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Characterization of field homomorphisms through Pexiderized functional equations

Eszter Gselmann[a], Gergely Kiss[b] and Csaba Vincze[b]

[a]Institute of Mathematics, University of Debrecen, Debrecen, Hungary; [b]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Science, Budapest, Hungary

ABSTRACT

The aim of this paper is to prove characterization theorems for field homomorphisms. More precisely, the main result investigates the following problem. Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field and $f_1, \dots, f_n : \mathbb{K} \to \mathbb{C}$ additive functions. Suppose further that equation

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \qquad (x \in \mathbb{K})$$

is also satisfied. Then the functions $f_1, \dots, f_n$ are linear combinations of field homomorphisms from $\mathbb{K}$ to $\mathbb{C}$.

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1. Introduction

The study of additive mappings from a ring into another ring which preserve squares was initiated by G. Ancochea in [1] in connection with problems arising in projective geometry. Later, these results were strengthened by (among others) Kaplansky [10] and Jacobson–Rickart [9].

Let R, R' be rings, the mapping $\varphi : R \to R'$ is called a *homomorphism* if

$$\varphi(a + b) = \varphi(a) + \varphi(b) \qquad (a, b \in R)$$

and

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad (a, b \in R)\,.$$

Furthermore, the function $\varphi : R \to R'$ is an *anti-homomorphism* if

$$\varphi(a + b) = \varphi(a) + \varphi(b) \qquad (a, b \in R)$$

and

$$\varphi(ab) = \varphi(b)\varphi(a) \qquad (a, b \in R)\,.$$

CONTACT Eszter Gselmann gselmann@science.unideb.hu Institute of Mathematics, University of Debrecen, Pf. 12, Debrecen, 4010 Hungary
Dedicated to the 70th birthday of Professor Mikl'os Laczkovich

Henceforth, $\mathbb{N}$ will denote the set of the positive integers. Let $n \in \mathbb{N}, n \geq 2$ be fixed. The function $\varphi : R \to R'$ is called an *n-homomorphism* if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad (a, b \in R)$$

and

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n) \quad (a_1, \ldots, a_n \in R)\,.$$

The function $\varphi : R \to R'$ is called an n-Jordan homomorphism if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad (a, b \in R)$$

and

$$\varphi(a^n) = \varphi(a)^n \quad (a \in R)\,.$$

Finally, we remark that in case $n = 2$ we speak about homomorphisms and Jordan homomorphisms, respectively. It was G. Ancochea who firstly dealt with the connection of Jordan homomorphisms and homomorphisms, see [1]. The results of G. Ancochea were generalized and extended in several ways, see for instance [9, 10, 26]. The concept of n-homomorphisms was introduced in Hejazian et al. [7]. Furthermore, the notion of n-Jordan homomorphisms was dealt with firstly in Herstein [8]. From the above definitions, it immediately follows that every n-homomorphism is an n-Jordan homomorphism. The converse, however, does not hold in general.

Let $n \in \mathbb{N}$, we say that a ring R is *of characteristic larger than n* if $n!x = 0$ implies that $x = 0$. The ring R is termed to be a *prime ring* if

$$a, b \in R \quad \text{and} \quad aRb = \{0\}$$

imply that either $a = 0$ or $b = 0$. In 1956 I.N. Herstein proved the following.

Theorem 1.1 (Herstein [8]): *If φ is a Jordan homomorphism of a ring R* ***onto*** *a prime ring R' of characteristic different from 2 and 3 then either φ is a homomorphism or an anti-homomorphism.*

In [8] not only Jordan homomorphisms but also n-Jordan mappings were considered. Concerning this the following statement was verified.

Theorem 1.2 (Herstein [8]): *Let φ be an n-Jordan homomorphism from a ring R* ***onto*** *a prime ring R' of characteristic larger than n. Suppose further that R has a unit element. Then $\varphi = \varepsilon\tau$ where τ is either a homomorphism or an anti-homomorphism and ε is an $(n-1)$st root of unity lying in the centre of R'.*

Clearly, homomorphisms, anti-homomorphisms and Jordan homomorphisms from a field $\mathbb{K}_1$ to a field $\mathbb{K}_2$ coincide.

Additive functions play central role in the theory of functional equations and also in theory of (commutative) algebra [12, 16, 19, 24, 25]. It is an important question that how morphisms can be characterized among additive mappings in general. In his seminal paper F. Halter–Koch [4] proved the following characterization.

Theorem 1.3 (Halter–Koch [4]): *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be fields containing $\mathbb{Q}$, $n \in \mathbb{Z} \setminus \{0,1\}$, $l \in \mathbb{N}$, and let $f, g : \mathbb{K}_1 \to \mathbb{K}_2$ be additive functions which are assumed to be injective if $n < 0$. Denote the n^{th} power of f by f^n. Suppose that f and g satisfy the functional equation*

$$g(x^{ln}) = f^n(x^l)$$

for all $x \in \mathbb{K}_1 \setminus \{0\}$. Then either $f = g = 0$, or $e = f(1) \neq 0, e^{-1}f : \mathbb{K}_1 \to \mathbb{K}_2$ is a field homomorphism, and $g = e^{n-1}f$.

Later, in the series of paper [4–6] Halter–Koch and Reich proved several characterization theorems concerning derivations as well as field homomorphisms.

Let $\mathbb{K}$ be a field containing $\mathbb{Q}$, $n \in \mathbb{Z} \setminus \{0\}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Q})$ and let $f, g \colon \mathbb{K} \to \mathbb{K}$ be additive functions so that

$$f\left(\frac{ax^n + b}{cx^n + d}\right) = \frac{x^{n-1}g(x)}{(cx^n + d)^2}, \tag{1}$$

respectively,

$$f\left(\frac{ax^n + b}{cx^n + d}\right) = \frac{ag(x)^n + b}{cg(x)^n + d} \tag{2}$$

holds for all possible values of x. In [4], it is proved that Equation (1) (under a mild condition) implies for the function g that the function $G \colon \mathbb{K} \to \mathbb{K}$ defined by

$$G(x) = g(x) - g(1)x \quad (x \in \mathbb{K})$$

is a derivation. Furthermore, in [6], the authors succeed to prove that Equation (2) furnishes the mapping $g(1)^{-1} \cdot g \colon \mathbb{K} \to \mathbb{K}$ to be a field automorphism.

The main purpose of this work is to put the previous investigations (such as for example Theorem 1.3 whenever $\mathbb{K}_2 = \mathbb{C}$) into a unified framework and to prove characterization theorems for field homomorphisms. The problem to be studied reads as follows.

Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field and let $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{C}$ be additive functions. Suppose further that we are given positive integers $p_1, \ldots, p_n, q_1, \ldots, q_n, N$ so that

$$\begin{array}{rl} p_i \neq p_j & \text{for} \quad i \neq j \\ q_i \neq q_j & \text{for} \quad i \neq j \\ 1 < p_i \cdot q_i = N & \text{for } i = 1, \ldots, n. \end{array} \tag{$\mathscr{C}$}$$

Suppose also that equation

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \tag{3}$$

is satisfied, where $f_i^{q_i}$ denote the $q_i{}^{th}$ power of f_i.

Throughout this paper, we always assume that the field $\mathbb{K}$ has characteristic 0 (about the problem on other fields we refer to Open problem 4 in Section 5). In what follows, we show that Equation (3) along with condition ($\mathscr{C}$) is suitable to characterize homomorphisms acting between the fields $\mathbb{K}$ and $\mathbb{C}$.

Remark 1.1: Obviously, solving functional Equation (3) is meaningful without condition ($\mathscr{C}$). At the same time, we have to point out that without this condition we cannot expect in general that all the solutions are linear combinations of homomorphisms or it can happen that the general problem can be reduced to the above formulated problem.

Indeed, if conditions

$$1 < p_i \cdot q_i = N \quad \text{for } i = 1, \ldots, n$$

are not satisfied, then the homogeneous terms of the same degree can be collected together, provided that $\mathbb{K}$ is of characteristic zero (in such a situation we have $\mathbb{Q} \subset \mathbb{K}$). To show this, assume that

$$\begin{aligned} p_i q_i &= N_1 \quad i = 1, \ldots, k_1 \\ p_i q_i &= N_2 \quad i = k_1 + 1, \ldots, k_2 \\ &\vdots \\ p_i q_i &= N_{j+1} \quad i = k_j + 1, \ldots, n \end{aligned}$$

where the positive integers $N_1, \ldots, N_{j+1}$ are different. Let $r \in \mathbb{Q}$ and $x \in \mathbb{K}$ be arbitrary and substitute rx in place of x in Equation (3) to get

$$\begin{aligned} 0 &= \sum_{i=1}^{n} f_i^{q_i}\left((rx)^{p_i}\right) = \sum_{i=1}^{n} r^{p_i q_i} f_i^{q_i}\left(x^{p_i}\right) \\ &= r^{N_1} \sum_{i=1}^{k_1} f_i^{q_i}\left(x^{p_i}\right) + r^{N_2} \sum_{i=k_1+1}^{k_2} f_i^{q_i}\left(x^{p_i}\right) + \cdots + r^{N_{j+1}} \sum_{i=k_j+1}^{n} f_i^{q_i}\left(x^{p_i}\right). \end{aligned}$$

Observe that the right-hand side of this identity is a polynomial of r for any fixed $x \in \mathbb{K}$, that has infinitely many zeros. This yields, however, that this polynomial cannot be nonzero, providing that all of its coefficients have to be zero, i.e.

$$\begin{aligned} \sum_{i=1}^{k_1} f_i^{q_i}\left(x^{p_i}\right) &= 0 \\ \sum_{i=k_1+1}^{k_2} f_i^{q_i}\left(x^{p_i}\right) &= 0 \\ &\vdots \\ \sum_{i=k_j+1}^{n} f_i^{q_i}\left(x^{p_i}\right) &= 0 \end{aligned}$$

This means that in such a situation the original problem can be split into several problems, where condition ($\mathscr{C}$) already holds.

On the other hand, if condition

$$\begin{aligned} p_i &\neq p_j \quad \text{for} \quad i \neq j \\ q_i &\neq q_j \quad \text{for} \quad i \neq j \end{aligned}$$

is not satisfied then in general we cannot expect that the solutions are linear combinations of field homomorphisms. Namely, in such a situation *arbitrary* additive functions can occur as solution, even in the simplest cases.

To see this, let $p, q \in \mathbb{N}$ be arbitrarily fixed and let $a\colon \mathbb{K} \to \mathbb{C}$ be an *arbitrary* additive function. Furthermore, assume that for the complex constants $\alpha_1, \ldots, \alpha_n$, identity

$$\alpha_1^q + \cdots + \alpha_n^q = 0$$

holds and consider the additive functions

$$f_i(x) = \alpha_i a(x) \qquad (x \in \mathbb{K}) .$$

Clearly, equation

$$\sum_{i=1}^{n} f_i(x^p)^q = 0$$

is fulfilled for all $x \in \mathbb{K}$. At the same time, in general, we cannot state that any of these functions is a linear combination of field homomorphisms.

2. Theoretical background

In this section, we collect some results concerning multiadditive functions, polynomials and exponential polynomials and differential operators. This collection highlights the main theoretical ideas that we follow subsequently. Here we use the notations and the terminology of Székelyhidi [22, 23].

Functional equations satisfied by additive functions may have some interest not only in the theory of functional equations but also in the theory of (commutative) algebra because the fundamental notions such as derivations and automorphisms are additive functions satisfying some additional equations as well. It is a crucial problem that how these morphisms can be characterized among additive mappings in general. The aim of our former paper [3] was to provide multivariate and univariate characterization theorems for (higher order) derivations. The main difficulty (there and also here) is that the investigated functional equations contains several unknown (additive) functions, but we have only one independent variable in our equation. Therefore, the first step is that we have to broaden the number of variables in the investigated equation, this is the so-called **symmetrization method**.

2.1. The symmetrization method

Definition 2.1: Let G, S be commutative semigroups, $n \in \mathbb{N}$ and let $A\colon G^n \to S$ be a function. We say that A is *n-additive* if it is a homomorphism of G into S in each variable. If $n = 1$ or $n = 2$ then the function A is simply termed to be *additive* or *bi-additive*, respectively .

The *diagonalization* or *trace* of an n-additive function $A\colon G^n \to S$ is defined as

$$A^*(x) = A(x, \dots, x) \quad (x \in G).$$

As a direct consequence of the definition each n-additive function $A\colon G^n \to S$ satisfies

$$\begin{aligned} &A(x_1, \dots, x_{i-1}, kx_i, x_{i+1}, \dots, x_n) \\ &\quad = kA(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \quad (x_1, \dots, x_n \in G) \end{aligned}$$

for all $i = 1, \dots, n$, where $k \in \mathbb{N}$ is arbitrary. The same identity holds for any $k \in \mathbb{Z}$ provided that G and S are groups, and for $k \in \mathbb{Q}$, provided that G and S are linear spaces over the rationals. For the diagonalization of A, we have

$$A^*(kx) = k^n A^*(x) \quad (x \in G).$$

One of the most important theoretical results concerning multiadditive functions is the so-called *Polarization formula*, that briefly expresses that every n-additive symmetric function is *uniquely* determined by its diagonalization under some conditions on the domain as well as on the range. Suppose that G is a commutative semigroup and S is a commutative group. The action of the *difference operator* Δ on a function $f\colon G \to S$ is defined by the formula

$$\Delta_y f(x) = f(x+y) - f(x) \quad \big(x, y \in G\big).$$

Note that the addition in the argument of the function is the operation of the semigroup G and the subtraction means the inverse of the operation of the group S.

Theorem 2.1 (Polarization formula): *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y_1, \dots, y_m \in G$ we have*

$$\Delta_{y_1, \dots, y_m} A^*(x) = \begin{cases} 0 & \text{if} \quad m > n \\ n!A(y_1, \dots, y_m) & \text{if} \quad m = n. \end{cases}$$

Corollary 2.1: *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y \in G$*

$$\Delta_y^n A^*(x) = n!A^*(y).$$

Lemma 2.1: *Let $n \in \mathbb{N}$ and suppose that the multiplication by $n!$ is surjective in the commutative semigroup G or injective in the commutative group S. Then for any symmetric,n-additive function $A\colon G^n \to S$, $A^* \equiv 0$ implies that A is identically zero, as well.*

The polarization formula plays the central role in the investigation of functional equations characterizing homomorphisms.

2.2. Polynomial and exponential functions

In what follows $(G, \cdot)$ is assumed to be a commutative group.

Definition 2.2: *Polynomials* are elements of the algebra generated by additive functions over G. Namely, if n is a positive integer, $P\colon \mathbb{C}^n \to \mathbb{C}$ is a (classical) complex polynomial in n variables and $a_k\colon G \to \mathbb{C}$ $(k = 1, \dots, n)$ are additive functions, then the function

$$x \longmapsto P(a_1(x), \dots, a_n(x))$$

is a polynomial and, also conversely, every polynomial can be represented in such a form.

Remark 2.1: We recall that the elements of $\mathbb{N}^n$ for any positive integer n are called (n-dimensional) *multi-indices*. Addition, multiplication and inequalities between multi-indices of the same dimension are defined component-wise. Further, we define x^α for any n-dimensional multi-index α and for any $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$ by

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$$

where we always adopt the convention $0^0 = 0$. We also use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$. With these notations any polynomial of degree at most N on the commutative semigroup G has the form

$$p(x) = \sum_{|\alpha| \le N} c_\alpha a(x)^\alpha \qquad (x \in G)\,,$$

where $c_\alpha \in \mathbb{C}$ and $a = (a_1, \dots, a_n)\colon G \to \mathbb{C}^n$ is an additive function. Furthermore, the *homogeneous term of degree* k of p is

$$\sum_{|\alpha| = k} c_\alpha a(x)^\alpha .$$

Lemma 2.2 (Lemma 2.7 of [22]): *Let G be a commutative group, n be a positive integer and let*

$$a = (a_1, \dots, a_n)\,,$$

where $a_1, \dots, a_n$ are linearly independent complex-valued additive functions defined on G. Then the monomials $\{a^\alpha\}$ for different multi-indices are linearly independent.

Definition 2.3: A function $m\colon G \to \mathbb{C}$ is called an *exponential* function if it satisfies

$$m(xy) = m(x)m(y) \qquad \big(x, y \in G\big)\,.$$

Furthermore, on an *exponential polynomial*, we mean a linear combination of functions of the form $p \cdot m$, where p is a polynomial and m is an exponential function.

It is worth to note that an exponential function is either nowhere zero or everywhere zero.

Definition 2.4: Let G be an Abelian group and $V \subseteq \mathbb{C}^G$ a set of functions. We say that V is *translation invariant* if for every $f \in V$ the function $\tau_g f \in V$ also holds for all $g \in G$, where

$$\tau_g f(h) = f(hg) \quad (h \in G).$$

The following lemma will be useful in the proof of Theorem 4.2.

Lemma 2.3 (Lemma 6. of [17]): *Let G be an Abelian group, and let V be a translation invariant linear subspace of all complex-valued functions defined on G. Suppose that $\sum_{i=1}^n p_i \cdot m_i \in V$, where $p_1, \dots, p_n : G \to \mathbb{C}$ are nonzero polynomials and $m_1, \dots, m_n : G \to \mathbb{C}$ are distinct exponentials for every $i = 1, \dots, n$. Then $p_i \cdot m_i \in V$ and $m_i \in V$ for every $i = 1, \dots, n$.*

2.2.1. Algebraic independence

As a remarkable ingredient of our argument, we recall a theorem of Reich and Schwaiger [20]. The original statement was formulated for functions defined on $\mathbb{C}$ (with respect to addition).

Theorem 2.2: *Let k, l, N be positive integers such that $k, l \leq N$. Let $m_1, \dots, m_k \colon \mathbb{C} \to \mathbb{C}$ be distinct nonconstant exponential functions, $a_1, \dots, a_l \colon \mathbb{C} \to \mathbb{C}$ additive functions that are linearly independent over $\mathbb{C}$. Then the functions $m_1, \dots, m_k, a_1, \dots, a_l$ are algebraically independent over $\mathbb{C}$.*

In particular, let $P_s \colon \mathbb{C}^l \to \mathbb{C}$ be a classical complex polynomial of l variables for all multi-index s satisfying $|s| \leq N$. Then the identity

$$\sum_{s\colon\, |s| \leq N} P_s(a_1, \dots, a_l) m_1^{s_1} \cdots m_k^{s_k} = 0 \tag{4}$$

implies that all polynomials P_s vanish identically ($|s| \leq N$).

Now we just focus on the last part of the statement. Most of the original argument works without changes for functions defined on any Abelian group. For an arbitrary field $\mathbb{K}$, we denote by $\mathbb{K}^\times$ (resp. $\mathbb{K}^+$) the *multiplicative (resp. additive) group* of $\mathbb{K}$.

(A) Let G be an Abelian group. If the additive functions $a_1, \dots, a_l : G \to \mathbb{C}$ are linearly independent over $\mathbb{C}$, then any system of terms $a_1^{s_1} \cdots a_l^{s_l}$ are also linearly independent over $\mathbb{C}$ for different nonzero multi-indices $(s_1, \dots, s_l) \in \mathbb{N}^l$. Note that $s_1 = \dots = s_l = 0$ provides the constant functions. This statement is nothing but Lemma 2.2.

(B) Nonconstant exponentials $m_1, \dots, m_k : G \to \mathbb{C}$ are algebraically independent if and only if $m_1^{s_1} \cdots m_k^{s_k} \neq 1$ for any $(s_1, \dots, s_k) \in \mathbb{N}^k$. The latter is not necessarily holds in general. Indeed, for the n-ordered cyclic group $\mathbb{Z}_n$ (with respect to addition) the statement is not true since $\varphi^n \equiv 1$ for every character $\varphi : \mathbb{Z}_n \to \mathbb{C}$.
In our case, when $G = \mathbb{K}^\times$ and the functions are additive on $\mathbb{K}^+$ the analogue holds. Obviously, exponential functions on $\mathbb{K}^\times$ that are additive on $\mathbb{K}^+$ are the field homomorphisms from $\mathbb{K}$ to $\mathbb{C}$. Therefore, none of them are constant.
Let $\varphi_1, \dots, \varphi_k$ be distinct field homomorphisms. To show that $\varphi_1^{s_1} \cdots \varphi_k^{s_k} \neq 1$ for any

nonzero multi-index $(s_1, \dots, s_k) \in \mathbb{N}^k$ is enough to find a nonzero witness element $h \in \mathbb{K}$ such that $\varphi_1^{s_1} \cdots \varphi_k^{s_k}(h) \neq 1$. As a special case ($J' = \emptyset$), we get it from the following statement.

Lemma 2.4 ([15, Lemma 3.3]): *Let $\mathbb{K}$ be a field of characteristic 0, let $\varphi_1, \dots, \varphi_k \colon \mathbb{K} \to \mathbb{C}$ be distinct homomorphisms for a positive integer k. Then there exists an element $0 \neq h \in \mathbb{K}$ such that*

$$\prod_{j \in J} \varphi_j(h) \neq \prod_{j' \in J'} \varphi_{j'}(h),$$

whenever J and J' are distinct multisets of the elements $1, \dots, k$.

(C) Using (A) and (B) and following the steps of the proof of Theorem 2.2 (which is Theorem 6. in [20]), we get that if $a_1, \dots, a_l$ are linearly independent and $m_1, \dots, m_k$ are nonconstant exponential functions, then Equation (4) holds if and only if every $P_s(a_1, \dots, a_l) \cdot m_1^{s_1} \cdots m_k^{s_k} = 0$ for all $s = (s_1, \dots, s_k)$, $|s| \leq N$.

Applying (A)–(C) we get the following statement.

Theorem 2.3: *Let $\mathbb{K}$ be a field of characteristic 0 and k, l, N be positive integers such that $k, l \leq N$. Let $m_1, \dots, m_k \colon \mathbb{K}^\times \to \mathbb{C}$ be distinct exponential functions that are additive on $\mathbb{K}^+$, let $a_1, \dots, a_l \colon \mathbb{K}^\times \to \mathbb{C}$ be additive functions that are linearly independent over $\mathbb{C}$ and let $P_s \colon \mathbb{C}^l \to \mathbb{C}$ be classical complex polynomials of l variables for all $|s| \leq N$. Then the equation*

$$\sum_{s\colon |s| \leq N} P_s(a_1, \dots, a_l) m_1^{s_1} \cdots m_k^{s_k} = 0 \tag{5}$$

implies that all polynomials P_s vanish identically ($|s| \leq N$).

2.3. Levi–Cività equations

As we will see in the next section, the so-called Levi–Cività functional equation will have a distinguished role in our investigations. Thus, below the most important statements will be summarized.

Theorem 2.4 (Theorem 10.1 of [22]): *Any finite-dimensional translation invariant linear space of continuous complex-valued functions on a topological Abelian group is spanned by exponential polynomials.*

In view of this theorem, if $(G, \cdot)$ is an Abelian group, then any function $f \colon G \to \mathbb{C}$ satisfying the so-called *Levi–Cività functional equation*, that is,

$$f(x \cdot y) = \sum_{i=1}^{n} g_i(x) h_i(y) \qquad \big(x, y \in G\big) \tag{6}$$

for some positive integer n and functions $g_i, h_i \colon G \to \mathbb{C}$ $(i = 1, \dots, n)$, is an exponential polynomial of order at most n. Indeed, Equation (6) expresses the fact that all the translates of the function f belong to the same finite-dimensional translation invariant linear space, namely

$$\tau_y f \in \operatorname{lin}\left(g_1, \dots, g_n\right)$$

holds for all $y \in G$.

Obviously, if the functions $h_1, \dots, h_n$ are linearly independent, then $g_1, \dots, g_n$ are linear combinations of the translates of f, hence they are exponential polynomials of order at most n, too. Moreover, they are built up from the same additive and exponential functions as the function f.

Before presenting the solutions of Equation (6), we introduce some notions.

Remark 2.2: Let $k, n, n_1, \dots, n_k$ be positive integers with $n = n_1 + \dots + n_k$ and let for $j = 1, \dots, k$ the complex polynomials $P_j, Q_{i,j}$ of $n_j - 1$ variables and of degree at most $n_j - 1$ be given, $i = 1, \dots, n; j = 1, \dots, k$. For any $j = 1, \dots, k$ and for arbitrary multi-indices $I_j = (i_1, \dots, i_{n_j-1})$ and $J_j = (j_1, \dots, j_{n_j-1})$, we define the $n_j \times n_j$ matrix $M_j(P; I_j, J_j)$ and the $n_j \times n$ matrix $N_j(Q; I_j)$ as follows: for any choice of $p, q = 0, 1, \dots, n_j - 1$ the $(n_j - p, n_j - q)$ element of $M_j(P; I_j, J_j)$ is given by

$$M_j(P; I_j, J_j)_{(n_j-p, n_j-q)} = \begin{cases} \dfrac{1}{p!q!} \partial_{i_1} \cdots \partial_{i_p} \partial_{j_1} \cdots \partial_{j_q} P_j\,(0, \dots, 0) & \text{for } p + q < n_j \\ 0 & \text{otherwise} \end{cases}$$

and for any choice of $p = 1, 2, \dots, n_j, q = 1, 2, \dots, n$ the (p, q) element of $N_j(Q; I_j)$ is given by

$$N_j(Q; I_j)_{p,q} = \frac{1}{(n_j - p)!} \partial_{i_1} \cdots \partial_{i_{n_j-p}} Q_{q,p}(0, \dots, 0).$$

Then let us define the $n \times n$ block matrices $M(P; I_1, \dots, I_k, J_1, \dots, J_k)$ and $N(Q; I_1, \dots, I_k)$ by

$$\begin{aligned} &M\left(P; I_1, \dots, I_k, J_1, \dots, J_k\right) \\ &= \begin{pmatrix} \boxed{M_1(P, I_1, J_1)} & 0 & \dots & 0 \\ 0 & \boxed{M_2(P, I_2, J_2)} & 0 & \dots \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & & \boxed{M_k(P, I_k, J_k)} \end{pmatrix} \end{aligned}$$

and

$$N\left(Q; I_1, \dots, I_k\right) = \begin{pmatrix} \boxed{N_1(Q; I_1)} \\ \vdots \\ \boxed{N_k(Q; I_k)} \end{pmatrix}.$$

The idea of using Levi–Cività equations rely on Theorem 10.4 of [22] which is the following.

Theorem 2.5: *Let G be an Abelian group, n be a positive integer and $f, g_i, h_i \colon G \to \mathbb{C}$ $(i = 1, \ldots, n)$ be functions so that both the sets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ are linearly independent. The functions $f, g_i, h_i \colon G \to \mathbb{C}$ $(i = 1, \ldots, n)$ form a non-degenerate solution of equation* (6) *if and only if*

(a) *there exist positive integers $k, n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$;*
(b) *there exist different nonzero complex exponentials $m_1, \ldots, m_k$;*
(c) *for all $j = 1, \ldots, k$ there exists linearly independent sets of complex additive functions*

$$\left\{a_{j,1}, \ldots, a_{j,n_j-1}\right\};$$

(d) *there exist polynomials $P_j, Q_{i,j}, R_{i,j} \colon \mathbb{C}^{n_j-1} \to \mathbb{C}$ for all $i = 1, \ldots, n; j = 1, \ldots, k$ in $n_j - 1$ complex variables and of degree at most $n_j - 1$;*

so that we have

$$f(x) = \sum_{j=1}^{k} P_j\left(a_{j,1}(x), \ldots, a_{j,n_j-1}(x)\right) m_j(x)$$

$$g_i(x) = \sum_{j=1}^{k} Q_{i,j}\left(a_{j,1}(x), \ldots, a_{j,n_j-1}(x)\right) m_j(x)$$

and

$$h_i(x) = \sum_{j=1}^{k} R_{i,j}\left(a_{j,1}(x), \ldots, a_{j,n_j-1}(x)\right) m_j(x)$$

for all $i = 1, \ldots, n$. Furthermore,

$$M\left(P; I_1, \ldots, I_k, J_1, \ldots, J_k\right) = N\left(Q; I_1, \ldots, I_k\right) N\left(R; J_1, \ldots, J_k\right)^T$$

holds for any choice of the multi-indices $I_j, J_j \in \mathbb{N}^{n_j-1}$ $(j = 1, \ldots, k)$, here ^{T} denotes the transpose of a matrix.

In [21], Shulman used representation theory to investigate a multivariate extension of the Levi–Cività equation. In order to quote her results, we need the following notions.

Remark 2.3: The notion of exponential polynomials can be formulated not only in the framework of the theory of functional equations but also in that of representation theory. This point of view can be really useful in many cases. Let G be a (not necessarily commutative) topological group and $\mathscr{C}(G)$ be the set of all continuous complex-valued functions on G. A function $f \in \mathscr{C}(G)$ is called an exponential polynomial function (or a *matrix function*) if there is a continuous representation π of G on a finite-dimensional topological space X such that

$$f(g) = \langle \pi(g)x, y \rangle \quad \left(g \in G\right),$$

where $x \in X$ and $y \in X^*$.

The minimal dimension of such representations is called the *degree* or the *order* of the exponential polynomial.

Furthermore, $f \in \mathscr{C}(G)$ is an exponential polynomial of degree less than n if it is contained in an invariant subspace $\mathscr{L} \subset \mathscr{C}(G)$ with $\dim(\mathscr{L}) \leq n$.

Definition 2.5: Let G be a group. We say that $f\colon G \to \mathbb{C}$ is a *local exponential polynomial* if its restriction to any finitely generated subgroup $H \subset G$ is an exponential polynomial on H.

A function $f \in \mathscr{C}(G)$ is an *almost exponential polynomial* if for any finite subset E of G, there is a finite-dimensional subspace $\mathscr{L}_E \subset \mathscr{C}(G)$, containing f and invariant for all operators τ_g as g runs through E, where

$$\tau_g f(h) = f(hg) \qquad (h \in G)\,.$$

Remark 2.4: It is an immediate consequence of the above definitions that any exponential polynomial is an almost exponential polynomial. Furthermore, if f is an almost exponential polynomial, then it is a local exponential polynomial, too. Clearly, for finitely generated topological groups, all these three notions coincide. At the same time, in general, these notions are different, even in case of discrete commutative groups, see [21].

Definition 2.6: Let G be a group and $n \in \mathbb{N}, n \geq 2$. A function $F\colon G^n \to \mathbb{C}$ is said to be *decomposable* if it can be written as a finite sum of products $F_1 \cdots F_k$, where all F_i depend on disjoint sets of variables.

Remark 2.5: Without loss of generality, we can suppose that $k = 2$ in the above definition, that is, decomposable functions are those mappings that can be written in the form

$$F(x_1, \dots, x_n) = \sum_E \sum_j A_j^E B_j^E$$

where E runs through all non-void proper subsets of $\{1, \dots, n\}$ and for each E and j the function A_j^E depends only on variables x_i with $i \in E$, while B_j^E depends only on the variables x_i with $i \notin E$.

Theorem 2.6: *Let G be a group and $f \in \mathscr{C}(G)$ and $n \in \mathbb{N}, n \geq 2$ be fixed. If the mapping*

$$G^n \ni (x_1, \dots, x_n) \longmapsto f\,(x_1 \cdots x_n)$$

is decomposable then f is an almost exponential polynomial function.

Remark 2.6: Recently, Laczkovich proved in [18] that if G is a commutative topological semigroup with unit, then a continuous function $f\colon G \to \mathbb{C}$ is an exponential polynomial if and only if there is an $n \geq 2$ such that $f(x_1 \cdots x_n)$ is decomposable.

2.4. Derivations and differential operators

Similarly as before, $\mathbb{K}$ denotes a field and $\mathbb{K}^\times$ stands for the multiplicative subgroup of $\mathbb{K}$.

In this section, we introduce differential operators acting on fields which have important role in our investigation.

Definition 2.7: A *derivation* on $\mathbb{K}$ is a map $d : \mathbb{K} \to \mathbb{K}$ such that equations

$$d(x+y) = d(x) + d(y) \quad \text{and} \quad d(xy) = d(x)y + xd(y) \tag{7}$$

are fulfilled for every $x, y \in \mathbb{K}$.

We say that the map $D : \mathbb{K} \to \mathbb{C}$ is a *differential operator of order n* if D can be represented as

$$D = \sum_{j=1}^{M} c_j d_{j,1} \circ \ldots \circ d_{j,k_j}, \tag{8}$$

where $c_j \in \mathbb{C}$ and $d_{i,j}$ are derivations on $\mathbb{K}$ and $k_j \leq n$ which fulfilled as equality for some j. If $k = 0$ then we interpret $d_1 \circ \ldots \circ d_k$ as the identity function id on $\mathbb{K}$.

Remark 2.7: Since the compositions $d_1 \circ \ldots \circ d_k$ span a linear space over $\mathbb{C}$, without loss of generality we may assume that each term of (8) are linearly independent. Equivalently we may fix a basis $\mathscr{B}$ of compositions. We also fix that the identity map id is in $\mathscr{B}$. We note that a differential operator of order n contains a composition of length n.

If a function m is additive on $\mathbb{K}$ and exponential on $\mathbb{K}^\times$, then m is clearly a field homomorphism. In our case this can be extended to $\mathbb{C}$ as an automorphism of $\mathbb{C}$ by [16, Theorem 14.5.1]. Now we concentrate on the subfields of $\mathbb{C}$ that has finite transcendence degree over $\mathbb{Q}$.

Lemma 2.5: *Let $\mathbb{K} \subset \mathbb{C}$ be field of finite transcendence degree and $\varphi : \mathbb{K} \to \mathbb{C}$ an injective homomorphism. Then there exists an automorphism ψ of $\mathbb{C}$ such that $\psi|_{\mathbb{K}} = \varphi$.*

Further relations can be presented between the exponential polynomials defined on $\mathbb{K}^\times$ and differential operators on $\mathbb{K}$. The connection was first realized in [13] and the connection between the degrees and orders was settled in [14]. Clearly every differential operator is additive on $\mathbb{K}$ and this additional property is a substantial part of the following statement.

Theorem 2.7: *Suppose that the transcendence degree of the field $\mathbb{K}$ over $\mathbb{Q}$ is finite. Let $f, m : \mathbb{K} \to \mathbb{C}$ be additive functions such that m is exponential on $K^\times$, too. Let φ be an extension of m to $\mathbb{C}$ as an automorphism of $\mathbb{C}$. Then the following are equivalent.*

(i) *$f = p \cdot m$ on $\mathbb{K}^\times$, where $p \cdot m$ is a local exponential polynomial on $\mathbb{K}^\times$.*
(ii) *$f = p \cdot m$ on $\mathbb{K}^\times$, where $p \cdot m$ is an almost exponential polynomial on $\mathbb{K}^\times$.*
(iii) *$f = p \cdot m$ on $\mathbb{K}^\times$, where p is a polynomial on $\mathbb{K}^\times$.*
(iv) *There exists a unique differential operator D on $\mathbb{K}$ such that $f = \varphi \circ D$ on $\mathbb{K}$.*

In this case, p is a polynomial of degree n if and only if D is a differential operator of order n.

Proof: The equivalence of (*i*), (*iii*) and (*iv*) follows from [13, Theorem 4.2]. Remark 2.4 implies the equivalence of (*ii*) with the others. The last part of the statement follows from [14, Corollary 1.1.]. ■

3. Preparatory statements

At first glance, Equation (3) itself seem not really restrictive for the functions $f_1, \dots, f_n$. At the same time, our results show that these additive functions are in fact very special, i.e. they are linear combinations of field homomorphisms from the field $\mathbb{K}$ to $\mathbb{C}$. This is caused by the additivity assumption on the involved functions, and this is the property that can effectively be combined with the theory of (exponential) polynomials on semigroups. More precisely, with the aid of the following lemma, we will be able to broaden the number of the variables appearing in Equation (3) from one to N.

Lemma 3.1: *Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field, $f_1, \dots, f_n \colon \mathbb{K} \to \mathbb{C}$ additive functions. Suppose further that we are given natural numbers $p_1, \dots, p_n, q_1, \dots, q_n$ such that they fulfill condition $(\mathscr{C})$. If*

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \tag{9}$$

is satisfied for any $x \in \mathbb{K}$, then we also have

$$\sum_{i=1}^{n} \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} f_i\left(x_{\sigma(1)} \cdots x_{\sigma(p_i)}\right) \cdots f_i\left(x_{\sigma(N-p_i+1)} \cdots x_{\sigma(N)}\right) = 0 \tag{10}$$

for any $x_1, \dots, x_N \in \mathbb{K}$, here $\mathscr{S}_N$ denotes the symmetric group of order N.

Proof: Suppose that $n \in \mathbb{N}$, $\mathbb{K}$ is a field, $f_1, \dots, f_n \colon \mathbb{K} \to \mathbb{C}$ are additive functions and define the function $F \colon \mathbb{K}^N \to \mathbb{C}$ through

$$\begin{aligned} &F(x_1, \dots, x_N) \\ &\quad = \sum_{i=1}^{n} \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} f_i\left(x_{\sigma(1)} \cdots x_{\sigma(p_i)}\right) \cdots f_i\left(x_{\sigma(N-p_i+1)} \cdots x_{\sigma(N)}\right) \quad (x_1, \dots, x_N \in \mathbb{K}). \end{aligned}$$

It is clear that F is a symmetric function, moreover, due to the additivity of the functions $f_1, \dots, f_n$, it is N-additive. Furthermore, in view of Equation (9),

$$F(x, \dots, x) = \sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \quad (x \in \mathbb{K}).$$

Therefore, the polarization formula immediately yields that the mapping F is identically zero on $\mathbb{K}^N$. ■

Equation (3) with two unknown functions

Now we investigate the case when $n = 2$. This case was also studied by Halter–Koch [4] in a special situation (when $n = p$ and $m = q$ (see Theorem 1.3).

Proposition 3.1: *Let $n, m, p, q \in \mathbb{N}$ be arbitrarily fixed so that $n \cdot m = p \cdot q > 1$ and $m \neq p$. Let $\mathbb{K}$ be a field and suppose that for additive functions $f, g \colon \mathbb{K} \to \mathbb{C}$ the functional equation*

$$f^m\left(x^n\right) = g^p\left(x^q\right) \quad (x \in \mathbb{K}) \tag{11}$$

is fulfilled. Then, and only then there exists a homomorphism $\varphi \colon \mathbb{K} \to \mathbb{C}$ so that

$$f(x) = f(1) \cdot \varphi(x) \quad \text{and} \quad g(x) = g(1) \cdot \varphi(x)$$

furthermore, we also have $f(1)^m - g(1)^p = 0$.

Proof: Let $N = n \cdot m = p \cdot q$. According to Lemma 3.1, we have that the symmetric N-additive function $F \colon \mathbb{K}^N \to \mathbb{C}$ defined by

$$\begin{aligned} F\left(x_1, \ldots, x_N\right) = \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} &\left[f\left(x_{\sigma(1)} \cdots x_{\sigma(n)}\right) \cdots f\left(x_{\sigma(N-n+1)} \cdots x_{\sigma(N)}\right)\right. \\ &\left. - g\left(x_{\sigma(1)} \cdots x_{\sigma(q)}\right) \cdots g\left(x_{\sigma(N-q+1)} \cdots x_{\sigma(N)}\right)\right] \quad (x_1, \ldots, x_N \in \mathbb{K}) \end{aligned}$$

is identically zero due to the fact that

$$F\left(x, \ldots, x\right) = f^m(x^n) - g^p(x^q) = 0 \quad (x \in \mathbb{K}).$$

From this, we get $F(1, 1, 1, \ldots, 1) = 0$ which implies

$$f^m(1) - g^p(1) = 0. \tag{12}$$

By appropriate substitution, $F(x, 1, 1, \ldots, 1) = 0$ clearly follows for any $x \in \mathbb{K}$, or equivalently

$$f^{m-1}(1)f(x) - g^{p-1}(1)g(x) = 0 \quad (x \in \mathbb{K}). \tag{13}$$

If $g^{p-1}(1) = 0$ and $f^{m-1}(1) \neq 0$, then $f \equiv 0$ would follow, which is impossible. A similar argument shows that $f^{m-1}(1) = 0$ and $g^{p-1}(1) \neq 0$ is also impossible. This means that either $g^{p-1}(1) \neq 0$ and $f^{m-1}(1) \neq 0$ or $g^{p-1}(1) = 0$ and $f^{m-1}(1) = 0$.

If $g^{p-1}(1) \neq 0$ and $f^{m-1}(1) \neq 0$ then

$$F(x, y, 1, \ldots, 1) = 0 \quad \left(x, y \in \mathbb{K}\right),$$

implies that there exist constants $c_1, c_2, d_1, d_2 \in \mathbb{Q}$ so that $c_1 + c_2 = d_1 + d_2 = 1$ and $c_1 \neq d_1$ (since $p \neq m$) such that

$$\begin{aligned} &c_1 f^{m-1}(1)f(xy) + c_2 f^{m-2}(1)f(x)f(y) \\ &\quad - d_1 g^{p-1}(1)g(xy) - d_2 g^{p-2}(1)g(x)g(y) = 0 \quad \left(x, y \in \mathbb{K}\right). \end{aligned} \tag{14}$$

Applying Equations (12) and (13), we get that

$$\begin{aligned} g^{p-2}(1)g(x)g(y) &= \frac{(g^{p-1}(1)g(x))(g^{p-1}(1)g(y))}{g^p(1)} \\ &= \frac{(f^{m-1}(1)f(x))(f^{m-1}(1)f(y))}{f^m(1)} = f^{m-2}(1)f(x)f(y), \end{aligned}$$

and we can eliminate g from Equation (14)

$$\begin{aligned} &c_1 f^{m-1}(1)f(xy) + c_2 f^{m-2}(1)f(x)f(y) - d_1 f^{m-1}(1)f(xy) - d_2 f^{m-2}(1)f(x)f(y) \\ &\quad = (c_1 - d_1)f^{m-1}(1)f(xy) + (c_2 - d_2)f^{m-2}(1)f(x)f(y) = 0. \end{aligned}$$

Since $c_1 + c_2 = d_1 + d_2 = 1, c_1 \neq d_1$ and $f(1) \neq 0$, it follows that $c_1 - d_1 = -(c_2 - d_2) \neq 0$ and the last expression can be reduced to

$$f(1)f(xy) = f(x)f(y).$$

Taking $\varphi(x) = f(x)/f(1)$ for all $x \in \mathbb{K}$, we get that $\varphi(xy) = \varphi(x)\varphi(y)$ (i.e. φ is multiplicative). Also φ is additive since f is additive. Thus φ is an injective homomorphism of $\mathbb{K}$. A similar argument shows that $g(x) = g(1)\psi(x)$, where ψ is an injective homomorphism of $\mathbb{K}$. Substituting this into Equation (11), we get that

$$f^m(1)\varphi^N = g^p(1)\psi^N.$$

Using Equation (12) and a symmetrization process, $\varphi = \psi$ follows and we get

$$f(x) = f(1)\varphi(x) \quad \text{and} \quad g(x) = g(1)\varphi(x) \quad (x \in \mathbb{K})$$

with a certain homomorphism $\varphi \colon \mathbb{K} \to \mathbb{C}$ and $f(1)^m - g(1)^n = 0$.

Finally, if $g(1)^{q-1} = 0$ and $f(1)^{m-1} = 0$, then $g(1) = f(1) = 0$ and we have two alternatives. Either $f \equiv 0$ and $g \equiv 0$ or at least one of them is non-identically zero, say $f \not\equiv 0$.

The first case clearly yields a solution to Equation (3).

Now we show that the latter case is not possible. Without loss of generality we may assume that $m < p$. Then

$$0 = F(\underbrace{x, \ldots x}_{m}, 1, \ldots, 1) = C \cdot f(x)^m,$$

for some positive constant C. Indeed each other summand stemming from f contain at least one term of $f(1)$ in the product, similarly each product of g's contains $g(1)$. Therefore $f(x) = 0$ for all $x \in \mathbb{K}$, contradicting our assumption. ■

4. Main results

Firstly, we show that every solution of Equation (3) is an almost exponential polynomial of the group $\mathbb{K}^\times$.

Theorem 4.1: *Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field, $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{C}$ additive functions. Suppose further that we are given positive integers $p_1, \ldots, p_n, q_1, \ldots, q_n, N$ so that they fulfill condition ($\mathscr{C}$). If*

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \tag{15}$$

holds for all $x \in \mathbb{K}$, then the functions $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{C}$ are almost exponential polynomials of the Abelian group $\mathbb{K}^\times$.

Proof: Without loss of generality we can (and we also do) assume that the parameters $q_1, \dots, q_n$ are arranged in a strictly increasing order, that is,

$$q_1 < q_2 < \cdots < q_n$$

holds and (due to condition $(\mathscr{C})$) we have that

$$p_1 > p_2 > \cdots > p_n$$

is also fulfilled.

We will show by induction on n that all the mappings $f_1, \dots, f_n$ are almost exponential polynomials. Since the multiadditive mapping F is identically zero on $\mathbb{K}^N$, we have that

$$\begin{aligned}
&\sum_{\sigma \in \mathscr{S}_N} f_1\left(x_{\sigma(1)} \cdots x_{\sigma(p_1)}\right) \cdots f_1\left(x_{\sigma(N-p_1+1)} \cdots x_{\sigma(N)}\right) \\
&\quad = -\sum_{i=2}^{n} \sum_{\sigma \in \mathscr{S}_N} f_i\left(x_{\sigma(1)} \cdots x_{\sigma(p_i)}\right) \cdots f_i\left(x_{\sigma(N-p_i+1)} \cdots x_{\sigma(N)}\right) \\
&\quad \times (x_1, \dots, x_N \in \mathbb{K})
\end{aligned}$$

Let us keep all the variables $x_{p_1+1}, \dots, x_N$ be fixed, while the others are arbitrary. Then the above identity yields that either f_1 is identically zero or f_1 is decomposable. Due to Theorem 2.6, in any cases, we have that f_1 is an almost exponential polynomial function. Therefore, for any finitely generated subgroup $H \subset \mathbb{K}^{\times}$, the function $f_1|_H$ is an exponential polynomial. In other words for any finitely generated subgroup $H \subset \mathbb{K}^{\times}$, the mapping $f_1|_H$ is not only decomposable but also fulfills a certain multivariate Levi–Cività functional equation.

Assume now that there exists a natural number k with $k \leq n-1$ so that all the mappings $f_1, \dots, f_k$ are almost exponential polynomials. Then, again due to the fact that $F \equiv 0$, we have that

$$\begin{aligned}
&\sum_{\sigma \in \mathscr{S}_N} f_{k+1}\left(x_{\sigma(1)} \cdots x_{\sigma(p_{k+1})}\right) \cdots f_{k+1}\left(x_{\sigma(N-p_{k+1}+1)} \cdots x_{\sigma(N)}\right) \\
&\quad = -\sum_{i=1}^{k} \sum_{\sigma \in \mathscr{S}_N} f_i\left(x_{\sigma(1)} \cdots x_{\sigma(p_i)}\right) \cdots f_i\left(x_{\sigma(N-p_i+1)} \cdots x_{\sigma(N)}\right) \\
&\qquad - \sum_{i=k+2}^{n} \sum_{\sigma \in \mathscr{S}_N} f_i\left(x_{\sigma(1)} \cdots x_{\sigma(p_i)}\right) \cdots f_i\left(x_{\sigma(N-p_i+1)} \cdots x_{\sigma(N)}\right) \\
&\qquad \times (x_1, \dots, x_N \in \mathbb{K}).
\end{aligned}$$

Let us keep all the variables $x_{p_{k+1}+1}, \dots, x_N$ be fixed, while the others are arbitrary. Then, in view of Theorem 2.6, this equation yields that either f_{k+1} is identically zero or f_{k+1} is an almost exponential polynomial, due to the fact that the first summand on the right-hand side is an almost exponential polynomial by induction, while the other summand consists only of decomposable terms. ■

Remark 4.1: Note that if

$$f_i(x) = a_i f(x) \quad (x \in \mathbb{K})$$

holds for all $i = 1, \dots, n$ with certain complex constants $a_1, \dots, a_n$ (assuming that at least one of them is nonzero), then we immediately get that there exists a homomorphism $\varphi\colon \mathbb{K} \to \mathbb{C}$ such that

$$f_i(x) = f_i(1)\varphi(x) \quad (x \in \mathbb{K}).$$

To see this suppose that the conditions of the previous theorem are satisfied. Then due to Lemma 3.1, we have that the mapping $F\colon \mathbb{K}^N \to \mathbb{C}$ defined by

$$F(x_1, \dots, x_N) \\ = \sum_{i=1}^{n} \frac{1}{N!} \sum_{\sigma \in \mathscr{S}_N} f_i\left(x_{\sigma(1)} \cdots x_{\sigma(p_i)}\right) \cdots f_i\left(x_{\sigma(N-p_i+1)} \cdots x_{\sigma(N)}\right) \quad (x_1, \dots, x_N \in \mathbb{K})$$

is identically zero.

From this, we immediately conclude that for any $x \in \mathbb{K}$

$$F(x, 1, \dots, 1) = 0$$

holds, that is,

$$\sum_{i=1}^{n} f_i(x) f_i^{q_i - 1}(1) = 0 \quad (x \in \mathbb{K}). \tag{16}$$

Again, due to the fact that F has to be identically zero, we also have

$$F(x, y, 1, \dots, 1) = 0 \quad \left(x, y \in \mathbb{K}\right),$$

i.e.

$$\sum_{i=1}^{n} \left[c_i f_i(xy) + d_i f_i(x) f_i(y)\right] = 0 \quad \left(x, y \in \mathbb{K}\right) \tag{17}$$

with certain constants $c_i, d_i \in \mathbb{C}$.

Indeed, in the special case

$$f_i(x) = a_i f(x) \quad (x \in \mathbb{K}),$$

Equation (17) yields that

$$\sum_{i=1}^{n} \left[c_i a_i f(xy) + d_i a_i^2 f(x) f(y)\right] = 0 \quad \left(x, y \in \mathbb{K}\right),$$

that is, f satisfies the Pexider equation

$$\alpha f(xy) = \beta f(x) f(y) \quad \left(x, y \in \mathbb{K}\right).$$

This means that f is a constant multiple of a multiplicative function. Since f has to be additive too, this multiplicative function has to be in fact a homomorphism. All in all, we have that the additive function $f\colon \mathbb{K} \to \mathbb{C}$ fulfills equation

$$\sum_{i=1}^{n} a_i^{q_i} f\left(x^{p_i}\right)^{q_i} = 0 \quad (x \in \mathbb{K})$$

with certain complex constants $a_1, \dots, a_n$ if and only if there exists a homomorphism such that

$$f(x) = f(1)\varphi(x) \quad (x \in \mathbb{K}),$$

moreover we also have

$$\sum_{i=1}^{n} a_i^{q_i} f(1)^{q_i} = 0.$$

As a consequence of Theorems 4.1 and 2.3 we have the following.

Theorem 4.2: *Let $\mathbb{K} \subset \mathbb{C}$ be a field of finite transcendence degree over $\mathbb{Q}$. Then the additive solutions f_i of Equation* (15) *under condition ($\mathscr{C}$) are of the form*

$$f_i = \sum_{j=1}^{n-1} P_{i,j}\varphi_j,$$

where $P_{i,j}$'s are polynomials on $\mathbb{K}^\times$ and $\varphi_j\colon \mathbb{K} \to \mathbb{C}$ are field homomorphisms. Moreover,

$$\tilde{f}_i(x) = P_{i,j_i}\varphi_{j_i}(x) \quad \left(x \in \mathbb{K}, j_i \in \{1, \dots, n-1\}\right).$$

is also a solution of (15) *and all $\tilde{f}_i$ are additive.*

Proof: By Theorem 4.1, the solutions $f_i\colon \mathbb{K} \to \mathbb{C}$ of (3) are almost exponential polynomials of the Abelian group $\mathbb{K}^\times$. Since $\mathbb{K}$ is a field of finite transcendence degree, by Remark 2.4 all f_i's are exponential polynomials. Since the linear space spanned by $f_1, \dots, f_n$ is of dimension at most $n-1$ (c.f. identity (16)), it follows that each f_i satisfies a certain Levi–Cività functional equation involving at most $n-1$ linearly independent term. Thus by Theorem 2.5 there are nonnegative integers $k, l \leq n-1$ and distinct (nonconstant) exponential functions $m_1, \dots, m_k\colon \mathbb{K}^\times \to \mathbb{C}$, further additive functions $a_1, \dots, a_l\colon \mathbb{K}^\times \to \mathbb{C}$ that are linearly independent over $\mathbb{C}$ and classical complex polynomials $P_{i,1}, \dots, P_{i,k}\colon \mathbb{C}^l \to \mathbb{C}$ with $\deg P_{i,j} \leq n-1$ be such that

$$f_i = \sum_{j=1}^{k} P_{i,j}(a_1, \dots, a_l)m_j. \tag{18}$$

Substituting f_i to (3), we have

$$0 = \sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{k} P_{i,j}(a_1, \dots, a_l)m_j\right)^{q_i}(x^{p_i}). \tag{19}$$

Since $m_1, \dots, m_k$ are distinct (nonconstant) exponentials, the coefficients of the terms $m_1^{a_1} \cdots m_k^{a_k}$ in the expansion must be 0. Taking all terms that contains only m_j as an exponential in the product. By this reduction, we get that

$$\sum_{i=1}^{n}(P_{i,j}m_j)^{q_i}(x^{p_i}) = 0 \tag{20}$$

holds for all $j = 1, \dots, k$.

The additive functions with respect to addition on $\mathbb{K}$ constitute a linear space that is translation invariant with respect to multiplication on $\mathbb{K}^{\times}$. By Lemma 2.3, we get that if $\sum_{j=1}^{k}(P_{i,j}m_j)$ is additive (with respect to addition on $\mathbb{K}$), then $P_{i,j} \cdot m_j$ and m_j are additive for every $j = 1, \dots, k$. The first implies that $\tilde{f}_i$ is additive. Since m_j is additive on $\mathbb{K}$ that has finite transcendence degree and multiplicative on $\mathbb{K}^{\times}$, by Lemma 2.5 m_j can be extended as an automorphism ϕ_j of $\mathbb{C}$. These imply the statement. ■

Remark 4.2: It is worth to note that the role of homomorphism m lost its importance. By Theorem 4.2 for finding a solution of (3) it is enough to find all solutions of (20) separately for every $j = 1, \dots, k$. Since $N = p_1q_1 = \dots = p_nq_n$ and $m_j \neq 0$, Equation (20) is equivalent to

$$\sum_{i=1}^{n} P_{i,j}^{q_i}(x^{p_i}) = 0. \tag{21}$$

Conversely, if (21) holds and $P_{i,j} \cdot \varphi(x)$ is additive, then $f_i = P_{i,j}\varphi$ is an additive solution of (20), where φ is an arbitrary homomorphism.

Remark 4.3: Our next aim is to prove Theorem 4.3. If we omit the condition of additivity of f_i then we can easily find solutions that are neither homomorphisms, nor differential operators as it can be seen in Example 4.1.

Example 4.1: To illustrate this, let us consider the following equation on a field $\mathbb{K}$.

$$f(x^4) + g^2(x^2) + h^4(x) = 0 \quad (x \in \mathbb{K}), \tag{22}$$

where $f, g, h \colon \mathbb{K} \to \mathbb{K}$ denote the unknown (not necessarily additive) functions.

Let a be an additive function on the group $\mathbb{K}^{\times}$. Consider the functions f, g and h defined through

$$\begin{aligned} f(x) &= -(20 + 4a(x) + a^2(x))x, \\ g(x) &= 2(1 + a(x))x, \\ h(x) &= 2x, \end{aligned}$$

that clearly provide a solution for (22). Indeed, using $a^k(x^l) = l^k \cdot a^k(x)$ for all $l, k \in \mathbb{N}$ we have

$$\begin{aligned} f(x^4) &= -(20 + 4a(x^4) + a^2(x^4))x^4 = -(20 + 16a(x) + 16a^2(x))x^4, \\ g^2(x^2) &= (2(1 + a(x^2)))^2x^4 = (4 + 16a(x) + 16a^2(x))x^4, \\ h^4(x) &= 16x^4. \end{aligned}$$

On the other hand, it does not satisfy (16). Clearly, $f(1) = -20, g(1) = 2, h(1) = 2$ and

$$\begin{aligned} &f(x) + g(1)g(x) + h^3(1)h(1) = (-20 - a(x) - a^2(x) + 4 + 4a(x) + 8)x \\ &\quad = \left(-a^2(x) + 3a(x) - 8\right) x \neq 0. \end{aligned}$$

This is caused by the fact that at least one of the function f, g and h is not additive. It is easy to check that g and h are additive on $\mathbb{K}$, but f is not.

Theorem 4.3: *Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field, $f_1, \dots, f_n \colon \mathbb{K} \to \mathbb{C}$ additive functions. Suppose further that we are given positive integers $p_1, \dots, p_n, q_1, \dots, q_n, N$ such that they fulfill condition ($\mathscr{C}$). If*

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \tag{23}$$

holds for all $x \in \mathbb{K}$, then there exist homomorphisms $\varphi_1, \dots, \varphi_{n-1} \colon \mathbb{K} \to \mathbb{C}$ and $\alpha_{i,j} \in \mathbb{C}$ $(i = 1, \dots, n; j = 1, \dots, n-1)$ so that

$$f_i(x) = \sum_{j=1}^{n-1} \alpha_{i,j}\varphi_j(x) \quad (x \in \mathbb{K}). \tag{24}$$

Moreover $\alpha_{i,j}\varphi_j$ gives also a solution of (23).

Proof: Let us assume first that $\mathbb{K} \subset \mathbb{C}$ be a field of finite transcendence degree over $\mathbb{Q}$. By Theorem 4.2, we can restrict our attention on the solutions $f_i = P_i \cdot \varphi$. Namely,

$$0 = \sum_{i=1}^{n} (P_i \cdot \varphi)(x^{p_i})^{q_i} = \varphi(x^N) \cdot \sum_{i=1}^{n} P_i^{p_i}(x^{p_i})^{q_i} \quad (x \in \mathbb{K}).$$

Clearly φ has no special role in the previous equation (see Remark 4.2), thus $\varphi \equiv id$ can be assumed along the proof. Therefore, the solutions are $f_i = P_i \cdot id$. By Theorem 2.7 we can identify f_i with a derivation D_i defined with (8), where the degree of P_i is the same as the order of D_i. Let us denote the maximal degree of all P_i by M. Note that D_i can be uniquely written in terms of the elements of the basis $\mathscr{B}$ defined as in Remark 2.7.

Let the elements of $\mathscr{B}$ be the functions $x, d_1, \dots, d_k, \dots, d_{i_1} \circ \cdots \circ d_{i_s}(x)$ that are linearly independent over $\mathbb{C}$ for all $i_1, \dots, i_s < n$. Since every composition is an additive function on $\mathbb{K}$, by Theorem 2.2, we get that the elements of $\mathscr{B}$ are also algebraically independent.

Now fix i such that D_i has maximal order M and q_i is the smallest possible. Thus it contains a term $d_{j_1} \circ \cdots \circ d_{j_M} \in \mathscr{B}$. Then we have that

$$\begin{aligned} &d_{j_1} \circ \cdots \circ d_{j_M}(x^{p_i}) \\ &\quad = p_i x^{p_i - 1}(d_{j_1} \circ \cdots \circ d_{j_M})(x) + p_i(p_i - 1)x^{p_i - 2} d_1(x)(d_2 \circ \cdots \circ d_{j_M})(x) + \dots \end{aligned}$$

Let us assume that $M > 1$. Since $x, d_{j_1} \circ \cdots \circ d_{j_M}(x) \in \mathscr{B}$ and they are distinct, the coefficient of

$$x^{q_i(p_i - 1)}(d_{j_1} \circ \cdots \circ d_{j_M}(x))^{q_i} \tag{25}$$

uniquely determined and it must vanish. In $D_i(x^{p_i})^{q_i}$ we have only the term of (25) with nonzero coefficient. Since q_i was minimal, $D_j(x^{p_j})^{q_j}$ does not contain the product (25), if

$j \neq i$. In such a situation however this term cannot vanish, contradicting to the algebraic independence. This leads to the fact that $M = 1$, i.e. every $D_i(x) = c_i \cdot x$, for some complex constant c_i.

This clearly implies in general that every solution can be written as

$$f_i(x) = \sum_{j=1}^{n-1} c_{i,j}\varphi_j(x),$$

for some constants $c_{i,j} \in \mathbb{C}$ and field homomorphisms $\varphi_1, \dots \varphi_{n-1} \colon \mathbb{K} \to \mathbb{C}$.

Now let $\mathbb{K}$ be an arbitrary field of characteristic 0 and assume that the statement is not true. Then by Theorem 4.1, there exist almost exponential polynomial solutions defined on $\mathbb{K}^{\times}$ such that

$$f_i = \sum_{j=1}^{n-1} P_{i,j}\varphi_j \not\equiv \sum_{j=1}^{n-1} \alpha_{i,j}\varphi_j.$$

Then there exists a finite set $S \subset \mathbb{K}$ which guarantees this. The field generated by S over $\mathbb{Q}$ is isomorphic to field $\mathbb{K} \subset \mathbb{C}$ of finite transcendence degree. Let us denote this isomorphism by $\Phi : \mathbb{Q}(S) \to \mathbb{K}$. The previous argument provides that $f_i \circ \Phi$ satisfy (24). Since Φ^{-1} is also an isomorphism, f_i satisfies (24), as well. This contradicts our assumption and finishes the proof. ■

Remark 4.4: Here we note that the proof of Theorem 4.3 essentially uses the fact that the field $\mathbb{K}$ has characteristic 0, that we assume throughout the whole paper.

The following example illustrates a special case when not all of f_i are of the form $c \cdot \varphi$. Theorem 4.4 is devoted to show that this is in some sense the exceptional case.

Example 4.2: Let $\mathbb{K}$ be a field and $f, g, h \colon \mathbb{K} \to \mathbb{C}$ be additive functions such that

$$f(x^4) + g^2(x^2) + h^4(x) = 0$$

holds for all $x \in \mathbb{K}$. According to the symmetrization method define the 4-additive function $F \colon \mathbb{K}^4 \to \mathbb{C}$ through

$$\begin{aligned} F(x_1, x_2, x_3, x_4) = &f(x_1x_2x_3x_4) + \frac{1}{3}\left\{g(x_1x_2)g(x_3x_4) + g(x_1x_3)g(x_2x_4)\right. \\ &\left.+g(x_1x_4)g(x_2x_3)\right\} + h(x_1)h(x_2)h(x_3)h(x_4) \quad (x \in \mathbb{K}). \end{aligned}$$

The above equation yields that the trace of F is identically zero, thus F itself is identically zero, too. From this, we immediately get that

$$F(x, 1, 1, 1) = h^3(1)\,h(x) + g(1)\,g(x) + f(x) \quad (x \in \mathbb{K}),$$

that is, the functions f, g, h are linearly dependent. Using this, we also have that

$$0 = F(x, y, 1, 1) = -3h^3(1)\,h(xy) - 2g(1)\,g(xy) + 3h^2(1)\,h(x)\,h(y) + 2g(x)\,g(y)$$

has to be fulfilled by any $x, y \in \mathbb{K}$.

Define the functions $\chi, \varphi_1, \varphi_2 \colon \mathbb{K} \to \mathbb{C}$ as

$$\begin{aligned} \chi(x) &= 3h(1)^3 h(x) + 2g(1)g(x) \\ \varphi_1(x) &= \sqrt{3}h(1)h(x) \qquad\qquad (x \in \mathbb{K}) \\ \varphi_2(x) &= \sqrt{2}g(x) \end{aligned}$$

to obtain the Levi–Cività equation

$$\chi(xy) = \varphi_1(x)\varphi_1(y) + \varphi_2(x)\varphi_2(y) \quad \big(x, y \in \mathbb{K}\big).$$

Using Theorem 2.5, we deduce that there are homomorphisms $\varphi_1, \varphi_2 \colon \mathbb{K} \to \mathbb{C}$ and complex constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ so that

$$\begin{aligned} g(x) &= \alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) \\ h(x) &= \beta_1\varphi_1(x) + \beta_2\varphi_2(x) \quad (x \in \mathbb{K}), \\ f(x) &= \gamma_1\varphi_1(x) + \gamma_2\varphi_2(x) \end{aligned}$$

where the above complex numbers will be determined from the functional equation.

Indeed, from one hand we have

$$\begin{aligned} -f(x^4) &= g^2(x^2) + h^4(x) = \big(\alpha_1\varphi_1(x^2) + \alpha_2\varphi_2(x^2)\big)^2 + (\beta_1\varphi_1(x) + \beta_2\varphi_2(x))^4 \\ &= \alpha_1\varphi_1(x)^4 + 2\alpha_1\alpha_2\varphi_1(x)^2\varphi_2(x)^2 + \alpha_2^2\varphi_2(x)^4 \\ &\quad + \beta_1^4\varphi_1(x)^4 + 4\beta_1^3\beta_2\varphi_1(x)^3\varphi_2(x) + 6\beta_1^2\beta_2^2\varphi_1(x)^2\varphi_2(x)^2 \\ &\quad + 4\beta_1\beta_2^3\varphi_1(x)\varphi_2(x)^3 + \beta_2^4\varphi_2(x)^4 \\ &= \big(\alpha_1^2 + \beta_1^4\big)\varphi_1(x)^4 + \big(2\alpha_1\alpha_2 + 6\beta_1^2\beta_2^2\big)\varphi_1(x)^2\varphi_2(x)^2 + \big(\alpha_2^2 + \beta_2^4\big)\varphi_2(x)^4 \\ &\quad + 4\beta_1^3\beta_2\varphi_1(x)^3\varphi_2(x) + 4\beta_1\beta_2^3\varphi_1(x)\varphi_2(x)^3 \end{aligned}$$

for all $x \in \mathbb{K}$.

On the other hand

$$-f(x^4) = -\gamma_1\varphi_1(x^4) - \gamma_2\varphi_2(x^4) = -\gamma_1\varphi_1(x)^4 - \gamma_2\varphi_2(x)^4 \quad (x \in \mathbb{K}).$$

Bearing in mind Lemma 2.2, after comparing the coefficients, we have especially that equations

$$\begin{aligned} \alpha_1^2 + \beta_1^4 &= -\gamma_1 \\ \alpha_2^2 + \beta_2^4 &= -\gamma_2 \\ \alpha_1\alpha_2 &= 0 \\ \beta_1\beta_2 &= 0 \end{aligned}$$

have to be fulfilled. This yields however that

$$\begin{aligned} f(x) &= -g(1)^2\varphi_1(x) - h^4(1)\varphi_2(x) \\ g(x) &= g(1)\varphi_1(x) \qquad\qquad (x \in \mathbb{K}). \\ h(x) &= h(1)\varphi_2(x) \end{aligned}$$

Without loss of generality we can (and we also do) assume that the parameters $q_1, \dots, q_n$ are arranged in a strictly increasing order, that is, $q_1 < q_2 < \dots < q_n$ holds.

Theorem 4.4: *Let $n \in \mathbb{N}$ be arbitrary and $\mathbb{K}$ a field. Assume that there are given positive integers p_i, q_i, N $(i = 1, \dots, n)$ so that condition $(\mathscr{C})$ is satisfied. Let $f_1, \dots, f_n$ be additive solutions of*

$$\sum_{i=1}^{n} f_i^{q_i}(x^{p_i}) = 0 \tag{26}$$

Then

$$f_i = \begin{cases} c_{i,j_i}\varphi_{j_i} & \text{if } i > 1 \quad \text{or} \quad q_1 \neq 1, \\ \sum_{j=1}^{n-1} c_{1,j}\varphi_j & \text{if } i = 1 \quad \text{and} \quad q_1 = 1, \end{cases} \tag{27}$$

where $\varphi_1, \dots \varphi_{n-1} \colon \mathbb{K} \to \mathbb{C}$ are arbitrary field homomorphisms and $\sum_{i=1}^{n-1} c_{i,j}^{q_i} = 0$ for all $j = 1, \dots, n$.

Proof: By Theorem 4.3, every solution

$$f_i(x) = \sum_{j=1}^{k} c_{i,j}\varphi_j(x) \qquad (x \in \mathbb{K}),$$

for some $c_{i,j} \in \mathbb{C}$ thus the statement for f_1 if $q_1 = 1$ is trivial.

We show the rest of the statement by using a descending process as follows.

Introducing the formal variables $x_1 = \varphi_1(x), \dots, x_k = \varphi_k(x)$, Equation (23) yields that

$$\sum_{i=1}^{n} \left(c_{i,1}x_1^{p_i} + \dots + c_{i,k}x_k^{p_i} \right)^{q_i} = 0. \tag{28}$$

By the polynomial theorem

$$\sum_{i=1}^{n} \sum_{J_{i,1}+\dots+J_{i,k}=q_i} \frac{q_i!}{J_{i,1}! \cdot \dots \cdot J_{i,k}!} c_{i,1}^{J_{i,1}} \cdot \dots \cdot c_{i,k}^{J_{i,k}} \cdot x_1^{J_{i,1}p_i} \cdot \dots \cdot x_k^{J_{i,k}p_i} = 0. \tag{29}$$

Since we have distinct homomorphisms it follows, by Theorem 2.3, that the coefficient of each monomial term of the polynomial in Equation (29) must be zero. Two addends belong to the same monomial term if and only if

$$J_{i,1}p_i = J_{j,1}p_j, \dots, J_{i,k}p_i = J_{j,k}p_j.$$

If $q_i \geq 2$ then we choose the values $J_{i,1} = 1, J_{i,2} = q_i - 1, J_{i,3} = \ldots = J_{i,k} = 0$. For each addend belonging to the same monomial term

$$\begin{aligned} p_i &= J_{j,1} p_j, \\ p_i(q_i - 1) &= J_{j,2} p_j, \\ J_{j,3} &= \ldots = J_{j,k} = 0. \end{aligned}$$

This means that p_j divides p_i or, in an equivalent way, q_i divides q_j. Without loss of generality, we can suppose that q_i is the maximal among the possible powers. Therefore, there is no any addend belonging to the same monomial term as $x_1^{p_i} x_2^{p_i(q_i-1)}$. Since $q_i \geq 2$ it follows that $c_{i,1} = 0$ or $c_{i,2} = 0$. Repeating the argument for arbitrary pair x_k and x_l we get that $c_{i,j} = 0$ except at most one. This immediately implies Equation (26).

Finally, condition

$$\sum_{i=1}^{n} c_{i,j}^{q_i} = 0$$

clearly follows from Theorem 4.2. ■

Example 4.3: Let $\mathbb{K}$ be a field. Illustrating the previous results, we consider all additive solutions $f_1, f_2, f_3, f_4 \colon \mathbb{K} \to \mathbb{C}$ of

$$f_1^2(x^6) + f_2^3(x^4) + f_3^4(x^3) + f_4^6(x^2) = 0 \quad (x \in \mathbb{K}). \tag{30}$$

with $f_i(1) \neq 0$ for $i = 1, 2, 3, 4$.

We distinguish two cases. If every f_i is of the form $c_i\varphi$, then

$$c_1^2 + c_2^3 + c_3^4 + c_4^6 = 0$$

and φ can be any homomorphism.

If not, then there are two different field homomorphisms φ_1, φ_2 such that

$$\begin{aligned} f_i &= c_i \varphi_1, \\ f_j &= c_j \varphi_2. \end{aligned}$$

for some $1 \leq i \neq j \leq 4$.

Practically, the only possible option is that $i_1, i_2 \in \{1, 2, 3, 4\}$ such that

$$\begin{aligned} f_{i_1} &= c_{i_1} \varphi_1, \\ f_{i_2} &= c_{i_2} \varphi_1. \end{aligned}$$

and for $j_1, j_2 \in \{1, 2, 3, 4\} \setminus \{i_1, i_2\}$ we have

$$\begin{aligned} f_{j_1} &= c_{j_1} \varphi_2, \\ f_{j_2} &= c_{j_2} \varphi_2. \end{aligned}$$

It also clearly follows that

$$c_{i_1}^{q_{i_1}} + c_{i_2}^{q_{i_2}} = 0$$
$$c_{j_1}^{q_{j_1}} + c_{j_2}^{q_{j_2}} = 0.$$

For instance, if $i_1 = 1, i_2 = 2, j_1 = 3, j_4 = 4$, then we get that

$$\begin{aligned} f_1 &= c_1\varphi_1, \\ f_2 &= c_2\varphi_1, \\ f_3 &= c_3\varphi_2, \\ f_4 &= c_4\varphi_2, \end{aligned}$$

where $c_1^2 + c_2^3 = c_3^4 + c_4^6 = 0$ and $\varphi_1, \varphi_2\colon \mathbb{K} \to \mathbb{C}$ are arbitrary field homomorphisms.

4.1. Summary

We can assume that $0 < q_1 < q_2 < \ldots < q_n$. As a consequence of Theorem 4.4 we get that for a given system of solutions f_i of (23) the index set $\mathscr{I} = \{1, \ldots, n\}$ can be decomposed into some subsets $\mathscr{I}_1, \ldots, \mathscr{I}_k$ $(k < n)$ such that

$$\bigcup_{j=1}^{k} \mathscr{I}_j = \mathscr{I}$$

and

$$\bigcap_{j=1}^{k} \mathscr{I}_j = \begin{cases} \emptyset & \text{if } q_1 \neq 1 \\ \{1\} & \text{if } q_1 = 1. \end{cases}$$

If $q_1 \neq 1$, then for every $\mathscr{I}_j$ $(j = 1, \ldots, k < n)$ there exists injective homomorphisms $\varphi_j\colon \mathbb{K} \to \mathbb{C}$ such that $f_i = c_i\varphi_j$ and $\sum_{i\in\mathscr{I}_j} c_i^{q_i} = 0$. If $q_1 = 1$, then $f_1 = \sum_{j=1}^{k} c_{1,j}\varphi_j$ and $f_i = c_i\varphi_j$ for all $1 \neq i \in \mathscr{I}_j$, and

$$c_{1,j} + \sum_{i\in\mathscr{I}_j, i\neq 1} c_i^{q_i} = 0.$$

Conversely, if there are given a partition $\mathscr{I}_j$ $(j = 1, \ldots, k)$ of $\{1, \ldots, n\}$ such that except maybe element 1, the sets are disjoint, then for every field homomorphism $\varphi_1, \ldots, \varphi_k\colon \mathbb{K} \to \mathbb{C}$ we get a solution of (23) as

$$f_i = \begin{cases} c_i\varphi_j & \text{if } i \in \mathscr{I}_j \text{ and either } i \neq 1 \text{ or } q_1 \neq 1 \\ \displaystyle\sum_{j=1}^{k} c_{1,j}\varphi_j & \text{if } q_1 = 1 \text{ and } i = 1, \end{cases}$$

where $\sum_{i\in\mathscr{I}_j} c_i^{q_i} = 0$ if $q_1 \neq 1$, otherwise $c_{1,j} + \sum_{i\in\mathscr{I}_j, i\neq 1} c_i^{q_i} = 0$. Additionally, we get that for every set $\mathscr{I}_j$ the system of $f_i = c_i\varphi_j$ $(i \in \mathscr{I}_j)$, where c_i satisfy the previous equation, is a solution of (23). Moreover, for every $j = 1, \ldots, k$

$$\sum_{i \in \mathscr{I}_j} f_i^{q_i}(x_i^p) = 0 \quad (x \in \mathbb{K})$$

also holds which is a sub-term of (23), thus it seems reasonable that we are just looking for solutions that do not satisfy any partial equation of (23).

We say that the system of functions $f_1, \ldots, f_n$ form an *irreducible* solution if it does not satisfy a sub-term of (23).

Corollary 4.1: *Under the assumptions of Theorem 4.4, let $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{C}$ be additive irreducible solutions of (23). Then for all $i = 1, \ldots, n$,*

$$f_i(x) = c_i \cdot \varphi(x) \qquad (x \in \mathbb{K}),$$

where $\varphi \colon \mathbb{K} \to \mathbb{C}$ is an arbitrary field homomorphism and $c_i \in \mathbb{C}$ satisfies

$$\sum_{i=1}^{n} c_i^{q_i} = 0.$$

4.2. Special cases

The following statement which is only about the *real-valued* solutions, is an easy observation which allows us to focus on the important cases henceforth.

Proposition 4.1: *Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field, $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{R}$ additive functions. Suppose further that we are given positive integers $p_1, \ldots, p_n, q_1, \ldots, q_n, N$ so that they fulfill condition $(\mathscr{C})$. If Equation (3) is satisfied for all $x \in \mathbb{K}$ by the functions $f_1, \ldots, f_n$ and the parameters fulfill*

$$q_i = 2k_i \quad (i = 1, \ldots, n)$$

with certain positive integers $k_1, \ldots, k_n$, then all the functions $f_1, \ldots, f_n$ are identically zero.

Proof: If the parameters fulfill

$$q_i = 2k_i \quad (i = 1, \ldots, n)$$

with certain positive integers $k_1, \ldots, k_n$, then Equation (3) can be rewritten as

$$\sum_{i=1}^{n} \left(f_i^{k_i}\left(x^{p_i}\right)\right)^2 = 0 \quad (x \in \mathbb{K}),$$

in other words, we received that the sum of nonnegative real numbers has to be zero, that implies that all the summands has to be zero for all $x \in \mathbb{K}$. Thus the functions $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{R}$ are identically zero. ∎

As an application of the results above, first we study the case

$$f_i(x) = a_i \cdot f(x) \quad (x \in \mathbb{K},\ i = 1, \ldots, n),$$

where $a_1, \ldots, a_n$ are given complex numbers so that at least one of them is nonzero.

Theorem 4.5: *Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field. Assume that there are given positive integers $p_1, \dots, p_n, q_1, \dots, q_n, N$ so that they fulfill condition $(\mathscr{C})$. The function $f\colon \mathbb{K} \to \mathbb{C}$ is an additive solution of*

$$\sum_{i=1}^{n} (a_i \cdot f)^{q_i} \left(x^{p_i}\right) = 0 \tag{31}$$

if and only if

$$f(x) = c \cdot \varphi(x), \tag{32}$$

where $\varphi\colon \mathbb{K} \to \mathbb{C}$ is a homomorphism and for the constant c equation

$$\sum_{i=1}^{n} (c \cdot a_i)^{q_i} = 0 \tag{33}$$

also has to be satisfied.

According to a result of Darboux [2], the only function $f\colon \mathbb{R} \to \mathbb{R}$ that is additive and multiplicative is of the form

$$f(x) = 0 \quad \text{or} \quad f(x) = x \quad (x \in \mathbb{R})\,.$$

From this, we get also that every homomorphism $f\colon \mathbb{R} \to \mathbb{C}$ is of the form

$$f(x) = \kappa \cdot x \quad (x \in \mathbb{R})\,,$$

where $\kappa \in \{0, 1\}$.

Corollary 4.2: *Let $n \in \mathbb{N}$ be arbitrary and assume that there are given positive integers $p_1, \dots, p_n, q_1, \dots, q_n, N$ so that they fulfill condition $(\mathscr{C})$. Let $f_1, \dots, f_n\colon \mathbb{R} \to \mathbb{C}$ be additive solutions of* (3)*. Then and only then, there are complex numbers $c_1, \dots, c_n$ with the property*

$$\sum_{i=1}^{n} c_i^{q_i} = 0$$

so that for all $i = 1, \dots, n$

$$f_i(x) = c_i \cdot x \quad (x \in \mathbb{R})\,.$$

The above corollary shows that for real functions every solution of Equation (3) is automatically continuous (in fact even analytic) without any regularity assumption.

5. Open problems and perspectives

In the last section of our paper, we list some open problems as well as we try to open up new perspectives concerning the investigated problem.

Definition 5.1: Let $(G, +)$ be an Abelian group and $n \in \mathbb{N}$, a function $f \colon G \to \mathbb{C}$ is termed to be a *(generalized) monomial of degree n* if it fulfills the so-called *monomial equation*, that is,

$$\Delta_y^n f(x) = n! f(y) \quad \left(x, y \in G\right).$$

Remark 5.1: Obviously generalized monomials of degree 1 are nothing else but additive functions. Furthermore, generalized monomials of degree 2 are solutions of the equation

$$\Delta_y^2 f(x) = n! f(y) \quad \left(x, y \in G\right),$$

which is equivalent to the so-called *square norm equation*, i.e.

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \left(x, y \in G\right).$$

In this case for the mapping $f \colon G \to \mathbb{C}$ the term *quadratic mapping* is used as well.

Proposition 5.1: *Let G be an Abelian group and $n \in \mathbb{N}$. A function $f \colon G \to \mathbb{C}$ is a generalized monomial of degree n, if and only if, there exists a symmetric, n-additive function $F \colon G^n \to \mathbb{C}$ so that*

$$f(x) = F(x, \ldots, x) \quad (x \in G).$$

Open Problem 5.1 (Higher order generalized monomial solutions): In this paper, we determined the additive solutions of Equation (3). It would be however interesting to determine the higher order monomial solutions of the equation in question. More precisely, the following problem would also be of interest. Let $n, k \in \mathbb{N}$ be arbitrary and $\mathbb{K}$ a field. Suppose further that we are given positive integers $p_1, \ldots, p_n, q_1, \ldots, q_n, N$ so that

$$\begin{aligned} p_i \neq\ & p_j \quad \text{for} \quad i \neq j \\ q_i \neq\ & q_j \quad \text{for} \quad i \neq j \\ 1 < p_i \cdot q_i =\ & N \quad \text{for} \quad i = 1, \ldots, n \end{aligned} \tag{$\mathscr{C}$}$$

Suppose also that equation

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \tag{34}$$

is satisfied.

Question 5.1: What are the generalized monomial solutions $f_1, \ldots, f_n \colon \mathbb{K} \to \mathbb{C}$ of degree k of (34) under the condition $(\mathscr{C})$?

Here the question is, whether we can say something more about these functions $f_1, \ldots, f_n$?

We remark that in case $k \geq 2$, we do not know whether such 'nice' representation for the functions $f_1, \ldots, f_n$ as in Theorem 4.3 can be expected.

At the same time, there are cases when the representation is 'nice' as well as previously. To see this, let us consider the following problem. Assume that for the quadratic function $f\colon \mathbb{K} \to \mathbb{C}$ we have

$$f(x^2) = f(x)^2 \quad (x \in \mathbb{K}).$$

Since f is a generalized monomial of degree 2, there exists a symmetric bi-additive function $F\colon \mathbb{K}^2 \to \mathbb{C}$ so that

$$F(x, x) = f(x) \quad (x \in \mathbb{K}).$$

Define the symmetric 4-additive mapping $\mathcal{F}\colon \mathbb{K}^4 \to \mathbb{C}$ through

$$\begin{aligned}\mathcal{F}(x_1, x_2, x_3, x_4) = {} & F(x_1x_2, x_3x_4) + F(x_1x_3, x_2x_4) + F(x_1x_4, x_2x_3) \\ & - F(x_1, x_2)F(x_3, x_4) - F(x_1, x_3)F(x_2, x_4) - F(x_1, x_4)F(x_2, x_3) \\ & \times (x_1, x_2, x_3, x_4 \in \mathbb{K}).\end{aligned}$$

Since

$$\mathcal{F}(x, x, x, x) = 3\left(F(x^2, x^2) - F(x, x)^2\right) = 3\left(f(x^2) - f(x)^2\right) = 0 \quad (x \in \mathbb{K}),$$

the mapping $\mathcal{F}$ has to be identically zero on $\mathbb{K}^4$. Therefore, especially

$$0 = \mathcal{F}(1, 1, 1, 1) = 3F(1, 1) - 3F(1, 1)^2,$$

yielding that either $F(1, 1) = 0$ or $F(1, 1) = 1$. Moreover,

$$0 = \mathcal{F}(x, 1, 1, 1) = 3F(x, 1) - 3F(1, 1)\, F(x, 1) \quad (x \in \mathbb{K}),$$

from which either $F(1, 1) = 1$ or $F(x, 1) = 0$ follows for any $x \in \mathbb{K}$.

Using that

$$0 = \mathcal{F}(x, x, 1, 1) = F(x^2, 1) - F(1, 1)\; F(x, x) + 2F(x, x) - 2F^2(x, 1) \quad (x \in \mathbb{K}),$$

we obtain that

$$(F(1, 1) - 2)\, F(x, x) = F(x^2, 1) - 2F^2(x, 1) \quad (x \in \mathbb{K}).$$

Now, if $F(1, 1) = 0$, then according to the above identities $F(x, 1) = 0$ would follow for all $x \in \mathbb{K}$. Since $\mathcal{F}(x, x, 1, 1) = 0$ is also fulfilled by any $x \in \mathbb{K}$, this immediately implies that

$$-2f(x) = -2F(x, x) = F(x^2, 1) - F(x, 1)^2 = 0 \quad (x \in \mathbb{K}),$$

i.e. f is identically zero.

In case $F(1,1) \neq 0$, then necessarily $F(1,1) = 1$ from which

$$-F(x,x) = F(x^2,1) - 2F(x,1)^2 \quad (x \in \mathbb{K}).$$

Define the non-identically zero additive function $a\colon \mathbb{K} \to \mathbb{C}$ by

$$a(x) = F(x,1) \quad (x \in \mathbb{K})$$

to get that

$$f(x) = F(x,x) = -F(x^2,1) + 2F(x,1)^2 = 2a(x)^2 - a(x^2) \quad (x \in \mathbb{K}).$$

Since $\mathcal{F}(x,x,x,x) = 0$ has to hold, the additive function $a\colon \mathbb{K} \to \mathbb{C}$ has to fulfill identity

$$-a(x^4) + a^2(x^2) + 4a^2(x)a(x^2) - 4a^4(x) = 0 \quad (x \in \mathbb{K})$$

too.

In what follows, we will show that the additive function a is of a rather special form. Indeed,

$$0 = \mathcal{F}(x,y,z,1) \quad \left(x,y,z \in \mathbb{K}\right)$$

means that a has to fulfill equation

$$a(x)a(yz) + a(y)a(xz) + a(z)a(xy) = 2a(x)a(y)a(z) + a(xyz) \quad \left(x,y,z \in \mathbb{K}\right)$$

Let now $z^* \in \mathbb{K}$ be arbitrarily fixed to have

$$a(x)a(yz^*) + a(y)a(xz^*) + a(z^*)a(xy) = 2a(x)a(y)a(z^*) + a(xyz^*) \quad \left(x,y,z \in \mathbb{K}\right).$$

Define the additive function $A\colon \mathbb{K} \to \mathbb{C}$ by

$$A(x) = a(xz^*) - a(z^*)a(x) \quad (x \in \mathbb{K})$$

to receive that

$$A(xy) = a(x)A(y) + a(y)A(x) \quad \left(x,y \in \mathbb{K}\right),$$

which is a special convolution type functional equation. Due to Theorem 12.2 of [22], we get that

(a) the function A is identically zero under any choice of z^*, implying that a has to be multiplicative. Note that a is additive, too. Thus, for the quadratic mapping $f\colon \mathbb{K} \to \mathbb{C}$ there exists a homomorphism $\varphi\colon \mathbb{K} \to \mathbb{C}$ such that

$$f(x) = \varphi(x)^2 \quad (x \in \mathbb{K}).$$

(b) or there exists multiplicative functions $m_1, m_2\colon \mathbb{K} \to \mathbb{C}$ and a complex constant α such that

$$a(x) = \frac{m_1(x) + m_2(x)}{2} \quad (x \in \mathbb{K})$$

and

$$A(x) = \alpha\left(m_1(x) - m_2(x)\right) \quad (x \in \mathbb{K}).$$

Due to the additivity of a, in view of the definition of the mapping A, we get that A is additive, too.This however means that both the maps $m_1 + m_2$ and $m_1 - m_2$ are additive, from which the additivity of m_1 and m_2 follows, yielding that they are in fact homomorphisms.Since

$$F(x,x) = f(x) = 2a(x)^2 - a(x^2) \quad (x \in \mathbb{K}),$$

we obtain for the quadratic function $f\colon \mathbb{K} \to \mathbb{C}$ that there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{K} \to \mathbb{C}$ such that

$$f(x) = \varphi_1(x)\varphi_2(x) \quad (x \in \mathbb{K}).$$

Summing up, we received the following: identity

$$f(x^2) = f(x)^2 \quad (x \in \mathbb{K})$$

holds for the quadratic function $f\colon \mathbb{K} \to \mathbb{C}$ if and only if there exists homomorphisms $\varphi_1, \varphi_2\colon \mathbb{K} \to \mathbb{C}$ such that

$$f(x) = \varphi_1(x)\varphi_2(x) \quad (x \in \mathbb{K}).$$

Open Problem 5.2 (Not necessarily additive solutions): Motivated by the above open problem as well as Remark 4.3, we can also pose the question below.

Let $n \in \mathbb{N}$ be arbitrary, $\mathbb{K}$ a field, $f_1, \ldots, f_n\colon \mathbb{K} \to \mathbb{C}$ be generalized or exponential polynomials. Suppose further that we are given natural numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ so that they fulfill condition $(\mathscr{C})$. Suppose also that equation

$$\sum_{i=1}^{n} f_i^{q_i}\left(x^{p_i}\right) = 0 \tag{35}$$

is satisfied.

In Example 4.1, we gave a non-additive solution of (22). Namely,

$$\begin{aligned} f(x) &= -(20 + 4a(x) + a^2(x))x, \\ g(x) &= 2(1 + a(x))x, \\ h(x) &= 2x \end{aligned}$$

The functions f, g and h are exponential polynomial solutions of (22). Thus it is clear that Theorem 4.4 do not hold without additivity of f, g and h.

Question 5.2: How can we characterize the solutions of (35) that are (exponential) polynomial on $\mathbb{K}^{\times}$?

Open Problem 5.3 (Solutions on rings and on fields of finite characteristic): As it already appears in the definition of homomorphisms, the natural domain and also the natural range of the functions in (35) are rings.

On the other hand, it is easy to see that there is no nontrivial field homomorphism from $\mathbb{K}$ to $\mathbb{C}$, if the characteristic of $\mathbb{K}$ is finite. The careful reader can also deduce using our methods, that already (35) has no solution in this case. At the same time, it can be easily seen that the equation has solutions if the functions $f_i\colon \mathbb{K} \to \mathbb{L}$ are constant multiple of a field homomorphism where $\mathbb{K}$ and $\mathbb{L}$ has the same characteristic.

Question 5.3: What are the additive (higher ordered/ exponential polynomial) solutions of Equation (3) in case when the functions $f_1, \ldots, f_n$ are defined between (not necessarily commutative) rings?

Open Problem 5.4 (Regular solutions in case $\mathbb{K} = \mathbb{C}$): To pose our last open problem, here we recall the following. Concerning homomorphisms, instead of $\mathbb{R}$, in $\mathbb{C}$ the situation is completely different, see Kestelman [11], since we have the following.

Proposition 5.2: *The only continuous endomorphisms $f\colon \mathbb{C} \to \mathbb{C}$ are $f \equiv 0$, $f \equiv \mathrm{id}$ or*

$$f(x) = \overline{x} \qquad (x \in \mathbb{C}).$$

These endomorphisms are referred to as trivial endomorphisms.

Concerning *nontrivial endomorphisms* we quote here the following.

Proposition 5.3:
(i) *There exist nontrivial automorphisms of $\mathbb{C}$.*
(ii) *If $f\colon \mathbb{C} \to \mathbb{C}$ is a nontrivial automorphism, then $f|_{\mathbb{R}}$ is discontinuous.*
(iii) *If $f\colon \mathbb{C} \to \mathbb{C}$ is a nontrivial automorphism, then the closure of the set $f(\mathbb{R})$ is the whole complex plane.*
(iv) *If $f\colon \mathbb{C} \to \mathbb{C}$ is a nontrivial automorphism, then $f(\mathbb{R})$ is a proper subfield of $\mathbb{C}$, $\mathrm{card}(f(\mathbb{R})) = \mathfrak{c}$ and either the planar (Lebesgue) measure of $f(\mathbb{R})$ is zero or $f(\mathbb{R}) \subsetneq \mathbb{C}$ is a saturated non-measurable set.*

As we saw above the continuous endomorphisms of $\mathbb{C}$ are of really pleasant form. This immediately implies that the *continuous* solutions of Equation (3) in case $\mathbb{K} = \mathbb{C}$ also have the same beautiful structure.

In some special cases, the continuity assumption can be weakened (e.g. to measurability) to guarantee the same result.

At the same time, we can formalize the following question.

Question 5.4: Is it possible to substitute the regularity assumption by an additional *algebraic* supposition for the unknown functions that would imply the same consequence?

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