehrtz, Szaniszló and Tokorno [6] states that the $\lambda_{3,2,1}$–number of trees with maximum degree of Δ is at most $2\Delta + 5$, and that this upper bound is tight. The properties of $L(3, 2, 1)$–labelings have been analyzed further in [7].

These notions extend in a natural way to larger distances, too. Let $j_1, j_2, ..., j_s \in \mathbb{N}$ be any integers. It is traditionally assumed that $j_1 \geq j_2 \geq ... \geq j_s$, due to motivation and practical considerations from frequency assignment, although one would get nontrivial theoretical problems without this restriction, too. An $L(j_1, j_2, ..., j_s)$–labeling of a graph $G = (V, E)$ is an assignment $\varphi : V \to \{0, 1, 2, ...\}$ such that $|\varphi(u) - \varphi(v)| \geq j_i$ for all pairs of vertices u, v whose distance in G is equal to i $(i = 1, 2, ..., s)$. The *span* of an $L(j_1, j_2, ..., j_s)-$labeling φ is the largest label assigned by φ to the vertices. In analogy to $\lambda_{2,1}$, $\lambda_{j_1,j_2,...,j_s} = \min_\varphi \max_{v\in V}(\varphi(v))$ is defined as the smallest possible span taken over all $L(j_1, j_2, ..., j_s)$–labelings of G.

Since components can be labeled independently of each other, it is immediately seen that if G is a disconnected graph with components $G_1, G_2, ..., G_k$,

then for any positive integers $j_1, j_2, ..., j_s$ we have
$\lambda_{j_1,j_2,...,j_s}(G) = \max_{1\leq i\leq k}(\lambda_{j_1,j_2,...,j_s}(G_i))$.

For this reason I restrict my attention to connected graphs in this work.

General properties with two levels of constraints Here I present some results from [5] and [8].

Definition An induced subgraph H of a graph G is a subgraph of it that contains all of the edges whose endpoints are both in H.

- For an induced subgraph H of the graph G the inequality $\lambda_{j_1,j_2}(H) \leq \lambda_{j_1,j_2}(G)$ holds with each j_1 and j_2. If H is a subgraph of G, but not induced, then the inequality holds only with the restriction that $j_1 \geq j_2$.

Definition The smallest number of colors needed to color the vertex set of a graph G so that no two adjacent vertices get the same color, is called the chromatic number of G and it is denoted by $\chi(G)$.

- $\chi(G) - 1 \leq \lambda_{j_1,1}(G) \leq j_1 \cdot (\chi(G) - 1)$ ($\chi(G)$ denotes the chromatic number of G, $\lambda_1(G)$ in the present terminology.)

- $\lambda_{j_1,1}(G) \geq \Delta + j_1 - 1$, where Δ denotes the maximum degree of G. Moreover, if $\lambda_{j_1,1}(G) = \Delta + j_1 - 1$ and $j_1 \geq 2$, then $f(v) = 0$ or $\Delta + j_1 - 1$ for any $L(j_1, 1)-$labeling f of G and any major vertex v. We call a vertex v major if $\deg(v) = \Delta$. Consequently, the graph cannot contain a set of 3 major vertices such that any two of them are distance at most 2 apart.

- The graph $S_n = \{v\} + \overline{K_n}$ is called a star. For the star $\lambda_{2,1}(S_n) = n + 1$ and $\lambda_{3,2,1}(S_n) = 2n + 1$ hold. If f_2 is an $(n + 1) - L(2, 1)-$labeling of S_n, then either $f_2(v) = 0$ or $f_2(v) = n + 1$ holds; similarly, if f_3 is a $(2n + 1) - L(3, 2, 1)-$labeling of S_n, then either $f_3(v) = 0$ or $f_3(v) = 2n + 1$ holds.

Some consequences are the following statements: $\lambda_{2,1}(G) \geq \Delta + 1$ for any graph G, and if $\lambda_{2,1}(G) = \Delta + 1$ and f_2 is a $(\Delta + 1) - L(2,1)-$labeling of G, then every vertex $v \in V(G)$ of maximum degree gets label either 0 or $\Delta + 1$. Similarly, $\lambda_{3,2,1}(G) \geq 2\Delta + 1$ for any graph G, and if $\lambda_{3,2,1}(G) = 2\Delta + 1$ and f_3 is a $(2\Delta + 1) - L(3,2,1)-$labeling of G, then every vertex $v \in V(G)$ of maximum degree gets label either 0 or $2\Delta + 1$. From this statement it follows that if there are three vertices $v_1, v_2, v_3 \in V(G)$, each of maximum degree, and $dist(v_i, v_k) \leq 2$ for each i and k (respectively, $dist(v_i, v_k) \leq 3$ for each i and k), then $\lambda_{2,1}(G) \geq \Delta + 2$ ($\lambda_{3,2,1}(G) \geq 2\Delta + 2$, respectively). These values and statements can be easily generalized for the $L(j_1, j_2)-$ and for the $L(j_1, j_2, j_3)-$labeling numbers.

- For any graph G and any positive integers j_1, j_2, j_3 and c the following statements hold:
  $\lambda_{c \cdot j_1, c \cdot j_2}(G) = c \cdot \lambda_{j_1, j_2}$ and $\lambda_{cj_1, cj_2, cj_3}(G) = c \cdot \lambda_{j_1, j_2, j_3}(G)$.

- $\lim_{j_1 \to \infty} \frac{\lambda_{j_1+1,1}(G)}{\lambda_{j_1,1}(G)} = 1$.

- For an induced subgraph H of the graph G the inequality
  $\lambda_{j_1, j_2, j_2}(H) \leq \lambda_{j_1, j_2, j_3}(G)$ holds.

- For any graph G and integers $j_1 \geq j_2 \geq j_3$ the inequality
  $\lambda_{j_1, j_2}(G) \leq \lambda_{j_1, j_2, j_3}(G)$ holds.

- If $l_i \leq j_i$, $(i = 1, 2, 3)$, then $\lambda_{l_1, l_2, l_3}(G) \leq \lambda_{j_1, j_2, j_3}(G)$ holds.

- If f is a $k - L(j_1, j_2, j_3)-$labeling of G, then $f' : V(G) \to \{0, ..., k\}$, $f'(v) = k - f(v)$ is also a proper $k - L(j_1, j_2, j_3)-$labeling of G.

- If the diameter of a graph G is d, where $1 \leq d \leq 3$, then
  $\lambda_{3,2,1}(G) \geq (4 - d) \cdot (|V(G)| - 1)$.

1.1 Some known bounds and exact values for $\lambda_{2,1}$, λ_{j_1,j_2}, $\lambda_{3,2,1}$ and λ_{j_1,j_2,j_3}

1.1.1 Trees

Definition A cycle is a closed walk with no repetitions of vertices and edges, other than the repetition of the starting and ending vertex.

Definition A graph is called a tree if it containes no cycles.

In a tree any two of its vertices are connected by exactly one simple path.

- $\Delta + 1 \leq \lambda_{2,1}(T) \leq \Delta + 2$ [2]

In [9] Chang and Kuo present an algorithm for the decision between $\Delta + 1$ and $\Delta + 2$. This algorithm is applicable for $\lambda_{j_1,1}(T)$, whose value is between $\Delta + j_1 - 1$ and $min\{2\Delta + j_1 - 2, \Delta + 2j_1 - 2\}$ [10]. But to extend it to determine $\lambda_{j_1,j_2}(T)$ with general j_1 and j_2 is NP-hard.

However, Georges and Mauro [11] gave the following bounds:

- $j_1 + (\Delta - 1)j_2 \leq \lambda_{j_1,j_2}(T) \leq j_1 + (2\Delta - 2)j_2$, if $\frac{j_1}{j_2} \geq \Delta$

In another paper [12] they determined the values for $\frac{j_1}{j_2} < \Delta$:

- $\lambda_{j_1,1,1}(T) = \begin{cases} (\Delta + j_1 - 2) + \Delta, & \text{if} \quad \Delta \geq j_1 \\ (\Delta + j_1 - 2) + j_1, & \text{if} \quad \Delta < j_1 \end{cases}$.
- If T is an $n-$ary tree and its height is at least 3, then $\lambda_{3,2,1}(T) = 2n+5$.
- $2\Delta + 1 \leq \lambda_{3,2,1}(T) \leq 2\Delta + 3$.

Lemma If T is a rooted tree with root v, then there is an f, $(2\Delta + 3) - L(3,2,1)-$labeling of T, in which $f(u) \equiv d(u,v)$ $(mod\, 2)$ for every $u \in V(T)$.

1.1.2 Paths

Definition A walk is a sequence $v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, ..., \{v_{n-1}, v_n\}, v_n$ of vertices and edges between consecutive vertices.

Definition A path is an open walk with no repetitions of vertices and edges. P_n denotes the path on n vertices.

Here the most important values [6]:

- $\lambda_{2,1}(P_n) = \begin{cases} 0 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \text{ or } 4 \\ 4 & \text{if } n \geq 5 \end{cases}.$

- $\lambda_{j_1,j_2}(P_n) = \begin{cases} 0 & \text{if } n = 1 \\ j_1 & \text{if } n = 2 \\ j_1 + j_2 & \text{if } n = 3 \text{ or } 4 \\ j_1 + 2j_2 & \text{if } n \geq 5 \text{ and } j_1 \geq 2j_2 \\ 2j_1 & \text{if } n \geq 5 \text{ and } j_1 \leq 2j_2 \end{cases}.$

- $\lambda_{j_1,1,1}(P_n) = \begin{cases} j_1 & \text{if } n = 2 \\ j_1 + 1 & \text{if } n = 3 \text{ or } 4 \\ j_1 + 2 & \text{otherwise} \end{cases}.$

- $\lambda_{3,2,1}(P_n) = \begin{cases} 3, & \text{if } \quad n = 2 \\ 5, & \text{if } \quad n = 3 \quad \text{or} \quad 4 \\ 6, & \text{if } \quad n = 5,\ 6 \quad \text{or} \quad 7 \\ 7, & \text{if } \quad n \geq 8 \end{cases}.$

1.1.3 Cyles of order $n \geq 3$

C_n denotes the cycle on n vertices.

- [2] $\lambda_{2,1}(C_n) = 4$.
- [11] If $\frac{j_1}{j_2} > 2$, then
$$\lambda_{j_1,j_2}(C_n) = \begin{cases} 2j_1 & \text{if } n \text{ odd}, n \geq 3 \\ j_1 + 2j_2 & \text{if } n \equiv 0 \ (mod\, 4) \\ 2j_1 & \text{if } n \equiv 2 \ (mod\, 4) \text{ and } \frac{j_1}{j_2} \leq 3 \\ j_1 + 3j_2 & \text{if } n \equiv 2 \ (mod\, 4) \text{ and } \frac{j_1}{j_2} > 3 \end{cases}.$$
- [11] If $\frac{j_1}{j_2} \leq 2$, then
$$\lambda_{j_1,j_2}(C_n) = \begin{cases} 2j_1 & \text{if } n \equiv 0 \ (mod\, 3) \\ 4j_2 & \text{if } n = 5 \\ j_1 + 2j_2 & \text{otherwise} \end{cases}.$$
- [6] $\lambda_{j_1,1,1}(C_n) = \begin{cases} j_1 + 2, & \text{if } \quad n \equiv 0 \quad (mod\, 4) \\ j_1 + 3, & \text{if } \quad n \equiv 2 \quad (mod\, 4), \quad \text{or } n = 11 \text{ and } j_1 = 2 \\ 6, & \text{if } \quad n = 7 \quad \text{and} \quad j_1 = 2 \\ 2j_1 & \text{otherwise} \end{cases}.$
- [6] $\lambda_{3,2,1}(C_n) = \begin{cases} 6, & \text{if } n = 3 \\ 7, & \text{if } n \text{ even} \\ 8, & \text{if } n \text{ odd and } n \neq 3;\ 7 \\ 9, & \text{if } n = 7 \end{cases}.$

1.1.4 Wheels

Definition The wheel W_n is obtained from C_n by adding a new vertex adjacent to all vertices in C_n.

- $\lambda_{2,1}(W_n) = \begin{cases} 6 & \text{if } n = 3 \text{ or } 4 \\ n+1 & \text{if } n \geq 5 \end{cases}$.

1.1.5 Planar graphs

Definition A graph is called planar if it can be drawn in the plane in such a way that no edges cross each other.

Definition A graph is called outerplanar if it is planar and it can be drawn in such a way that all of its vertices belong to the unbounded face of the drawing.

- [13] If G is outerplanar, then $\lambda_{2,1}(G) \leq \Delta + 8$.
- [14] The above bound was improved to $\Delta + 2$ for $\Delta \geq 8$ and to 10 otherwise.
- [13] If G is outerplanar and chordal (see definition in the subsection "Chordal graphs"), then $\lambda_{2,1}(G) \leq \Delta + 6$.
- [13] If G is planar, then $\lambda_{2,1}(G) \leq 3\Delta + 28$.
- [15] $\lambda_{2,1}(G) \leq 8\Delta - 13$, if $\Delta \geq 5$.
  (If $\Delta \leq 8$, then $8\Delta - 13 < 3\Delta + 28$ holds.)
- [13] If G is planar and chordal, then $\lambda_{2,1}(G) \leq 3\Delta + 22$.
- [16] General j_1 and j_2:
  $$\lambda_{j_1,j_2}(G) \leq \begin{cases} (2j_2 - 1)\Delta + 4j_1 + 4j_2 - 4, & \text{if } g(G) \geq 7 \\ (2j_2 - 1)\Delta + 6j_1 + 12j_2 - 9, & \text{if } g(G) \geq 6 \\ (2j_2 - 1)\Delta + 6j_1 + 24j_2 - 15, & \text{if } g(G) \geq 5 \end{cases},$$
  where $g(G)$ denotes the girth of G, the length of a shortest cycle contained in it.

1.1.6 Chordal graphs

Definition A graph is called chordal if each induced cycle, contained in it, has at most three vertices.

- [10] If G is a chordal graph with maximum degree Δ, then $\lambda_{j_1,1}(G) \leq \frac{(2j_1+\Delta-1)^2}{4}$.

Definition A clique is a set of pairwise adjacent vertices in the graph.

A famous property of chordal graphs [17] is that their vertex set has an ordering $V(G) = \{v_1, ..., v_n\}$ such that for any i, the neighbors of v_i in $\{v_{i+1}, ..., v_n\}$ form a clique, B_i. And conversely, graphs with this property are proved to be chordal. The mentioned ordering is called simplicial order or perfect elimination order.

It is easy to see that the following graph class is a subset of that of the chordal graphs.

Definition Given a positive integer t, $t-$trees are the graphs that arise from a $t-$clique (i.e. K_t) by 0 or more iterations of adding a new vertex joined to a $t-$clique in the graph. [18] (A tree is a 1-tree.)

- [10] If G is a $t-$tree with maximum degree Δ, then $\lambda_{j_1,1}(G) \leq (2j_1 - 1 + \Delta - t) \cdot t$.

Definition A graph is a partial $t-$tree if it is a subgraph of a $t-$tree. The treewidth of a graph is the minimum value t for which the graph is a partial $t-$tree. Another definition of treewidth can be introduced as follows. The two definitions are proved to be equivalent.

Definition tree decomposition of a graph $G = (V, E)$ is a tree T with vertices X_1, ..., X_n, where each X_i is a subset of V, satisfying the following properties:

- The union of all sets X_i equals V. That is, each graph vertex is contained in at least one tree vertex.
- If X_i and X_l both contain a vertex v, then all vertices X_k of the tree in the (unique) path between X_i and X_l contain v as well. Equivalently, the tree vertices containing vertex v form a connected subtree of T.
- For every edge (vw) in the graph, there is a subset X_i that contains both v and w. That is, vertices are adjacent in the graph only when the corresponding subtrees have a vertex in common.

The width of a tree decomposition is the size of its largest set X_i minus 1. The treewidth $tw(G)$ of a graph G is the minimum width among all possible tree decompositions of G. In this definition, the size of the largest set is diminished by 1 in order to make the treewidth of a tree equals 1. Equivalently, the treewidth of G is 1 less than the size of the largest clique in the chordal graph containing G with the smallest clique number. A chordal graph with this clique size may be obtained by adding to G an edge between every two vertices that both belong to at least one of the sets X_i.

- [13] For a graph of treewidth t, we have $\lambda_{2,1} \leq (\Delta + 2) \cdot t$.

Definition A vertex set is called independent if no two of them are adjacent.

Definition split graph is a graph G whose vertex set can be split into two sets K and S, such that K induces a clique and S induces an independent set in G. (Split graphs are chordal.)

- [13] If G is a split graph, then $\lambda_{2,1}(G) \leq \Delta^{1,5} + 2\Delta + 2$, and for any Δ there is a split graph with $\lambda_{2,1} \geq \frac{1}{3} \cdot \sqrt{\frac{2}{3}} \cdot \Delta^{1,5}$. (The essential point of this result is that it gives a nonlinear lower bound.)

Definition An $n-$sun is a chordal graph that contains a Hamiltonian cycle $(x_1, y_1, x_2, y_2, ..., x_n, y_n, x_1)$ in which each x_i has degree 2 and the vertices y_i form an $n-$clique.

A *sun − free* (SF)/ *odd − sun − free* (OSF)/ *3 − sun − free* (3SF) chordal graph is a chordal graph containing no $n-$sun with $n \geq 3$/ odd $n \geq 3$/ $n = 3$ as an induced subgraph.

- [10] If G is an OSF-chordal graph, then $\lambda_{j_1,1} \leq j_1 \cdot \Delta$.
- If G is an SF-chordal graph, then $\lambda_{j_1,1} \leq \Delta + (2j_1 - 2)(\chi(G) - 1)$.

Definition A graph is called interval graph if its vertices can be represented by an interval of the real line such that two vertices are adjacent if and only if the two corresponding intervals intersect.

A *unit interval graph* is an interval graph whose interval representation contains only intervals of the same length. The following bounds are known for unit interval graphs:

- [19] $\lambda_{2,1}(G) \leq 2\chi(G)$.
- [20] $\lambda_{j_1,j_2}(G) \leq j_1(\chi(G)-1)+j_2$ if $j_1 > 2j_2$, and $2j_2 \cdot \chi(G)$, if $j_1 \leq 2j_2$.
- [20]$\lambda_{j_1,j_2}(G) \geq max\{j_1(\chi(G)-1), j_2 \cdot \lambda_{1,1}(G)\}$ A linear-time algorithm is proposed that can $L(j_1, j_2)-$label a given unit interval graph using the largest label no more than the bound above.

1.1.7 Cartesian product of graphs

The Cartesian product $(G\Box H)$ of the graphs G and H is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and edge set

$$\begin{aligned} E(G\Box H) &= \{((v,w),(v',w'))| \\ &\quad ((v = v') \wedge ((w,w') \in E(H))) \vee (((v,v') \in E(G)) \wedge (w = w'))\} \end{aligned}$$

From [11] and [21] we have the following statements:

Let P_i be a path of order p_i, for $i = 1, 2, ..., n, n \geq 2$. $P := P_1\Box P_2\Box ...\Box P_n$. Then

- If $p_i \geq 3$ for every i, and $p_i \geq 4$ for at least two distinct i-s, then $\lambda_{2,1}(P) = 2n + 2$.
- Suppose $p_n = 2$ and $p_i \geq 3$, $1 \leq i \leq n - 1$ and if $p_i \geq 4$ for at least two distinct i-s, then $\lambda_{2,1}(P) = 2n + 1$.
- Suppose $p_i \geq 5$ for each i. Then $\lambda_{j_1,j_2}(P) = j_1 + (4n - 2)j_2$ if $\frac{j_1}{j_2} \geq 2n$, and $2j_1 + 2(n - 1)j_2 \leq \lambda_{j_1,j_2}(P) \leq 2j_1 + (2n - 1)j_2$ if $\frac{j_1}{j_2} < 2n$.

P is called an $n-$cube Q_n, if $p_i = 2$ for all i. We have the following bounds for the $L(2, 1)-$labeling number of Q_n:

- [2] $\lambda_{2,1}(Q_n) \leq 2n + 1$.
- [15] $\lambda_{2,1}(Q_n) \geq n + 3$.
- Exactly: $\lambda_{2,1}(Q_3) = 6$, $\lambda_{2,1}(Q_4) = 7$, $\lambda_{2,1}(Q_5) = 8$. So the lower bound is tight for $n = 3, 4$ and 5.
- [21] $\lambda_{2,1}(Q_n) \leq 2^k - 1$ for $n \leq 2^k - k - 1$.
- [21]$\lambda_{2,1}(Q_n) \leq 2^k + 2^{k-q+1} - 2$ for $1 \leq q \leq k$ and $n \leq 2^k - q$.
- [21] $lim\,inf\ \frac{\lambda_{2,1}(Q_n)}{n} = 1$.
- [21] In general $\lambda_{2,1}(Q_n) \leq 2n$ for all $n \geq 2$. (Notice that this is an improvement of the result from the first one from [2].)
- [2] $2n + 3 \leq \lambda_{3,2,1}(Q_n) \leq 4n + 3$ if $n \geq 3$.

1.1.8 The cross-product of paths and cycles

Definition The cross-product of the graphs $G_1, G_2, ..., G_k$ is the graph, denoted by $G_1 \times G_2 \times ... \times G_k$, with vertex set

$$V(G_1 \times G_2 \times ... \times G_k) = V(G_1) \times V(G_2) \times ... \times V(G_k)$$

and edge set

$$\begin{aligned} E(G_1 \times G_2 \times ... \times G_k) &= \{(u_1, u_2, ..., u_k)(v_1, v_2, ..., v_k) \mid \\ & u_l, v_l \in V(G_l), (\forall l : 1 \le l \le k), \\ & (\exists i : (u_i v_i \in E(G_i)), \\ & (u_j = v_j, \, if \, (j \neq i)))\}. \end{aligned}$$

Since the structure of cross-products of most graphs, in general they are not easy to deal with, the labeling problem has been studied especially for paths and cycles, but these of general length. A lot of values have been determined exactly, here the most general ones are presented:

- [22] $\lambda_{3,2,1}(P_2 \times P_n) = \begin{cases} 7 & \text{if } n = 2 \\ 8 & \text{if } n = 3;\ 4 \\ 9 & \text{if } n \ge 5 \end{cases}.$
- [22] $\lambda_{3,2,1}(P_m \times P_n) \le 11$ if $n \ge m \ge 3$,
  and $\lambda_{3,2,1}(P_3 \times P_n) \le 10$ if $n = 4$ or 5.
- [22] $\lambda_{3,2,1}(P_3 \times P_n) = \begin{cases} 9 & \text{if } n = 3 \\ 10 & \text{if } n = 4;\ 5 \\ 11 & \text{if } n \ge 6 \end{cases}.$
- [22] If $n \ge m \ge 4$ then $\lambda_{3,2,1}(P_m \times P_n) = 11$.
- [23] $\lambda_{3,2,1}(C_m \times P_n) = 11$ if $m \equiv 0 \ (mod\, 4)$ and $n \ge 3$.
- [23] $\lambda_{3,2,1}(C_m \times C_n) = 11$ if $m \equiv 0 \ (mod\, 4)$ and $n \equiv 0 \ (mod\, 12)$.

1.1.9 Generalized Petersen graphs

The generalized Petersen graph of order n is the 3-regular graph with $2n$ ($n \ge 3$) vertices consisting of two disjoint $n-$cycles, called inner and outer cycles, such that each vertex on the outer cycle is adjacent to a (necessarily unique) vertex on the inner cycle. This definition was introduced by Georges and Mauro. They gave the following bounds in [24]:

- $\lambda_{2,1}(G) \leq 9$, if G is a generalized Petersen graph.
- $\lambda_{2,1}(G) \leq 8$, if G is a generalized Petersen graph of order greater than 12.
- $\lambda_{2,1}(G) \leq 7$, if G is a generalized Petersen graph of order $3 \leq n \leq 12$ but $\neq 5$.

1.2 Bounds on other parameters

1.2.1 Bounds from the chromatic number

- [2] If G is a graph with n vertices, then $\lambda_{2,1}(G) \leq n + \chi(G) - 2$.
- [2] For any graph G, $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$.
- [2] If G is a graph with diameter 2, then $\lambda_{2,1}(G) \leq \Delta^2$.

Later on some reductions were made on the second bound. These are the following in order:

$\Delta^2 + \Delta$ in [9], $\Delta^2 + \Delta - 1$ in [25] and $\Delta^2 + \Delta - 2$ in [26].

A generalization of the same bound was given in [10], and this is the following: $\lambda_{j_1,1}(G) \leq \Delta^2 + (j_1 - 1)\Delta$.

1.2.2 Bounds on the path covering number

A path covering of G, denoted by $C(G)$, is a collection of vertex-disjoint paths in G such that each vertex in $V(G)$ is incident to a path in $C(G)$. A minimum path covering of G is a path covering of G with minimum cardinality, and the path covering number $c(G)$ of G is the cardinality of a minimum path covering of G. We observe that there exists a Hamiltonian path in G if and only if $c(G) = 1$. (A Hamiltonian path is a path that visits all vertices in the graph exactly once.) We know the following from [8]:

Definition The complement of a graph G is the graph with the same vertex set in which two vertices are adjacent if and only if they are non-adjacent in G.

- $\lambda_{2,1}(G) \leq n-1$ if and only if $c(\overline{G}) = 1$ where $\overline{G}$ denotes the complement of G.
- Let r be an integer, $r \geq 2$. Then $\lambda_{2,1}(G) = n + r - 2$ if and only if $c(\overline{G}) = r$.

1.3 Related problems

1.3.1 The size

Let $G(n, k)$ be the collection of all graphs with order n (the number of its vertices) and $L(2,1)-$labeling number $\lambda_{2,1} = k$. The following results [27] show the exact values of the minimum and maximum sizes (the number of the edges) of graphs in $G(n, k)$. These are denoted by $s_m(n, k)$ and $s_M(n, k)$, respectively.

- $$s_m(n,k) = \begin{cases} 0 & \text{if } k = 0, \\ \frac{(k-3n+2)(n-k-1)}{2} & \text{if } k \text{ is even and } n < k < \frac{6n-4}{5}, \\ \frac{(k-3n+2)(n-k-1)}{2} & \text{if } k \text{ is odd and } n < k < \frac{6n-1}{5}, \\ \frac{k(k+2)}{8} & \text{if } k \text{ is even, } n < k, \text{ and k} > \frac{6\text{n}-4}{5}, \\ \frac{(k-1)(k+5)}{8} & \text{if } k \text{ is odd, } n < k, \text{ and k} > \frac{6\text{n}-1}{5}, \\ k-1 & \text{if } 2 \leq k \leq n. \end{cases}$$

- Let $n = a(k+1) + r,\ a > 0,\ 0 \le r < k+1$. Then $s_M(n,k) =$

$$\begin{cases} 0 & \text{if } k = 0, \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } k = 2 \text{ and } n \ge 2, \\ a\left(\binom{k+1}{2} - k\right) + \binom{r}{2} & \text{if } 2r-2 < k \text{ and } 2 < k \le n-1, \\ a\left(\binom{k+1}{2} - k\right) + \binom{r}{2} - 2r + k + 2 & \text{if } 2r-2 \ge k \text{ and } 2 < k \le n-1, \\ \binom{n}{2} - 2n + k + 2 & \text{if } n-1 \le k \le 2n-2, k \ge 3. \end{cases}$$

1.3.2 The edge span

Definition [28] The edge span of a labeling f is defined as $max\{|f(u) - f(v)| : \{u,v\} \in E(G)\}$. The $L(j_1, j_2, ..., j_s)-$edge span of G, $\beta_{j_1,j_2,...,j_s}(G)$, is the minimum edge span over all $L(j_1, j_2, ..., j_s)-$labelings on G.

Yeh calculated the following edge span values in [28]:

- $\beta_{j_1,j_2,j_3}(P_n) = j_1$,where $n \ge 2$.
- $\beta_{2,1}(C_3) = 4$ and $\beta_{2,1}(C_n) = 3$ for $n \ge 4$.
- $\beta_{j_1,j_2,1}(C_n) = j_1 + j_2$, if $j_2 \le j_1 \le 3j_2$ where $n \ge 4$.
- Let T be a tree with maximum degree Δ. Then $\beta_{2,1}(T) = \left\lceil \frac{\Delta}{2} \right\rceil + 1$.
- $\beta_{j_1,j_2,j_3}(K_n) = (n-1) \cdot j_1 = \lambda_{j_1,j_2,j_3}(K_n)$.
- Let $K = K_{n_1,n_2,...,n_k}$ be a complete $k-$partite graph, where $n_1 \ge n_2 \ge ... \ge n_k$. Then $\beta_{2,1}(K) = \left\lceil \frac{n_1}{2} \right\rceil + n_2 + ... + n_k + k - 2$.

The $\beta_{j_1,j_2,...,j_s}(G)$ corresponds not necessarily to an optimal $L(j_1, j_2, ..., j_s)-$labeling. Let $\beta^*_{j_1,j_2,...,j_s}(G)$ denote the edge span over all $L(j_1, j_2, ..., j_s)-$labelings of G with optimal labeling numbers. (Clearly, $\beta_{j_1,...,j_s}(G) \le \beta^*_{j_1,...,j_s}(G)$ for any graph.)

$P_n \times P_m$ gives an example for the above statement:

$\beta_{2,1}(P_n \times P_m) = 3$ and $\beta^*_{2,1}(P_n \times P_m) = 5$ for $n \geq m \geq 2$.

An interesting question is, whether one can characterize those graphs for which equality holds.

Examples: $\beta_{2,1}(\Gamma_\square) = \beta^*_{2,1}(\Gamma_\square) = 5$ and $\beta_{2,1}(\Gamma_\triangle) = \beta^*_{2,1}(\Gamma_\triangle) = 7$, where $\Gamma_\square$ and $\Gamma_\triangle$ are the square and the triangular lattices. More about them is in Chapter 5.

1.3.3 Critical graphs

We call a graph G $\lambda_{j_1,j_2,...,j_s}-$critical if $\lambda_{j_1,j_2,...,j_s}(G) > \lambda_{j_1,j_2,...,j_s}(H)$ for every proper subgraph H of G.

This topic has not been studied well so far. Some results are in [29].

1.3.4 $(p, q)-$total labeling

The notion of $(p, q)-$total labeling is introduced by Havet and Yu [30].

Definition The $(p, q)-$total labeling of the graph G is an assignment f from the vertex set and the edge set to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if x is a vertex and y is an edge incident to x, and $|f(x) - f(y)| \geq q$ if x and y are a pair of adjacent vertices or a pair of adjacent edges for all x and y in $V(G) \cup E(G)$.

The $(p, q)-$total labeling problem asks the minimum k among all possible assignments $V(G) \cup E(G) \to \{0, ..., k\}$. This minimum value is the $(p, q)-$total labeling number and is denoted by $\lambda^T_{p,q}(G)$.

It is a special case of $L(p, q)-$labeling, because a $(p, q)-$total labeling of G corresponds to an $L(p, q)-$labeling of the incidence graph of G where the incidence graph of G is the graph obtained from G by replacing each edge (v_i, v_j) with two edges (v_i, v_{ij}) and (v_{ij}, v_j) after introducing one new vertex v_{ij}.

Some bounds for $(p,1)$-total labeling numbers

- [30] $\lambda^T_{p,1}(G) \geq \Delta + p - 1$
- [30] $\lambda^T_{p,1}(G) \geq \Delta + 1$ if $p \geq \Delta$

Definition The smallest number of colors needed to color each edge of a graph G such that no two edges incident on the same vertex get the same color, is called the chromatic index of G.

- [30] $\lambda^T_{p,1}(G) \leq min\{2\Delta + p - 1, \chi(G) + \chi'(G) + p - 2\}$ for any graph G where $\chi(G)$ and $\chi'(G)$ denote the chromatic number and the chromatic index of G, respectively. A consequence of this is that $\lambda^T_{p,1}(G) \leq \Delta + p + 3$ for any planar graph G (by the Four-Color Theorem).
- [30] $\lambda^T_{p,1}(K_n) \leq n + 2p - 2$
- [31] $\lambda^T_{p,1}(G) \leq \Delta + p + s$ for any $s-$degenerate graph (by $\chi(G) \leq s+1$ and $\chi'(G) \leq \Delta + 1$), where an $s-$degenerate graph G is a graph which can be reduced to a trivial graph by successive removal of vertices with degree at most s.
- [32] $\lambda^T_{p,1}(G) \leq \Delta + p + 1$ for any outerplanar graph other than an odd cycle.

(p,q)-total labeling of trees The next upper and lower bounds are from [33]:

Upper bounds If $p = q + r$ for $r \in \{0, 1, ..., q-1\}$ and $\Delta > 1$ (respectively, $\Delta = 1$) then $\lambda^T_{p,q}(T) \leq p + (\Delta - 1)q + r$ holds and this bound is tight (respectively, $\lambda^T_{p,q}(T) = p + q$). If $p \geq 2q$ then $\lambda^T_{p,q}(T) \leq p + \Delta q$ holds and this bound is tight. If $p \geq \Delta q$ then $\lambda^T_{p,q}(T) = p + \Delta q$.

Lower bounds If $q \leq p < (\Delta - 1)q$ then $\lambda^T_{p,q}(T) \geq p + (\Delta - 1)q$ holds and this bound is tight. If $p = (\Delta - 1)q + r$ for $r \in \{0, 1, ..., q - 1\}$ then $\lambda^T_{p,q}(T) \geq p + (\Delta - 1)q + r$ holds and this bound is tight. If $p \geq \Delta q$ then $\lambda^T_{p,q}(T) = p + \Delta q$.

- The $(p, q)-$total labeling problem with $p \leq \frac{3q}{2}$ for trees can be solved in linear time. If $\Delta \geq 2$, we have $\lambda^T_{p,q}(T) \in \{p+(\Delta-1)q,\ p+(\Delta-1)q+r\}$. If $\frac{3q}{2} > p > q$ and $\Delta \geq 4$ then $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ holds if and only if no two vertices with degree Δ are adjacent.

- In the case of $p = 2q$, the condition that no two vertices with degree Δ are adjacent, is sufficient for $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$, while in the case of $p > \frac{3q}{2}$ and $p \neq 2q$, this condition is not sufficient.

- For any two nonnegative integers p and q the $L(p, q)-$labeling problem for trees can be solved in polynomial time if $\Delta = \mathrm{O}(log^{\frac{1}{3}}|I|)$ where $|I| = max\{|V(T)|, log(p)\}$. Particularly, if Δ is a fixed constant, it is solvable in linear time.

1.3.5 Algorithmic complexity

It was already known from [2] that the decision version of $L(2, 1)$ – that is, the input consists of a graph G and an integer k, and the question is whether $\lambda_{2,1}(G) \leq k$ holds – on unrestricted input graphs is NP-complete (This is obtained by a double reduction from the HAMILTONIAN PATH problem. This problem asks whether a graph has a Hamiltonian path or not.). But the algorithmic complexity of distance-constrained labeling with two levels of conditions is an interesting issue even for trees. Restricting the input graphs to trees, the $L(2, 1)-$labeling problem becomes solvable in polynomial time using an algorithm of Chang and Kuo [9]. With some modifications, this algorithm can handle $L(j_1, 1)-$labelings [10], too, and not only on trees but also on the slightly wider class of graphs which can be transformed to a tree by removing at most p edges for some fixed p.

In contrast to this, Fiala, Golovach and Kratochvíl proved that the decision version of the $L(j_1, j_2)-$labeling problem is NP-complete on trees whenever j_1 is not a multiple of j_2. (Otherwise it is reducible to $L(\frac{j_1}{j_2}, 1)$, which is solvable efficiently as mentioned above.)

The $L(2,1)-$labeling problem still remains NP-hard for planar graphs, bipartite graphs, split graphs and chordal graphs [34], even for graphs of treewidth 2 [35]. In the case of planar graphs, determining the existence of a $k-L(2,1)-$labeling is NP-hard for $k \geq 4$, while it can be done in polynomial time for $k \leq 3$ [36].

We know that the $L(2,1)-$labeling number of a star $K_{1,p}$ is $p+1$. In a labeling of this span the central vertex gets either 0 or $p+1$, and exactly one of its neighbors gets the other label. Hence the $L(2,1)-$labeling number of a $p-$regular graph ($p \geq 3$) is bigger than $p+1$. But the decision question, whether a $p-$regular graph admits an $L(2,1)-$labeling of span at most $p+2$, is NP-complete for any integer $p \geq 3$ [37].

1.3.6 Radio number

An interesting particular case of $L(j_1, j_2, ..., j_s)-$labeling is called *radio labeling*, defined with the condition $j_1 = \text{diam}(G)$, $j_2 = \text{diam}(G) - 1$,..., $j_{\text{diam}(G)} = 1$, where $\text{diam}(G)$ stands for the diameter of G. In other words, denoting the distance of two vertices u and v by $dist(u,v)$, the assignment $\varphi : V \to \{0, 1, 2, ...\}$ is required to satisfy the inequality

$$|\varphi(u) - \varphi(v)| \geq \text{diam}(G) + 1 - dist(u,v)$$

for all vertex pairs u, v. The *radio number* of G, denoted by $rn(G)$, is the minimum span of a radio labeling of G. That is, $rn(G) = \lambda_{d,d-1,...,1}(G)$, where $d = \text{diam}(G)$.

The notion of radio number was introduced by Chartrand, Erwin, Harary and Zhang in [38], and has been studied for many kinds of graphs, including paths and cycles [39], powers of paths and cycles [40, 41, 42], spider graphs

[43] and some Cartesian product of some graphs [44, 45, 46, 47, 48, 49]. A brief summary of the known results with the related references is given by the subsection 7.4 of the survey [50].

In the present context the most important citations are Liu's paper [43] for a general lower bound on the radio number of trees, and the work of Li, Mak ang Zhou [51] who determined the radio number of complete m-ary (rooted) trees.

Part III

New theoretical results in distance-constrained labeling

One structure I dealt with is the level-wise regular tree. I focused on the radio labeling of trees. The structure of level-wise regular trees is favorable to study since it helps getting sharp upper bounds. I used exactly this feature in order to determine the values that I present with the associated proofs in Section 2. I used a similar trick for an other graph class as well, namely, for the unit interval graphs. Section 3 is about my results according to them.

2 Radio labeling of level-wise regular trees

Level-wise regular trees As mentioned above, Li, Mak and Zhou determined the radio number of complete m-ary rooteed trees. Every such tree has even diameter, moreover the degree of its root is smaller (by 1) than that of all the other non-leaf vertices. Therefore one of our goals was to prove a result, analogous to the one of [51], for trees in which all internal vertices have the same degree and the diameter is unrestricted. We establish this by considering a more general class of trees. In this way our theorem also includes that of for $m \geq 3$ as a particular case.

It is well known that every tree $T = (V, E)$ has a central vertex r or a central edge $r'r''$, depending on the parity of the diameter $\mathrm{diam}(T)$. Setting $L_0 = \{r\}$ if $\mathrm{diam}(T)$=$2h$ is even, and $L_0 = \{r'r''\}$ if $\mathrm{diam}(T)$=$2h+1$ is odd, every vertex of T is at distance at most $h = \left\lfloor \frac{1}{2}\mathrm{diam}(T) \right\rfloor$ apart from L_0. Define the level sets of T as $L_i = \{v \in V \mid dist(v, L_0) = i$, for $1 \leq i \leq h$.

The vertices $v \in L_i$ will be referred to as $i-vertices$ for $i = 0, 1, 2, ..., h$. The value h represents the height of the structure with respect to the central level L_0. This h is either the radius of T (if diam(T) is even) or the radius minus 1 (if diam(T) is odd).

We say that T is level-wise regular if all $i-$vertices have the same degree, say m_i, for every $i = 0, 1, 2, ..., h$. In particular, a complete m-ary tree is represented with the values $m_0 = m$, $m_1 = m_2 = ... = m_{h-1} = m + 1$ and $m_h = 1$, while internally $m-$regular complete trees are represented with $m_0 = m_1 = ... = m_{h-1} = m$ and $m_h = 1$. Note that all leaves are at the same distance from L_0 in every level-wise regular tree.

We always have $m_h = 1$ by definition, hence a level-wise regular tree of height h is characterized by an ordered $h-$tuple $(m_0, m_1, ..., m_{h-1})$. We use the notation $T^1_{m_0,m_1,...,m_{h-1}}$ for the tree uniquely identified by $(m_0, m_1, ..., m_{h-1})$ with $|L_0| = 1$ (having even diameter $2h$), and $T^2_{m_0,m_1,...,m_{h-1}}$ for the tree identified by $(m_0, m_1, ..., m_{h-1})$ with $|L_0| = 2$ (having odd diameter $2h + 1$). In either case, the superscript indicates the cardinality of L_0.

We determined the exact value of $\lambda_{d,d-1,...,1}(T^p_{m_0,m_1,...,m_{h-1}})$ with $p = 1, 2$ for every $d \geq 1$ and for all level-wise regular trees in which $m_i \geq 3$ holds for all $0 \leq i \leq h - 1$ where $d = \text{diam}(T)$ and $h = \left\lfloor \frac{d}{2} \right\rfloor$. In particular, for internally regular complete trees we have:

Theorem 1 Let $d \geq 3$ and $m \geq 3$ be integers, and let $h = \left\lfloor \frac{d}{2} \right\rfloor$. Then for the internally $(m+1)$-regular complete trees with diameter d and height h we have:

(a) If $d = 2h$ then the complete tree T with a central vertex and parameters $m_0 = m+1$ and $m_1 = ... = m_{h-1} = m$ has $\lambda_{d,d-1,...,1}(T) = 1 + \sum_{i=0}^{\frac{d}{2}-1}((m + 1) \cdot m^i \cdot (d - 1 - 2i)) = m^h + \frac{4m^{h+1}-2hm^2-4m+2h}{(m-1)^2}$.

(b) If $d = 2h + 1$ then the complete tree T with a central edge and parameters $m_0 = m_1 = ... = m_{h-1} = m$ has $\lambda_{d,d-1,...,1}(T) = \sum_{i=0}^{\frac{d-1}{2}}(2 \cdot m^i \cdot (d - 2i)) - d = 2m^h + \frac{6m^{h+1}-2m^h-(2h+1)m^2-4m+2h+1}{(m-1)^2}$.

It is interesting to compare these formulas with the one derived in [51]; I shall put some comments of this kind in the concluding section.

The cited results on paths and complete binary trees indicate that allowing $m_i = 2$ changes the problem in a substantial way and leads to different formulas. Nevertheless, we stated and proved the lower bound under the weaker restriction $m_i \geq 2$ (that is written in a later paragraph), because some combinations of larger degrees may still allow the formula to be tight. It remains an open problem for future research to analyze which h-tuples $(m_0, m_1, ..., m_{h-1})$ correspond to cases of equality.

2.1 Lower bounds from weighted powers of graphs

The aim of the first part of this section is to indicate a way how lower bounds on $rn(G)$ and more generally on $\lambda_{j_1,j_2,...,j_d}(G)$ can be obtained. The second part applies the idea for level-wise regular trees. In a later section it is proved that the derived bounds are tight in many cases.

Let the ordered $d-$tuple $\mathbf{j}$=$(j_1, j_2, ..., j_d)$ of integers be given, and let $G = (V, E)$ be a graph. The d^{th} power of graph G, denoted by G^d, is traditionally defined as the graph whose vertex set is V and two vertices are adjacent in G^d if and only if their distance in G is at most d. Denoting by E^d the edge set of G^d, we define the weight function $w_j : E^d \to \{j_1, j_2, ..., j_d\}$ as $w_j(u, v) = j_i \iff dist(u, v) = i$ for each edge $(uv) \in E^d$. Hence, the edge weights precisely express the lower bounds on the differences between vertex labels, as prescribed by $\mathbf{j}$.

Once the values of $j_1, j_2, ..., j_d$ are understood, we shall simplify the notation from w_j to w by writing $G^d_w = (V, E^d, w)$, and call G^d_w the *weighted power graph* of G. In fact the precise term would be "weighted d^{th} power graph with respect to $j_1, j_2, ..., j_d$" but the parameters are assumed to be given throughout.

Lower bounds for radio labeling The relevance of G^d_w in the context of radio labeling is shown by the following assertion.

Proposition 2 For every graph G, the value of $rn(G)$ is at least as large as the minimum weighted length of a Hamiltonian path in G_w^d, where d is the diameter of G.

Proof Observe that in a radio labeling no two vertices can get the same label. For this reason, the weighted power graph for radio labeling is a *complete* graph equipped with positive edge weights. Every radio labeling of G defines a total order on the vertex set by increasing labels, and hence we obtain a Hamiltonian path of G in a natural way by this order. Moreover, consecutive vertices differ in their labels by at least as much as the weight of the edge joining them.

It is important to note that equality does not always hold. For instance, if $P = v_1v_2v_3v_4v_5$ is the path of length 4, then $rn(P) = 10$ holds as a particular case of the formula $rn(P_{2k+1}) = 2k^2 + 2$ from [39] . On the other hand, since the weight of an edge (v_iv_j) is equal to $n - |i - j|$, the Hamiltonian path $v_3v_5v_1v_4v_2$ in P_w^4 has weight 3+1+2+3=9. The point is that Hamiltonian paths take only the consecutive vertex pairs into account, while in a radio labeling on all pairs. Indeed, the subpath $v_5v_1v_4$ has length 3, but v_5 and v_4 should differ by at least 4 in label.

We next observe that the lower bound in Proposition 2 can be refined to a tight estimate.

Proposition 3 For every graph G, the value of $rn(G)$ is equal to the smallest possible weighted length of a longest directed path taken over all transitive orientations of G_w^d, where d is the diameter of G.

Proof Let $\varphi : V \to \{0, 1, ..., rn(G)\}$ be a minimum-span radio labeling of graph $G = (V, E)$. Since no two vertices can get the same label, φ defines a natural ordering on V. We index the vertices in the increasing order of labels, that is $0 = \varphi(v_1) < \varphi(v_2) < ... < \varphi(v_n) = rn(G)$. Orienting each edge of G_w^d from smaller index to larger one, the weighted length of any oriented path is at most the difference of labels of its two ends, therefore no path longer than $rn(G)$ can occur.

Conversely, let $v_1, v_2, ..., v_n$ be the vertex order generated by a transitive orientation of G_w^d, and suppose that the maximum weighted length of a directed path in this orientation is ℓ. Define $\varphi(v_1) = 0$ and compute the vertex labels recursively by the rule

$$\varphi(v_i) = d + 1 + \max_{1 \le j \le i-1} (\varphi(v_j) - dist(v_i, v_j)) \tag{3}$$

for $i = 2, 3, ..., n$. This is a radio labeling of G because the separation constraint is respected between any two vertices. Moreover, a path of weighted length $\varphi(v_n)$ exists; it can be identified by backtracking. Indeed, each v_i with $i > 1$ attains equality in (3) for some $j = j_i < i$, and therefore making one such edge (v_j, v_i) for each i, there exists a monotone decreasing path from v_n to v_1. Consequently, we have $rn(G) \le \varphi(v_n) \le \ell$.

2.2 Lower bound for level-wise regular trees

Given an h-tuple $(m_0, m_1, ..., m_{h-1})$, let us use the simplified notation T^1 and T^2 for the level-wise regular trees $T^1 = T^1_{m_0,m_1,...,m_{h-1}}$, $T^2 = T^2_{m_0,m_1,...,m_{h-1}}$.

Under the stricter assumption $m_i \ge 3$ for all $0 \le i < h$, Propositions 2 and 3 will turn out to be equivalent for each T^1 and T^2. Using those propositions, we derive the following general lower bound:

Theorem 4 If $h \ge 1$ and $m_0, m_1, ..., m_{h-1} \ge 2$, then

$$\lambda_{d,d-1,...,1}(T^1) \ge (d+1)(n-1)+1-2\cdot\sum_{i=1}^{h}(m_0\cdot i\cdot \prod_{0<j<i}(m_j-1)) \tag{4}$$

and

$$\lambda_{d,d-1,...,1}(T^2) \ge d(n-1)-4\cdot\sum_{i=1}^{h}(i\cdot \prod_{0\le j<i}(m_j-1)) \tag{5}$$

for $d = 2h$ and $d = 2h + 1$, respectively.

Proof Although the ideas are very similar in the arguments for d even and odd, there are some differences and it is convenient to split the proof into two parts according to the parity of d. In either case, we denote

$$n = \sum_{i=0}^{h} |L_i|,$$

the number of vertices.

Case 1: $d = 2h$

We have $|L_0| = 1$ and

$$|L_i| = m_0 \cdot \prod_{0<j<i} (m_j - 1)$$

for $i = 1, ..., h$. Moreover, the distance between an i'-vertex v' and an i''-vertex v'' has the upper bound

$$dist(v', v'') \leq i' + i'', \tag{6}$$

therefore the edge $v'v''$ in $(T^1)^d_w$ has weight at least $d+1-(i'+i'')$. We define

$$\ell(v) = \ell_i = \frac{d+1}{2} - i = h + \frac{1}{2} - i$$

for every $i-$vertex v, for any $i = 0, 1, ..., h$.

Let $P = v_1v_2...v_n$ be any Hamiltonian path of $(T^1)^d$. Due to inequality (6), the weight of any edge (v_jv_{j+1}) for two consecutive vertices in P is at least $\ell(v_j) + \ell(v_{j+1})$. Internal vertices of P occur in two such pairs, while the two ends occur in just one pair each. Consequently,

$$\begin{aligned}\lambda_{d,d-1,\ldots,1}(T^1) &\geq \min_{v_1 v_2 \ldots v_n} (\sum_{j=1}^{n} 2\ell(v_j)) - \ell(v_1) - \ell(v_n) \\ &= (\sum_{i=1}^{h} (d+1-2i) \cdot |L_i|) - \ell_0 - \ell_1) \\ &= (d+1)n - d - 2 \cdot \sum_{i=1}^{h} (m_0 \cdot i \cdot \prod_{0<j<i} (m_j - 1)\end{aligned}$$

where minimum in the first line is taken over all permutations $(v_1, v_2, ..., v_n)$ of the vertices. This completes the proof of (4).

Case 2: $d = 2h + 1$

In this case $|L_o| = 2$ and

$$|L_i| = 2 \cdot \prod_{j=0}^{i-1} (m_j - 1)$$

for $i = 1, ..., h$. Moreover, since the deepest level is $L_{\frac{d-1}{2}}$ instead of $L_{\frac{d}{2}}$, also the upper bound on the distance between an i'-vertex v' and an i''-vertex v'' is slightly different:

$$dist(v', v") \leq i' + i" + 1, \tag{7}$$

the "+1" term being due to the central edge. For this reason, the edge $(v'v'')$ in $(T_2)^d_w$ now has weight at least $d - (i' + i'')$. We therefore define

$$\ell(v) = \ell_i = \frac{d}{2} - i = h + \frac{1}{2} - i$$

for every i-vertex v, for any $i = 0, 1, ..., h$. Notice that the formula is unchanged as a function of h, but it is somewhat different when viewed as a function of d.

The next observations are analogous to those for T^1 above. Let $P = v_1v_2...v_n$ be any Hamiltonian path of $(T^2)^d$. Due to inequality (7), the weight of any edge (v_jv_{j+1}) for two consecutive vertices in P is at least $\ell(v_j)+\ell(v_{j+1})$. Internal vertices of P occur in two such pairs, while the two ends occur in just one pair each. Note that we now have two 0-vertices. Consequently,

$$\begin{aligned} \lambda_{d,d-1,...,1}(T^2) &= \min_{v_1v_2...v_n} (\sum_{j=1}^{n} 2\ell(v_j)) - \ell(v_1) - \ell(v_n) \\ &\geq \left(\sum_{i=0}^{h} (d-2i) \cdot |L_i| \right) - 2\ell_0 \\ &= dn - d - 2 \cdot \sum_{i=1}^{h} \left(2i \cdot \prod_{j=0}^{i-1} (m_j - 1) \right) \end{aligned}$$

where minimum in the first line is taken over all permutations $(v_1, v_2, ..., v_n)$ of the vertices.

This proves (5) and also completes the proof of the theorem.

2.3 Tightness of the lower bound

This subsection is about the proof that the lower bounds presented in the previous subsection can be attained with equality with a suitable permutation of the vertices, whenever a complete tree does not contain vertices of degree 2.

Proof of tightness

Theorem 5 If $m_i \geq 3$ for all $0 \leq i < h$, then equality holds in the inequalities (4) and (5) of Theorem 4.

Proof We construct suitable vertex orders attaining equality for both d even and odd. In fact the case of even d will be crucial, from which we can build a permutation for odd d, too.

With standard terminology, for an i-vertex $v \in L_i$ the unique neighbor of v in L_{i-1} is its parent (if $1 \le i \le h$) and its neighbors in L_{i+1} are its children (if $0 \le i \le h-1$). We say that a vertex u is an *ancestor* of v if u is on the path from v to the root. In particular, by this definition, v is considered to be an ancestor of itself, too.

We recall that $\ell_i = h + \frac{1}{2} - i$ has been defined in the proof of Theorem 4; it is $\frac{d+1}{2} - i$ if d is even, and $\frac{d}{2} - i$ if d is odd.

Case 1: $d = 2h$

We prove that there exists an order of the vertices $v_1, v_2, ..., v_n$ such that the labeling φ defined with the rules

$$\varphi(v_1) = 0, \qquad \varphi(v_i) = \varphi(v_{i-1}) + \ell(v_{i-1}) + \ell(v_i) \text{ for } i = 2, ..., n \tag{8}$$

is an $L(d, d-1, ..., 1)-$labeling of $T = T^1_{m_0, m_1, ..., m_{h-1}}$. The general scheme of the order is

$$L_0 \;-\; L_h \;-\; L_{h-1} \;-\; ... \;-\; L_2 \;-\; L_1.$$

The crucial point is how to permute the vertices inside each level L_i in a way that the distance constraints are respected by all vertex pairs.

Viewing T as a rooted tree with root L_0, let us mark the edges joining each $v \in L_i \, (1 \le i \le h-1)$ to its children in L_{i+1} with the integers $0, 1, ..., m_i - 2$; from the root to L_1 the marking ranges from 0 to $m_0 - 1$. Then each $v \in L_i$ $(1 \le i \le h)$ is represented by the sequence

$$\mathbf{a}(v) = (a_{i-1}, a_{i-2}, ..., a_1, a_0) = (a_{i-1}(v), a_{i-2}(v), ..., a_1(v), a_0(v))$$

of marks along the path from v to L_0. From this, we denote $m'_0 = m_0$ and $m'_i = m_i - 1$ for $i = 1, ..., h-1$, and associate v with the number

$$
\begin{aligned}
s(v) &= a_0 + a_1 m_0 + a_2 m_0(m_1-1) + ... + a_{i-1} m_0(m_1-1)\cdots(m_{i-1}-1) \\
&= \sum_{k=0}^{i-1}(a_k \cdot \prod_{0\le j\le k-1} m'_j)
\end{aligned}
$$

which establishes a bijection between the elements of L_i and the nonnegative integers ranging from 0 to $m'_0 \cdot m'_1 \cdots m'_{i-1} - 1$. We then list each level i in increasing order of $s(v)$.

The vertex order obtained in this way satisfies the following important separation properties:

- any m_0 non-root vertices are mutually separated by the root in T;
- any $m_0(m_1 - 1)$ consecutive vertices of $L_2 \cup ... \cup L_h$ have mutually distinct ancestors in L_2;
- in general, if two vertices v_p and v_q in $\cup_{i\le j\le h} L_j$ have the same ancestor in L_i, then $|p-q| \ge \prod_{k=0}^{i-1} m'_k$.

In other words, in the vertex order defined above, any $|L_i|$ consecutive vertices of $L_i \cup L_{i+1} \cup ... \cup L_h$ have mutually distinct ancestors in L_i.

The vertices are then labeled recursively by the rule (8). Between two consecutive i-vertices it means difference $d + 1 - 2i$, and between the last vertex of L_{i+1} and the first vertex of L_i it means difference $d - 2i$.

Consider any two vertices, say an i'-vertex v' and an i''-vertex v''. Assume that their lowest common ancestor z is an i-vertex; that is, the ancestors of v' and v'' in L_{i+1} are distinct. (In particular, it is allowed that $z \in \{v', v''\}$ and $i \in \{i', i''\}$.) Due to the separation property above and the assumption $m_i \ge 3$, the difference of labels of v' and v'' is at least

$$
\ell_{i'} + \ell_{i''} + |L_i| - 1 \ge d + 1 - i' - i'' + (3 \cdot 2^{i-1} - 1) \ge d + 1 - i' - i'' + 2i,
$$

while the distance of v' and v'' in T is precisely $i'+i''-2i$. Thus, φ is a radio labeling indeed. Since the span of φ is equal to the lower bound in (4), the labeling is optimal. This completes the proof for d even.

Case 2: $d = 2h + 1$

Removing the central edge from $T^2_{m_0,m_1,...,m_{h-1}}$ we obtain two isomorphic complete trees, say T' and T'', of even diameter $d-1 = 2h$. Compared to the case of T^1, the difference is that now the roots of T' and T'' have degree $m_0 - 1$, which is allowed to be as small as 2. We construct the vertex order of $T^2_{m_0,m_1,...,m_{h-1}}$ as follows:

- the sequence begins with the root of T' and ends with the root of T'';
- the vertices of T' and T'' alternate;
- the subsequence consisting of the vertices of T' is in the order

  $$L_0 \ - \ L_h \ - \ L_{h-1} \ - \ ... \ - \ L_2 \ - \ L_1$$

  with the separation property that any $\prod_{j=0}^{i-1}(m_j-1)$ consecutive vertices in $L_i \cup L_{i+1} \cup ... \cup L_h$ from T' have mutually distinct ancestors in L_i;
- the subsequence consisting of the vertices of T'' is in the inverse order

  $$L_1 \ - \ L_2 \ - \ ... \ - \ L_{h-1} \ - \ L_h \ - \ L_0$$

  satisfying the same separation property as prescribed for T'.

By the explicit construction for even diameter, such an order exists. We index the vertices as $v_1, v_2, ..., v_n$ along this order. Also here we apply the rule (8) to define the labeling φ.

To show that φ is a radio labeling, observe first that if $v' \in T'$ is an i'-vertex and v"$\in T''$ is an i''-vertex, then their distance is precisely $i' + i'' + 1$, just because they are on different sides of the central edge of L_0. So they have to satisfy the inequality

$$|\varphi(v') - \varphi(v'')| \geq (d+1) - (i' + i'' + 1) = \ell(v') + \ell(v'').$$

This requirement obviously holds by (8).

Suppose that the i'-vertex v' and the i''-vertex v'' belong to the same branch of T, say without loss of generality that both of them are in T'. Assume that their lowest common ancestor is an i-vertex. Then their distance is $i' + i'' - 2i$ and hence they should satisfy

$$|\varphi(v') - \varphi(v'')| \geq (d+1) - (i' + i'' - 2i) = \ell(v') + \ell(v'') + 2i + 1.$$

Now again, the terms $\ell(v') + \ell(v'')$ are ensured by (8), thus it will suffice to show that there are at least $2i + 1$ vertices between v' and v'' in the vertex order. Since the vertices alternate between T' and T'', we need i intermediate vertices from T'. The requirement holds indeed, as implied by the separation property: $(\prod_{j=0}^{i-1}(m_j - 1)) - 1 \geq 2^i - 1 \geq i$ for any integer i. This completes the proof of the theorem.

2.4 A further open problem

It remains an interesting open problem to extend the study to labelings in which the constraints involve only a range of distances smaller than the diameter:

Problem Let $m \geq 3$, $d \geq 3$ and $h > \frac{d}{2}$ be given integers. Determine $\lambda_{d,d-1,\ldots,1}(T)$ for the trees $T = T^1_{m,\ldots,m}$ and $T = T^2_{m,\ldots,m}$ of height h.

In the following an approach is sketched which yields only a suboptimal solution for the radio number but may be useful in attacking the problem above.

Alternative vertex orders and larger diameter

The following construction is for the internally regular complete trees $T = T^1_{m,\ldots,m}$ with even diameter $d = 2h$. This arrangement of vertices is based on an approach totally different from the one given in the previous section. It yields a span $\frac{d}{2}$ larger than optimal, but seems to offer a higher degree of flexibility and may turn out to be useful in handling related questions like the above problem.

Here the general principle of ordering the vertices is inserting them level by level. The numerical basis of the insertion procedure is the following observation. Independently of the actual height, the number of leaves equals $m-2$ times the number of internal vertices, plus 2; that is,

$$|L_i| = 2 + (m-2) \cdot \sum_{j=0}^{i-1} |L_j|$$

for any $1 \le i \le h$. Then the first few steps of the insertions are as follows:

- First, list the $m+1$ vertices from the levels 0 and 1, $L_0 \cup L_1$, so that the root vertex from L_0 is placed at the beginning.
- Between any two 1-vertices insert exactly $m-2$ vertices of L_2 which are pairwise separated by the root from each other and from the two 1-vertices.
- Between the root and the first 1-vertex, as well as after the last 1-vertex, insert $m-1$ vertices of L_2 which are pairwise separated by the root from each other and from the 1-vertex.
- To describe the general step, it is convenient to introduce the following terminology: In a linear arrangement of the vertices, which begins with L_0, an *i-interval* means a segment starting and also ending with a vertex from $\cup_{1 \le j \le i} L_j$ and containing no vertices from $\cup_{1 \le j \le i} L_j$ other than the first and the last one in the segment; and we also use the term

i-interval for the closing segment which starts with the last vertex of $\cup_{1\le j\le i} L_j$ in the given vertex order. So the number of i-intervals is precisely $\sum_{1\le j\le i} |L_j|$ for every $i = 1, 2, ..., h$. When inserting the next level L_{i+1} into the sequence, the quantitative condition is:

- The first and the last $i-$intervals contain $m-1$ vertices from L_{i+1}, and each of the other $i-$intervals contains $m-2$ vertices from L_{i+1}.

We prescribe the following general separation property, which has to hold for all pairs k, l of the parameters $(1 \le l \le k \le h)$.

- If two vertices are on levels at most k and their lowest common ancestor is on level l, then they are in different $(k-l)-$intervals.

It can be verified by an argument along the lines of Subsection 2.3 that this requirement is sufficient to ensure that the rule (8) defines a radio labeling from such a vertex order, whenever $m \ge 3$.

2.5 Internally regular trees vs. complete $m-$ary trees

It is interesting to make a comparison between the radio numbers of internally regular trees and complete $m-$ary trees. Li, Mak and Zhou proved that the complete $m-$ary tree of height h – that is, the complete tree with a central vertex and with $m_0 = m$ and $m_1 = ... = m_{h-1} = m+1$ has radio number

$$\lambda_{d,d-1,...,1}(T_{m,...,m}) = m^h + \frac{3m^{h+1} - m^h - 2hm^2 + (2h-3)m + 1}{(m-1)^2}. \tag{9}$$

This graph has m branches originating from the central vertex, while the internally $(m+1)-$regular tree in (1) has $m+1$ branches of the same shape. Multiplying the radio number of (9) by $m+1$ and the one of (1) by m, the difference is just 1 (the latter is smaller). On the other hand, the internally regular complete tree in (2) with odd diameter consists of two disjoint copies of the complete $m-$ary tree of the same height, plus an edge joining the roots of those subtrees. Here, instead of saving, the span in (2) requires an additive term $2h-1$ exceeding the double of (9).

The analogous question is of interest more generally for non-constant sequences $(m_0, m_1, ..., m_{h-1})$ of vertex degrees, too.

2.6 Algorithm

In this subsection I present an algorithm for a decision problem. The input is a tree T and an integer λ. The question is whether the tree can be $L(j, j-1, ..., 1)$–labeled with integers not bigger than λ.

First, set up a postorder sequence of the vertices. Record a list for every vertex. We call the subtree induced by a vertex and its descendants the subtree of this vertex. And we call the subtree induced by a vertex and its descendants not deeper than $j-1$ from it the $(j-1)-$subtree of the vertex. In a list there should be stored all proper $L(j, j-1, ..., 1)-$labelings of the $(j-1)-$subtree of the actual vertex which can be extended for its whole subtree. All information needed can be derived from the descendants, because they were visited in postorder.

- Test every combination of the proper labelings of the $(j-1)-$subtrees of the children and every label $(0, 1, ..., \lambda)$ for the actual vertex. Save the combinations which label the complete subtree properly, into the list of the actual vertex $j-1$ depth.

- If a list of any vertex is left empty, stop the algorithm and the answer is NO. Namely if there is a subtree that cannot be labeled by the integers $0, 1, ..., \lambda$ then the whole tree cannot be labeled either. But if no such a vertex exists, the algorithm does not stop before the testing of the root and also its list becomes non-empty and then the tree has a proper $L(j, j-1, ..., 1)-$labeling. Therefore the answer is YES.

In the case of answer YES the proper labeling (labelings) can be given based on the lists. If we consider the maximum degree Δ as a parameter then the size of a $(j-1)-$subtree is bounded. Thus, the number of steps in the algorithm is at most $C \cdot \lambda^{(\Delta^j)}$ with an appropriate constant C.

3 $L(j, j{-}1, ..., 2, 1)$–labeling of unit interval graphs

Interval graphs and specially unit interval graphs have been introduced in 1.1.6. Based on the proof for their $L(2,1)-$labeling number I gave an upper bound for their $L(j, j-1, ..., 2, 1)-$labeling number. In this chapter I present this result.

3.1 Circular $L(j, j-1, ..., 2, 1)-$labeling of paths

For determining the bound for unit interval graphs an upper bound for the circular $L(j, j-1, ..., 2, 1)-$labeling number of paths is also needed. Circular labeling means that the possible labels are the natural numbers on a circle from 0 to $k-1$ (actually, the value of k corresponds to 0), and the length of both sections on the circle between two vertices v and u have to be at least $j_{dist(u,v)}$. A proper circular labeling of the graph G on the circle of circumference k is called a $k-$circular $L(j, j-1, ..., 2, 1)-$labeling of G. $\lambda^C_{j,j-1,...,2,1}(G)$ is the smallest k, for which G has a $k-$circular $L(j,j-1, ..., 2, 1)$–labeling.

Proposition 1 If j is odd then $\lambda^C_{j,j-1,...,2,1}(P) \leq \frac{(j+1)^2}{2}$ and if j is even then $\lambda^C_{j,j-1,...,2,1}(P) \leq \frac{j\cdot(j+3)}{2}$, where P is a path of arbitrary length.

Proof We get a proper $k-$circular labeling of G ($k = \frac{(j+1)^2}{2}$ or $\frac{\mathrm{j\cdot(j+3)}}{2}$) when giving labels to the vertices in sequence in such a way that

$$c(v) = c(\text{left neighbor of } v) + j\ (mod\ \lambda)$$

for every vertex v.

For showing the correctness of the above method we must see that the difference of labels of any two vertices at distance x is at least $j+1-x$. There are several cases:

1. $c(v_2) = c(v_1)+xj$ (no reduction between v_1 and v_2):$x \geq 1 \Longrightarrow x(j+1) \geq j+1 \Longrightarrow xj \geq j+1-x$.

2. $c(v_2) = c(v_1) + xj - k$ and $xj < k$: $c(v_2) < c(v_1) \Longrightarrow |c(v_2) - c(v_1)| = c(v_1) - c(v_2) = k - xj$. If j is odd then $k = \frac{(j+1)^2}{2}$ holds: $\frac{j+1}{2} \geq 1 \Longrightarrow \frac{j+1}{2} \cdot (j+1-2x) \geq j+1-2x \Longrightarrow \frac{(j+1)^2}{2} - (j+1)x \geq j+1-2x \Longrightarrow k - jx \geq j+1-x$. If j is even then $k = \frac{j \cdot (j+3)}{2} = \frac{j^2+3j}{2} > \frac{j^2+2j+1}{2} = \frac{(j+1)^2}{2}$ holds. Because of the inequality for odd j also the inequality for even j will be correct, namely in the last inequality $k - jx > \frac{(j+1)^2}{2} - jx \geq j+1-x$.

3. $k \leq xj < 2k$: $c(v_2) = c(v_1) + xj - k$ holds also in this case, but $c(v_2) \geq c(v_1) \Longrightarrow |(c(v_2) - c(v_1)| = c(v_2) - c(v_1) = xj - k$: If j is odd then $xj \geq \frac{(j+1)^2}{2} = \frac{j^2+2j+1}{2} \Longrightarrow x \geq \left\lceil \frac{j+2+\frac{1}{j}}{2} \right\rceil$ holds, because $x \in \mathbb{N} \Longrightarrow x \geq \frac{j+3}{2} \Longrightarrow x - \frac{j+1}{2} \geq \frac{2}{2} = 1 \Longrightarrow x(j+1) - \frac{(j+1)^2}{2} \geq j+1 \Longrightarrow xj - \frac{(j+1)^2}{2} \geq j+1-x$. If j is even then $xj \geq \frac{j \cdot (j+3)}{2} \Longrightarrow x \geq \frac{j+3}{2}$ holds. We use the inequality for odd j again, and the statement is proved.

4. $xj \geq 2k \Longrightarrow xj \geq (j+1)^2$ and $xj \geq j(j+3) \Longrightarrow x > j$. In this case there is no restriction.

So the difference of the labels of each pair of vertices meets the requirements.

3.2 An upper bound for the $L(j, j-1, ..., 2, 1)$−labeling number of unit interval graphs

Unit interval graphs are perfect, so $\chi = \omega$. I am going to give a lower and an upper bound as a function of χ.

The lower bound is the $L(j, j-1, ..., 2, 1)$−labeling number of the complete graph K_χ since each unit interval graph with clique number $\omega = \chi$ contains it as a subgraph, and K_χ is a unit interval graph itself (this is important because of the tightness). In this graph any two vertices are adjacent, hence the difference is at least j between any two labels, so the labeling number is $j \cdot (\chi - 1)$.

For giving an upper bound one might examine a graph that contains each unit interval graph with clique number χ as a subgraph. There exists such a graph, namely the unit interval graph in which each χ vertices in the simplicial order form a clique. In this graph an arbitrary vertex is adjacent to those that are at least 1 and at most $(\chi - 1)$ away in the simplicial order, its second neighbors are the vertices at least χ and at most $(2\chi - 2)$ away, and in general its i^{th} neighbors are at least $(i-1)(\chi-1)+1$ and at most $i \cdot (\chi - 1)$ away from it.

A proper $L(j, j-1, ..., 2, 1)-$labeling procedure

- Since unit interval graphs are chordal, there exists a simplicial order of the vertices of all such graphs. First, divide the simplicial order into sections which cover it overlap-free and completely. Each section is formed by χ or $\chi + 1$ vertices. I discuss below how many vertices a particular section contains. In each section the labels are in decreasing order, the last label is at most $\frac{j^2}{2} - 1$ or $\frac{(j-1)(j+2)}{2} - 1$ depending on the parity of j. The difference of two consecutive labels is exactly $\frac{j^2}{2}$ or $\frac{(j-1)(j+2)}{2}$. So, each section corresponds to a residue class of the $L(j-1, j-2, ..., 2, 1)-$circular labeling number of the path of arbitrary length. For simplicity, let the representative element of a residue class be the smallest one.

- A residue class is assigned to each section as follows: The first one shall belong to 0, the next one to $(j-1)$, and in general a section belonging to x is followed by one belonging to $(x + j - 1)\ mod\ \frac{j^2}{2}$ or $(x + j - 1)\ mod\ \frac{(j-1)(j+2)}{2}$.

- The length of a section: If the representative (the minimum) element of a residue class belonging to a section is greater than the one of that belonging to the previous (say the *mod* operation does not results in reduction), then let this section contain χ vertices, otherwise (when the *mod* operation results in reduction) $\chi + 1$ vertices.

The biggest label used It is easy to see that the biggest label used is about $\frac{j^2}{2} \cdot \chi$ or $\frac{(j-1)(j+2)}{2} \cdot \chi$, respectively. The deviation depends on two things: On the one hand, what residue classes occur, in other words which is the greatest representative element at all. On the other hand, what is the greatest element representing a section of length $\chi + 1$, since the greatest label in this section is the greatest at all. This is true, because the greatest label of any section of length $\chi + 1$ is greater than the greatest element of any section of length χ. If the corresponding representative element is x, then the greatest label is $\frac{j^2}{2} \cdot \chi + x$ or $\frac{(j-1)(j+2)}{2} \cdot \chi + x$. Reduction occurs between two sections if and only if the representative element of the second one is smaller than $j - 1$. Depending on the parity of j this is the following:

If j is odd: In this case $2 \mid (j-1)$, so $\gcd(j-1, \frac{(j-1)(j+2)}{2}) = \frac{j-1}{2}$, because $\frac{j+2}{2}$ is not an integer, while $\frac{j-1}{2}$ is. Hence, the representative elements of the occurring residue classes are the integer multiples of $\frac{j-1}{2}$. $\frac{j-1}{2}$ itself is the largest one of them, which is smaller than $j - 1$. Consequently, the largest label used is $\frac{(j-1)(j+2)}{2} \cdot \chi + \frac{j-1}{2}$.

If j is even: In this case $2 \nmid (j-1)$, so $\gcd(j-1, \frac{j^2}{2}) = 1$, hence each integer between 0 and $\frac{j^2}{2} - 1$ occurs as a representative element. $j - 2$ is the largest one of them, which is smaller than $j - 1$. So, the largest label used is $\frac{j^2}{2} \cdot \chi + j - 2$.

An example: $L(3,2,1)$–labeling Our bound for the circular labeling number of a path is $\frac{(3-1)(3+2)}{2} = 5$. The labels of a unit interval graph with

$\omega = \chi$ are the following in the simplicial order:

$5\chi - 5$	$5\chi - 10$	...	0	$5\chi - 3$	$5\chi - 8$	...	2	$5\chi - 1$	$5\chi - 6$	...	4
	χ pcs	Rc :0			χ pcs	Rc :2			χ pcs	Rc :4	

$5\chi + 1$	$5\chi - 4$	...	1	$5\chi - 2$	$5\chi - 7$	...	3	5χ	$5\chi - 5$	...	0
	$\chi + 1$ pcs	Rc :1			χ pcs	Rc :3			$\chi + 1$ pcs	Rc :0	

It has to be shown that the above labeling procedure results in a proper labeling, so the constraint is met for each pair of vertices. This can be done as follows: Let an arbitrary vertex be given, it has a label assigned to it. We check how far the vertices are that have labels differring by at most $j - 1$ from the label of the given vertex. Because of the symmetry it is enough to examine only those vertices, which come later in the order. If a vertex is the x^{th} one in its section then in any other section the vertex closest in label is the x^{th} or the $(x+1)^{\text{th}}$ one. This smallest difference is equal to the difference of the representative elements of the sections. If the two sections are the a^{th} and the b^{th} one, then there are at least $(b - a) \cdot \chi - 1$ vertices between the two examined vertices in the simplicial order. The exact value is depending on the number of reductions between them. In other words, it means that the second vertex is the $(b-a) \cdot \chi^{\text{th}}$ following one from the first vertex in the simplicial order. Since $(b - a) \cdot \chi > (b - a) \cdot (\chi - 1)$, the graph distance of the vertices is greater than $b - a$.

The representative elements of their sections have been selected so that their difference is at least $(j - 1) + 1 - (b - a) = j - (b - a)$. So the difference is big enough. This argument is generally valid for all vertex pairs, hence the constraint is met for any two vertices. This verifies the property of the labeling scheme.

Part IV

Application of Combinatorial Optimization Methods

The distance-constrained labeling problem introduced so far is an interesting graph theoretical problem. However, it cannot model a practical frequency assignment problem properly because the discrete graph distances do not reflect the real distances between transmitters. Hence it can only serve as a coarse approximation. This fact was my motivation to study the topic from the application point of view. In the following chapter I describe my experiences.

4 New model for the frequency assignment problem

4.1 Linear programming and integer programming

The following introduction of the two combinatorial optimization methods are based on the presentation in [52]. The goal of a linear programming problem is to find a vector $x \in \mathbb{R}^n$ that fulfills all given inequalities in the system $Ax \leq b$ and maximizes a certain objective function $c^T x$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are vectors. This problem is denoted as linear program (LP), its standard form is

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \ . \\ & x \in \mathbb{R}^n \end{array}$$

A vector $x \in \mathbb{R}^n$ that satisfies $Ax \leq b$, is called a feasible solution. A feasible solution that is maximal, is called an optimal solution.

The difference between a linear programming problem and an integer programming problem is small but significant. Namely, the entries of the solution vector x have to take integer values instead of reals. Formally, an integer programming problem (IP) consists of finding a vector $x \in \mathbb{Z}^n$ that fulfills all given inequalities in the system $Ax \leq b$ and maximizes a certain objective function $c^T x$, that is

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \in \mathbb{Z}^n \end{array} .$$

The third condition is the integrality constraint. This makes the problem much harder. The linear programming variant is namely solvable in polynomial time, while an integer programming problem is in general NP-hard [53]. Anyway, there are a few exact and heuristic algorithms that handle the problem quite well, making some calculations possible.

4.2 Models for computing distance-constrained labelings

Here I start with an integer programming model for the classical problem. Let the label of vertex v be represented by the integer variable $c(v)$.

4.2.1 Integer programming formulation of the $L(j_1, j_2, ..., j_s)$–problem

The problem of finding an optimum $L(j_1, j_2, ..., j_s)-$labeling can be stated as follows.

min L

1. $L - c(v) \geq 0 \qquad \forall\, v \in V(G)$

2. $|c(v) - c(u)| \geq j_{dist(u,v)} \qquad \forall\, u, v \in V(G)$ with $dist(u,v) \leq s$

3. $c(v) \geq 0 \qquad \forall\, v \in V(G)$

4. $c(v) \in \mathbb{Z} \qquad \forall\, v \in V(G)$

Inequalities (1) model the min-max objective function and so $min\lambda_{j_1,j_2,\ldots,j_s}(G)$ is computed. The distance constraints (2) for vertex pairs u and v can be linearized in the standard way with additional binary variables z_{uv} as

$$\begin{aligned} c(v) - c(u) + M \cdot z_{uv} &\geq j_{dist(u,v)} && \forall\, u, v \in V(G) \text{ with } dist(u,v) \leq s \\ c(u) - c(v) + M \cdot (1 - z_{uv}) &\geq j_{dist(u,v)} && \forall\, u, v \in V(G) \text{ with } dist(u,v) \leq s \\ z_{uv} \in \{0,1\} & && \forall\, u, v \in V(G) \end{aligned}$$

The number M has to be chosen big enough. Obviously $M \geq j_1 + \lambda_{j_1,j_2,\ldots,j_s}(G)$ has to hold. If no good upper bound on $\lambda_{j_1,j_2,\ldots,j_s}(G)$ is known, we just set $M = n \cdot j_1$.

4.3 New model

A possibility for making the model more practical would be to insert artificial vertices to increase the distance between vertices closer to reality. This way we preserve the discrete nature of the model, but the graph size is increased.

One could also allow real values as labels (leaving the graph and the distance unchanged). Since proper integer solutions remain feasible, the minimum span is not bigger than in the classical model. Actually, this is the LP relaxation of the model above.

It would be most precise to consider the complete graph on the transmitters and to take the Euclidean distance between transmitters as edge weights. Since the graph is finite, there is also only a finite number of occurring distances, and constraints can be formulated as above relative to a sequence $L(j_{dist_1}, j_{dist_2}, \ldots, j_{dist_t})$.

For the *Euclidean model* we can basically adopt the integer model with two little changes. First, the variables $c(v)$ are now continuous. Second, the Euclidean distances have to be transformed to suitable right hand side values for the inequalities (2). This will be accomplished by a function $\ell(dist(u, v))$. The formulation of the Euclidean approach is the following.

$\min L$

1. $L - c(v) \geq 0 \qquad \forall\, v \in V(G)$
2. $c(v) - c(u) + M \cdot z_{uv} \geq \ell(dist(u, v)) \quad \forall\, u, v \in V(G)$ with $dist(u, v) \leq s$
3. $c(u) - c(v) + M \cdot (1 - z_{uv}) \geq \ell(dist(u, v)) \forall\, u, v \in V(G)$ with $dist(u, v) \leq s$
4. $c(v) \geq 0 \qquad \forall\, v \in V(G)$
5. $z_{uv} \in \{0, 1\} \qquad \forall\, u, v \in V(G)$.

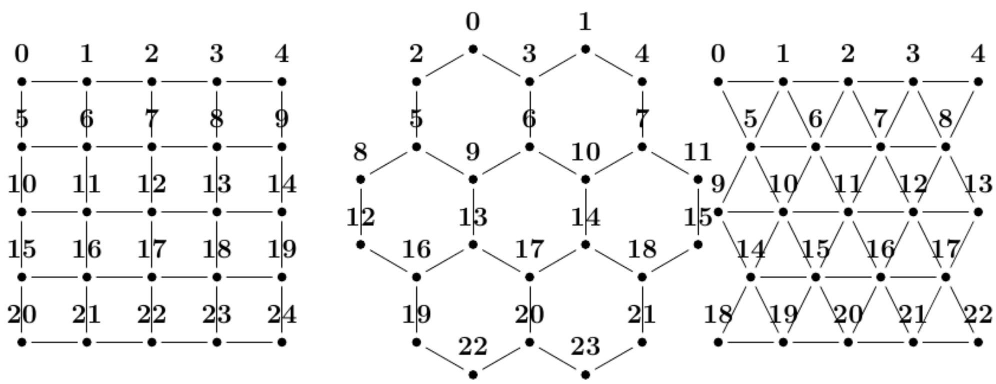
4.4 Test instances

I am not aware of any research on solving distance-constrained labeling problems to optimality, so I took own generated benchmark problems. The goal was to test the graph and the Euclidean model on instances which somehow resemble the practical situation for real frequency assignment problems.

In radio and mobile networks large areas are usually covered by polygons which together form lattices. In practice three lattices play a prominent role: hexagonal, triangular and square lattices. Experiments have shown that the covering by hexagons is the most economical one. In this case the transmitters are assumed to at the centers of the hexagons and two transmitters are adjacent in the graph if and only if the corresponding hexagons share a common edge. The graph constructed this way is a triangular lattice.

In our experiments we performed computations with the Euclidean model and with the graph model for three problem types arising from lattice graphs.

The three lattice types shown in the figure were considered. For the graph version the classical graph distance was taken, (e.g., the distance between 0 and 21 in the triangular lattice is 5). For the Euclidean version coordinates were given to the vertices such that every edge shown in the figure has length 1. So the coordinates for the vertices 0, 1 and 2 are $(-\frac{\sqrt{3}}{2}, +\frac{3}{2}), (+\frac{\sqrt{3}}{2}, +\frac{3}{2}), (-\sqrt{3}, +1)$ in the hexagonal lattice, $(-2, +2), (-1, +2), (0, +2)$ in the square lattice and $(-2, +2), (-1, +2), (0, +2)$ in the triangular lattice. (E.g., the distance between 0 and 1 in the hexagonal lattice is $\sqrt{3}$.)



a) Square Lattice b) Hexagonal Lattice c) Triangular Lattice

The difference of the distance between two vertices in the Euclidean and in the graph model can be small or 0, but also relatively high. Two vertices with graph distance 4 could, for example, have Euclidean distance 4, $\sqrt{10}$ or $\sqrt{8}$.

With the help of the transformation function ℓ we tried to have a similar range of spans for the graph and the corresponding Euclidean model. Of course, ℓ has to be monotonically decreasing. We experimented with two variants for ℓ which in addition have the property that the values for integer distances are preserved. i.e. $\ell(dist(u, v)) = j_{dist(u,v)}$ for $dist(u, v)$ integer. E.g., this is satisfied by defining $\ell(dist(u, v)) = j + 1 - dist(u, v)$ for $L(j, j-1, ..., 1)$-labelings. Since the graph distances are not smaller than the Euclidean distances, the spans for the graph problems will be higher in general.

4.5 First computational experiments

For our computation we considered the square lattice on 25 vertices, the hexagonal lattice on 24 vertices and the triangular lattice on 23 vertices (as depicted in the figure). The minimum spans have been computed with ILOG CPLEX Version 12.4.

Linear transformation We first considered $L(2,1)$- and $L(3,2,1)$-labelings for the classical model and used $\ell_2(dist(u,v)) = 3 - dist(u,v)$ and $\ell_3(dist(u,v)) = 4 - dist(u,v)$ in the respective Euclidean model. The distance of adjacent vertices is 1 in both cases, the distance between non-adjacent vertices can be the same as their graph distance, but also considerably smaller. Because of this, the constraints for most vertex pairs are stronger in the Euclidean model.

Table 1 gives the spans and running times (in min:sec or sec(s)) as obtained with CPLEX. The spans between the models differ as expected. Surprisingly, the running times for the Euclidean model are considerably higher.

Lattice	$\ell_2(dist) = 3 - dist$	$L(2,1)$	$\ell_3(dist) = 4 - dist$	$L(3,2,1)$
Hexagonal	6.42 (1.69 s)	5 (0.08 s)	16.62 (30:38.7)	9 (0.29 s)
Triangular	9.78 (10.96 s)	8 (0.34 s)	21.91 (1731:23.0)	18 (2:47.3)
Square	8.64 (4.92 s)	6 (0.2 s)	19.91 (463:13.8)	11 (0.68 s)

Table 1 Spans and running times for the linear function

Stepwise transformation With this type of function we want to map the Euclidean distances to integer values, again preserving the value if the distance is integer already.

Table 2 shows the possible Euclidean distances for the three lattices and the corresponding graph distances.

For the hexagonal lattice here are some vertex pairs with Euclidean distance $3\sqrt{3}$ and graph distance 6, and some with Euclidean distance 5 and graph distance 7. For these pairs the graph distance is longer although the Euclidean is shorter. For the square lattice there are for example pairs with Euclidean distance $2\sqrt{2}$ and graph distance 4, and some with Euclidean distance 3 and graph distance 3. For the triangular lattice there are no such exceptions.

Hexagonal		Triangular		Square	
Euclidean	Graph	Euclidean	Graph	Euclidean	Graph
1	1	1	1	1	1
$\sqrt{3}$	2	$2, \sqrt{3}$	2	$2, \sqrt{2}$	2
$2, \sqrt{7}$	3	$3, \sqrt{7}$	3	$3, \sqrt{5}$	3
$3, 2\sqrt{3}$	4	$4, \sqrt{13}, 2\sqrt{3}$	4	$4, \sqrt{10}, 2\sqrt{2}$	4
$4, \sqrt{13}, \sqrt{19}$	5	$\sqrt{19}, \sqrt{21}$	5	$\sqrt{13}, \sqrt{17}$	5
$\sqrt{21}, 3\sqrt{3}$	6	$2\sqrt{7}$	6	$2\sqrt{5}, 3\sqrt{2}$	6
$5, 2\sqrt{7}$	7			5	7
				$4\sqrt{2}$	8

Table 2 Euclidean and graph distances for the three lattices

Let $f(x)$ denote the graph distance corresponding to the Euclidean distance (in the respective lattice). For $L(2,1)$- and $L(3,2,1)$-labelings only the graph distances 1,2 and 3 have to be taken into account. For the respective pairs we set $\ell_2(dist(u,v)) = 3 - f(dist(u,v))$ for $L(2,1)$-labelings and $\ell_3(dist(u,v)) = 4 - f(dist(u,v))$ for $L(3,2,1)$-labelings.

Table 3 shows spans and running times for these step functions. Although the constraints are the same for each vertex pair in this case, the calculations are still much slower in the Euclidean model. However, compared with the previous function they have improved a lot.

Lattice	$\ell_2(dist) = 3 - dist$	$L(2,1)$	$\ell_3(dist) = 4 - dist$	$L(3,2,1)$
Hexagonal	5 (0.6 s)	5 (0.08 s)	9 (1.6)	9 (0.29 s)
Triangular	8 (7.9 s)	8 (0.34 s)	18 (19:30.4)	18 (2:47.3)
Square	6 (6.5 s)	6 (0.2 s)	11 (1:08.9)	11 (0.68 s)

Table 3 Spans and running times for the step function

4.6 Model improvements

The computation times are not really satisfactory, so we examined several possibilities for improving the model. They will be discussed in this subsection. All computation have been performed for the linear transformation function.

4.6.1 Reducing the value of M

It is well-known that whenever a linear model contains a so-called *big* M, it is usually advantageous to find the smallest possible value for M. A speed-up of the computations can be expected. However, this is not guaranteed and the effect can also be converse.

Since the optimum values from the classical model are available, it is easy to get good M values for the Euclidean model. In the case $\ell_2(dist) = 3 - dist$ we set $M_2^* = 2 \cdot \lambda_{2,1}$, while $M_3^* = 3 \cdot \lambda_{3,2,1}$ in the case $\ell_3(dist) = 4 - dist$.

Table 4 gives the running times and comparisons with those of the computations using the previous M value.

One can see that the modification works fairly well for the triangular lattice, but rather poorly for the hexagonal lattice. For the square lattice we have mixed results.

Lattice	$\ell_2(dist)$	$\ell_3(dist)$
Hexagonal	1.82 (107.7%)	110:51.9 (361.8%)
Triangular	8.88 (81.0%)	984:24.0 (56.9%)
Square	7.25 (147.4%)	353:42.8 (76.4%)

Table 4 Effect of new setting of M

Changing the order of the vertices In the optimum search algorithm of the CPLEX also the order of the vertices plays a role. Therefore one may want to try whether a new order would cause any improvement. We considered reasonable to set up an order in which the successive vertices are not close to each other in the lattices. For this we changed the numbering according to the following table.

Square:			Triangular:			Hexagonal:		
1	→	1	1	→	1	1	→	1
2	→	23	2	→	6	2	→	4
3	→	20	3	→	11	3	→	16
4	→	12	4	→	16	4	→	20
5	→	9	5	→	21	5	→	24
6	→	6	6	→	13	6	→	22
7	→	3	7	→	18	7	→	14
8	→	25	8	→	3	8	→	18
9	→	17	9	→	8	9	→	9
10	→	14	10	→	5	10	→	11
11	→	11	11	→	10	11	→	8
12	→	8	12	→	15	12	→	12
13	→	5	13	→	20	13	→	3
14	→	22	14	→	23	14	→	5
15	→	19	15	→	17	15	→	2
16	→	16	16	→	2	16	→	6
17	→	13	17	→	7	17	→	19
18	→	10	18	→	12	18	→	23
19	→	2	19	→	9	19	→	15
20	→	24	20	→	14	20	→	13
21	→	21	21	→	19	21	→	17
22	→	18	22	→	22	22	→	21
23	→	15	23	→	4	23	→	7
24	→	7				24	→	10
25	→	4						

In practice, we only had to shuffle the rows and columns of the distance matrix properly. Unfortunately, in the end no improvement was observed.

4.6.2 Strengthening the model

Valid inequalities The structure of valid inequalities is the same as that of the other inequalities in the formulation. So, in a linear programming formulation the left side is a linear combination of the variables, while the right side is an integer/real number. In opposite to the defining inequalities however, the formulation is right also without them. But every solution of the input problem satisfies them. So, the inequalities in the above models are straightforward and we are interested in strengthening them. Let G' be a node induced subgraph of G. If we consider feasible labelings for G' then any constraint on the labels for G' is valid for all subgraphs of G isomorphic to G'. In the following we have chosen the smallest possible sum of the labels for G' as constraint, i.e., if S is the smallest sum of feasible labels for G' then the inequality $\sum_{v \in V(H)} c(v) \geq S$ is valid for all subgraphs H of G isomorphic to G'. We have experimented with several types of subgraphs.

Stars as subgraphs A star is a graph $G = (V, E)$ such that $E = \{vw | w \in V \setminus \{v\}\}$ for some $v \in V$ (center vertex of the star) and suppose $|V| = m$. The sum of labels for a star is a lower bound for the sum of labels of every subgraph containing a star. Consider an $L(j_1, j_2, ...)$-labeling. For the star m labels have to be chosen such that the gap between the labels of any two non-central vertices is at least j_2. We distinguish three cases.

1. The label of the center is the smallest one. Then the labels are 0, j_1, $j_1 + j_2$, $j_1 + 2j_2$, $j_1 + 3j_2$,..., $j_1 + (m-2)j_2$ summing up to $(m-1) \cdot j_1 + \sum_{i=1}^{m-2} i \cdot j_2$.

2. The label of the center is the greatest one. Then the labels are 0, j_2, $2j_2$,..., $(m-2)j_2$, $(m-2)j_2 + j_1$ with sum $(\sum_{i=1}^{m-2} i \cdot j_2) + (m-2)j_2 + j_1$.

3. The central label is the $(k+1)st$ greatest label for some$k \geq 1$. In this case the labels are 0, j_2, $2j_2$,..., $(k-1)j_2$, $(k-1)j_2 + j_1$, $(k-1)j_2 + 2j_1$,

$k \cdot j_2 + 2j_1$,..., $(m-3)j_2 + 2j_1$ and their sum is
$(\sum_{i=1}^{m-3} i \cdot j_2) + 2(k-1)j_2 + (2(m-k-1)+1) \cdot j_1$.

Since $j_1 \geq j_2$, an easy calculation shows that the smallest sum is obtained in case 2 and is equal to $\frac{1}{2}(m+1)(m-2) \cdot j_2 + j_1$. So for every vertex $u \in V$ we can add its associated *star inequality*

$$c(u) + \sum_{i \in N(u)} c(i) \geq \frac{(deg(u)+2)(deg(u)-1)}{2} \cdot j_2 + j_1$$

to the models (where $N(u)$ denotes the set of neighbors of u and $deg(u)$ is the degree of u.)

Sublattices A second possibility is to associate inequalities with small sublattices of the lattices we considered in our computational experiments. We considered the triangles with side length 1, 2 and 3, the hexagons with side length 1 and 2, trapezes as half of a hexagon, squares with side length 1, 2 and 3, and small square lattices with 4 and 9 vertices. Table 5 gives the smallest sums of labels for these subgraphs for the labelings $L(2,1)$ and $L(3,2,1)$.

We examined the effect of the addition of these small subgraph inequalities. The results are presented in Table 6 for the hexagonal, in Table 7 for the triangular, and in Table 8 for the square lattice. For the square lattice we only added non-overlapping squares which is only a small subset of possible squares.

For the hexagonal lattice we observe a running time improvement of about 10-20%. In the triangular case, time reduces to about 50% in two cases for the $L(3,2,1)$-labeling. Also in one case for the square lattice a considerable improvement is achieved.

Sublattice	$\ell_2(dist) = 3 - dist$	$\ell_3(dist) = 4 - dist$
Hexagon with side length 1	16.6077	31.6077
Hexagon with side length 2	3	10.8231
Trapeze with side length 1	7.0718	13.0718
Triangle with side length 1	6	9
Triangle with side length 2	3	6
Triangle with side length 3	0	3
Square 1×1 (4 nodes)	10.3431	16.3431
Square 2×2 (4 nodes)	2.68629	8.68629
Square 3×3 (4 nodes)	0	2
Square lattice 2×2 (9 nodes)	29.1922	59.4033
Square lattice 1×2 (6 nodes)	16.2273	30.283

Table 5 Minimum label sums for sublattices

Hexagonal lattice	$\ell_2(dist) = 3 - dist$	$\ell_3(dist) = 4 - dist$
Without any subgraph-inequality	1.69	30:38.7
With the subgraph-inequalities		
Hexagons with side length 1	1.37	26:28.7
Hexagons with side length 1 and 2	1.75	27:21.7
Trapezes	1.77	25:29.2

Table 6 Effect of subgraph inequalities for the hexagonal lattice

Triangular lattice	$\ell_2(dist) = 3 - dist$	$\ell_3(dist) = 4 - dist$
Without any subgraph-inequality	10.96	1731:23.0
With the subgraph-inequalities		
Triangels with side length 1	12.59	782:13.0
Triangels with side length 1,2 and 3	10.02	1537:08.1
Hexagons with side length 1	11.9	706:52.0

Table 7 Effect of subgraph inequalities for the triangular lattice

Square lattice	$\ell_2(dist) = 3 - dist$	$\ell_3(dist) = 4 - dist$
Without any subgraph-inequality	4.92	463:13.8
With the subgraph-inequalities		
Squares with side length 1	12.76	632:25.5
Squares with side length 1,2 and 3 *	7.11	326:59.4
Rectangle with side length 1×2	6.22	476:52.8

* inequalities added only for non-overlapping squares (substantially fewer than existing)

Table 8 Effect of subgraph inequalities for the square lattice

Part V

Another type of graph partition

5 Edge decompositions

So far I dealt with the labeling of the vertices. The labeling gives a partition on the vertex set of the graph. In the case of radio labeling this partition is trivial, but in general not necessarily. This section deals with a problem, in which the edge set of the graph is partitioned in a specified way.

In general, an edge decomposition of a graph $G = (V, E)$ is a collection of graphs $G_i = (V_i, E_i)$ such that each G_i is a subgraph of G, any two G_i, G_j $(i \neq j)$ are edge-disjoint, and their union contains all edges of G.

The study of edge decompositions (as well as the theory of balanced incomplete block designs and related areas, see [54] with more than 2200 references) started with the famous paper [55] of Kirkman in 1847.

Still, after more than one and a half centuries, quite recently Bondy and Szwarcfiter [56] introduced a natural side condition which has led to an interesting new direction.

Among several results, we solve one of the open problems stated in [56].

5.1 The problems

Given a graph F, determine the maximum number $ex^*(n, F)$ of edges in a graph G of order n such that the edge set of G can be decomposed into edge-disjoint induced subgraphs isomorphic to F.

As mentioned above this issue was addressed in a paper of Bondy and Szwarcfiter [56]. On the other hand, nearly three decades earlier, with a very different approach, Frankl and Füredi [57] considered a closely related problem on hypergraph packing. They introduced a function $f(n, F)$ whose definition is more technical but always satisfies the inequality $f(n, F) \leq$

$ex^*(n,F)$. Hence, lower bounds on their problem are also lower bounds on $ex^*(n,F)$, while upper bounds on $ex^*(n,F)$ are also upper bounds of the problem of [57] .

For some good reasons, to be explained below, we studied the complementary function

$$\overline{ex^*}(n,F) := \binom{n}{2} - ex^*(n,F).$$

5.2 Earlier results

The fundamental result of Wilson [58] states that every sufficiently large complete graph admits an edge decomposition into complete subgraphs of given order whenever two obvious necessary divisibility conditions hold. Since every complete subgraph necessarily is induced, $\overline{ex^*}(n,K_p) = \mathrm{O}(n)$ holds for every fixed $p \geq 3$, and $\overline{ex^*}(n,K_p)$ oscillates between 0 and $cn + \mathrm{O}(1)$ for some $c = c(p)$. (Of course, $\overline{ex^*}(n,K_2) = 0$ holds for all n.)

It is also easily observed (as noted first in [53]) that if F' is obtained from F by adding an isolated vertex, then $\overline{ex^*}(n,F') \leq \overline{ex^*}(n-1,F) + n - 1$, thus we lose at most a linear additive term if any fixed number of isolates are added to F. For this reason we may assume that F does not have isolated vertices.

In general, Cohen and Tuza [59] proved that

$$\overline{ex^*}(n,F) = \mathrm{o}(n^2) \tag{1}$$

holds for all non-edgeless graphs F as n gets large. Due to the connection between the problems [56] and [57], the same asymptotic upper bound can be deduced from the results of Frankl and Füredi, too. In comparison, the methods in [57] are probabilistic, whilst the results of [59] are partly constructive, applying the properties of various classes of Kneser graphs.

Because of (1), the main problem is to determine the order of magnitude of $\overline{ex^*}(n,F)$ for a given F as a function of n. Several estimates have been proved in [56] and [59]:

- $\overline{ex^*}(n, F) = \Theta(n)$ if F is a complete equipartite non-complete graph (and in particular if $F = C_4$) or F is a star or $F = K_4 - e$ ([56]);
- $\overline{ex^*}(n, F) = \Theta(n^{\frac{3}{2}})$ if $F = 2K_2$ or $F = C_6$ (lower bounds in [56], constructive upper bounds in [59]).

In some cases, more precise or even exact results are known, but here we prefer to emphasize growth order.

5.3 The new results

We gave a general lower bound, namely,

Theorem 1 There exists a constant $c > 0$ with the following property: If F is a graph without isolated vertices, and F is not a complete multipartite graph, then

$$\overline{ex^*}(n, F) \geq (c - \mathrm{o}(1))n^{\frac{3}{2}} \qquad \text{as } n \to \infty.$$

Proof Let F be any isolate-free graph satisfying the assumptions of the theorem. Denote by p the number of vertices and by q the number of edges in F. Since F is not complete murtipartite, it contains some vertex w and an edge (yz) such that (wy) and (wz) are non-edges. Indeed, the complement of F contains some connected component of order at least 3 which is not a complete graph, and then this component contains an induced path $ywz \cong P_3$, a proper choice for the three vertices named above in F.

Let $G = (V, E)$ be a graph of order n, which is extremal for $ex^*(n, F)$; and let $H = \overline{G}$ its complement. By the theorem of Cohen and Tuza [59], for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon, F)$ such that, for every $n > n_0$, G has more than $(\frac{1}{2} - \epsilon)n^2$ edges. As a consequence, the edge set of G is decomposed into more than $\frac{1-2\epsilon}{2q}n^2$ copies of F. In each copy, vertex w is mapped to some vertex of G. Let k_v denote the number of copies of F in which w is mapped to vertex $v \in V$. Then we have

$$\sum_{v \in V} k_v > \frac{1-2\epsilon}{2q} n^2.$$

Choosing now $\epsilon = \frac{1}{10}$, it follows that at least $\frac{n}{5q}$ among the n terms on the left side are not smaller than $\frac{n}{5q}$. This specifies a set $X \subset V$ such that

$$|X| \geq \frac{n}{5q} \qquad \text{and} \qquad k_x \geq \frac{n}{5q} \qquad \text{for all } x \in X.$$

The copies of the edge (yz) appear in the non-neighborhoods of the copies of w. This requires at least k_x distinct edges in the complementary neighborhood $N_H(x)$, implying

$$\binom{d_H(x)}{2} \geq k_x;$$

$$d_H(x) > \sqrt{2k_x} \geq \sqrt{0.4\frac{n}{q}}$$

for every $x \in X$. Consequently,

$$\overline{ex^*}(n, F) = |E(H)| = \frac{1}{2}\sum_{v \in V} d_H(v) \geq \frac{1}{2}\sum_{x \in X} d_H(x) > \frac{1}{\sqrt{10q}} n^{\frac{3}{2}}.$$

This inequality proves the theorem.

Corollary 2 Every graph F containing the path P_4 or the matching $2K_2$ or the paw $K_4 - P_3$ as an induced subgraph, satisfies $\overline{ex^*}(n, F) \geq cn^{\frac{3}{2}}$ for some constant $c > 0$.

Combining these lower bounds with the constructions of [59], the following cases solve the problem of Bondy and Szwarzfiter [56].

Corollary 3 We have $\overline{ex^*}(n, P_4) = \Theta(n^{\frac{3}{2}})$ and $\overline{ex^*}(n, K_4 - P_3) = \Theta(n^{\frac{3}{2}})$.

For graphs containing an induced $K_4 - P_3$, the lower bound $cn^{\frac{3}{2}}$ for a slightly different problem was proved in [57, Proposition 2.4]. For paths, until now only a linear lower bound was known in general, and $\Theta(n^{\frac{3}{2}})$ was proved only for *regular* graphs decomposable into induced copies of F (see [56]).

We conjecture that no other growth function occurs as $\overline{ex^*}(n, F)$ which would lie strictly between $\Theta(n)$ and $\Theta(n^{\frac{3}{2}})$.

Conjecture 4 If F is a complete multipartite graph, then $\overline{ex^*}(n, F) = \mathrm{O}(n)$.

As mentioned above, linear upper bound was known previously for complete equipartite graphs, for stars and for $K_{2,1,1}$. We prove the following further cases. The first one is very simple, while the other one is our second main result in this chapter.

Proposition 5 If $F = K_{a,b}$ with $a \geq 2$ and $b \geq 1$, then $\overline{ex^*}(n, F) = \mathrm{O}(n)$.

Proof It is easy to decompose $K_{ab,ab}$ into induced copies of $K_{a,b}$, as follows. We partition the first vertex class into b disjoint sets of size a, and the second vertex class into a disjoint sets of size b. The combinations of those sets yield ab copies of $K_{a,b}$, which together partition the edge set of $K_{ab,ab}$.

Suppose next that n is of the form $n = kab$ for some integer $k \geq 2$. We replace each edge of $K_{\frac{n}{ab}}$ with an independent set of cardinality ab, and substitute the above decomposition of $K_{ab,ab}$ into the image of each edge of $K_{\frac{n}{ab}}$. In this way a graph of order n is obtained, which admits a decomposition into induced copies of $K_{a,b}$, and its complement has as few as $\frac{ab-1}{2} \cdot n$ edges.

Finally, if $n \equiv r \ (mod\ ab)$, then we make the same construction on $n' := n - r$ vertices and insert r isolates. This graph is decomposable into induced copies of $K_{a,b}$, and its complement has fewer than $\frac{3}{2}abn$ edges.

Theorem 6 If $F = K_{a,b,c}$ is a complete tripartite graph, but F is not complete, then $\overline{ex^*}(n, F) = \mathrm{O}(n)$.

Proof Let $F = K_{a,b,c}$. We carry out a construction in several steps which will yield the complete tripartite graph K_{a^2bc,ab^2c,abc^2}. Not alone this graph, but also the steps leading to it, will be essential in the sense that they simultaneously maintain two edge partitions: one into copies of $K_{a,b,c}$ and the other into copies of $K_{abc,abc}$, with a strong interrelation between the two.

We start with three disjoint sets A, B, C of equal cardinality $|A| = |B| = |C| = abc$, partitioned into sets of cardinalities a, b and c, respectively:

$$A = \cup_{j=1}^{b} \cup_{k=1}^{c} A_{j,k}, \qquad B = \cup_{i=1}^{a} \cup_{k=1}^{c} B_{i,k}, \qquad C = \cup_{i=1}^{a} \cup_{j=1}^{b} C_{i,j}.$$

Our approach is to start with an initial construction and extend it incrementally, making it denser in each step.

- Packing of $K_{a,b,c}$ into $K_{abc,abc,abc}$.

For every triplet (i, j, k) with $1 \leq i \leq a$, $1 \leq j \leq b$, $1 \leq k \leq c$, define the vertex set

$$V_{i,j,k} = A_{j,k} \cup B_{i,k} \cup C_{i,j}.$$

We use each $V_{i,j,k}$ to insert a copy of $K_{a,b,c}$ with vertex classes $A_{j,k}$, $B_{i,k}$, $C_{i,j}$ inside $A \cup B \cup C$. It is immediate to verify that the copies determined by $V_{i,j,k}$ and $V_{i',j',k'}$ are edge-disjoint for any two ordered triplets $(i, j, k) \neq (i', j', k')$, because they share vertices in at most one vertex class. (For example, changing the value of i modifies $V_{i,j,k}$ in both B and C.)

If we fix the first subscript i for the moment, and let j run from 1 to b and also let k run from 1 to c, the corresponding $c-$element sets have a union of cardinality bc in C. Consequently, the subgraph between B and C, whose edges are covered with the copies of $K_{a,b,c}$, is the vertex-disjoint union of a copies of $K_{bc,bc}$.

Analogously, fixing the second subscript j, and letting i, k run over their range, we see that the subgraph composed from the copies of $K_{a,b,c}$ between A and C is the vertex-disjoint union of b copies of $K_{ac,ac}$. In the same way, the edges, which are covered between A and B, form the union of c vertex-disjoint copies of $K_{ab,ab}$.

For reference in the next step, we denote this construction by $G[A, B, C]$.

- Saturation of edges between A and B in a star-like extension.

We use copies of $G[A, B, C]$ as building blocks in the following way: We take c graphs isomorphic to $G[A, B, C]$, denoted as

$$G[A, B, C^{k'}] \qquad\qquad (1 \le k' \le c),$$

where the sets $C^1, ..., C^c$ are mutually disjoint, but A and B are common in all those copies of $G[A, B, C]$. Moreover, the vertices of A occur in a different order in each $G[A, B, C^{k'}]$, in such a way that the corresponding vertex sets determining the copies of $K_{a,b,c}$ are

$$V_{i,j,k}^{k'} = A_{j+k'-1,k} \cup B_{i,k} \cup C_{j,k}^{k'},$$

where $j + k' - 1$ in the subscript of A is meant cyclically modulo c. This yields the complete bipartite graph $K_{abc,abc}$ between A and B. The edges from A to each $C^{k'}$ form b disjoint copies of $K_{ac,ac}$; and similarly, from B to each $C^{k'}$ we have a disjoint copies of $K_{bc,bc}$.

It should be emphasized that the second subscripts in the sets $A_{j,k}$ have not been permuted. As a consequence, the copies of $K_{ac,ac}$ between A and any $C^{k'}$ define the same vertex partition of A into b sets of cardinality ac. This property is essential for later use.

For reference in the next step, we denote this construction by $G[A, B, C^*]$.

- Saturation of edges between A and C^*.

Here we use copies of $G[A, B, C^*]$ as building blocks. We stick them together on the set $A \cup C^*$, creating b copies $B^1, ..., B^b$ of B, so that the next graph is built from the subgraphs

$$G[A, B^{j'}, C^*] \qquad\qquad (1 \le j' \le b),$$

where the sets $B^1, ..., B^b$ are mutually disjoint. We again take different vertex orders inside A for the different values of j', to make each $(A, C^{k'})$ complete bipartite, namely isomorphic to $K_{a,b,c}$. This can be done, because the $(ac)-$element vertex classes of the copies of $K_{ac,ac}$ define the same partition of $C^{k'}$ for all j'.

For reference in the next step, we denote this construction by $G[A, B^*, C^*]$.

- Saturation of edges between B^* and C^*.

Here we use copies of $G[A, B^*, C^*]$ as building blocks. We take a graphs isomorphic to $G[A, B^*, C^*]$, denoted as

$$G[A^{i'}, B^*, C^*] \qquad\qquad (1 \le i' \le a),$$

where the sets $A^1, ..., A^a$ are mutually disjoint, but B^* and C^* are common in all those copies. Now the vertex order in the set B^* is fixed to be the same in all the a copies of $G[A, B^*, C^*]$; but the order inside C^* is chosen to be different in each copy, so that the union of the resulting graphs $G[A^{i'}, B^*, C^*]$ creates $K_{abc,abc}$ between each $B^{j'}$ and $C^{k'}$. This can be done similarly to the preceding steps, by choosing a vertex orders so that any one of them takes the same order inside all $C^{k'}$.

In this way we obtain a graph, which we denote by $G[A^*, B^*, C^*]$. As we indicated at the very beginning of the proof already, this graph is isomorphic to K_{a^2bc,ab^2c,abc^2}; but in fact it is more than that. The procedure above describes edge decompositions of $G[A^*, B^*, C^*]$ into induced subgraphs isomorphic to $K_{abc,abc}$, as well as into induced subgraphs isomorphic to $K_{a,b,c}$. Moreover, each copy of $K_{a,b,c}$ is embedded into a copy of some $K_{abc,abc,abc}$ in $G[A^*, B^*, C^*]$.

- Construction of dense induced packing.

Let us denote by $G^* \cong K_{a,b,c}$ the graph obtained from $G[A^*, B^*, C^*]$ by contracting each of the sets $A^{i'}, B^{j'}$ and $C^{k'}$ to a distinct single vertex and joining two of the new vertices by an edge if their preimages induce $K_{abc,abc}$ in $G[A^*, B^*, C^*]$.

Let n be arbitrarily given, and define $n' := \lfloor \frac{n}{abc} \rfloor$. We now apply Wilson's theorem to G^* and n'. If n' is sufficiently large and satisfies some simple conditions, which only depend on G^*(more explicitly on divisibility conditions expressed in terms of the fixed integers a, b, c), then $K_{n'}$ has an edge decomposition into subgraphs $G_1^*, G_2^*, ...$ isomorphic to G^*. Replace each vertex of $K_{n'}$ with an independent set of abc vertices, and add further $(n - abcn')$ isolates. In this way we obtain a graph G of order n. Then each G_ℓ^* becomes an induced subgraph G_ℓ of G, isomorphic to $G[A^*, B^*, C^*]$. Based on the procedure of constructing $G[A^*, B^*, C^*]$, every G_ℓ^* is decomposable into induced copies of $K_{a,b,c}$, and this is an induced decomposition of G, too, because the "induced subgraph" relation is transitive.

Disregarding small values of n, the number of isolated vertices in G is less than abc times the gap occuring between two consecutive values of n', which are feasible for G^*-decomposition. Moreover, omitting the isolates from G, the complementary degree of each vertex becomes precisely $abc - 1$. Thus, the overall number of non-edges in G is at most $K \cdot n$ for some constant $K = K(a, b, c)$. This completes the proof of the theorem.

Proposition 7 If F is a complete multipartite graph, but not complete, then $\overline{ex^*}(n, F) = \Omega(n)$. (This means that one can replace $O(n)$ with $\Theta(n)$ in the previous results, $\overline{ex^*}(n, F)$ has a linear lower bound.)

Proof Let F be a complete multipartite graph, say with q edges. If $F = K_{1,q}$ is a star, then the exact value of $\overline{ex^*}(n, F)$ is known by [56, Theorem 3]; the complement of the extremal graph is the union of complete graphs K_q, with one additional K_r if n is not a multiple of q and $n \equiv r\ (mod\, q)$. In this case we have $\overline{ex^*}(n, F) = \frac{q-1}{2}n + \mathrm{O}(1)$.

If F is not a star, then either it has more than two vertex classes or it is bipartite with at least two vertices in each class. Thus, in either case, F contains a vertex pair, say $\{v_0, w_0\}$, such that $(v_0 w_0)$ is a non-edge, moreover v_0 and w_0 have at least two common neighbors in F. Also $q \geq 4$ holds.

Let $G = (V, E)$ be a graph of order n, which is extremal for $ex^*(n, F)$; and let $H = \overline{G}$ be its complement. Then H has $\overline{ex^*}(n, F)$ edges; let us denote $m := ex^*(n, F) = |E(G)|$ and $\overline{m} := \overline{ex^*}(n, F) = |E(H)| = \binom{n}{2} - m$. Each of the $\overline{m}$ edges in H is the image of $\{v_0, w_0\}$ in at most $\frac{n-2}{2}$ copies of F. Consequently, using also the fact that $q > 2$, we obtain:

$$m \leq q \cdot \frac{n-2}{2} \cdot \overline{m},$$

$$\binom{n}{2} = m + \overline{m} < q \cdot \frac{n-1}{2} \cdot \overline{m},$$

$$\overline{m} > \frac{n}{q}.$$

This completes the proof.

6 Summary

During my Ph.D. work I studied different graph partitions. The largest part of my studies is made up by the so-called distance-constrained labeling, where graphs have to be labeled under some restrictions on the distance between the vertices. Depending on the constraint this problem can be quite simple, but also very complicated. Simplicity means here that we can label general and relatively big graphs optimally or nearly optimally. Conversely, there are several constraints, under which giving even a near optimal labeling takes exponential time. Therefore, it is appropriate to start the studies under such restrictions on simple graph classes.

The constraint that I have studied the most, the $L(j, j-1, ..., 2, 1)$–labeling is special in the sense that the required differences form an arithmetic sequence, but it is also general, due to the arbitrary length of the sequence. Therefore, it is a constraint, under which the optimal labeling is difficult for general graphs. Our theoretical investigations were carried out on trees. In our paper, accepted for publication in February 2015, a strict upper bound for the $L(j, j-1, ..., 2, 1)-$labeling number of trees of diameter at most j is given. Also a procedure is presented, which labels the given trees within this bound. Although we conjecture that the bound is only slightly larger for trees of larger diameter, this remains an open problem. As well as to estimate how much this bound deviates from the optimum in the case of trees, significantly different from those, for which the bound is strict. However, our procedure is a good base also for these ones.

Under the same constraint I also studied the labeling of unit interval graphs. I report the results also on these issues in this thesis.

Chapter 4 examines the problem in perspectives of practice and application. The content can be divided into two parts. Firstly, the need arises naturally to estimate the labeling number of arbitrary graphs under arbitrary constraints. This can be done by various combinatorial optimization methods. I chose linear programming, since the problem is pretty easily to formalize as a linear programming task and with an appropriate software good estimates or even exact values can be obtained. The other question relates to the practical use. The graph theoretical problem is a good model of the channel assignment, but it is not precize. However, in the precize model even good estimates can be given in much slower time. According to the practice, the comparisons were performed on the three lattice graphs, that are most characteristic of the radio transmitter location. Finally, it had to be concluded that the price of the precision is too high, namely the calculations were much slower even after all modifications for improvement, than in the original model.

In the last unit of the thesis an other kind of partition comes into focus. The labeling is a partition, since transmitters using the same frequences have to be chosen, according to the specified constraint. However, in this chapter not the vertices, but the edges are partioned. The way, how it has to be done, is that each set containes edges forming a subgraph isomorphic to a predefined graph. The matter is the maximum number of edges a graph can have, if there exist this partition on its edge set.

Chapters 2-6 contain apart from the references only new results obtained partially or wholly by me. The content of Chapters 2 and Chapter 6 have been already published/accepted and that of Chapter 4 is under preparation.

As a final conclusion, I can say that the initial goals have been successfully achieved, an extensive analysis of the graph partitions, especially labelings, in several aspects has been presented in this Thesis.

Összegzés

A doktoranduszi munkám során különböző gráfpartíciókkal foglalkoztam. Kutatásaim legnagyobb részét az úgynevezett távolság szerint korlátozott címkézések tették ki, ahol a gráfokat a csúcspárok távolságából adódó megszorítások mellett kell megcímkézni. A korlátozó feltételtől függően a probléma lehet viszonylag egyszerű, de akár nagyon bonyolult is. Itt az egyszerűség azt jelenti, hogy viszonylag nagy méretű általános gráfokat meg tudunk címkézni optimálisan vagy közel az optimumhoz. Ugyanakkor vannak olyan feltételek, amelyek mellett még egy közel optimális címkézés megadásához is exponenciálisan hosszú időre van szükségünk. Ennek megfelelően érdemes az ilyen feltételek vizsgálatát egyszerű gráfosztályokra vonatkozóan kezdeni.

Az általam leginkább vizsgált feltétel, az $L(j, j-1, ..., 2, 1)-$címkézés speciális atekintetben, hogy a megkövetelt különbségek egy számtani sorozatot alkotnak, ugyanakkor általános is a sorozat tetszőleges hossza miatt. Így ez egy olyan feltétel, amely mellett bonyolult az általános gráfok optimális címkézése. Az elméleti vizsgálatainkat fákra vonatkozóan végeztük. A cikkünkben, melyet 2015 februárjában fogadtak el publikálásra, megadunk egy éles felső korlátot a legfeljebb j átmérőjű fák $L(j, j-1, ..., 2, 1)-$címkézési számára. Szintén szerepel benne egy eljárás, amely mentén meg is címkézhetőek a fák ezen korláton belül. Ugyan az a sejtésünk, hogy a korlát csupán egy kicsivel nagyobb a nagyobb átmérőjű fák esetén, ez a probléma egyelőre nyitott maradt. Mint ahogyan az is, hogy mekkora eltérés van egy-egy olyan fa valós paramétere és a korlát között, amely nagyban különbözik azoktól, amelyekre a korlát éles. Mindenesetre az eljárásunk ezen fák esetére is jó kiindulási alapot ad.

Azonos korlátozó feltétel mellett az egységintervallum-gráfok címkézését is vizsgáltam. Az ezekre vonatkozó eredményeimet is részletezem ebben az értekezésben.

A 4. fejezet a gyakorlat és az alkalmazás szempontjából vizsgálja a problémát. Tartalmilag két részre osztható a fejezet. Egyrészt, természetesen adódik az igény tetszőleges gráfok tetszőleges korlátozó feltétel melletti címkézésére. Ehhez különböző kombinatorikus optimalizálási módszereket használhatunk. Én a lineáris programozást választottam, mivel a probléma meglehetősen egyszerűen formalizálható lineáris programozási feladatként, illetve megfelelő szoftverrel jó becsléseket, sőt akár pontos értékeket is kaphatunk. A másik kérdés a gyakorlati alkalmazáshoz kapcsolódik. A gráfelméleti probléma jól modellezi a frekvenciakiosztást, de nem precíz. Ugyanakkor a precíz modellben csupán jó becsléseket is sokkal lassabban kaphatunk. A valósághoz igazodva, az összehasonlítások azon a 3 hálógráfon lettek elvégezve, amelyek a rádióadók elhelyezkedését tekintve a leginkább relevánsak. Végül azt a következtetést kellett levonni, hogy a precizitás ára túl nagy, ugyanis a számolások még a javítást célzó módosítások után is sokkal lassabban futottak le, mint az eredeti modell esetén.

Az értekezés utolsó egységében egy másik fajta partíció kerül a középpontba. A címkézés egy partíció abban az értelemben, hogy egyes frekvenciákon (azonos) sugárzó adókat válogatunk össze egy bizonyos korlátozó feltételnek megfelelően. Ez a fejezet azonban nem a csúcsok, hanem az élek partícionálásáról szól. A végeredménynek úgy kell kinéznie, hogy minden halmaz egy előre meghatározott gráffal izomorf részgráf éleit tartalmazza. A kérdés pedig az, hogy legfeljebb hány éle lehet egy gráfnak ahhoz, hogy azokat lehessen ilyen módon partícionálni.

A 2-6. fejezetek a hivatkozások kivételével csak új eredményeket tartalmaznak, amelyek teljesen vagy részben az én munkámból születtek. A 2. és a 6. fejezet tartalma már el van fogadva publikálásra/publikálva van, a 4. fejezetének közlése pedig kidolgozás alatt áll.

Végső következtetésként azt mondhatom, hogy a kitűzött célokat sikerült elérni, főként a címkézések tekintetében, a gráfpartíciók egy több szemszögből történő, átfogó vizsgálatát tudom bemutatni ebben az értekezésben.

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The results in Part IV have been computed with ILOG CPLEX 12.4 (CPLEX Optimizer, www.cplex.com).

Publications

The thesis is based on the following publications

1. Veronika Halász, Zsolt Tuza: Asymptotically optimal induced decompositions, Applicable Analysis and Discrete Mathematics 8 (2014), pp. 320-329, DOI: 10.2298/AADM140718009H

2. Veronika Halász, Zsolt Tuza: Distance-constrained labeling of complete trees, Discrete Mathematics 338 (2015), pp. 1398-1406, DOI: 10.1006/j.disc.2015.02.016

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Index