Explicit formulas for Laplace transforms of certain functionals of some time inhomogeneous diffusions

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Abstract

We consider a process $(X_t^{(\alpha)})_{t\in[0,T)}$ given by the SDE $\mathrm{d}X_t^{(\alpha)} = \alpha b(t)X_t^{(\alpha)}\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t$, $t\in[0,T)$, with initial condition $X_0^{(\alpha)} = 0$, where $T\in(0,\infty]$, $\alpha\in\mathbb{R}$, $(B_t)_{t\in[0,T)}$ is a standard Wiener process, $b:[0,T)\to\mathbb{R}\setminus\{0\}$ and $\sigma:[0,T)\to(0,\infty)$ are continuously differentiable functions. Assuming $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b(t)}{\sigma(t)^2}\right) = -2K\frac{b(t)^2}{\sigma(t)^2}$, $t\in[0,T)$, with some $K\in\mathbb{R}$, we derive an explicit formula for the joint Laplace transform of $\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s$ and $(X_t^{(\alpha)})^2$ for all $t\in[0,T)$ and for all $\alpha\in\mathbb{R}$. Our motivation is that the maximum likelihood estimator (MLE) $\widehat{\alpha}_t$ of α can be expressed in terms of these random variables. As an application, we show that in case of $\alpha = K$, $K\neq 0$,

$$\sqrt{I_K(t)}\,(\widehat{\alpha}_t - K) \stackrel{\mathcal{L}}{=} -\frac{\operatorname{sign}(K)}{\sqrt{2}}\frac{\int_0^1 W_s\,\mathrm{d}W_s}{\int_0^1 (W_s)^2\,\mathrm{d}s}, \qquad \forall\, t\in(0,T),$$

where $I_K(t)$ denotes the Fisher information for α contained in the observation $(X_s^{(K)})_{s\in[0,\,t]}$, $(W_s)_{s\in[0,1]}$ is a standard Wiener process and $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution. We also prove asymptotic normality of the MLE $\widehat{\alpha}_t$ of α as $t\uparrow T$ for $\operatorname{sign}(\alpha-K) = \operatorname{sign}(K)$, $K\neq 0$. As an example, for all $\alpha\in\mathbb{R}$ and $T\in(0,\infty)$, we study the process $(X_t^{(\alpha)})_{t\in[0,T)}$ given by the SDE $\mathrm{d}X_t^{(\alpha)} = -\frac{\alpha}{T-t}X_t^{(\alpha)}\,\mathrm{d}t + \mathrm{d}B_t$, $t\in[0,T)$, with initial condition $X_0^{(\alpha)} = 0$. In case of $\alpha>0$, this process is known as an α-Wiener bridge, and in case of $\alpha = 1$, this is the usual Wiener bridge.

1 Introduction

Several contributions have already been appeared containing explicit formulae for Laplace transforms of functionals of diffusion processes, see, e.g., Borodin and Salminen [7], Liptser and Shiryaev

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[22, Sections 7.7 and 17.3], Arató [2], Yor [27], Deheuvels and Martynov [9], Deheuvels, Peccati and Yor [10], Mansuy [24], Albanese and Lawi [1], Kleptsyna and Le Breton [19], [20], Hurd and Kuznetsov [16] and Gao, Hannig, Lee and Torcaso [15] (the latter one is about the Laplace transform of the squared L^2-norm of some Gauss processes). These formulae play an important role in theory of parameter estimation. Most of the literature concern time homogeneous diffusion processes.

To describe our aims, let us start with the usual Ornstein–Uhlenbeck process $(Z_t^{(\alpha)})_{t\geqslant 0}$ given by the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}Z_t^{(\alpha)} = \alpha\, Z_t^{(\alpha)}\,\mathrm{d}t + \mathrm{d}B_t, & t \geqslant 0, \\ Z_0^{(\alpha)} = 0, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $(B_t)_{t\geqslant 0}$ is a standard Wiener process. An explicit formula is available for the Laplace transform of the random variable $\int_0^t (Z_s^{(\alpha)})^2\,\mathrm{d}s$, $t \geqslant 0$, namely, for all $t \geqslant 0$ and $\mu > 0$,

$$\mathsf{E}\exp\left\{-\mu\int_0^t (Z_s^{(\alpha)})^2\,\mathrm{d}s\right\} = \left(\frac{\mathrm{e}^{-\alpha t}\sqrt{\alpha^2+2\mu}}{\sqrt{\alpha^2+2\mu}\cosh(t\sqrt{\alpha^2+2\mu}) - \alpha\sinh(t\sqrt{\alpha^2+2\mu})}\right)^{\frac{1}{2}}, \tag{1.1}$$

see, e.g., Liptser and Shiryaev [22, Lemma 17.3] or Gao, Hannig, Lee and Torcaso [15, Theorem 4].

Kleptsyna and Le Breton [19, Proposition 3.2] presented an extension of the above mentioned result for fractional Ornstein–Uhlenbeck type processes.

In case of a time homogeneous diffusion process $(H_t)_{t\geqslant 0}$, Albanese and Lawi [1] and Hurd and Kuznetsov [16] recently addressed the question whether it is possible to compute the Laplace transform

$$\mathsf{E}\left[\mathrm{e}^{-\int_0^t \phi(H_s)\,\mathrm{d}s} q(H_t)\right], \qquad t > 0,$$

in an analitically closed form, where $\phi, q : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions. These papers provided a number of interesting cases when the Laplace transform can be evaluated in terms of special functions, such as hypergeometric functions. Their methods are based on probabilistic arguments involving Girsanov theorem, and alternatively on partial differential equations involving Feynman–Kac formula.

As new results, in case of some time inhomogeneous diffusion processes, we will derive an explicit formula for the joint Laplace transform of certain functionals of these processes using the ideas of Florens-Landais and Pham [14, Lemma 4.1], and see also Liptser and Shiryaev [22, Lemma 17.3]. Let $T \in (0, \infty]$ be fixed. Let $b : [0, T) \to \mathbb{R}$ and $\sigma : [0, T) \to \mathbb{R}$ be continuously differentiable functions. Suppose that $\sigma(t) > 0$ for all $t \in [0, T)$, and $b(t) \neq 0$ for all $t \in [0, T)$ (and hence $b(t) > 0$ for all $t \in [0, T)$ or $b(t) < 0$ for all $t \in [0, T)$). For all $\alpha \in \mathbb{R}$, consider the process $(X_t^{(\alpha)})_{t\in[0,T)}$ given by the SDE

$$\begin{cases} \mathrm{d}X_t^{(\alpha)} = \alpha\, b(t) X_t^{(\alpha)}\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, & t \in [0, T), \\ X_0^{(\alpha)} = 0. \end{cases} \tag{1.2}$$

The SDE (1.2) is a special case of Hull–White (or extended Vasicek) model, see, e.g., Bishwal [5, page 3]. Assuming

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b(t)}{\sigma(t)^2}\right) = -2K\frac{b(t)^2}{\sigma(t)^2}, \qquad t \in [0, T), \tag{1.3}$$

with some $K \in \mathbb{R}$, we derive an explicit formula for the joint Laplace transform of

$$\int_0^t \frac{b(s)^2}{\sigma(s)^2} (X_s^{(\alpha)})^2 \, \mathrm{d}s \qquad \text{and} \qquad (X_t^{(\alpha)})^2 \tag{1.4}$$

for all $t \in [0, T)$ and for all $\alpha \in \mathbb{R}$, see Theorem 2.2.

We note that, using Lemma 11.6 in Liptser and Shiryaev [22], not assuming condition (1.3), one can derive the following formula for the Laplace transform of $\int_0^t \frac{b(s)^2}{\sigma(s)^2} (X_s^{(\alpha)})^2 \, \mathrm{d}s$,

$$\mathsf{E} \exp \left\{ -\mu \int_0^t \frac{b(s)^2}{\sigma(s)^2} (X_s^{(\alpha)})^2 \, \mathrm{d}s \right\} = \exp \left\{ \int_0^t \sigma(s)^2 \gamma_t(s) \, \mathrm{d}s \right\}, \qquad \mu > 0,$$

for all $t \in [0, T)$, where $\gamma_t : [0, t] \to \mathbb{R}$ is the unique solution of the Riccati differential equation

$$\begin{cases} \frac{\mathrm{d}\gamma_t}{\mathrm{d}s}(s) = 2\mu \frac{b(s)^2}{\sigma(s)^2} - 2\alpha b(s) \gamma_t(s) - \sigma(s)^2 \gamma_t(s)^2, \qquad s \in [0, t], \\ \gamma_t(t) = 0. \end{cases} \tag{1.5}$$

As a special case of our formula for the joint Laplace transform of (1.4), under the assumption (1.3), we have an explicit formula for the Laplace transform of $\int_0^t \frac{b(s)^2}{\sigma(s)^2} (X_s^{(\alpha)})^2 \, \mathrm{d}s$, $t \in [0, T)$, see Theorem 2.2 with $\nu = 0$. We suspect that, under the assumption (1.3), the Riccati differential equation (1.5) may be solved explicitly.

We note that Deheuvels and Martynov [9] considered weighted Brownian motions $W_\gamma(t) := t^\gamma W_t$, $t \in (0, 1]$, with $W_\gamma(0) := 0$, and weighted Brownian bridges $B_\gamma(t) := t^\gamma W_t - t^{\gamma+1} W_1$, $t \in (0, 1]$, with $B_\gamma(t) := 0$, and with exponent $\gamma > -1$, where $(W_t)_{t \geqslant 0}$ is a standard Wiener process, and they explicitly calculated the Laplace transforms of the quadratic functionals $\int_0^1 W_\gamma(s)^2 \, \mathrm{d}s$ and $\int_0^1 B_\gamma(s)^2 \, \mathrm{d}s$ by means of Karhunen–Loève expansions. Deheuvels, Peccati and Yor [10] derived similar results for weighted Brownian sheets and bivariate weighted Brownian bridges. Motivated by Theorems 1.3 and 1.4 in Deheuvels and Martynov [9] and Theorem 4.1 in Deheuvels, Peccati and Yor [10], we conjecture that our explicit formula in Theorem 2.2 for the joint Laplace transform of (1.4) may be expressed as an infinite product containing the eigenvalues of the integral operator associated with the covariance function of $(X_t^{(\alpha)})_{t \in [0,T)}$. Assumption (1.3) may play a crucial role in the calculation of these eigenvalues and also for deriving a (weighted) Karhunen–Loève expansion for $(X_t^{(\alpha)})_{t \in [0,T)}$. Once a (weighted) Karhunen–Loève expansion is available for $(X_t^{(\alpha)})_{t \in [0,T)}$, one may derive the Laplace transform of $\int_0^t \frac{b(s)^2}{\sigma(s)^2} (X_s^{(\alpha)})^2 \, \mathrm{d}s$, $t \in [0, T)$, as an infinite product. We note that this approach can be carried through in the special case of a so-called α-Wiener bridge with $\alpha = 1/2$ (introduced and discussed later on). Finally, we also remark that Gao, Hannig, Lee and Torcaso [15] used the same approach via Karhunen–Loève expansions for calculating the Laplace transform of the squared L^2-norm of some Gauss processes such as Ornstein-Uhlenbeck processes, time-changed Wiener bridges and integrated Wiener processes.

In Remark 2.4 we give a third possible explanation for the role of the assumption (1.3).

The random variables in (1.4) appear in the maximum likelihood estimator (MLE) $\widehat{\alpha}_t$ of α based on an observation $(X_s^{(\alpha)})_{s \in [0,t]}$. This is the reason why it is useful to calculate their joint Laplace transform explicitly. For a more detailed discussion, see Sections 3 and 4.

It is known that, under some conditions on b and σ (but without assumption (1.3)), the distribution of the MLE $\widehat{\alpha}_t$ of α normalized by Fisher information can converge to the standard normal distribution, to the Cauchy distribution or to the distribution of $c \int_0^1 W_s \, \mathrm{d}W_s \,/\, \int_0^1 (W_s)^2 \, \mathrm{d}s$, where $(W_s)_{s\in[0,1]}$ is a standard Wiener process, and $c = 1/\sqrt{2}$ or $c = -1/\sqrt{2}$, see Luschgy [23, Section 4.2] and Barczy and Pap [4]. As an application of the joint Laplace transform of (1.4), under the conditions $\int_0^T \sigma(s)^2 \, \mathrm{d}s < \infty$ and

$$b(t) = \frac{\sigma(t)^2}{-2K \int_t^T \sigma(s)^2 \, \mathrm{d}s}, \qquad t \in [0,T), \tag{1.6}$$

with some $K \neq 0$ (note that in this case condition (1.3) is satisfied), we give an alternative proof for

$$\sqrt{I_\alpha(t)}\Big(\widehat{\alpha}_t - \alpha\Big) \stackrel{\mathcal{L}}{\longrightarrow} \begin{cases} \mathcal{N}(0,1) & \text{if } \operatorname{sign}(\alpha - K) = \operatorname{sign}(K), \\ -\frac{\operatorname{sign}(K)}{\sqrt{2}} \frac{\int_0^1 W_s \, \mathrm{d}W_s}{\int_0^1 (W_s)^2 \, \mathrm{d}s} & \text{if } \alpha = K, \end{cases} \qquad \text{as } t \uparrow T,$$

where $I_\alpha(t)$ denotes the Fisher information for α contained in the observation $(X_s^{(\alpha)})_{s\in[0,t]}$, $(W_s)_{s\in[0,1]}$ is a standard Wiener process and $\stackrel{\mathcal{L}}{\longrightarrow}$ denotes convergence in distribution, see Theorem 3.6. In fact, in case of $\alpha = K$, for all $t \in (0,T)$,

$$\sqrt{I_K(t)}\,(\widehat{\alpha}_t - K) \stackrel{\mathcal{L}}{=} -\frac{\operatorname{sign}(K)}{2\sqrt{2}} \frac{(W_1)^2 - 1}{\int_0^1 (W_s)^2 \, \mathrm{d}s} = -\frac{\operatorname{sign}(K)}{\sqrt{2}} \frac{\int_0^1 W_s \, \mathrm{d}W_s}{\int_0^1 (W_s)^2 \, \mathrm{d}s},$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution, see Theorem 3.6. We note that in case of $\operatorname{sign}(\alpha - K) = -\operatorname{sign}(K)$, one can prove $\sqrt{I_\alpha(t)}\,(\widehat{\alpha}_t - \alpha) \stackrel{\mathcal{L}}{\longrightarrow} \zeta$ as $t \uparrow T$, where ζ is a random variable with standard Cauchy distribution, see, e.g., Luschgy [23, Section 4.2] or Barczy and Pap [4]. The proof in this case is based on a martingale limit theorem, and we do not know whether one can find a proof using the explicit form of the joint Laplace transform of (1.4).

By Barczy and Pap [4, Corollaries 9 and 11], under the conditions $\int_0^T \sigma(s)^2 \, \mathrm{d}s < \infty$ and (1.6), we have for all $\alpha \neq K$, $K \neq 0$, the MLE $\widehat{\alpha}_t$ of α is asymptotically normal with an appropriate *random* normalizing factor, see also Remark 3.10. In case of $\alpha = K$, $K \neq 0$, under the above conditions, we determine the distribution of this randomly normalized MLE using the joint Laplace transform of (1.4), see Theorem 3.9. As a by-product of this result, giving a counterexample, we show that Remark 1.47 in Prakasa Rao [25] contains a mistake, see Remark 3.11.

Using the explicit form of the Laplace transform we also prove strong consistency of the MLE of α for all $\alpha \in \mathbb{R}$, see Theorem 3.12.

As an example, for all $\alpha \in \mathbb{R}$ and $T \in (0,\infty)$, we study the process $(X_t^{(\alpha)})_{t\in[0,T)}$ given by the SDE

$$\begin{cases} \mathrm{d}X_t^{(\alpha)} = -\frac{\alpha}{T-t} X_t^{(\alpha)} \, \mathrm{d}t + \mathrm{d}B_t, \qquad t \in [0,T), \\ X_0^{(\alpha)} = 0. \end{cases} \tag{1.7}$$

In case of $\alpha > 0$, this process is known as an α-Wiener bridge, and in case of $\alpha = 1$, this is the usual Wiener bridge. As a special case of the explicit form of the joint Laplace transform of (1.4), we obtain the joint Laplace transform of $\int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2} \, \mathrm{d}u$ and $(X_t^{(\alpha)})^2$ for all $t \in [0,T)$, see

Theorem 4.1. As a special case of this latter formula we get the Laplace transform of $\int_0^t \frac{(B_u)^2}{(T-u)^2}\,\mathrm{d}u$, $t \in [0,T)$, which was first calculated by Mansuy [24, Proposition 5], see Remark 2.8. Finally, we remark that in case of $\alpha > 0$ unweighted and weighted Karhunen–Loève expansions are available for the α-Wiener bridge $(X_t^{(\alpha)})_{t\in[0,T)}$ on $[0,T]$ and $[0,S]$ with $0 < S < T$, respectively, see Barczy and Iglói [3]. Further, using the weighted Karhunen–Loève expansion, one can also get the Laplace transform of $\int_0^t \frac{(X_s^{(1/2)})^2}{(T-s)^2}\,\mathrm{d}s$, $t \in [0,T)$, see Barczy and Iglói [3, Proposition 3.1], i.e., in the special case of an α-Wiener bridge with $\alpha = 1/2$ the approach using Karhunen–Loève expansions mentioned earlier can be carried through.

2 Laplace transform

Let $T \in (0,\infty]$ be fixed. Let $b : [0,T) \to \mathbb{R}$ and $\sigma : [0,T) \to \mathbb{R}$ be continuously differentiable functions. Suppose that $\sigma(t) > 0$ for all $t \in [0,T)$, and $b(t) \neq 0$ for all $t \in [0,T)$ (and hence $b(t) > 0$ for all $t \in [0,T)$ or $b(t) < 0$ for all $t \in [0,T)$). For all $\alpha \in \mathbb{R}$, consider the SDE (1.2). Note that the drift and diffusion coefficients of the SDE (1.2) satisfy the local Lipschitz condition and the linear growth condition (see, e.g., Jacod and Shiryaev [17, Theorem 2.32, Chapter III]). By Jacod and Shiryaev [17, Theorem 2.32, Chapter III], the SDE (1.2) has a unique strong solution

$$(2.1)\qquad X_t^{(\alpha)} = \int_0^t \sigma(s)\exp\left\{\alpha\int_s^t b(u)\,\mathrm{d}u\right\}\mathrm{d}B_s, \qquad t\in[0,T).$$

Note that $(X_t^{(\alpha)})_{t\in[0,T)}$ has continuous sample paths by the definition of strong solution, see, e.g., Jacod and Shiryaev [17, Definition 2.24, Chapter III]. For all $\alpha \in \mathbb{R}$ and $t \in (0,T)$, let $\mathsf{P}_{X^{(\alpha)},t}$ denote the distribution of the process $(X_s^{(\alpha)})_{s\in[0,t]}$ on $\big(C([0,t]),\mathcal{B}(C([0,t]))\big)$, where $C([0,t])$ and $\mathcal{B}(C([0,t]))$ denote the set of all continuous real valued functions defined on $[0,t]$ and the Borel σ-field on $C([0,t])$, respectively. The measures $\mathsf{P}_{X^{(\alpha)},t}$ and $\mathsf{P}_{X^{(\beta)},t}$ are equivalent for all $\alpha,\ \beta \in \mathbb{R}$ and for all $t \in (0,T)$, and

$$(2.2)\quad \frac{\mathrm{d}\mathsf{P}_{X^{(\alpha)},t}}{\mathrm{d}\mathsf{P}_{X^{(\beta)},t}}\left(X^{(\beta)}\big|_{[0,t]}\right) = \exp\left\{(\alpha-\beta)\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\,\mathrm{d}X_s^{(\beta)} - \frac{\alpha^2-\beta^2}{2}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s\right\},$$

see, e.g., Liptser and Shiryaev [21, Theorem 7.19]. Note also that for all $s \in [0,T)$, $X_s^{(\alpha)}$ is normally distributed with mean 0 and with variance

$$(2.3)\qquad V(s;\alpha) := \mathsf{E}\big(X_s^{(\alpha)}\big)^2 = \int_0^s \sigma(u)^2\exp\left\{2\alpha\int_u^s b(v)\,\mathrm{d}v\right\}\mathrm{d}u, \qquad s\in[0,T),$$

and then, by the conditions on b and σ, $V(s;\alpha) > 0$ for all $s \in (0,T)$.

The next lemma is about the solutions of the differential equation (DE) (1.3).

2.1 Lemma. *Let $T \in (0,\infty]$ be fixed and let $b : [0,T) \to \mathbb{R}\setminus\{0\}$ and $\sigma : [0,T) \to (0,\infty)$ be continuously differentiable functions. The DE (1.3) leads to a Bernoulli type DE having solutions*

$$(2.4)\qquad b(t) = \frac{\sigma(t)^2}{2\left(K\int_0^t \sigma(s)^2\,\mathrm{d}s + C\right)}, \qquad t\in[0,T),$$

where $C \in \mathbb{R}$ is such that the denominator $K\int_0^t \sigma(s)^2\,\mathrm{d}s + C \neq 0$ for all $t \in [0,T)$.

Proof. The DE (1.3) can be written in the form

$$\frac{b'(t)\sigma(t) - 2b(t)\sigma'(t)}{\sigma(t)^3} = -2K\frac{b(t)^2}{\sigma(t)^2}, \qquad t \in [0,T),$$

which is equivalent to the Bernoulli type DE

$$b'(t) - 2b(t)(\ln(\sigma(t)))' = -2Kb(t)^2, \qquad t \in [0,T).$$

Since $b(t) \neq 0$ for all $t \in [0,T)$, we get

$$b'(t)b(t)^{-2} - 2(\ln(\sigma(t)))'b(t)^{-1} = -2K, \qquad t \in [0,T).$$

Let $u(t) := b(t)^{-1}$, $t \in [0,T)$. Then

$$-u'(t) - 2(\ln(\sigma(t)))'u(t) = -2K, \qquad t \in [0,T), \tag{2.5}$$

which is an inhomogeneous linear differential equation. The homogeneous linear DE $v'(t) + 2(\ln(\sigma(t)))'v(t) = 0$ has solutions $v(t) = 2C\sigma(t)^{-2}$, $t \in [0,T)$, $C \in \mathbb{R}$, and hence

$$u(t) = 2K\frac{\int_0^t \sigma(s)^2\,\mathrm{d}s}{\sigma(t)^2}, \qquad t \in [0,T),$$

is a particular solution of the inhomogeneous linear DE (2.5), which yields the assertion. $\square$

Now we derive an explicit formula for the joint Laplace transform of $\int_0^t \frac{b(s)^2}{\sigma(s)^2}\big(X_s^{(\alpha)}\big)^2\,\mathrm{d}s$ and $\big(X_t^{(\alpha)}\big)^2$ for all $t \in [0,T)$ under the assumption (2.4) on b and σ. We use the same technique (sometimes called Novikov's method, see, e.g., Arató [2]) as in the proof of Lemma 4.1 in Florens-Landais and Pham [14] or see also the proof of Lemma 17.3 in Liptser and Shiryaev [22].

2.2 Theorem. *Let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE (1.2) where b is given by (2.4). Then for all $\mu > 0$, $\nu \geqslant 0$, and $t \in [0,T)$, we have*

$$\mathsf{E}\exp\left\{-\mu\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s - \nu[X_t^{(\alpha)}]^2\right\}$$

$$= \frac{B_{K,C}(t)^{\frac{K-\alpha}{4}}}{\sqrt{\cosh\left(\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}\ln(B_{K,C}(t))\right) - \frac{\alpha-K-4\nu\left(K\int_0^t\sigma(s)^2\,\mathrm{d}s+C\right)}{\sqrt{2\mu+(\alpha-K)^2}}\sinh\left(\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}\ln(B_{K,C}(t))\right)}},$$

where

$$B_{K,C}(t) := \begin{cases} \left(1 + \frac{K}{C}\int_0^t \sigma(s)^2\,\mathrm{d}s\right)^{\frac{1}{K}} & \textit{if } K \neq 0, \\ \exp\left\{\frac{1}{C}\int_0^t \sigma(s)^2\,\mathrm{d}s\right\} & \textit{if } K = 0, \end{cases} \qquad t \in [0,T).$$

For the proof of Theorem 2.2 we need two lemmas. The first one can be considered as a preliminary version of Theorem 2.2, the second one is about the variance of $X_t^{(\alpha)}$.

2.3 Lemma. *Let* $(X_t^{(\alpha)})_{t\in[0,T)}$ *be the process given by the SDE* (1.2). *If assumption* (1.3) *is satisfied with some* $K \in \mathbb{R}$ *and if* $\operatorname{sign}(b) = \pm\mathbb{1}_{[0,T)}$, *then for all* $\mu > 0$, $\nu \geqslant 0$ *and* $t \in [0,T)$, *we have*

(2.6)

$$\mathsf{E}\exp\left\{-\mu\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s - \nu[X_t^{(\alpha)}]^2\right\} = \left(\frac{\exp\left\{-A^{\pm}_{\mu,\alpha,K}\int_0^t b(s)\,\mathrm{d}s\right\}}{1+\left(2\nu - A^{\pm}_{\mu,\alpha,K}\frac{b(t)}{\sigma(t)^2}\right)V(t;\alpha - A^{\pm}_{\mu,\alpha,K})}\right)^{\frac{1}{2}},$$

where $A^{\pm}_{\mu,\alpha,K} := \alpha - K \mp \sqrt{2\mu + (\alpha-K)^2}$.

Proof. For all $\mu > 0$, $\nu \geqslant 0$ and $t \in [0,T)$, let

$$\Psi_t(\alpha,\mu,\nu) := \mathsf{E}\left(\exp\left\{-\mu\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s - \nu[X_t^{(\alpha)}]^2\right\}\right).$$

Heuristically, using (2.2), we have for all $\alpha, \beta \in \mathbb{R}$, $\mu > 0$, $\nu \geqslant 0$ and $t \in (0,T)$,

$$\begin{aligned}\Psi_t(\alpha,\mu,\nu) &= \mathsf{E}\left[\exp\left\{-\mu\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s - \nu[X_t^{(\beta)}]^2\right\}\frac{\mathrm{d}\mathsf{P}_{X^{(\alpha)},t}}{\mathrm{d}\mathsf{P}_{X^{(\beta)},t}}\left(X^{(\beta)}|_{[0,t]}\right)\right]\\ &= \mathsf{E}\left[\exp\left\{-\mu\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s - \nu[X_t^{(\beta)}]^2 + (\alpha-\beta)\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\,\mathrm{d}X_s^{(\beta)}\right.\right.\\ &\qquad\left.\left. - \frac{\alpha^2-\beta^2}{2}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s\right\}\right].\end{aligned} \tag{2.7}$$

In what follows, using Theorem 1 in Delyon and Hu [11], we give a precise derivation of (2.7). For all $t \in (0,T)$, let $g : [0,t]\times\mathbb{R}\to\mathbb{R}$, $h : [0,t]\times\mathbb{R}\to\mathbb{R}$ and $\sigma : [0,t]\times\mathbb{R}\to\mathbb{R}$ be defined by

$$g(u,x) := \alpha b(u)x, \qquad h(u,x) := (\beta-\alpha)\frac{b(u)}{\sigma(u)}x, \qquad \sigma(u,x) := \sigma(u), \quad \forall\ (u,x)\in[0,t]\times\mathbb{R}.$$

Then g, h and σ are locally Lipschitz functions with respect to the second variable. Let $f : C([0,t])\times C([0,t])\to\mathbb{R}$,

$$f(x,w) := \exp\left\{-\mu\int_0^t \frac{b(s)^2}{\sigma(s)^2}x(s)^2\,\mathrm{d}s - \nu[x(t)]^2\right\}, \qquad \forall\ (x,w)\in C([0,t])\times C([0,t]).$$

Using Theorem 1 in Delyon and Hu [11] with the above choices of g, h, σ and f, we obtain for all $\alpha, \beta\in\mathbb{R}$, $\mu > 0$, $\nu \geqslant 0$ and $t \in (0,T)$,

$$\begin{aligned}\Psi_t(\alpha,\mu,\nu) = \mathsf{E}\Bigg[\exp\Bigg\{&-\mu\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\beta)})^2\,\mathrm{d}u - \nu[X_t^{(\beta)}]^2 - (\beta-\alpha)\int_0^t \frac{b(u)}{\sigma(u)}X_u^{(\beta)}\,\mathrm{d}B_u\\ &- \frac{(\beta-\alpha)^2}{2}\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\beta)})^2\,\mathrm{d}u\Bigg\}\Bigg].\end{aligned}$$

By the SDE (1.2), we conclude (2.7).

We check that for all $\beta\in\mathbb{R}$ and $t\in[0,T)$,

$$\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\,\mathrm{d}X_s^{(\beta)} = \frac{1}{2}\left(\frac{b(t)}{\sigma(t)^2}(X_t^{(\beta)})^2 - \int_0^t\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b(s)}{\sigma(s)^2}\right)\right](X_s^{(\beta)})^2\,\mathrm{d}s - \int_0^t b(s)\,\mathrm{d}s\right). \tag{2.8}$$

By Itô's rule (see, e.g., Liptser and Shiryaev [21, Theorem 4.4]), we get

$$
\begin{aligned}
\mathrm{d}\left(\frac{b(t)}{\sigma(t)^2}X_t^{(\beta)}\right) &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b(t)}{\sigma(t)^2}\right)\right] X_t^{(\beta)}\,\mathrm{d}t + \frac{b(t)}{\sigma(t)^2}\,\mathrm{d}X_t^{(\beta)} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b(t)}{\sigma(t)^2}\right)\right] X_t^{(\beta)}\,\mathrm{d}t + \beta\frac{b(t)^2}{\sigma(t)^2}X_t^{(\beta)}\,\mathrm{d}t + \frac{b(t)}{\sigma(t)}\,\mathrm{d}B_t, \quad t\in[0,T).
\end{aligned}
\tag{2.9}
$$

Now we verify that $(X_t^{(\beta)})_{t\in[0,T)}$ and $\left(\frac{b(t)}{\sigma(t)^2}X_t^{(\beta)}\right)_{t\in[0,T)}$ are continuous semimartingales adapted to the filtration induced by B. Consider the decomposition

$$
X_t^{(\beta)} = \exp\left\{\beta\int_0^t b(u)\,\mathrm{d}u\right\}\int_0^t \sigma(s)\exp\left\{-\beta\int_0^s b(u)\,\mathrm{d}u\right\}\mathrm{d}B_s, \qquad t\in[0,T).
$$

Here the deterministic function $\exp\left\{\beta\int_0^t b(u)\,\mathrm{d}u\right\}$, $t\in[0,T)$, is monotone and hence has a finite variation over each finite interval of $[0,T)$, and then, by Jacod and Shiryaev [17, Proposition 4.28, Chapter I], it is a semimartingale. Since

$$
\int_0^t \sigma(s)\exp\left\{-\beta\int_0^s b(u)\,\mathrm{d}u\right\}\mathrm{d}B_s, \qquad t\in[0,T),
$$

is a martingale with respect to the filtration induced by B, using Theorem 4.57 in Chapter I in Jacod and Shiryaev [17] with the function $f(x,y):=xy$, $x,y\in\mathbb{R}$, we have $(X_t^{(\beta)})_{t\in[0,T)}$ is a continuous semimartingale adapted to the filtration induced by B. Similarly as above, using that by our assumptions, $\frac{b(t)}{\sigma(t)^2}$, $t\in[0,T)$, is continously differentiable, and hence has a finite variation over each finite interval of $[0,T)$, one can get $\left(\frac{b(t)}{\sigma(t)^2}X_t^{(\beta)}\right)_{t\in[0,T)}$ is a continuous semimartingale adapted to the filtration induced by B. Moreover, by (2.9), the cross-variation process of the continuous martingale parts of the processes $(X_t^{(\beta)})_{t\in[0,T)}$ and $\left(\frac{b(t)}{\sigma(t)^2}X_t^{(\beta)}\right)_{t\in[0,T)}$ equals

$$
\int_0^t \sigma(s)\frac{b(s)}{\sigma(s)}\,\mathrm{d}s = \int_0^t b(s)\,\mathrm{d}s, \qquad t\in[0,T).
$$

Hence, by integration by parts formula (see, e.g., Karatzas and Shreve [18, page 155]), we have

$$
\begin{aligned}
\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\,\mathrm{d}X_s^{(\beta)} &= \frac{b(t)X_t^{(\beta)}}{\sigma(t)^2}X_t^{(\beta)} - \int_0^t X_s^{(\beta)}\,\mathrm{d}\left(\frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\right) - \int_0^t b(s)\,\mathrm{d}s \\
&= \frac{b(t)}{\sigma(t)^2}(X_t^{(\beta)})^2 - \int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b(s)}{\sigma(s)^2}\right)\right](X_s^{(\beta)})^2\,\mathrm{d}s \\
&\quad - \int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\,\mathrm{d}X_s^{(\beta)} - \int_0^t b(s)\,\mathrm{d}s, \qquad t\in[0,T),
\end{aligned}
$$

which gives us (2.8).

Then, using condition (1.3), we have

$$
\begin{aligned}
&\Psi_t(\alpha,\mu,\nu)\\
&= \mathsf{E}\left[\exp\left\{-\frac{1}{2}(2\mu+\alpha^2-\beta^2)\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s - \frac{1}{2}\left(2\nu - \frac{(\alpha-\beta)b(t)}{\sigma(t)^2}\right)(X_t^{(\beta)})^2\right.\right.\\
&\qquad\left.\left. - \frac{\alpha-\beta}{2}\int_0^t b(s)\,\mathrm{d}s - \frac{\alpha-\beta}{2}\int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b(s)}{\sigma(s)^2}\right)\right](X_s^{(\beta)})^2\,\mathrm{d}s\right\}\right] \qquad (2.10)\\
&= \mathsf{E}\left[\exp\left\{-\frac{1}{2}(2\mu+\alpha^2-\beta^2-2K(\alpha-\beta))\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s\right.\right.\\
&\qquad\left.\left. - \frac{1}{2}\left(2\nu - \frac{(\alpha-\beta)b(t)}{\sigma(t)^2}\right)(X_t^{(\beta)})^2 - \frac{\alpha-\beta}{2}\int_0^t b(s)\,\mathrm{d}s\right\}\right].
\end{aligned}
$$

We choose $\beta\in\mathbb{R}$ such that $2\mu+\alpha^2-\beta^2-2K(\alpha-\beta)=0$. Namely, let

$$\beta := K \pm \sqrt{2\mu+(\alpha-K)^2}, \qquad \text{if} \quad \operatorname{sign}(b) = \pm\mathbb{1}_{[0,T)}.$$

Then

$$\Psi_t(\alpha,\mu,\nu) = \exp\left\{-\frac{\alpha-\beta}{2}\int_0^t b(s)\,\mathrm{d}s\right\}\mathsf{E}\left[\exp\left\{-\frac{1}{2}\left(2\nu - \frac{(\alpha-\beta)b(t)}{\sigma(t)^2}\right)(X_t^{(\beta)})^2\right\}\right]. \qquad (2.11)$$

The Laplace transform of a normally distributed random variable ξ with mean 0 and with variance $D>0$ is

$$\mathsf{E}(\mathrm{e}^{-s\xi^2}) = \frac{1}{\sqrt{1+2sD}}, \qquad s\geqslant 0. \qquad (2.12)$$

Since for all $t\in[0,T)$, $X_t^{(\beta)}$ is normally distributed with mean 0 and with variance $V(t;\beta)$, using (2.12) we have for all $t\in[0,T)$,

$$\mathsf{E}\left[\exp\left\{-\frac{1}{2}\left(2\nu-(\alpha-\beta)\frac{b(t)}{\sigma(t)^2}\right)(X_t^{(\beta)})^2\right\}\right] = \frac{1}{\sqrt{1+\left(2\nu-(\alpha-\beta)\frac{b(t)}{\sigma(t)^2}\right)V(t;\beta)}}.$$

For this we have to check that

$$\frac{1}{2}\left(2\nu-(\alpha-\beta)\frac{b(t)}{\sigma(t)^2}\right)\geqslant 0, \qquad t\in[0,T).$$

This is satisfied, since $\nu\geqslant 0$ and for all $\alpha\in\mathbb{R}$, $\mu>0$, we have

$$\alpha-\beta = \alpha-K\mp\sqrt{2\mu+(\alpha-K)^2} = A^{\pm}_{\mu,\alpha,K},$$

and hence $(\alpha-\beta)b(t)\leqslant 0$ for all $t\in[0,T)$ in both cases. □

2.4 Remark. Note that in Lemma 2.3 we do not use the explicit solutions of the DE (1.3) given in Lemma 2.1, since we wanted to demonstrate the role of condition (1.3) in the proof of Theorem 2.3. By this condition, the process $\int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{b(s)}{\sigma(s)^2}\right)\right](X_s^{(\beta)})^2\,\mathrm{d}s$, $t\in[0,T)$, has the form $-2K\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s$, $t\in[0,T)$, and hence $\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\beta)}\,\mathrm{d}X_s^{(\beta)}$, can be expressed in terms of only the random variables $\left(X_t^{(\beta)}\right)^2$ and $\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s$, see formula (2.8). As a consequence, in the calculation of $\Psi_t(\alpha,\mu,\nu)$ in the proof of Theorem 2.3, by the special choice of β, one can get rid of the stochastic integral $\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\beta)})^2\,\mathrm{d}s$, see (2.10) and (2.11).

In the next lemma we calculate explicitly the variance $V(t;\alpha)$ of $X_t^{(\alpha)}$ for all $t \in [0,T)$.

2.5 Lemma. *Let* $(X_t^{(\alpha)})_{t\in[0,T)}$ *be the process given by the SDE* (1.2)*, where* b *is given by* (2.4)*. Then*

$$V(t;\alpha) = \begin{cases} \frac{C}{\alpha-K}\left(B_{K,C}(t)^\alpha - B_{K,C}(t)^K\right) & \text{if } \alpha \neq K, \\ C B_{K,C}(t)^K \ln(B_{K,C}(t)) & \text{if } \alpha = K, \end{cases}$$

where $B_{K,C}(t)$*,* $t \in [0,T)$*, is defined in Theorem 2.2.*

Proof. First let us suppose that $b(t) > 0$ for all $t \in [0,T)$. Then C is positive, since by $b(0) > 0$, $K\int_0^0 \sigma(u)^2\,\mathrm{d}u + C$ should be positive. If $\alpha \neq K$ and $K \neq 0$, by (2.3), we have for all $t \in [0,T)$,

$$\begin{aligned} V(t;\alpha) &= \int_0^t \left(\frac{K\int_0^t \sigma(u)^2\,\mathrm{d}u + C}{K\int_0^s \sigma(u)^2\,\mathrm{d}u + C}\right)^{\frac{\alpha}{K}} \sigma(s)^2\,\mathrm{d}s \\ &= \frac{1}{K-\alpha}\left(K\int_0^t \sigma(u)^2\,\mathrm{d}u + C\right)^{\frac{\alpha}{K}} \left(\left(K\int_0^t \sigma(u)^2\,\mathrm{d}u + C\right)^{\frac{K-\alpha}{K}} - C^{\frac{K-\alpha}{K}}\right), \end{aligned}$$

which yields the assertion in case of $\alpha \neq K$, $K \neq 0$.

The other cases can be handled similarly.

Let us suppose now that $b(t) < 0$ for all $t \in [0,T)$. For all $\beta \in \mathbb{R}$, let us consider the process $(N_t^{(\beta)})_{t\in[0,T)}$ given by the SDE

$$\begin{cases} \mathrm{d}N_t^{(\beta)} = \beta\,\widetilde{b}(t) N_t^{(\beta)}\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, & t \in [0,T), \\ N_0^{(\beta)} = 0, \end{cases}$$

where $\widetilde{b}(t) := -b(t)$, $t \in [0,T)$. Then, by uniqueness of a strong solution, the process $(X_t^{(\alpha)})_{t\in[0,T)}$ given by the SDE (1.2) and the process $(N_t^{(-\alpha)})_{t\in[0,T)}$ coincide and hence $V(t;\alpha) = V_{N^{(-\alpha)}}(t)$, $t \in [0,T)$, where $V_{N^{(-\alpha)}}(t) := \mathsf{E}(N_t^{(-\alpha)})^2$, $t \in [0,T)$. Moreover, $V_{N^{(-\alpha)}}(t)$, $t \in [0,T)$, is given by the formulae in the present Lemma 2.5 with (α, K, C) is replaced by $(-\alpha, -K, -C)$. Since these formulae are invariant under the above defined replacement, we have the assertion. □

Proof of Theorem 2.2. First we check that for all $K \in \mathbb{R}$,

$$\int_0^t b(s)\,\mathrm{d}s = \frac{1}{2}\ln(B_{K,C}(t)), \qquad t \in [0,T). \tag{2.13}$$

If $K \neq 0$, then

$$\begin{aligned} \int_0^t b(s)\,\mathrm{d}s &= \int_0^t \frac{\sigma(s)^2}{2\left(K\int_0^s \sigma(u)^2\,\mathrm{d}u + C\right)}\,\mathrm{d}s = \frac{1}{2K}\ln\left(K\int_0^t \sigma(u)^2\,\mathrm{d}u + C\right) - \frac{1}{2K}\ln C \\ &= \frac{1}{2}\ln\left(1 + \frac{K}{C}\int_0^t \sigma(u)^2\,\mathrm{d}u\right)^{\frac{1}{K}} = \frac{1}{2}\ln(B_{K,C}(t)), \qquad t \in [0,T), \end{aligned}$$

and if $K = 0$, then $\int_0^t b(s)\,\mathrm{d}s = \int_0^t \frac{\sigma(s)^2}{2C}\,\mathrm{d}s = \frac{1}{2}\ln(B_{K,C}(t))$, $t \in [0,T)$. By Lemmas 2.3 and 2.5, using also (2.13), for all $\mu > 0$, $\nu \geqslant 0$, and $t \in [0,T)$, we have

$$\psi_t(\alpha,\mu,\nu) = \left(\frac{\exp\left\{ -\frac{A^{\pm}_{\mu,\alpha,K}}{2}\ln(B_{K,C}(t)) \right\}}{1 + \left(2\nu - \frac{A^{\pm}_{\mu,\alpha,K}}{2\left(K\int_0^t \sigma(s)^2\,\mathrm{d}s + C\right)}\right)\frac{C}{\pm\sqrt{2\mu+(\alpha-K)^2}}\left(B_{K,C}(t)^{K\pm\sqrt{2\mu+(\alpha-K)^2}} - B_{K,C}(t)^K\right)} \right)^{\frac{1}{2}}$$

$$= \left(\frac{B_{K,C}(t)^{-\frac{\alpha-K}{2}}}{D} \right)^{\frac{1}{2}},$$

where

$$\begin{aligned}
D &:= B_{K,C}(t)^{\mp\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}} \\
&\quad + \frac{4\nu\left(K\int_0^t \sigma(s)^2\,\mathrm{d}s + C\right) - A^{\pm}_{\mu,\alpha,K}}{\pm 2\sqrt{2\mu+(\alpha-K)^2}} \left(B_{K,C}(t)^{\pm\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}} - B_{K,C}(t)^{\mp\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}} \right) \\
&= \left(\frac{1}{2} \pm \frac{4\nu\left(K\int_0^t \sigma(s)^2\,\mathrm{d}s + C\right) - \alpha + K}{2\sqrt{2\mu+(\alpha-K)^2}} \right) B_{K,C}(t)^{\pm\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}} \\
&\quad + \left(\frac{1}{2} \mp \frac{4\nu\left(K\int_0^t \sigma(s)^2\,\mathrm{d}s + C\right) - \alpha + K}{2\sqrt{2\mu+(\alpha-K)^2}} \right) B_{K,C}(t)^{\mp\frac{\sqrt{2\mu+(\alpha-K)^2}}{2}},
\end{aligned}$$

which yields the assertion. □

2.6 Remark. Note that formula (2.6) in Lemma 2.3 for the joint Laplace transform of (1.4) depends on the sign of the function $\mathrm{sign}(b)$, but in Theorem 2.2 it turned out that the sign is indifferent. We also remark that the case $b(t) < 0$, $t \in [0,T)$, can be traced back to the case $b(t) > 0$, $t \in [0,T)$, using the same arguments that are written for the case $b(t) < 0$, $t \in [0,T)$, at the end of the proof of Lemma 2.5. The point is that the formulae in Theorem 2.2 are invariant under the replacement of (α, b, K, C) with $(-\alpha, -b, -K, -C)$.

In the next two remarks we consider special cases of Theorem 2.2.

2.7 Remark. As a special case of Theorem 2.2, one can get back formula (1.1) due to Liptser and Shiryaev [22, Lemma 17.3], and also the well-known Cameron–Martin formula for a standard Wiener process. Namely, let $T := \infty$, $b(t) := 1$, $t \geqslant 0$, and $\sigma(t) := 1$, $t \geqslant 0$. Let us consider the process $(X^{(\alpha)}_t)_{t\in[0,T)}$ given by the SDE (1.2), which is the usual Ornstein–Uhlenbeck process starting from 0. Clearly, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b(t)}{\sigma(t)^2}\right) = 0$, $t > 0$, and hence Theorem 2.2 with $\nu = 0$, $K = 0$ and with $C = \frac{1}{2}$ implies (1.1). With $\alpha = 0$, we get back the Cameron–Martin formula for a standard Wiener process,

$$\mathsf{E}\exp\left\{ -\mu \int_0^t (B_u)^2\,\mathrm{d}u \right\} = \frac{1}{\sqrt{\cosh(t\sqrt{2\mu})}}, \qquad t \geqslant 0, \quad \mu > 0,$$

see, e.g., Liptser and Shiryaev [21, formula (7.147)].

2.8 Remark. Let $T \in (0, \infty)$, $b(t) := -\frac{1}{T-t}$, $t \in [0, T)$, and $\sigma(t) := 1$, $t \in [0, T)$. Let us consider the process $(X_t^{(\alpha)})_{t\in[0,T)}$ given by the SDE (1.2). Hence condition (2.4) is satisfied with $K := \frac{1}{2}$ and $C := -\frac{T}{2}$, and clearly, $B_{K,C}(t) = (1 - t/T)^2$, $t \in [0, T)$. Then Theorem 2.2 with $\nu = 0$ and $\alpha = 0$ implies that for all $\mu > 0$ and $t \in [0, T)$,

$$\mathsf{E}\exp\left\{-\frac{\mu}{2}\int_0^t \frac{(B_u)^2}{(T-u)^2}\,\mathrm{d}u\right\}$$

$$= \frac{\left(1-\frac{t}{T}\right)^{\frac{1}{4}}}{\sqrt{\cosh\left(\ln\left(1-\frac{t}{T}\right)\sqrt{\mu+\frac{1}{4}}\right) + \frac{1}{2\sqrt{\mu+\frac{1}{4}}}\sinh\left(\ln\left(1-\frac{t}{T}\right)\sqrt{\mu+\frac{1}{4}}\right)}}.$$

An easy calculation shows that for all $\mu > 0$ and $t \in [0, T)$,

$$\mathsf{E}\exp\left\{-\frac{\mu}{2}\int_0^t \frac{(B_u)^2}{(T-u)^2}\,\mathrm{d}u\right\} = \frac{\left(\frac{T-t}{T}\right)^{\frac{1+\sqrt{4\mu+1}}{4}}}{\sqrt{1-\frac{1+\sqrt{4\mu+1}}{2\sqrt{4\mu+1}}\left(1-\left(1-\frac{t}{T}\right)^{\sqrt{4\mu+1}}\right)}}.$$

This is the corrected formula of Proposition 5 in Mansuy [24], which contains a misprint.

3 Maximum likelihood estimation via Laplace transform

As a special case of (2.2), the measures $\mathsf{P}_{X^{(\alpha)},t}$ and $\mathsf{P}_{X^{(0)},t}$ are equivalent for all $\alpha \in \mathbb{R}$ and for all $t \in (0, T)$, and

$$\frac{\mathrm{d}\mathsf{P}_{X^{(\alpha)},t}}{\mathrm{d}\mathsf{P}_{X^{(0)},t}}\left(X^{(\alpha)}\big|_{[0,t]}\right) = \exp\left\{\alpha\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\alpha)}\,\mathrm{d}X_s^{(\alpha)} - \frac{\alpha^2}{2}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s\right\}.$$

Here $\mathsf{P}_{X^{(0)},t}$ is nothing else but the Wiener measure on $\big(C([0,t]), \mathcal{B}(C([0,t]))\big)$.

For all $t \in (0, T)$, the maximum likelihood estimator $\widehat{\alpha}_t$ of the parameter α based on the observation $(X_s^{(\alpha)})_{s\in[0,t]}$ is defined by

$$\widehat{\alpha}_t := \arg\max_{\alpha\in\mathbb{R}} \ln\left(\frac{\mathrm{d}\mathsf{P}_{X^{(\alpha)},t}}{\mathrm{d}\mathsf{P}_{X^{(0)},t}}\left(X^{(\alpha)}\big|_{[0,t]}\right)\right).$$

The following lemma due to Barczy and Pap [4, Lemma 1] guarantees the existence of a unique MLE of α.

3.1 Lemma. *For all* $\alpha \in \mathbb{R}$ *and* $t \in (0, T)$, *we have*

$$\mathsf{P}\left(\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s > 0\right) = 1.$$

By Lemma 3.1, for all $t \in (0, T)$, there exists a unique maximum likelihood estimator $\widehat{\alpha}_t$ of the parameter α based on the observation $(X_s^{(\alpha)})_{s\in[0,t]}$ given by

$$\widehat{\alpha}_t = \frac{\int_0^t \frac{b(s)}{\sigma(s)^2}X_s^{(\alpha)}\,\mathrm{d}X_s^{(\alpha)}}{\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s}, \qquad t \in (0, T).$$

To be more precise, by Lemma 3.1, for all $t \in (0,T)$, the MLE $\widehat{\alpha}_t$ exists P-almost surely. Using the SDE (1.2) we obtain

$$\widehat{\alpha}_t - \alpha = \frac{\int_0^t \frac{b(s)}{\sigma(s)} X_s^{(\alpha)} \, \mathrm{d}B_s}{\int_0^t \frac{b(s)^2}{\sigma(s)^2} (X_s^{(\alpha)})^2 \, \mathrm{d}s}, \qquad t \in (0,T). \tag{3.1}$$

For all $t \in (0,T)$, the Fisher information for α contained in the observation $(X_s^{(\alpha)})_{s\in[0,\,t]}$, is defined by

$$I_\alpha(t) := \mathsf{E}\left(\frac{\partial}{\partial\alpha} \ln\left(\frac{\mathrm{d}\mathsf{P}_{X^{(\alpha)},\,t}}{\mathrm{d}\mathsf{P}_{X^{(0)},\,t}}\left(X^{(\alpha)}\big|_{[0,\,t]}\right)\right)\right)^2 = \int_0^t \frac{b(s)^2}{\sigma(s)^2}\, \mathsf{E}\big(X_s^{(\alpha)}\big)^2 \, \mathrm{d}s,$$

where the last equality follows by the SDE (1.2) and Karatzas and Shreve [18, Proposition 3.2.10]. Note that, by the conditions on b and σ, $I_\alpha : (0,T) \to (0,\infty)$ is an increasing function. Now we calculate the Fisher information $I_\alpha(t)$, $t \in (0,T)$, explicitly.

3.2 Lemma. *Let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE (1.2), where b is given by (2.4). Then for all $t \in (0,T)$,*

$$I_\alpha(t) = \begin{cases} \frac{1}{4(\alpha-K)^2}(B_{K,C}(t)^{\alpha-K} - 1) - \frac{1}{4(\alpha-K)} \ln(B_{K,C}(t)) & \text{if } \alpha \neq K, \\ \frac{1}{8}(\ln(B_{K,C}(t)))^2 & \text{if } \alpha = K, \end{cases}$$

where $B_{K,C}(t)$, $t \in [0,T)$, is defined in Theorem 2.2.

Proof. First let us suppose that $b(t) > 0$ for all $t \in [0,T)$. Then C is positive, since by $b(0) > 0$, $K\int_0^0 \sigma(u)^2 \, \mathrm{d}u + C$ should be positive. In case of $\alpha \neq K$ and $K \neq 0$, by Lemma 2.5, we get for all $t \in (0,T)$,

$$\begin{aligned} I_\alpha(t) &= \int_0^t \frac{\sigma(s)^2}{4\left(K\int_0^s \sigma(u)^2 \, \mathrm{d}u + C\right)^2} V(s;\alpha) \, \mathrm{d}s = \int_0^t \frac{\sigma(s)^2}{4C(\alpha-K)} \big(B_{K,C}(s)^{\alpha-2K} - B_{K,C}(s)^{-K}\big) \, \mathrm{d}s \\ &= \int_0^t \frac{\sigma(s)^2}{4C(\alpha-K)} \left(\left(1 + \frac{K}{C}\int_0^s \sigma(u)^2 \, \mathrm{d}u\right)^{\frac{\alpha-2K}{K}} - \left(1 + \frac{K}{C}\int_0^s \sigma(u)^2 \, \mathrm{d}u\right)^{-1}\right) \mathrm{d}s, \end{aligned}$$

which yields the assertion in case of $\alpha \neq K$ and $K \neq 0$.

The other cases can be handled similarly.

The case $b(t) < 0$, $t \in [0,T)$, can be handled similarly to what is written for the case $b(t) < 0$, $t \in [0,T)$, at the end of the proof of Lemma 2.5. The point is that the formulae in the present Lemma 3.2 are invariant under the replacement of (α, b, K, C) with $(-\alpha, -b, -K, -C)$. $\square$

Later on we intend to prove limit theorems for the MLE $\widehat{\alpha}_t$ of α normalized by Fisher information $I_\alpha(t)$. For proving these limit theorems, condition $\lim_{t\uparrow T} I_\alpha(t) = \infty$ plays a crucial role. In what follows we examine under what additional conditions on b and σ, $\lim_{t\uparrow T} I_\alpha(t) = \infty$ is satisfied.

3.3 Lemma. *Let* $(X_t^{(\alpha)})_{t\in[0,T)}$ *be the process given by the SDE* (1.2)*, where* b *is given by* (2.4)*. In case of* $K \neq 0$*,*

$$\lim_{t\uparrow T} I_\alpha(t) = \infty \qquad \Longleftrightarrow \qquad \lim_{t\uparrow T} \int_0^t \sigma(u)^2 \, \mathrm{d}u = \begin{cases} \infty & \text{if } \frac{C}{K} > 0, \\ -\frac{C}{K} & \text{if } \frac{C}{K} < 0. \end{cases}$$

In case of $K = 0$*, we have* $\lim_{t\uparrow T} I_\alpha(t) = \infty$ *holds if and only if* $\lim_{t\uparrow T} \int_0^t \sigma(u)^2 \, \mathrm{d}u = \infty$*.*

Proof. First we note that $C \neq 0$, since by $b(0) \neq 0$, $K \int_0^0 \sigma(u)^2 \, \mathrm{d}u + C$ should be not zero. Now we check that for all $K \in \mathbb{R}$,

$$\lim_{t\uparrow T} I_\alpha(t) = \infty \qquad \Longleftrightarrow \qquad \lim_{t\uparrow T} B_{K,C}(t)^{\alpha-K} \in \{0, \infty\}. \tag{3.2}$$

If $\alpha \neq K$, by Lemma 3.2, we get

$$\begin{aligned} I_\alpha(t) &= \frac{1}{4(\alpha-K)^2}\Big(\exp\left\{(\alpha-K)\ln(B_{K,C}(t))\right\} - (\alpha-K)\ln(B_{K,C}(t)) - 1\Big) \\ &= \frac{1}{4(\alpha-K)^2} f\left(\ln(B_{K,C}(t)^{\alpha-K})\right), \end{aligned}$$

where $f(x) := \mathrm{e}^x - x - 1$, $x \in \mathbb{R}$. Using that the function $\int_0^t \sigma(u)^2 \, \mathrm{d}u$, $t \in [0,T)$, is monotone increasing, we have $\lim_{t\uparrow T} B_{K,C}(t)$ exists. Hence

$$\lim_{t\uparrow T} I_\alpha(t) = \infty \qquad \Longleftrightarrow \qquad \lim_{t\uparrow T} \ln(B_{K,C}(t)^{\alpha-K}) \in \{-\infty, \infty\},$$

which implies (3.2). A similar argument shows that (3.2) is valid also in case of $\alpha = K$. Hence, by the definition of $B_{K,C}(t)$, we have in case of $K \neq 0$,

$$\lim_{t\uparrow T} I_\alpha(t) = \infty \qquad \Longleftrightarrow \qquad \lim_{t\uparrow T} \left(1 + \frac{K}{C}\int_0^t \sigma(s)^2 \, \mathrm{d}s\right)^{\frac{\alpha-K}{K}} \in \{0, \infty\},$$

and in case of $K = 0$,

$$\lim_{t\uparrow T} I_\alpha(t) = \infty \qquad \Longleftrightarrow \qquad \lim_{t\uparrow T} \exp\left\{\frac{\alpha}{C}\int_0^t \sigma(s)^2 \, \mathrm{d}s\right\} \in \{0, \infty\}.$$

This implies the assertion. $\square$

Note that if the function $b : [0,T) \to \mathbb{R} \setminus \{0\}$ is given by (2.4) and if we suppose also that $K \neq 0$, $\frac{C}{K} < 0$, then, by Lemma 3.3, we have

$$C = -K \lim_{t\uparrow T} \int_0^t \sigma(u)^2 \, \mathrm{d}u =: -K \int_0^T \sigma(u)^2 \, \mathrm{d}u \in \mathbb{R} \setminus \{0\}, \tag{3.3}$$

and hence

$$b(t) = \frac{\sigma(t)^2}{2\left(K \int_0^t \sigma(u)^2 \, \mathrm{d}u - K \int_0^T \sigma(u)^2 \, \mathrm{d}u\right)} = \frac{\sigma(t)^2}{-2K \int_t^T \sigma(u)^2 \, \mathrm{d}u}, \qquad t \in [0,T),$$

which is nothing else but the form (1.6) of b. Moreover, by Lemma 3.3, we have $\lim_{t\uparrow T} I_\alpha(t) = \infty$ holds in this case.

In all what follows we will suppose that the function b is given by (1.6) with some $K \neq 0$, where $\int_0^T \sigma(u)^2 \, \mathrm{d}u < \infty$, and in this case, as an application of the explicit form of the joint Laplace transform of (1.4), we will give a complete description of the asymptotic behavior of the MLE $\widehat{\alpha}_t$ of α as $t \uparrow T$. In the other cases (for which $\lim_{t\uparrow T} I_\alpha(t) = \infty$) the asymptotic behavior of the MLE $\widehat{\alpha}_t$ as $t \uparrow T$ may be worked out using the same arguments as follows, but we do not consider these cases.

For our later purposes, we examine the asymptotic behavior of $I_\alpha(t)$ as $t \uparrow T$.

3.4 Lemma. *Let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE (1.2), where b is given by (1.6) with some $K \neq 0$ and we suppose that $\int_0^T \sigma(s)^2 \, \mathrm{d}s < \infty$. Then in case of $\operatorname{sign}(\alpha - K) = -\operatorname{sign}(K)$,*

$$\lim_{t\uparrow T} \frac{I_\alpha(t)}{\frac{1}{4(K-\alpha)^2}\left(\frac{\int_0^T \sigma(s)^2\,\mathrm{d}s}{\int_t^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{K-\alpha}{K}}} = 1,$$

in case of $\alpha = K$,

$$\lim_{t\uparrow T} \frac{I_\alpha(t)}{\frac{1}{8K^2}\left(\ln\left(\int_t^T \sigma(s)^2\,\mathrm{d}s\right)\right)^2} = 1,$$

and in case of $\operatorname{sign}(\alpha - K) = \operatorname{sign}(K)$,

$$\lim_{t\uparrow T} \frac{I_\alpha(t)}{\frac{1}{4K(K-\alpha)}\ln\left(\int_t^T \sigma(s)^2\,\mathrm{d}s\right)} = 1.$$

The next lemma is about the asymptotic behavior of the Laplace transform of the denominator in (3.1).

3.5 Lemma. *Let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE (1.2), where b is given by (1.6) with some $K \neq 0$ and we suppose that $\int_0^T \sigma(s)^2 \, \mathrm{d}s < \infty$. Then*

$$\frac{1}{I_\alpha(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2 \, \mathrm{d}u \xrightarrow{\mathcal{L}} \begin{cases} (W_1)^2 & \text{if } \operatorname{sign}(\alpha - K) = -\operatorname{sign}(K), \\ 2\int_0^1 (W_s)^2\,\mathrm{d}s & \text{if } \alpha = K, \\ 1 & \text{if } \operatorname{sign}(\alpha - K) = \operatorname{sign}(K), \end{cases} \tag{3.4}$$

as $t \uparrow T$, where $(W_s)_{s\in[0,1]}$ is a standard Wiener process. In fact, in case of $\alpha = K$, for all $t \in (0, T)$,

$$\frac{1}{I_K(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(K)})^2 \, \mathrm{d}u \stackrel{\mathcal{L}}{=} 2\int_0^1 (W_s)^2 \, \mathrm{d}s, \qquad t \in (0, T). \tag{3.5}$$

Proof. We will show that for all $\mu > 0$,

$$\lim_{t\uparrow T} \mathsf{E} \exp\left\{-\frac{\mu}{I_\alpha(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2 \, \mathrm{d}u\right\} = \begin{cases} \frac{1}{\sqrt{1+2\mu}} & \text{if } \operatorname{sign}(\alpha - K) = -\operatorname{sign}(K), \\ \frac{1}{\sqrt{\cosh(2\sqrt{\mu})}} & \text{if } \alpha = K, \\ \mathrm{e}^{-\mu} & \text{if } \operatorname{sign}(\alpha - K) = \operatorname{sign}(K). \end{cases} \tag{3.6}$$

In fact, in case of $\alpha = K$, we prove that for all $t \in (0,T)$ and $\mu \geqslant 0$,

$$\mathsf{E}\exp\left\{-\frac{\mu}{I_K(t)}\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(K)})^2\,\mathrm{d}u\right\} = \frac{1}{\sqrt{\cosh(2\sqrt{\mu})}}. \tag{3.7}$$

First we suppose that $K < 0$. Then we have $b(t) > 0$, $t \in [0,T)$, and the function b satisfies the DE (1.3). By (3.3),

$$B_{K,C}(t) = \left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{1}{K}}, \qquad t \in [0,T), \quad K \neq 0, \tag{3.8}$$

and hence, by Theorem 2.2, for all $\alpha \in \mathbb{R}$, $\mu > 0$ and $t \in (0,T)$, we get

$$\mathsf{E}\exp\left\{-\frac{\mu}{I_\alpha(t)}\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2\,\mathrm{d}u\right\} = \frac{1}{\sqrt{C_{\mu,\alpha,K}(t)}} \tag{3.9}$$

where

$$\begin{aligned} C_{\mu,\alpha,K}(t) := &\left(\frac{1}{2} + \frac{\alpha - K}{2\widetilde{A}_{\mu,\alpha,K}(t)}\right)\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K-\widetilde{A}_{\mu,\alpha,K}(t)}{2K}} \\ &+ \left(\frac{1}{2} - \frac{\alpha - K}{2\widetilde{A}_{\mu,\alpha,K}(t)}\right)\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K+\widetilde{A}_{\mu,\alpha,K}(t)}{2K}}, \end{aligned}$$

and

$$\widetilde{A}_{\mu,\alpha,K}(t) := \sqrt{\frac{2\mu}{I_\alpha(t)} + (\alpha - K)^2}.$$

Now we consider the case $K < 0$ and $\alpha > K$. Using that $\lim_{t\uparrow T} I_\alpha(t) = \infty$ and $\alpha - K > 0$, we have $\lim_{t\uparrow T}\widetilde{A}_{\mu,\alpha,K}(t) = \alpha - K$. Then, using Lemma 3.4 and that $\lim_{x\downarrow 0} x^x = 1$, an easy calculation shows that

$$\begin{aligned} &\lim_{t\uparrow T}\left(\frac{1}{2} + \frac{\alpha - K}{2\widetilde{A}_{\mu,\alpha,K}(t)}\right)\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K-\widetilde{A}_{\mu,\alpha,K}(t)}{2K}} \\ &\qquad = \lim_{t\uparrow T}\left(\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{-1+\sqrt{8\mu\left(\frac{\int_0^T \sigma(s)^2\,\mathrm{d}s}{\int_t^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{K}}+1}}\right)^{\frac{-\alpha+K}{2K}} \\ &\qquad = \lim_{t\uparrow T}\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{8\mu\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}}{1+\sqrt{8\mu\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}+1}}\cdot\frac{-\alpha+K}{2K}} = 1. \end{aligned}$$

Moreover,

$$\lim_{t\uparrow T}\left(\frac{1}{2}-\frac{\alpha-K}{2\widetilde{A}_{\mu,\alpha,K}(t)}\right)\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K+\widetilde{A}_{\mu,\alpha,K}(t)}{2K}}$$

$$=\lim_{t\uparrow T}\left(\frac{1}{2}-\frac{1}{2\sqrt{8\mu\left(\frac{\int_0^T\sigma(s)^2\,\mathrm{d}s}{\int_t^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{K}}+1}}\right)\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{2K}\left(1+\sqrt{8\mu\left(\frac{\int_0^T\sigma(s)^2\,\mathrm{d}s}{\int_t^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{K}}+1}\right)}$$

$$=\lim_{t\uparrow T}\frac{4\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{2K}\left(-1+\sqrt{8\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}+1}\right)}}{\sqrt{8\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}+1}\left(1+\sqrt{8\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}+1}\right)}=2\mu,$$

since the denominator tends to 2 as $t\uparrow T$, and

$$\lim_{t\uparrow T}\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{2K}\left(-1+\sqrt{8\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}+1}\right)}$$

$$=\lim_{t\uparrow T}\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{\alpha-K}{2K}\frac{8\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}}{1+\sqrt{8\mu\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-\alpha+K}{K}}+1}}}=1.$$

Hence, by (3.9), we have (3.6) in case of $K<0$ and $\alpha>K$. By (2.12), for all $\mu>0$, we have

$$\mathsf{E}(\mathrm{e}^{-\mu(W_1)^2})=\frac{1}{\sqrt{1+2\mu}},$$

and the unicity of Laplace transform implies (3.4) in case of $K<0$ and $\alpha>K$.

Now we consider the case $K<0$ and $\alpha=K$. For all $t\in(0,T)$ and $\mu>0$, by (3.9), we get

$$\mathsf{E}\exp\left\{-\frac{\mu}{I_K(t)}\int_0^t\frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s\right\}=\frac{1}{\sqrt{\frac{1}{2}\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\sqrt{\frac{\mu}{2K^2I_K(t)}}}+\frac{1}{2}\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{-\sqrt{\frac{\mu}{2K^2I_K(t)}}}}}$$

$$=\frac{1}{\sqrt{\frac{1}{2}\mathrm{e}^{-2\sqrt{\mu}}+\frac{1}{2}\mathrm{e}^{2\sqrt{\mu}}}}=\frac{1}{\sqrt{\cosh(2\sqrt{\mu})}},$$

where the last but one equality follows from the fact, by Lemma 3.2, in case of $K<0$ and $\alpha=K$ we have

$$\sqrt{I_K(t)}=\frac{1}{2\sqrt{2}K}\ln\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right),\qquad t\in(0,T),\tag{3.10}$$

and from the fact that $x^{\frac{1}{\ln x}} = \mathrm{e}$ for all $x > 0$. By formula (1.9.3) in Borodin and Salminen [7, Part II, Section 1], we get

$$\mathsf{E} \exp\left\{ -2\mu \int_0^1 (W_u)^2 \,\mathrm{d}u \right\} = \frac{1}{\sqrt{\cosh(2\sqrt{\mu})}}, \qquad \mu > 0,$$

and the unicity of Laplace transform implies (3.7) and (3.5) in case of $K < 0$ and $\alpha = K$.

Now we consider the case $K < 0$ and $\alpha < K$. Using that $\lim_{t\uparrow T} I_\alpha(t) = \infty$, we have $\lim_{t\uparrow T} \widetilde{A}_{\mu,\alpha,K}(t) = -(\alpha - K)$, since $\alpha - K < 0$. Then

$$\lim_{t\uparrow T} \left(\frac{1}{2} + \frac{\alpha - K}{2\widetilde{A}_{\mu,\alpha,K}(t)} \right) \left(\frac{\int_t^T \sigma(s)^2 \,\mathrm{d}s}{\int_0^T \sigma(s)^2 \,\mathrm{d}s} \right)^{\frac{\alpha - K - \widetilde{A}_{\mu,\alpha,K}(t)}{2K}} = 0.$$

Moreover, by Lemma 3.4, we get

$$\begin{aligned}
&\lim_{t\uparrow T} \left(\frac{1}{2} - \frac{\alpha - K}{2\widetilde{A}_{\mu,\alpha,K}(t)} \right) \left(\frac{\int_t^T \sigma(s)^2 \,\mathrm{d}s}{\int_0^T \sigma(s)^2 \,\mathrm{d}s} \right)^{\frac{\alpha - K + \widetilde{A}_{\mu,\alpha,K}(t)}{2K}} \\
&= \lim_{t\uparrow T} \left(\frac{\int_t^T \sigma(s)^2 \,\mathrm{d}s}{\int_0^T \sigma(s)^2 \,\mathrm{d}s} \right)^{\frac{\alpha - K + \sqrt{\frac{8\mu}{\frac{1}{K(K-\alpha)} \ln\left(\int_t^T \sigma(s)^2 \,\mathrm{d}s\right)} + (\alpha-K)^2}}{2K}} \\
&= \lim_{t\uparrow T} \left(\frac{\int_t^T \sigma(s)^2 \,\mathrm{d}s}{\int_0^T \sigma(s)^2 \,\mathrm{d}s} \right)^{-\frac{K-\alpha}{2K}\left(1 - \sqrt{\frac{8\mu K}{(K-\alpha)\ln\left(\int_t^T \sigma(s)^2 \,\mathrm{d}s\right)} + 1}\right)} \\
&= \lim_{t\uparrow T} \left(\frac{\int_t^T \sigma(s)^2 \,\mathrm{d}s}{\int_0^T \sigma(s)^2 \,\mathrm{d}s} \right)^{\frac{K-\alpha}{2K} \frac{\frac{8\mu K}{(K-\alpha)\ln\left(\int_t^T \sigma(s)^2 \,\mathrm{d}s\right)}}{1 + \sqrt{\frac{8\mu K}{(K-\alpha)\ln\left(\int_t^T \sigma(s)^2 \,\mathrm{d}s\right)} + 1}}} = \mathrm{e}^{2\mu},
\end{aligned}$$

since

$$\lim_{t\uparrow T} \left(\frac{\int_t^T \sigma(s)^2 \,\mathrm{d}s}{\int_0^T \sigma(s)^2 \,\mathrm{d}s} \right)^{\frac{1}{\ln\left(\int_t^T \sigma(s)^2 \,\mathrm{d}s\right)}} = \mathrm{e}.$$

Hence, by (3.9) and the unicity of Laplace transform, we have (3.6) and (3.4) in case of $K < 0$ and $\alpha < K$.

The case $K > 0$ can be handled in the same way as at the end of the proof of Lemma 3.2. □

3.6 Theorem. *Let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE* (1.2)*, where b is given by* (1.6) *with some $K \neq 0$ and we suppose that $\int_0^T \sigma(s)^2 \,\mathrm{d}s < \infty$. Then*

$$\sqrt{I_\alpha(t)}\,(\widehat{\alpha}_t - \alpha) \xrightarrow{\mathcal{L}} \begin{cases} \mathcal{N}(0,1) & \text{if } \operatorname{sign}(\alpha - K) = \operatorname{sign}(K), \\ -\frac{\operatorname{sign}(K)}{\sqrt{2}} \frac{\int_0^1 W_s \,\mathrm{d}W_s}{\int_0^1 (W_s)^2 \,\mathrm{d}s} & \text{if } \alpha = K, \end{cases}$$

as $t \uparrow T$, where $(W_s)_{s\in[0,1]}$ is a standard Wiener process. In fact, in case of $\alpha = K$, for all $t \in (0,T)$,

$$\sqrt{I_K(t)}\,(\widehat{\alpha}_t - K) \stackrel{\mathcal{L}}{=} -\frac{\operatorname{sign}(K)}{2\sqrt{2}} \frac{(W_1)^2 - 1}{\int_0^1 (W_s)^2 \,\mathrm{d}s} = -\frac{\operatorname{sign}(K)}{\sqrt{2}} \frac{\int_0^1 W_s \,\mathrm{d}W_s}{\int_0^1 (W_s)^2 \,\mathrm{d}s}. \tag{3.11}$$

Proof. First we suppose that $K<0$. Then we have $b(t)>0$, $t\in[0,T)$, and the function b satisfies the DE (1.3). By the SDE (1.2) and (2.8), we have for all $\alpha\in\mathbb{R}$ and $t\in[0,T)$,

$$\begin{aligned}\int_0^t \frac{b(s)}{\sigma(s)} X_s^{(\alpha)}\,\mathrm{d}B_s &= \int_0^t \frac{b(s)}{\sigma(s)^2} X_s^{(\alpha)}\,\mathrm{d}X_s^{(\alpha)} - \alpha \int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s \\ &= \frac{b(t)}{2\sigma(t)^2}(X_t^{(\alpha)})^2 - \frac{1}{2}\int_0^t b(s)\,\mathrm{d}s - (\alpha-K)\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s.\end{aligned} \tag{3.12}$$

Now let us suppose that $K<0$ and $\alpha<K$. By Lemma 3.4, $\lim_{t\uparrow T} I_\alpha(t)=\infty$ holds, and Lemma 3.5 implies that

$$\frac{1}{I_\alpha(t)}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(\alpha)})^2\,\mathrm{d}s \xrightarrow{\mathsf{P}} 1 \qquad \text{as } t\uparrow T,$$

where $\xrightarrow{\mathsf{P}}$ denotes convergence in probability. Indeed, if $K<0$ and $\alpha<K$, then the limit in (3.4) is 1, which is a constant, and hence convergence in distribution implies convergence in probability. Hence we can apply Theorem 4 in Barczy and Pap [4] with $Q(t):=\frac{1}{\sqrt{I_\alpha(t)}}$, $t\in(0,T)$, and $\eta:=1$, and then we have the assertion in case of $K<0$ and $\alpha<K$.

Now let us suppose that $K<0$ and $\alpha=K$. By (3.1) and (3.12), we get

$$\widehat{\alpha}_t - K = \frac{\frac{b(t)}{2\sigma(t)^2}\left(X_t^{(K)}\right)^2 - \frac{1}{2}\int_0^t b(s)\,\mathrm{d}s}{\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s}, \qquad t\in(0,T).$$

Then for all $t\in(0,T)$,

$$\sqrt{I_K(t)}\,(\widehat{\alpha}_t - K) = \frac{1}{2\sqrt{2}}\,\frac{\frac{1}{\sqrt{2I_K(t)}}\frac{b(t)}{\sigma(t)^2}\left(X_t^{(K)}\right)^2 - \frac{1}{\sqrt{2I_K(t)}}\int_0^t b(s)\,\mathrm{d}s}{\frac{1}{2I_K(t)}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s}.$$

To prove (3.11), it is enough to check that

$$\begin{aligned}&\left(\frac{1}{\sqrt{2I_K(t)}}\frac{b(t)}{\sigma(t)^2}\left(X_t^{(K)}\right)^2, \frac{1}{2I_K(t)}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s\right) \\ &\stackrel{\mathcal{L}}{=} \left((W_1)^2, \int_0^1 (W_s)^2\,\mathrm{d}s\right) \stackrel{\mathcal{L}}{=} \left(1+2\int_0^1 W_s\,\mathrm{d}W_s, \int_0^1 (W_s)^2\,\mathrm{d}s\right), \qquad t\in(0,T),\end{aligned} \tag{3.13}$$

and

$$\int_0^t b(s)\,\mathrm{d}s = \sqrt{2I_K(t)}, \qquad t\in(0,T). \tag{3.14}$$

Using that for all $\mu>0$ and $\nu\geqslant 0$,

$$\mathsf{E}\exp\left\{-\mu\int_0^1 (W_s)^2\,\mathrm{d}s - \nu[W_1]^2\right\} = \frac{1}{\sqrt{\cosh(\sqrt{2\mu}) + \frac{2\nu}{\sqrt{2\mu}}\sinh(\sqrt{2\mu})}},$$

(see, e.g., formula (1.9.3) in Borodin and Salminen [7, Part II, Section 1], or as a special case of our Theorem 2.2), to prove the first equality in distribution of (3.13), it is enough to verify that for all $\mu > 0$ and $\nu \geqslant 0$,

$$\mathsf{E}\exp\left\{-\frac{\mu}{2I_K(t)}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s - \frac{\nu}{\sqrt{2I_K(t)}}\frac{b(t)}{\sigma(t)^2}(X_t^{(K)})^2\right\}$$
$$= \frac{1}{\sqrt{\cosh(\sqrt{2\mu}) + \frac{2\nu}{\sqrt{2\mu}}\sinh(\sqrt{2\mu})}}, \qquad t \in (0,T).$$

By Theorem 2.2, we get for all $t \in (0,T)$,

$$\mathsf{E}\exp\left\{-\frac{\mu}{2I_K(t)}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s - \frac{\nu}{\sqrt{2I_K(t)}}\frac{b(t)}{\sigma(t)^2}(X_t^{(K)})^2\right\}$$
$$= \frac{1}{\sqrt{\left(\frac{1}{2}-\frac{\nu}{\sqrt{2\mu}}\right)\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{\sqrt{\frac{\mu}{4K^2I_K(t)}}} + \left(\frac{1}{2}+\frac{\nu}{\sqrt{2\mu}}\right)\left(\frac{\int_t^T\sigma(s)^2\,\mathrm{d}s}{\int_0^T\sigma(s)^2\,\mathrm{d}s}\right)^{-\sqrt{\frac{\mu}{4K^2I_K(t)}}}}}$$
$$= \frac{1}{\sqrt{\left(\frac{1}{2}-\frac{\nu}{\sqrt{2\mu}}\right)\mathrm{e}^{-\sqrt{2\mu}} + \left(\frac{1}{2}+\frac{\nu}{\sqrt{2\mu}}\right)\mathrm{e}^{\sqrt{2\mu}}}} = \frac{1}{\sqrt{\cosh(\sqrt{2\mu}) + \frac{2\nu}{\sqrt{2\mu}}\sinh(\sqrt{2\mu})}},$$

where the last but one equality follows from (3.10) and from the fact that $x^{\frac{1}{\ln x}} = \mathrm{e}$ for all $x > 0$. Hence, by the uniqueness of Laplace transform, for all $t \in (0,T)$, the joint distribution of

$$\frac{1}{2I_K(t)}\int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2\,\mathrm{d}s \qquad \text{and} \qquad \frac{1}{\sqrt{2I_K(t)}}\frac{b(t)}{\sigma(t)^2}(X_t^{(K)})^2$$

is the same as the joint distribution of $\int_0^1 (W_s)^2\,\mathrm{d}s$ and $(W_1)^2$. Finally, by Itô's formula,

$$\int_0^1 W_s\,\mathrm{d}W_s = \frac{1}{2}((W_1)^2 - 1),$$

and hence for all $t \in (0,T)$, we have (3.13). We note that $\frac{\int_0^1 W_s\,\mathrm{d}W_s}{\int_0^1 (W_s)^2\,\mathrm{d}s}$ is the limit distribution of the Dickey–Fuller statistic, see, e.g., the Ph.D. thesis of Bobkoski [6], or (7.14) and Theorem 9.5.1 in Tanaka [26].

Now we check (3.14). Since $K < 0$ and $\alpha = K$, using (3.10), we get for all $t \in (0,T)$,

$$\int_0^t b(s)\,\mathrm{d}s = \int_0^t \frac{\sigma(s)^2}{-2K\int_s^T\sigma(u)^2\,\mathrm{d}u}\,\mathrm{d}s = \frac{1}{2K}\ln\left(\frac{\int_t^T\sigma(u)^2\,\mathrm{d}u}{\int_0^T\sigma(u)^2\,\mathrm{d}u}\right) = \sqrt{2I_K(t)}.$$

Let us suppose now that $K > 0$. Then $b(t) < 0$ for all $t \in [0,T)$. The statement in this case can be obtained from the case $b(t) > 0$ for all $t \in [0,T)$, using the arguments at the end of the proof of Lemma 2.5. The point is that we need to consider the replacement of (α, b, K) with $(-\alpha, -b, -K)$ and, with the notations introduced in the proof of Lemma 2.5, to take into account that $\widehat{(-\alpha)}_t^{(N^{(-\alpha)})} = -\widehat{\alpha}_t^{(X^{(\alpha)})}$, $t \in (0,T)$. □

3.7 Remark. We note that Theorem 3.6 can be derived from our more general results, namely, from Barczy and Pap [4, Theorems 5 and 10]. We also remark that using these results one can also weaken the conditions on b and σ in Theorem 3.6.

3.8 Remark. In case of $\operatorname{sign}(\alpha - K) = -\operatorname{sign}(K)$, under the conditions of Theorem 3.6, one can prove that

$$\sqrt{I_\alpha(t)}\,(\widehat{\alpha}_t - \alpha) \stackrel{\mathcal{L}}{\longrightarrow} \zeta \qquad \text{as } t \uparrow T,$$

where ζ is a standard Cauchy distributed random variable, see, e.g., Luschgy [23, Section 4.2] or Barczy and Pap [4]. The proof in this case is based on a martingale limit theorem, and we do not know whether one can find a proof using the explicit form of the joint Laplace transform of (1.4). Lemma 3.5 implies only

$$\frac{1}{I_\alpha(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2} (X_u^{(\alpha)})^2 \,\mathrm{d}u \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{N}(0,1)^2 \qquad \text{as} \quad t \uparrow T. \tag{3.15}$$

However, using a martingale limit theorem, one can prove that the convergence in (3.15) holds almost surely (with some appropriate random variable ξ^2 as the limit). To be able to use Theorem 4 in Barczy and Pap [4], we need convergence in probability in (3.15). Hence the question is whether we can improve the convergence in distribution in (3.15) to convergence in probability using only the explicit form of the joint Laplace transform of (1.4). We do not know if one can find such a technique.

The next theorem is about the (asymptotic) behavior of the MLE of $\alpha = K$, $K \neq 0$ using an appropriate *random* normalizing factor.

3.9 Theorem. *Let $(X_t^{(K)})_{t\in[0,T)}$ be the process given by the SDE* (1.2)*, where b is given by* (1.6) *with some $K \neq 0$ and we suppose that $\int_0^T \sigma(s)^2 \,\mathrm{d}s < \infty$. Then for all $t \in (0,T)$,*

$$\left(\int_0^t \frac{b(u)^2}{\sigma(u)^2} (X_u^{(K)})^2 \,\mathrm{d}u\right)^{\frac{1}{2}} (\widehat{\alpha}_t - K) \stackrel{\mathcal{L}}{=} -\operatorname{sign}(K) \frac{\int_0^1 W_u \,\mathrm{d}W_u}{\left(\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}}} = -\frac{\operatorname{sign}(K)}{2} \frac{(W_1)^2 - 1}{\left(\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}}}.$$

Proof. First we suppose that $K < 0$. By (3.11) and (3.13), we have for all $\alpha \in \mathbb{R}$ and for all $t \in (0,T)$,

$$\begin{aligned}
\left(\int_0^t \frac{b(u)^2}{\sigma(u)^2} (X_u^{(K)})^2 \,\mathrm{d}u\right)^{\frac{1}{2}} (\widehat{\alpha}_t - K) &= \sqrt{I_K(t)}(\widehat{\alpha}_t - K) \left(\frac{1}{I_K(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2} (X_u^{(K)})^2 \,\mathrm{d}u\right)^{\frac{1}{2}} \\
&\stackrel{\mathcal{L}}{=} \frac{1}{\sqrt{2}} \frac{\int_0^1 W_u \,\mathrm{d}W_u}{\int_0^1 (W_u)^2 \,\mathrm{d}u} \left(2 \int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}} = \frac{\int_0^1 W_u \,\mathrm{d}W_u}{\left(\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}}}, \qquad t \in (0,T),
\end{aligned}$$

which implies the assertion using Itô's formula.

The case $K > 0$ can be handled in the same way as at the end of the proof of Theorem 3.6. □

3.10 Remark. We note that, by Barczy and Pap [4, Corollaries 9 and 11], under the conditions $\int_0^T \sigma(s)^2 \,\mathrm{d}s < \infty$ and (1.6), we have for all $\alpha \neq K$, $K \neq 0$, the MLE of α is asymptotically normal with a corresponding *random* normalizing factor, namely, for all $\alpha \neq K$, $K \neq 0$,

$$\left(\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2 \,\mathrm{d}u\right)^{\frac{1}{2}} (\widehat{\alpha}_t - \alpha) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \qquad \text{as} \quad t \uparrow T.$$

As a consequence of Theorem 3.9, giving an illuminating counterexample, we show that Remark 1.47 in Prakasa Rao [25] contains a mistake.

3.11 Remark. By giving a counterexample, we show that condition (1.5.26) in Remark 1.47 in Prakasa Rao [25] is not enough to assure (1.5.35) in Prakasa Rao [25]. By (3.1), we have for all $\alpha \in \mathbb{R}$ and $t \in (0,T)$,

$$\left(\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2 \,\mathrm{d}u\right)^{\frac{1}{2}} (\widehat{\alpha}_t - \alpha) = \frac{\frac{1}{\sqrt{I_\alpha(t)}} \int_0^t \frac{b(u)}{\sigma(u)} X_u^{(\alpha)} \,\mathrm{d}B_u}{\left(\frac{1}{I_\alpha(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2 \,\mathrm{d}u\right)^{1/2}}. \tag{3.16}$$

By Lemma 3.5 (under its conditions), we have

$$\frac{1}{I_K(t)} \int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(K)})^2 \,\mathrm{d}u \stackrel{\mathcal{L}}{=} 2\int_0^1 (W_u)^2 \,\mathrm{d}u, \qquad t \in (0,T).$$

Hence if Remark 1.47 in Prakasa Rao [25] were true, then we would have

$$\left(\frac{1}{\sqrt{I_K(t)}} \int_0^t \frac{b(s)}{\sigma(s)} X_s^{(K)} \,\mathrm{d}B_s, \frac{1}{I_K(t)} \int_0^t \frac{b(s)^2}{\sigma(s)^2}(X_s^{(K)})^2 \,\mathrm{d}s\right)$$
$$\xrightarrow{\mathcal{L}} \left(\left(2\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}} \xi,\; 2\int_0^1 (W_u)^2 \,\mathrm{d}u\right) \quad \text{as} \;\; t \uparrow T,$$

where ξ is a standard normally distributed random variable independent of $\int_0^1 (W_u)^2 \,\mathrm{d}u$. By (3.16) and continuous mapping theorem, we would have

$$\left(\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(K)})^2 \,\mathrm{d}u\right)^{\frac{1}{2}} (\widehat{\alpha}_t - K) \xrightarrow{\mathcal{L}} \frac{\left(2\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}} \xi}{\left(2\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}}} = \xi \qquad \text{as} \;\; t \uparrow T,$$

which is a contradiction, since, by Theorem 3.9, the limit distribution is

$$-\frac{\operatorname{sign}(K)}{2} \frac{(W_1)^2 - 1}{\left(\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}}}.$$

Note that this limit distribution can not be a standard normal distribution, see, e.g., Feigin [13, Section 2]. Indeed, in case of $K < 0$,

$$\mathsf{P}\left(-\frac{\operatorname{sign}(K)}{2} \frac{(W_1)^2 - 1}{\left(\int_0^1 (W_u)^2 \,\mathrm{d}u\right)^{\frac{1}{2}}} > 0\right) = \mathsf{P}((W_1)^2 > 1) = 2(1 - \Phi(1)),$$

which is not equal to $\mathsf{P}(\mathcal{N}(0,1) > 0) = \frac{1}{2}$. In case of $K > 0$, we can arrive at a contradiction similarly.

The next theorem is about the strong consistency of the MLE of α.

3.12 Theorem. *Let* $(X_t^{(\alpha)})_{t\in[0,T)}$ *be the process given by the SDE* (1.2)*, where* b *is given by* (1.6) *with some* $K \neq 0$ *and we suppose that* $\int_0^T \sigma(s)^2\,\mathrm{d}s < \infty$. *Then the maximum likelihood estimator of* α *is strongly consistent, i.e., for all* $\alpha \in \mathbb{R}$,

$$\mathsf{P}\Big(\lim_{t\uparrow T} \widehat{\alpha}_t = \alpha\Big) = 1.$$

Proof. First we suppose that $K < 0$. Then we have $b(t) > 0,\ t \in [0,T)$, and the function b satisfies the DE (1.3). We check that for all $\alpha \in \mathbb{R}$,

$$\mathsf{E}\exp\left\{-\lim_{t\uparrow T}\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2\,\mathrm{d}u\right\} = \lim_{t\uparrow T}\mathsf{E}\exp\left\{-\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2\,\mathrm{d}u\right\} = 0.$$

The first equality follows from monotone convergence theorem, and the second one can be derived as follows. Using (3.8) and Theorem 2.2 with $\mu := 1$ and $\nu := 0$, we get for all $t \in (0,T)$,

$$\mathsf{E}\exp\left\{-\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2\,\mathrm{d}u\right\} = \frac{1}{\sqrt{C_{\alpha,K}(t)}}$$

where

$$C_{\alpha,K}(t) := \left(\frac{1}{2} + \frac{\alpha-K}{2\sqrt{2+(\alpha-K)^2}}\right)\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-K+\alpha-\sqrt{2+(\alpha-K)^2}}{2K}}$$
$$+ \left(\frac{1}{2} - \frac{\alpha-K}{2\sqrt{2+(\alpha-K)^2}}\right)\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-K+\alpha+\sqrt{2+(\alpha-K)^2}}{2K}}.$$

In case of $\alpha - K \geqslant 0$, we have $\sqrt{2+(\alpha-K)^2} > \alpha - K$ and hence

$$\lim_{t\uparrow T}\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-K+\alpha-\sqrt{2+(\alpha-K)^2}}{2K}} = 0, \tag{3.17}$$

$$\lim_{t\uparrow T}\left(\frac{\int_t^T \sigma(s)^2\,\mathrm{d}s}{\int_0^T \sigma(s)^2\,\mathrm{d}s}\right)^{\frac{-K+\alpha+\sqrt{2+(\alpha-K)^2}}{2K}} = \infty. \tag{3.18}$$

In case of $\alpha - K < 0$, we have $\sqrt{2+(\alpha-K)^2} > -(\alpha-K)$ and hence (3.17) and (3.18) are satisfied again. Since

$$\frac{1}{2} - \frac{\alpha-K}{2\sqrt{2+(\alpha-K)^2}} = \frac{\sqrt{2+(\alpha-K)^2}-\alpha+K}{2\sqrt{2+(\alpha-K)^2}} > 0, \qquad \alpha \in \mathbb{R},$$

we get $\lim_{t\uparrow T} C_{\alpha,K}(t) = \infty$, and hence

$$\mathsf{P}\left(\lim_{t\uparrow T}\int_0^t \frac{b(u)^2}{\sigma(u)^2}(X_u^{(\alpha)})^2\,\mathrm{d}u = \infty\right) = 1, \qquad \alpha \in \mathbb{R}.$$

Then by a strong law of large numbers for continuous local martingales, see, e.g., Barczy and Pap [4, Theorem 15], we get the MLE of α is strongly consistent for all $\alpha \in \mathbb{R}$.

The case $K > 0$ can be handled in the same way as at the end of the proof of Theorem 3.6. □

Finally, we note that in this section we studied the MLE $\widehat{\alpha}_t$ of α based on a *continuous* observation $(X_s^{(\alpha)})_{s\in[0,t]}$ using the results on Laplace transforms presented in Section 2. However, a continuous observation of a diffusion process is only a mathematical idealization, in practice the observation is always discrete. Hence one can pose the question whether our results on the MLE of α based on continuous observations give some information also for discrete observations. Parameter estimation for discretely observed diffusion processes has been studied by many authors, for a detailed discussion and references see, e.g., Bishwal [5]. For discrete observations, one possible approach is to try to find a good approximation of the MLE of α based on continuous observations (for example, Itô type approximation for the stochastic integral in the numerator of (3.1) and usual rectangular approximation for the ordinary integral in the denumerator of (3.1)). In this paper we do not consider this question.

4 α-Wiener bridge

For $T \in (0,\infty)$ and $\alpha \in \mathbb{R}$, let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE (1.7). To our knowledge, these kinds of processes in the case of $\alpha > 0$ have been first considered by Brennan and Schwartz [8], and see also Mansuy [24]. In Brennan and Schwartz [8] the SDE (1.7) is used to model the arbitrage profit associated with a given futures contract in the absence of transaction costs. By (2.1), the unique strong solution of the SDE (1.7) is

$$X_t^{(\alpha)} = \int_0^t \left(\frac{T-t}{T-s}\right)^{\alpha} \mathrm{d}B_s, \qquad t \in [0,T).$$

Theorem 2.2 has the following consequence on the joint Laplace transform of $\int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2}\,\mathrm{d}u$ and $(X_t^{(\alpha)})^2$.

4.1 Theorem. *Let $(X_t^{(\alpha)})_{t\in[0,T)}$ be the process given by the SDE (1.7). For all $\mu > 0$, $\nu \geqslant 0$ and $t \in [0,T)$, we have*

$$\mathsf{E}\exp\left\{-\mu\int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2}\,\mathrm{d}u - \nu[X_t^{(\alpha)}]^2\right\}$$
$$= \frac{\left(1-\frac{t}{T}\right)^{(1-2\alpha)/4}}{\sqrt{\cosh\left(\frac{\sqrt{8\mu+(2\alpha-1)^2}}{2}\ln\left(1-\frac{t}{T}\right)\right) + \frac{1-2\alpha-4\nu(T-t)}{\sqrt{8\mu+(2\alpha-1)^2}}\sinh\left(\frac{\sqrt{8\mu+(2\alpha-1)^2}}{2}\ln\left(1-\frac{t}{T}\right)\right)}}.$$

Proof. Let $b(t) := -\frac{1}{T-t}$, $t \in [0,T)$, and $\sigma(t) := 1$, $t \in [0,T)$. Hence condition (2.4) is satisfied with $K := \frac{1}{2}$ and $C := -\frac{T}{2}$, and clearly,

$$B_{K,C}(t) = \left(1-\frac{t}{T}\right)^2, \qquad t \in [0,T).$$

By Theorem 2.2, we have the assertion. □

Theorem 3.6 has the following consequence on the asymptotic behavior of the maximum likelihood estimator $\widehat{\alpha}_t$ of α as $t \uparrow T$.

4.2 Theorem. *Let* $(X_t^{(\alpha)})_{t\in[0,T)}$ *be the process given by the SDE* (1.7)*. For each* $\alpha > \frac{1}{2}$*, the maximum likelihood estimator* $\widehat{\alpha}_t$ *of* α *is asymptotically normal, namely, for each* $\alpha > \frac{1}{2}$*,*

$$\sqrt{I_\alpha(t)}(\widehat{\alpha}_t - \alpha) \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{N}(0,1) \qquad \text{as } t \uparrow T.$$

If $\alpha = \frac{1}{2}$*, then the distribution of* $\sqrt{I_{1/2}(t)}\left(\widehat{\alpha}_t - \frac{1}{2}\right)$ *is the same for all* $t \in (0,T)$*, namely,*

$$\sqrt{I_{1/2}(t)}\left(\widehat{\alpha}_t - \frac{1}{2}\right) \stackrel{\mathcal{L}}{=} -\frac{1}{2\sqrt{2}}\frac{(W_1)^2 - 1}{\int_0^1 (W_s)^2\,\mathrm{d}s} = -\frac{1}{\sqrt{2}}\frac{\int_0^1 W_s\,\mathrm{d}W_s}{\int_0^1 (W_s)^2\,\mathrm{d}s},$$

where $(W_s)_{s\in[0,1]}$ *is a standard Wiener process.*

The following remark is about the asymptotic behavior of the MLE of α in case of $\alpha < \frac{1}{2}$. We note that up to our knowledge this case can not be handled using only Laplace transforms.

4.3 Remark. If $\alpha < \frac{1}{2}$, then

$$\sqrt{I_\alpha(t)}\left(\widehat{\alpha}_t - \alpha\right) \stackrel{\mathcal{L}}{\longrightarrow} \zeta \qquad \text{as } t \uparrow T,$$

where ζ is a standard Cauchy distributed random variable, see, e.g., Luschgy [23, Section 4.2] or Barczy and Pap [4].

Theorem 3.9 has the following consequence on the (asymptotic) behavior of the MLE of $\alpha = 1/2$ using a *random* normalization.

4.4 Theorem. *Let* $(X_t^{(\alpha)})_{t\in[0,T)}$ *be the process given by the SDE* (1.7)*. For all* $t \in (0,T)$*, we have*

$$\left(\int_0^t \frac{(X_u^{(1/2)})^2}{(T-u)^2}\,\mathrm{d}u\right)^{1/2}\left(\widehat{\alpha}_t - \frac{1}{2}\right) \stackrel{\mathcal{L}}{=} -\frac{\int_0^1 W_s\,\mathrm{d}W_s}{\left(\int_0^1 (W_s)^2\,\mathrm{d}s\right)^{1/2}} = -\frac{1}{2}\frac{(W_1)^2 - 1}{\left(\int_0^1 (W_s)^2\,\mathrm{d}s\right)^{1/2}}.$$

Finally, we note that Es-Sebaiy and Nourdin [12] studied the parameter estimation for so-called α-fractional bridges which are given by the SDE (1.7) replacing the standard Wiener process B by a fractional Wiener process.

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References

[1] C. Albanese and S. Lawi, Laplace transforms for integrals of Markov processes. *Markov Processes and Related Fields* **11**(4), 677–724 (2005).

[2] M. Arató, *Linear Stochastic Systems With Constant Coefficients.* A Statistical Approach: Lecture Notes in Control and Information Sciences **(45)**. Springer, 1982.

[3] M. Barczy and E. Iglói, Karhunen-Loève expansions of alpha-Wiener bridges. *Central European Journal of Mathematics* **9**(1), 65–84 (2011).

[4] M. Barczy and G. Pap, Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes. *Journal of Statistical Planning and Inference* **140**(6), 1576–1593 (2010).

[5] J. P. N. Bishwal, *Parameter Estimation in Stochastic Differential Equations.* Springer-Verlag, Berlin, Heidelberg, 2007.

[6] M. J. Bobkoski, *Hypothesis testing in nonstationary time series.* Ph.D. Dissertation, University of Wisconsin, 1983.

[7] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion – Facts and Formulae*, 2nd edition. Birkhäuser, 2002.

[8] M. J. Brennan and E. S. Schwartz, Arbitrage in stock index futures. *The Journal of Business* **63**(1), S7-S31 (1990).

[9] P. Deheuvels and G. Martynov, Karhunen–Loève expansions for weighted Wiener processes and Brownian bridges via Bessel functions. In: *Progress in Probability* vol. 55, pp. 57–93, Birkhäuser Verlag, Basel, 2003.

[10] P. Deheuvels, G. Peccati and M. Yor, On quadratic functionals of the Brownian sheet and related processes. *Stochastic Processes and their Applications* **116**(3), 493–538 (2006).

[11] B. Delyon and Y. Hu, Simulation of conditioned diffusion and application to parameter estimation. *Stochastic Processes and their Applications* **116**(11), 1660–1675 (2006).

[12] K. Es-Sebaiy and I. Nourdin, Parameter estimation for α-fractional bridges. *Arxiv* (2011). URL: http://arxiv.org/abs/1101.5790

[13] P. D. Feigin, Some comments concerning a curious singularity. *Journal of Applied Probability* **16**(2), 440–444 (1979).

[14] D. Florens-Landais and H. Pham, Large deviations in estimation of an Ornstein-Uhlenbeck model. *Journal of Applied Probability* **36**(1), 60–70 (1999).

[15] F. Gao, J. Hannig, T-Y. Lee and F. Torcaso, Laplace transforms via Hadamard factorization. *Electronic Journal of Probability* **8**, no. 13, 20 pp. (2003).

[16] T. R. Hurd and A. Kuznetsov, Explicit formulas for Laplace transforms of stochastic integrals. *Markov Processes and Related Fields* **14**(2), 277–290 (2008).

[17] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd edition. Springer-Verlag, Berlin, 2003.

[18] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd edition. Springer-Verlag, Berlin, Heidelberg, 1991.

[19] M. L. Kleptsyna and A. Le Breton, Statistical analysis of the fractional Ornstein–Uhlenbeck type process. *Statistical Inference for Stochastic Processes* **5**(3), 229–248 (2002).

[20] M. L. Kleptsyna and A. Le Breton, A Cameron-Martin type formula for general Gaussian processes – a filtering approach. *Stochastics and Stochastics Reports* **72**(3-4), 229–250 (2002).

[21] R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes I. General Theory*, 2nd edition. Springer-Verlag, Berlin, Heidelberg, 2001.

[22] R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes II. Applications*, 2nd edition. Springer-Verlag, Berlin, Heidelberg, 2001.

[23] H. Luschgy, Local asymptotic mixed normality for semimartingale experiments. *Probability Theory and Related Fields* **92**(2), 151–176 (1992).

[24] R. Mansuy, On a one-parameter generalization of the Brownian bridge and associated quadratic functionals. *Journal of Theoretical Probability* **17**(4), 1021–1029 (2004).

[25] B. L. S. Prakasa Rao, *Semimartingales and their Statistical Inference.* Chapman & Hall/CRC, 1999.

[26] K. Tanaka, *Time Series Analysis, Nonstationary and Noninvertible Distribution Theory.* Wiley Series in Probability and Statistics, 1996.

[27] M. Yor, *Exponential Functionals of Brownian Motion and Related Processes.* Springer-Verlag, Berlin, 2001.