# Affine extensions of loops

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## 1 Introduction

Most of the known examples of loops $L$ with strong relations to geometry have classical groups as the groups generated by their left translations ([7], [10], [9],[6], [8], Chapter 9, [12], Chapters 22 and 25, [4], [5]). These groups $G$ may be seen as subgroups of the stabilizer of 0 in the group of affinities of suitable affine spaces $\mathcal{A}_n$, and as the elements of the loops $L$ one can often take certain projective subspaces of the hyperplane at infinity of $\mathcal{A}_n$. The semidirect products $T \rtimes G$, where $T$ is the translation group of the affine space $\mathcal{A}_n$, have in many cases a geometric interpretation as motion groups of affine metric geometries. In the papers [4], [5] three dimensional connected differentiable loops are constructed which have the connected component of the motion group of the 3-dimensional hyperbolic or pseudo-euclidean geometry as the group topologically generated by the left translations and which are Bol, Bruck or left A-loops. The set of the left translations of these loops induces on the plane at infinity the set of left translations of a loop isotopic to the hyperbolic plane loop (cf. [12], Chapter 22, p. 280, [9], p. 189). This and the fact that, up to our knowledge, there are only few known examples of sharply transitive sections in affine metric motion groups, motivated us to seek a simple geometric procedure for an extension of a loop realized as the image $\Sigma^*$ of a sharply transitive section in a subgroup $G^*$ of the projective linear group $PGL(n-1, \mathbb{K})$ to a loop realized as the image of a sharply transitive section in a group $\Delta = T' \rtimes C$ of affinities of the $n$-dimensional space $\mathcal{A}_n = \mathbb{K}^n$ over a commutative field $\mathbb{K}$. Moreover, we desire that $T'$ is a large subgroup of affine translations and that $\alpha(C) = G^*$ holds for the canonical homomorphism $\alpha : GL(n, \mathbb{K}) \to PGL(n, \mathbb{K})$. We show that this goal can be achieved if in the $(n-1)$-dimensional projective

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hyperplane $E$ of infinity of $\mathcal{A}_n$ for $G^*$ there exists an orbit $\mathcal{O}$ of $m$-dimensional subspaces such that $\Sigma^*$ acts sharply transitively on $\mathcal{O}$, if there is a subspace of dimension $(n-1-m)$ having empty intersection with any element of $\mathcal{O}$ and if the restriction of $\alpha^{-1}$ to $\Sigma^*$ defines a bijection from $\alpha^{-1}(\Sigma^*)$ onto $\Sigma^*$.

In the third section we demonstrate that our construction successfully can be applied to sharply transitive sections in unitary and orthogonal groups $SU_{p_2}(n,F)$ of positive index $p_2$ over ordered pythagorean $n$-real fields $F$. In this way we obtain many non-isotopic topological loops. The groups generated by the left translations of these loops are semidirect products $T \rtimes C$, where $T$ is the full translation group of $\mathcal{A}_n$ and where $\alpha(C)$ is a non-solvable normal subgroup of $\alpha(SU_{p_2}(n,F))$.

In the last section we take for the groups $G$ unitary or orthogonal Lie groups of any positive index in order to obtain differentiable loops $L$ such that the group topologically generated by the left translations of $L$ is a pseudo-unitary motion group or the connected component of a pseudo-euclidean motion group.

# 2 Some basic notations of loop theory

A set $L$ with a binary operation $(x,y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b/a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \to L$ is a bijection of $L$ for any $a \in L$. Two loops $(L_1, \cdot)$ and $(L_2, *)$ are isotopic if there are three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \cdot y)$ holds for any $x, y \in L_1$. A loop $(L, \cdot)$ is called topological if $L$ is a topological space and the mappings $(x,y) \mapsto x \cdot y$, $(x,y) \mapsto x \backslash y$, $(x,y) \mapsto y/x : L^2 \to L$ are continuous. A loop $(L, \cdot)$ is called differentiable if $L$ is a $C^\infty$-differentiable manifold and the mappings $(x,y) \mapsto x \cdot y$, $(x,y) \mapsto x \backslash y$, $(x,y) \mapsto y/x : L^2 \to L$ are differentiable.

A loop $L$ is a Bol loop if the identity $x(y \cdot xz) = (x \cdot yx)z$ holds. A Bruck loop is a Bol loop $(L, \cdot)$ satisfying the automorphic inverse property, i.e. the identity $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ for all $x, y \in L$. A loop $L$ is a left A-loop if each $\lambda_{x,y} = \lambda_{xy}^{-1} \lambda_x \lambda_y : L \to L$ is an automorphism of $L$.

Let $G$ be the group generated by the left translations of $L$ and let $H$ be the stabilizer of $e \in L$ in the group $G$. The left translations of $L$ form a subset of $G$ acting on the cosets $\{xH; x \in G\}$ such that for any given cosets $aH$ and $bH$ there exists precisely one left translation $\lambda_z$ with $\lambda_z aH = bH$.

Conversely let $G$ be a group, let H be a subgroup of $G$ and let $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the subset $\sigma(G/H)$ generates $G$

and acts sharply transitively on the space $G/H$ of the left cosets $\{xH, x \in G\}$ (cf. [12], p. 18). We call such a section sharply transitive. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space $G/H$ or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. If $N$ is the largest normal subgroup of $G$ contained in $H$ then the factor group $G/N$ is isomorphic to the group generated by the left translations of $L(\sigma)$.
Two loops $L_1$ and $L_2$ having the same group $G$ as the group generated by the left translations and the same stabilizer $H$ of $e \in L_1, L_2$ are isomorphic if there is an automorphism of $G$ leaving $H$ invariant and mapping $\sigma_1(G/H)$ onto $\sigma_2(G/H)$. The automorphisms of a loop $L$ corresponding to a sharply transitive section $\sigma : G/H \to G$ are given by the automorphisms of $G$ leaving $H$ and $\sigma(G/H)$ invariant. If two loops are isotopic then the groups generated by their left translations are isomorphic ([13], Theorem III.2.7, p. 65). Loops $L$ and $L'$ having the same group $G$ generated by their left translations are isotopic if and only if there is a loop $L''$ isomorphic to $L'$ having $G$ again as the group generated by its left translations and there exists an inner automorphism $\tau$ of $G$ mapping $\sigma''(G/H)$ belonging to $L''$ onto the set $\sigma(G/H)$ corresponding to $L$ (cf. [12], Theorem 1.11. pp. 21-22).

# 3 Affine extensions

Let $G$ be a subgroup of the general linear group $GL(n, \mathbb{K})$ over a commutative field $\mathbb{K}$. Denote by $\alpha$ the canonical epimorphism from $GL(n, \mathbb{K})$ onto $PGL(n, \mathbb{K})$. The kernel $Z$ of $\alpha$ is the centre of $GL(n, \mathbb{K})$. Let $\tilde{H}$ be a subgroup of $G$ with $Z \cap G \le \tilde{H}$ such that for the pair $G^* = \alpha(G)$ and $H^* = \alpha(\tilde{H})$ there exists a sharply transitive section $\sigma^* : G^*/H^* \to G^*$ determining a loop $L^*$. Moreover, we assume that $\Sigma^* := \sigma^*(G^*/H^*)$ generates $G^*$ and that for the preimage $(\alpha|G)^{-1}(\Sigma^*) = \Sigma \subseteq G$ one has $\tilde{H} \cap \Sigma = \{1\}$. Then the mapping $\alpha$ induces a bijection from $\Sigma$ onto $\Sigma^*$.

We denote by $\mathcal{A}_n$ the $n$-dimensional affine space $\mathbb{K}^n$ and by $E$ the projective hyperplane of dimension $(n-1)$ at infinity of $\mathcal{A}_n$. Let $U^*$ be an $m$-dimensional subspace of $E$ having $H^*$ as the stabilizer of $U^*$ in $G^*$. Let $\mathcal{X}$ be the set

$$\mathcal{X} = \{\gamma U^*; \gamma \in \Sigma^*\}.$$

The elements of $\mathcal{X}$ may be seen as the elements of $L^*$ such that $U^*$ is the identity of $L^*$ and the multiplication is given by $X^* \circ Y^* = \tau^*_{U^*,X^*}(Y^*)$ for all $X^*, Y^* \in \mathcal{X}$, where $\tau^*_{U^*,X^*}$ is the unique element of the sharply transitive set $\Sigma^*$ of the linear transformations of $E$ mapping $U^*$ onto $X^*$.

Let $A = T \rtimes S$ be the semidirect product consisting of affinities of $\mathcal{A}_n = \mathbb{K}^n$, where $T$ is the translation group of $\mathcal{A}_n$ and $S$ is the stabilizer of $0 \in \mathcal{A}_n$

isomorphic to the group $GL(n,\mathbb{K})$. We consider the group $G$ as a subgroup of $S$ in the group $\Theta = \mathbb{K}^n \rtimes G$ of affinities of $\mathcal{A}_n$. The subgroup $\tilde{H}$ of $S$ fixes the point $0 \in \mathcal{A}_n$ and the subspace $U^*$ of the hyperplane $E$. Let $U$ be the $(m+1)$-dimensional affine subspace containing 0 and intersecting $E$ in $U^*$. If $H$ is the stabilizer of $U$ in the group $\Theta$, then one has $\tilde{H} = H \cap \Theta_0$, where $\Theta_0$ is the stabilizer of the point 0 in $\Theta$.

Let $W$ be a subspace of $\mathcal{A}_n$ such that $W$ contains 0, has affine dimension $(n-m-1)$ and intersects any subspace of the set $\mathcal{Z} := \{\rho(U); \rho \in \Sigma\}$ only in 0. Let $T_W$ be the group of affine translations $x \mapsto x + w : \mathcal{A}_n \to \mathcal{A}_n$ with $w \in W$. Then $W$ intersects any subspace $\delta(Y)$, where $\delta \in T_W$ and $Y \in \mathcal{Z}$, in precisely one point. Moreover, the stabilizer of $\delta(Y)$ in $T_W$ consists only of the identity.

**Theorem 1.** *The subset $\Xi = T_W\Sigma = \{\tau\rho; \tau \in T_W, \rho \in \Sigma\}$ of the group $\Theta = T \rtimes G$ acts sharply transitively on the set*

$$\mathcal{U} = \{\psi(U); \psi \in \Xi\} = \{\psi(U); \psi \in \Theta\}.$$

*The elements of $\mathcal{U}$ can be taken as the elements of a loop $L_\Xi$ which has $U$ as the identity and for which the multiplication is defined by*

$$X \circ Y = \tau_{U,X}(Y) \quad \text{for all} \quad X, Y \in \mathcal{U},$$

*where $\tau_{U,X}$ is the unique element of $\Xi$ mapping $U$ onto $X$.*

*The set $\Xi$ is the set of the left translations of $L_\Xi$ and generates a group $\Delta$ which is a semidirect product $\Delta = T' \rtimes C$, where the normal subgroup $T'$ consists of translations of the affine space $\mathcal{A}_n$ and $C$ is a subgroup of $G$ with $\alpha(C) = G^*$.*

*There is a sharply transitive section $\sigma : \Delta/\hat{H} \to \Delta$ such that $\sigma(\Delta/\hat{H}) = \Xi$, the group $\hat{H}$ is the stabilizer of $U$ in $\Delta$ and the subgroup $T' \cap \hat{H}$ consists of all translations $x \mapsto x + u : \mathcal{A}_n \to \mathcal{A}_n$ with $u \in U$.*

*Proof.* Let $D_1$ and $D_2$ be elements belonging to $\mathcal{U}$. We show that there is precisely one element $\beta \in \Xi$ with $\beta(D_1) = D_2$. Let $D_1^* = D_1 \cap E$ and $D_2^* = D_2 \cap E$, where $E$ is the hyperplane at infinity of $\mathcal{A}_n$. Thus there exists precisely one element $\rho^* \in \Sigma^*$ and hence there exists precisely one element $\rho \in \Sigma$ with $\alpha(\rho) = \rho^*$ such that $\rho^*(D_1^*) = D_2^*$. The subspaces $\rho(D_1)$ and $D_2$ intersect $E$ in $D_2^*$. In the group $T_W$ there exists precisely one translation $\tau$ mapping the point $\rho(D_1) \cap W$ onto the point $D_2 \cap W$. Hence the element $\beta = \tau\rho$ is the only element in $\Xi$ mapping $D_1$ onto $D_2$ and the set $\Xi$ is a sharply transitive set on $\mathcal{U}$. It follows that the subspaces in $\mathcal{U}$ can be taken as the elements of a loop $L_\Xi$ having $U$ as the identity, such that the multiplication is defined as in the assertion of the theorem.

The group $\Delta$ generated by the left translations of $L_\Xi$ is a subgroup of $\Theta = T \rtimes G$. Let $\hat{H}$ be the stabilizer of $U$ in $\Delta$. Since $\Xi$ is the image of a sharply transitive section $\sigma : \Delta/\hat{H} \to \Delta$ we have $\Delta(U) = \Xi\hat{H}(U) = \Xi(U)$. Let $T_U$ be the group of affine translations $x \mapsto x + u : \mathcal{A}_n \to \mathcal{A}_n$ with $u \in U$. Since $W \oplus U = \mathbb{K}^n$ we have that $T = T_W \times T_U$. Thus one has $\Delta T(U) = \Delta T_W T_U(U) = \Delta T_W(U) = \Delta(U)$ since $T_W \leq \Delta$. For the group $\Lambda$ of dilatations $x \mapsto ax : \mathcal{A}_n \to \mathcal{A}_n$ with $a \in \mathbb{K}\backslash\{0\}$ we have that $T\Lambda$ is a normal subgroup of $\Theta\Lambda$ and $\Lambda(U) = U$. Moreover $\Theta(U) = \Delta T\Lambda(U)$ since the kernel of the restriction of $\alpha : GL(n, \mathbb{K}) \to PGL(n, \mathbb{K})$ to $G$ consists only of dilatations.

The group $\Delta$ contains a normal subgroup $N$ fixing the hyperplane $E$ at infinity pointwise. Since $\Sigma^*$ generates $G^*$ we see that $\Delta/N$ is isomorphic to $G^*$.

Let $T' = T \cap \Delta$. Then $\Delta$ is the semidirect product of $\Delta = T' \rtimes C$, where $C$ is the stabilizer of 0 in $\Delta$ and $CN/N$ is isomorphic to $G^*$. □

# 4 Applications

Let $R$ be an ordered pythagorean field and let $K = R(i)$ be the algebraic extension of $R$ such that $i^2 = -1$. Let $F \in \{R, K\}$ and let $V = F^n$ be an $n$-dimensional $F$-vector space for a fixed $n \geq 3$. The automorphism $a \mapsto \bar{a} : F \to F$ is the identity if $F = R$ or the involutory automorphism fixing $R$ elementwise and mapping $i$ onto $-i$ if $F = K$. Denote by $\mathcal{M}_n(F)$ the set of the $(n \times n)$-matrices over $F$. If $A = (a_{i,j})$ is a matrix in $\mathcal{M}_n(F)$ then $\bar{A}^t = (\bar{a}_{j,i})$. Let $\mathcal{H}(n, F)$ be the set of positive definite hermitian $(n \times n)$-matrices, i.e. the set

$$\mathcal{H}(n, F) = \{A \in \mathcal{M}_n(F); A = \bar{A}^t \text{ with } \bar{v}^t A v > 0 \text{ for all } v \in V\backslash\{0\}\}.$$

We assume that the field $R$ is $n$-real which means that the characteristic polynomial of every matrix in $\mathcal{H}(n, F)$ splits over $K$ into linear factors. Thus this polynomial splits into linear factors already over $R$ (cf. [8], p. 14). The class of $n$-real fields contains the class of totally real fields (cf. [8], p. 13), which is larger than the class of real closed fields and the class of hereditary euclidean fields. A hereditary euclidean field $k$ is an ordered field such that every formally real algebraic extension of $k$ has odd degree over $k$ (cf. [15], Satz 1.2 (3), p. 197).

The group

$$U(n, F) = \{B \in GL(n, F); B\bar{B}^t = I_n\},$$

where $I_n$ is the identity in $GL(n, F)$, is called the orthogonal group for $F = R$

and the unitary group for $F = K$. Let

$$J_{(p_1,p_2)} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$$

be the diagonal $(n \times n)$-matrix such that the first $p_1$ entries are 1 and the remaining $p_2$ entries are $-1$. We have $p_1 + p_2 = n$. The matrix $J_{(p_1,p_2)}$ defines a hermitian form on $F^n$ for $F = K$ and an orthogonal form for $F = R$ by

$$\bar{v}^t J v = \sum_{i=1}^{p_1} \bar{v}_i v_i - \sum_{j=p_1+1}^{n} \bar{v}_j v_j.$$

Let $p_2 > 0$. The unitary (orthogonal) group of index $p_2$ is the set

$$U_{p_2}(n, F) = \{A \in GL_n(F); \bar{A}^t J_{(p_1,p_2)} A = J_{(p_1,p_2)}\}.$$

Since the group $U_{p_2}(n, F)$ is isomorphic to the group $U_{(n-p_2)}(n, F)$ (cf. [14], Proposition 9.11, p. 153) we may assume that $p_1 \geq p_2$. Let

$$\Omega_{(p_1,p_2)}(F) = U_{p_2}(n, F) \cap U(n, F) \quad \text{and} \quad \Sigma_{(p_1,p_2)}(F) = U_{p_2}(n, F) \cap \mathcal{H}(n, F).$$

The group $\Omega_{(p_1,p_2)}(F)$ is the direct product $\Omega_{(p_1,p_2)}(F) = U(p_1, F) \times U(p_2, F)$, where $U(p_1, F)$ may be identified with the group $\begin{pmatrix} U(p_1, F) & 0 \\ 0 & I_{p_2} \end{pmatrix}$ and $U(p_2, F)$ may be identified with the group $\begin{pmatrix} I_{p_1} & 0 \\ 0 & U(p_2, F) \end{pmatrix}$; here $I_{p_i}$ is the identity in $GL(p_i, F)$ (cf. [8], Theorem 9.13, p. 123).

According to [8] (Theorem 9.11, p. 121) the set $\Sigma_{(p_1,p_2)}(F)$ is the image of a sharply transitive section $\sigma' : U_{p_2}(n, F)/\Omega_{(p_1,p_2)}(F) \to U_{p_2}(n, F)$ such that the corresponding loop $L_{(p_1,p_2)}$ is a Bruck loop.

The group $G_{(p_1,p_2)}$ generated by the set $\Sigma_{(p_1,p_2)}(F)$ of the left translations of $L_{(p_1,p_2)}$ is contained in the group $SU_{p_2}(n, F) := \{A \in U_{p_2}(n, F);\ \det A = 1\}$ (cf. [8], 9.14, p. 124). Thus the loop $L_{(p_1,p_2)}$ corresponds also to the section

$$\sigma : SU_{p_2}(n, F)/\Phi \to SU_{p_2}(n, F),$$

where $\Phi := (U(p_1, F) \times U(p_2, F)) \cap SU_{p_2}(n, F)$.

The kernel of the restriction of $\alpha : GL(n, F) \to PGL(n, F)$ to the group $SU_{p_2}(n, F)$ consists of the matrices $D_a = \mathrm{diag}(a, \ldots, a)$, $a \in F \backslash \{0\}$ and $a^n = 1$. Moreover one has $a\bar{a} = 1$ since any matrix $D_a$ satisfies $\bar{D}_a^t J_{(p_1,p_2)} D_a = J_{(p_1,p_2)}$. Thus any matrix $D_a$ is contained in $\Phi$ and $\alpha$ induces a bijection from $\Sigma_{(p_1,p_2)}(F)$ onto $\alpha(\Sigma_{(p_1,p_2)}(F))$. The set $\alpha(\Sigma_{(p_1,p_2)}(F))$ is the image of a sharply transitive section

$$\sigma^* : \alpha(SU_{p_2}(n, F))/\alpha(\Phi) \to \alpha(SU_{p_2}(n, F))$$

which corresponds to a Bruck loop $L^*_{(p_1,p_2)}$.

The elements of $\Sigma_{(p_1,p_2)}(F)$ are matrices $A \in SU_{p_2}(n,F)$ satisfying the relations $A = \bar{A}^t$ and $\bar{v}^t A v > 0$ for all $v \in V \backslash \{0\}$. With $A$ also $A^{-1}$ is contained in $\Sigma_{(p_1,p_2)}(F)$ ([8] 1.11, p. 16). Because of $B^{-1} = \bar{B}^t$ for all $B \in \Phi$ and $\bar{B}^t A B \in \Sigma_{(p_1,p_2)}(F)$ ([8] 1.11, p. 16) one has

$$B^{-1}AB \in \Sigma_{(p_1,p_2)}(F) \text{ for all } B \in \Phi \text{ and } A \in \Sigma_{(p_1,p_2)}(F). \tag{1}$$

Since $\sigma$ is a section every element $S$ of $SU_{p_2}(n,F)$ can be written in a unique way as $S = S_1 C$ with $S_1 \in \Sigma_{(p_1,p_2)}(F)$ and $C \in \Phi$. The set

$$\Sigma_{(p_1,p_2)}(F)^{G_{(p_1,p_2)}} = \{Y^{-1}XY;\ X \in \Sigma_{(p_1,p_2)}(F), Y \in G_{(p_1,p_2)}\}$$

is invariant with respect to the conjugation by the elements $S \in SU_{p_2}(n,F)$ :

$$S^{-1}Y^{-1}XYS = C^{-1}S_1^{-1}Y^{-1}XYS_1C =$$

$$[(C^{-1}S_1^{-1}C)(C^{-1}Y^{-1}C)](C^{-1}XC)[(C^{-1}YC)(C^{-1}S_1C)] \in \Sigma_{(p_1,p_2)}(F)^{G_{(p_1,p_2)}}.$$

Hence the group $G_{(p_1,p_2)}$, which is generated also by $\Sigma_{(p_1,p_2)}(F)^{G_{(p_1,p_2)}}$, is a normal non central subgroup of $SU_{p_2}(n,F)$. Then according to Théoréme 5 in [2] p. 70 the group $G_{(p_1,p_2)}$ coincides with $SU_{p_2}(n,F)$ if $F = K$. If $F = R$ and $(n,p_2) \neq (4,2)$ then the group $G_{(p_1,p_2)}$ contains the commutator subgroup $[SU_{p_2}(n,F)]' =: \mathcal{K}_{(n,p_2)}$ of $SU_{p_2}(n,F)$ ([3], p. 63 and pp. 58-59). If $F = R$ and $(n,p_2) = (4,2)$ then the commutator subgroup $\mathcal{K}_{(4,2)}$ is isomorphic to the direct product $PSL_2(R) \times PSL_2(R)$ ([3], p. 59). Since the hermitian matrices in the set $\Sigma_{(2,2)}(F)$ depend on 3 free parameters ([8], 9.12, p. 122) the group $G_{(2,2)}$ contains $\mathcal{K}_{(4,2)}$. Therefore in any case the group $G_{(p_1,p_2)}$ is a normal subgroup of $SU_{p_2}(n,F)$ containing $\mathcal{K}_{(n,p_2)}$.

The group $G_{(p_1,p_2)}$ leaves the value $\bar{v}^t J_{(p_1,p_2)} v$ invariant since

$$\bar{v}^t(\bar{A}^t J_{(p_1,p_2)} A)v = \bar{v}^t J_{(p_1,p_2)} v \text{ for } A \in SU_{p_2}(n,F).$$

We see the group $G_{(p_1,p_2)}$ as a subgroup of the stabilizer of the element 0 in the group $A$ of affinities of $\mathcal{A}_n = F^n$, and the group $\alpha(G_{(p_1,p_2)}) := G^*_{(p_1,p_2)}$ as a subgroup of the group $PGL(n,F)$ which acts on the $(n-1)$-dimensional projective hyperplane $E$ at infinity of $\mathcal{A}_n$.

We embed the affine space $\mathcal{A}_n$ into the $n$-dimensional projective space $P_n(F)$ such that $(x_1, \cdots, x_n) \mapsto F^*(1, x_1, \cdots, x_n)$, $x_i \in F$ for all $1 \leq i \leq n$ and $F^* = F \backslash \{0\}$. With respect to this embedding the hyperplane $E$ consists of the points $\{F^*(0, x_1, \cdots, x_n), x_i \in F, \text{not all } x_i = 0\}$. The cone in $\mathcal{A}_n$ which is described by the equation

$$(*) \qquad \sum_{i=1}^{p_1} \bar{x}_i x_i - \sum_{j=p_1+1}^{n} \bar{x}_j x_j = 0$$

intersects $E$ in a hyperquadric $C$; the points $\{F^*(0, x_1, \cdots, x_n)\}$ of $C$ satisfy the equation $(*)$. The hypersurface $C$ of $E$ divides the points of $E \backslash C$ into two regions $R_1$ and $R_2$. A point $F^*(0, x_1, \cdots, x_n)$ belongs to $R_1$ if and only if $\sum_{i=1}^{p_1} \bar{x}_i x_i > \sum_{j=p_1+1}^{n} \bar{x}_j x_j$. It belongs to $R_2$ if and only if $\sum_{i=1}^{p_1} \bar{x}_i x_i < \sum_{j=p_1+1}^{n} \bar{x}_j x_j$. The group $\alpha(SU_{p_2}(n,F)) = SU_{p_2}(n,F)/\Lambda'$, where $\Lambda'$ is the group of dilatations contained in $SU_{p_2}(n,F)$, leaves $R_1$, $R_2$ as well as $C$ invariant since for any $f \in F$ and $v \in V = F^n$ one has $(\bar{f}\bar{v}^t)J_{(p_1,p_2)}(fv) = (\bar{f}f)(\bar{v}^t J_{(p_1,p_2)} v)$ and $\bar{f}f > 0$. The group $\alpha(\Phi) = \Phi/(\Phi \cap \Lambda')$ leaves the subspace

$$W_1^* = \{(0, x_1, \ldots, x_{p_1}, 0, \ldots, 0); x_i \in F\} \subseteq E$$

as well as the subspace

$$W_2^* = \{(0, \ldots, 0, x_{p_1+1}, \ldots, x_n); x_i \in F\} \subseteq E$$

invariant. The intersection $W_1^* \cap W_2^*$ is empty since $W_i^* \subseteq R_i, i = 1, 2$.

Let $W_i, i = 1, 2$, be the $p_i$-dimensional affine subspace of $\mathcal{A}_n$ containing 0 such that $W_i \cap E = W_i^*$. Thus $W_1 \cap W_2 = \{0\}$. Let $\tilde{W}_j$ be a $p_j$-dimensional affine subspace of $\mathcal{A}_n$ such that $p_j = n - p_i$ and $\tilde{W}_j$ intersects $W_i$ only in the point 0. Thus $\tilde{W}_j$ intersects any subspace of the set

$$\mathcal{Z}_i = \{\rho(W_i), \rho \in G_{(p_1,p_2)}\} = \{\lambda(W_i), \lambda \in \Sigma_{(p_1,p_2)}(F)\},$$

where $i \neq j$, only in 0. Affine subspaces $\tilde{W}_j$ with these properties exist, one can take for instance $\tilde{W}_j = \rho(W_j) \in \mathcal{Z}_j$.

Let $\Theta$ be the semidirect product $\Theta = T \rtimes G_{(p_1,p_2)}$, where $T$ is the translation group of $\mathcal{A}_n$. According to Theorem 1 the set $\Xi_{(p_i,\tilde{W}_j)} = \{T_{\tilde{W}_j} \Sigma_{(p_1,p_2)}(F)\}$, $i \neq j$, acts sharply transitively on the set

$$\mathcal{U}_i = \{\psi(W_i); \psi \in \Xi_{(p_i,\tilde{W}_j)}\}.$$

Thus a loop $L_{(p_i,\tilde{W}_j)}$ is realized on $\mathcal{U}_i$.

The group $SU_{p_2}(n,K)$ acts irreducibly on the vector space $V = K^n$ and the commutator subgroup $\mathcal{K}_{(n,p_2)}$ of $SO_{p_2}(n,R)$ acts irreducibly on $V = R^n$ (cf. [1], Theorem 3.24, p. 136). Hence the group $\Delta$ generated by the left translations $\Xi_{(p_i,\tilde{W}_j)}$ of the loop $L_{(p_i,\tilde{W}_j)}$ contains all translations of the affine space $\mathcal{A}_n$. It follows that $\Delta$ is the semidirect product $\Delta = T \rtimes C$ of the translation group $T$ by a subgroup $C$ of the stabilizer of $0 \in \mathcal{A}_n$ in the group $A$ of affinities. If $F = K$ then $C$ is isomorphic to $SU_{p_2}(n,K)$ and the stabilizer $\hat{H}$ of $W_i$ in $\Delta$ is the semidirect product $T_{W_i} \rtimes \Phi$ since any element $g \in G_{(p_1,p_2)} = SU_{p_2}(n,K)$ has a unique representation as $g = g_1 g_2$ with $g_1 \in \Sigma_{(p_1,p_2)}(K)$ and $g_2 \in \Phi$. If $F = R$ then $C$ is a normal subgroup of $SO_{p_2}(n,R)$ containing $\mathcal{K}_{(n,p_2)}$ and the stabilizer $\hat{H}$ of $W_i$ in $\Delta$ is the semidirect product $T_{W_i} \rtimes \Gamma$, where $\Gamma = \Phi \cap C$.

**For $p_1 > p_2$ the loop $L_{(p_1,\tilde{W}_2)}$ is never isotopic to a loop $L_{(p_2,\tilde{W}_1)}$.** This follows from the fact that the stabilizer $H_k$, $k = 1,2$, of the identity of $L_{(p_k,\tilde{W}_l)}$ with $l \neq k$ in $\Delta$ contains the group $T_{W_k}$ as the largest normal subgroup consisting of affine translations. Since $T_{W_1}$ is not isomorphic to $T_{W_2}$ one has that $H_1$ is not isomorphic to $H_2$. (cf. [13], Theorem III.2.7, p. 65)

Now we consider the loops $L_{(p_i,W_j)}$ and $L_{(p_i,\tilde{W}_j)}$ for $W_j \neq \tilde{W}_j$. According to (1) the subspaces $W_1$ and $W_2$ are invariant under the subgroup $\Phi$ of the stabilizer of $0 \in \mathcal{A}_n$ in the group $A$ of affinities. Hence if $g \in \Phi$ then one has $g\Sigma_{(p_1,p_2)}(F)g^{-1} = \Sigma_{(p_1,p_2)}(F)$ and $gT_{W_k}g^{-1} = T_{W_k}$, $k = 1,2$, for the group $T_{W_k} = \{x \mapsto x + w_k; w_k \in W_k\}$. This yields $g\Xi_{(p_i,W_j)}g^{-1} = \Xi_{(p_i,W_j)}$. For $W_j \neq \tilde{W}_j$ the group $\Phi$ does not normalize the translation group $T_{\tilde{W}_j}$. Therefore

$$gT_{\tilde{W}_j}\Sigma_{(p_1,p_2)}(F)g^{-1} = (gT_{\tilde{W}_j}g^{-1})(g\Sigma_{(p_1,p_2)}(F)g^{-1}) =$$

$$(gT_{\tilde{W}_j}g^{-1})\Sigma_{(p_1,p_2)}(F) \neq \Xi_{(p_1,\tilde{W}_j)}$$

for suitable elements $g \in \Phi$. This means that not all elements of $\Phi$ induce automorphisms of $L_{(p_i,\tilde{W}_j)}$. Therefore **the loops $L_{(p_i,W_j)}$ and $L_{(p_i,\tilde{W}_j)}$ are not isomorphic if $W_j \neq \tilde{W}_j$.**

**Proposition 2.** *Any loop $L_{(p_i,\tilde{W}_j)}$ is a topological loop with respect to the topology induced on the set $\mathcal{U}$ by the topology on the set of the $p_i$-dimensional subspaces of $\mathcal{A}_n$ which is derived from the topology of the topological field $F$.*

*Proof.* Since $R$ is an ordered field, $R$ as well as $K = R(i)$ are topological fields with respect to the topology given by the ordering of $R$. Then the ring $\mathcal{M}_n(F)$ of $(n \times n)$-matrices over $F$ is a topological ring such that the open $\varepsilon$-neighbourhoods of $0 \in \mathcal{M}_n(F)$ consist of matrices $(c_{i,j})$ with $|c_{ij}| < \varepsilon$. The group $GL(n,F) \leq \mathcal{M}_n(F)$ is a topological group. Since the set $Z = \{\mathrm{diag}(a,\dots,a), a \in F\backslash\{0\}\}$ is a closed subgroup of $GL(n,F)$ the group $PGL(n,F) = GL(n,F)/Z$ is a topological group. The subgroups $SU_{p_2}(n,F)$ and $\Phi = (U(p_1,F) \times U(p_2,F)) \cap SU_{p_2}(n,F)$ are closed subgroups of $GL(n,F)$. Moreover $SU_{p_2}(n,F)Z/Z$ as well as $\Phi Z/Z$ are closed subgroups of $PGL(n,F)$.

The affine space $\mathcal{A}_n = F^n$ and the $(n-1)$-dimensional projective hyperplane $E$ carry topologies derived from the topology of the field $F$ (cf. [11], Chapter XI). The semidirect product $A = T \rtimes GL(n, F)$ is a topological group consisting of continuous affinities; it induces on the hyperplane $E$ a continuous group of projective collineations. Any subset of $A$ is a topological space with respect to the topology induced from $A$ and any subgroup of $A$ becomes a topological group in this manner.

Let $Q_1$ be a fixed $p_i$-dimensional subspace of $\mathcal{A}_n$ and let $\mathcal{Q}$ be the set of the affine $(n-p_i)$-dimensional affine subspaces with $|Q_1 \cap Q| = 1$ for $Q \in \mathcal{Q}$. The set $\mathcal{Q}$ also carries a topology determined by the topology of $F$. The set $\mathcal{Q}^*$ of intersections $Q^*$ of the affine subspaces $Q$ of $\mathcal{Q}$ with $E$ inherits the topology of the Graßmannian manifold of the $(n - p_i - 1)$-dimensional subspaces of the hyperplane $E$. The geometric operation $(Q, Q_1) \mapsto Q \cap Q_1 : \mathcal{Q} \to Q_1$ is continuous.

On the topological space $\Sigma_{(p_1,p_2)}(F)$ a topological Bruck loop $L_{(p_1,p_2)}$ is realized by the multiplication

$$A \circ B = \sqrt{AB^2A} \text{ for all } \; A, B \in \Sigma_{(p_1,p_2)}(F), \tag{2}$$

where $X \mapsto \sqrt{X}$ is the inverse map of the bijection $X \mapsto X^2 : \Sigma_{(p_1,p_2)}(F) \to \Sigma_{(p_1,p_2)}(F)$ (cf. [8] (1.14), p. 17 and (9.1) Theorem (4), p. 108, [12], p. 121). We denote by $[\rho(W_i)]^*$ with $\rho \in \Sigma_{(p_1,p_2)}(F)$ the intersection of the subspace $\rho(W_i)$ with the hyperplane $E$ and by $\mathcal{Z}_i^*$ the set $\{[\rho(W_i)]^*;\; \rho \in \Sigma_{(p_1,p_2)}(F)\}$. For the elements of the loop $L_{(p_i,\tilde{W}_j)}$ one can take the elements of the set

$$\mathcal{U}_{(p_i,\tilde{W}_j)} = \{\psi(W_i);\; \psi \in \Xi_{(p_i,\tilde{W}_j)}\} = \{\tau\rho(W_i);\; \tau \in T_{\tilde{W}_j}, \rho \in \Sigma_{(p_1,p_2)}(F)\}.$$

The subspace $\tilde{W}_j$ is homeomorphic to the group $T_{\tilde{W}_j}$, and the set $\mathcal{Z}_i$ is homeomorphic to $\Sigma_{(p_1,p_2)}(F)$. Any element $\tau\rho(W_i) \in \mathcal{U}_{(p_i,\tilde{W}_j)}$ is uniquely determined by $[\rho(W_i)]^*$ and $(\tau\rho(W_i)) \cap \tilde{W}_j$. The mapping

$$\omega : \tau\rho(W_i) \mapsto ((\tau\rho(W_i)) \cap \tilde{W}_j, [\rho(W_i)]^*)$$

from $\mathcal{U}_{(p_i,\tilde{W}_j)}$ onto the topological product $\tilde{W}_j \times \mathcal{Z}_i^*$ is a bijection such that

$$\omega^{-1} : (w, Z^*) \mapsto w \vee Z^*,$$

where $w \vee Z^*$ is the $p_i$-dimensional affine subspace containing $w \in \tilde{W}_j$ and intersecting $E$ in $Z^* \in \mathcal{Z}_i^*$. Since the geometric operations of joining and of intersecting of distinct subspaces are continuous maps, $\omega$ is a homeomorphism.

Let $(w_k, Z_k^*) \in \tilde{W}_j \times \mathcal{Z}_i^*$ with $k = 1, 2$ and let $\tau_k\rho_k(W_i)$ be the subspaces of $\mathcal{U}_{(p_i,\tilde{W}_j)}$ such that $\omega(\tau_k\rho_k(W_i)) = (w_k, Z_k^*)$. The multiplication given by

$$(w_1, Z_1^*) \circ (w_2, Z_2^*) = (w_3, Z_3^*), \tag{3}$$

where $Z_3^* = [\rho_1\rho_2(W_i)]^*$ and

$$w_3 = \tau_1[\rho_1(\tau_2\rho_2(W_i)) \cap \tilde{W}_j] = \tau_1\big[\big(\rho_1[\tau_2\rho_2(W_i) \cap \tilde{W}_j] \vee [\rho_1\rho_2(W_i)]^*\big) \cap \tilde{W}_j\big]$$

yields a topological loop. This loop is homeomorphic to $L_{(p_i,\tilde{W}_j)}$ since $[\rho_1\tau_2\rho_2(W_i)]^* = [\rho_1\rho_2(W_i)]^*$ and $[\tau_1\rho_1\tau_2\rho_2(W_i)]^* = [\rho_1\rho_2(W_i)]^*$. □

# 5 Special cases: $\mathbb{R}$ and $\mathbb{C}$

**Proposition 3.** *The loop $L_{(p_i,\tilde{W}_j)}$ is a differentiable loop diffeomorphic to $\mathbb{R}^d$, where $d = \varepsilon(p_j + p_1p_2)$, with $\varepsilon = 1$ if $F = \mathbb{R}$ and $\varepsilon = 2$ if $F = \mathbb{C}$.*

*If $F = \mathbb{C}$ then the group $\Delta$ generated by the left translations of $L_{(p_i,\tilde{W}_j)}$ is the semidirect product $\mathbb{C}^n \rtimes SU_{p_2}(n,\mathbb{C})$ and the stabilizer of $W_i$ in $\Delta$ is the semidirect product $\mathbb{C}^{p_i} \rtimes \Pi$, where $\Pi$ is an epimorphic image of the direct product $SU(p_1,\mathbb{C}) \times SU(p_2,\mathbb{C}) \times SO_2(\mathbb{R})$.*

*If $F = \mathbb{R}$ then $\Delta$ is the semidirect product $\mathbb{R}^n \rtimes SO_{p_2}(n,\mathbb{R})^\circ$, where $SO_{p_2}(n,\mathbb{R})^\circ$ is the connected component of $SO_{p_2}(n,\mathbb{R})$, and the stabilizer of $W_i$ in $\Delta$ is the semidirect product $\mathbb{R}^{p_i} \rtimes (SO(p_1,\mathbb{R}) \times SO(p_2,\mathbb{R}))$.*

*Proof.* Clearly the topological manifold $L_{(p_i,\tilde{W}_j)}$ carries the differentiable structure of the real differentiable manifold $\Xi_{(p_i,\tilde{W}_j)}$ which is the topological product of $T_{\tilde{W}_j}$ and $\Sigma_{(p_1,p_2)}(F)$.

According to Section 4 the group $\Delta$ topologically generated by the left translations $\Xi_{(p_i,\tilde{W}_j)}$ is the semidirect product $\Delta = F^n \rtimes C$, where $C$ contains the commutator subgroup of $SU_{p_2}(n,F)$.

If $F = \mathbb{C}$ then $C = SU_{p_2}(n,\mathbb{C})$ and the stabilizer $\hat{H}$ of $W_i$ in $\Delta$ is the semidirect product $T_{W_i} \rtimes \Phi$ with $\Phi = [U_{p_1}(\mathbb{C}) \times U_{p_2}(\mathbb{C})] \cap SU_{p_2}(n,\mathbb{C})$ which is a maximal compact subgroup of $SU_{p_2}(n,\mathbb{C})$ ([16], p. 28). The groups $SU_{p_2}(n,\mathbb{C})$ and $\Phi$ are connected therefore the groups $\Delta$ and $\hat{H}$ are connected. Since $\Delta$ is the topological product $\Xi_{(p_i,\tilde{W}_j)} \times \hat{H} = \Xi_{(p_i,\tilde{W}_j)} \times T_{W_i} \times \Phi$ it follows that the manifold $\Xi_{(p_i,\tilde{W}_j)}$ and hence the loop $L_{(p_i,\tilde{W}_j)}$ are diffeomorphic to an affine space.

If $F = \mathbb{R}$ then $C$ is a subgroup of $SO_{p_2}(n,\mathbb{R})$ containing the commutator subgroup $\mathcal{K}_{(n,p_2)}$. According to [3] p. 57 the factor group $SO_{p_2}(n,\mathbb{R})/\mathcal{K}_{(n,p_2)}$ has order 2. Hence $\mathcal{K}_{(n,p_2)}$ is the connected component of $SO_{p_2}(n,\mathbb{R})$. The group $\Phi = [O_{p_1}(\mathbb{R}) \times O_{p_2}(\mathbb{R})] \cap SO_{p_2}(n,\mathbb{R})$ is not connected since the factor group $O(p_i,\mathbb{R})/SO(p_i,\mathbb{R})$ has order 2 ([14], Corollary 9.37, p. 158) and the product $\alpha_1\alpha_2$ with $\alpha_i \in O(p_i,\mathbb{R})$, but $\alpha_i \notin SO(p_i,\mathbb{R})$ for $i = 1,2$, is an element of $SO_{p_2}(n,\mathbb{R})$. The group $SO_{p_2}(n,\mathbb{R})$ is homeomorphic to the topological product $\Sigma_{(p_1,p_2)}(\mathbb{R}) \times \Phi$. Since $SO_{p_2}(n,\mathbb{R})$ has two connected components and $\Phi$ is not connected the manifold $\Sigma_{(p_1,p_2)}(\mathbb{R})$ is connected. It follows that the group $C$ generated by $\Sigma_{(p_1,p_2)}(\mathbb{R})$ is connected and hence isomorphic to the connected component $SO_{p_2}(n,\mathbb{R})^\circ = \mathcal{K}_{(n,p_2)}$ of $SO_{p_2}(n,\mathbb{R})$. Thus the group $\Delta = T \rtimes C$ is connected. Moreover $\Delta$ is the topological product $\Xi_{(p_i,\tilde{W}_j)} \times \hat{H} = \Xi_{(p_i,\tilde{W}_j)} \times T_{W_i} \times (\Phi \cap \hat{H})$. Since $\Delta$, $\Xi_{(p_i,\tilde{W}_j)}$ and $T_{W_i}$ are connected, the group $\Phi \cap \hat{H}$ is connected and hence a maximal compact subgroup of $SO_{p_2}(n,\mathbb{R})$. This yields that $\Xi_{(p_i,\tilde{W}_j)}$ and $L_{(p_i,\tilde{W}_j)}$ are diffeomorphic to an affine space.

The group $\Delta$ is the topological product $\Xi_{(p_i,\tilde{W}_j)} \times \hat{H}$. Thus for the real dimension of $L_{(p_i,\tilde{W}_j)}$ one has

$$\dim L_{(p_i,\tilde{W}_j)} = \dim \Xi_{(p_i,\tilde{W}_j)} = \dim\Delta - \dim\hat{H}$$
$$\dim_{\mathbb{R}} T_{W_i} + \dim_{\mathbb{R}} T_{\tilde{W}_j} + \dim SU_{p_2}(n,F) - \dim_{\mathbb{R}} T_{W_i} - \dim(C \cap \hat{H}).$$

If $F = \mathbb{C}$ then the group $\Phi = C \cap \hat{H}$ is an epimorphic image of the direct product $SU(p_1,\mathbb{C}) \times SU(p_2,\mathbb{C}) \times SO_2(\mathbb{R})$ (cf. [16], p. 28). This yields
$\dim L_{(p_i,\tilde{W}_j)} = [(p_1+p_2)^2 - 1] + 2p_j - (p_1^2 - 1) - (p_2^2 - 1) - 1 = 2p_j + 2p_1p_2$
since the dimension of a unitary group $SU_k(m,\mathbb{C})$ is equal to $(m-1)^2 + 2(m-1)$ for $0 \le k \le m$ ([16], p. 26 and p. 28). It follows that $L_{(p_i,\tilde{W}_j)}$ is diffeomorphic to $\mathbb{R}^{2(p_j+p_1p_2)}$.

The group $\Delta$ is the semidirect product $\Delta = \mathbb{C}^n \rtimes C$, where $C$ is the group $SU_{p_2}(n,\mathbb{C})$ and the stabilizer $\hat{H}$ is the semidirect product $T_{W_i} \rtimes \Phi$, where $\Phi$ is an epimorphic image of $SU(p_1,\mathbb{C}) \times SU(p_2,\mathbb{C}) \times SO_2(\mathbb{R})$.

If $F = \mathbb{R}$ then $C \cap \hat{H} = SO(p_1,\mathbb{R}) \times SO(p_2,\mathbb{R})$ ([16], p. 31 and p. 38). It follows that

$$\dim L_{(p_i,\tilde{W}_j)} = \tfrac{1}{2}(p_1+p_2)(p_1+p_2-1) + p_j - \tfrac{1}{2}p_1(p_1-1) - \tfrac{1}{2}p_2(p_2-1) = p_j + p_1p_2.$$

Hence the loop $L_{(p_i,\tilde{W}_j)}$ is diffeomorphic to $\mathbb{R}^{p_j+p_1p_2}$.

The group $\Delta$ is the semidirect product $\Delta = \mathbb{R}^n \rtimes C$, where $C$ is the group $SO_{p_2}(n,\mathbb{R})^\circ$ and the stabilizer $\hat{H}$ of $W_i$ in $\Delta$ is the semidirect product $\mathbb{R}^{p_i} \rtimes (SO(p_1,\mathbb{R}) \times SO(p_2,\mathbb{R}))$.

The loop $L_{(p_i,\tilde{W}_j)}$ is diffeomorphic to the manifold $\tilde{W}_j \times \mathcal{Z}_i$ since $\mathcal{Z}_i$ is diffeomorphic to $\Sigma_{(p_1,p_2)}(\mathbb{R})$. The mapping $(x, D^*) \mapsto x \vee D^*$ assigning to a point $x \in \mathcal{A}_n = F^n$, $F \in \{\mathbb{R},\mathbb{C}\}$ and to an element $D^*$ of the Graßmannian manifold of the $(p_i - 1)$-dimensional $F$-subspaces of the hyperplane $E$ the affine subspace $D$ containing $x$ and intersecting $E$ in $D^*$ is differentiable. Also the mapping $D \to D \cap \tilde{W}_j$ assigning to a $p_i$-dimensional affine $F$-subspace $D$ of $\mathcal{A}_n$ the point $D \cap \tilde{W}_j$ is differentiable. Since the loop realized on $\Sigma_{(p_1,p_2)}(F)$ by the multiplication (2) is differentiable, the representation of $L_{(p_i,\tilde{W}_j)}$ on the manifold $\tilde{W}_j \times \mathcal{Z}_i$ by the multiplication (3) yields that $L_{(p_i,\tilde{W}_j)}$ is differentiable. □

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