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Invariance equations for two-variable means

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Baják Szabolcs

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témavezető

Invariance equations for two-variable means

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a matematika tudományágban

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Contents

1

Introduction

1.1. Preliminaries

In this section, we introduce the necessary notations and terminology. As usual, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of natural, rational and real numbers, respectively, $\mathbb{R}_+$ denotes the set of positive real numbers and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$.

Let $I \subseteq \mathbb{R}$ be a nonvoid open interval. A two-variable continuous function $M : I^2 \to I$ is called a *mean* on I if

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y) \qquad (x, y \in I)$$

holds. If both inequalities are strict whenever $x \neq y$, then M is called a *strict mean* on I. A mean M on I is said to be *symmetric* if $M(x, y) = M(y, x)$ holds for all $x, y \in I$. A mean M on $\mathbb{R}_+$ is called *homogeneous* if $M(tx, ty) = tM(x, y)$ holds for all $t,\ x,\ y \in \mathbb{R}_+$.

Classical examples for two-variable symmetric strict and homogeneous means on $\mathbb{R}_+$ are the *arithmetic*, *geometric* and *harmonic means*, which will be denoted by $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{H}$, respectively, i.e.,

$$\mathcal{A}(x, y) := \frac{x + y}{2}, \qquad \mathcal{G}(x, y) := \sqrt{xy}, \qquad \mathcal{H}(x, y) := \frac{2xy}{x + y}.$$

Another class of two-variable symmetric and homogeneous means on $\mathbb{R}_+$ is the class of *power means*, also called *Hölder means*. We will denote the power mean of exponent p by M_p and it is defined as

$$M_p(x, y) := \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{\frac{1}{p}}, & \text{if } p \neq 0, \\ \sqrt{xy}, & \text{if } p = 0. \end{cases}$$

It is easy to see that the class of power means contains the arithmetic, geometric and harmonic means by choosing the value of p as 1, 0 and -1, respectively.

The two-variable *Gini* and *Stolarsky means* (cf. [**26**], [**45**], [**46**]) are two substantial generalizations of the power means. These means play important roles in the theory of means and in various application of inequalities in mathematics.

For parameters $p, q \in \mathbb{R}$, the two-variable Gini mean $G_{p,q} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined, for $x, y \in \mathbb{R}_+$, by

$$G_{p,q}(x,y) = \begin{cases} \left(\dfrac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}} & \text{for } p \neq q, \\ \exp\left(\dfrac{x^p \ln x + y^p \ln y}{x^p + y^p}\right) & \text{for } p = q, \end{cases}$$

and the two-variable Stolarsky (or the difference) mean $S_{p,q} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the following:

$$S_{p,q}(x,y) := \begin{cases} \left(\dfrac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{\frac{1}{p-q}}, & \text{if } (p-q)pq(x-y) \neq 0, \\ \exp\left(-\dfrac{1}{p} + \dfrac{x^p \log x - y^p \log y}{x^p - y^p}\right), & \text{if } p = q,\ pq(x-y) \neq 0, \\ \left(\dfrac{x^p - y^p}{p(\log x - \log y)}\right)^{\frac{1}{p}}, & \text{if } q = 0,\ p(x-y) \neq 0, \\ \left(\dfrac{x^q - y^q}{q(\log x - \log y)}\right)^{\frac{1}{q}}, & \text{if } p = 0,\ q(x-y) \neq 0, \\ \sqrt{xy}, & \text{if } p = q = 0, \\ x, & \text{if } x = y. \end{cases}$$

For positive numbers x and y, their power mean of exponent p can also be obtained as $G_{p,0}(x,y)$ and as $S_{2p,p}(x,y)$ (in particular, $G_{1,0} = S_{2,1}$, $G_{0,0} = S_{0,0}$ and $G_{-1,0} = S_{-2,-1}$ are the arithmetic, geometric and harmonic means, respectively), thus power means are contained in both classes. Alzer and Ruscheweyh ([**1**]) proved that the means that are simultaneously Gini and Stolarsky means are exactly the power means.

Another possible generalization of the power means is the class of the so-called *quasi-arithmetic means* ([**28**]). If $I \subset \mathbb{R}$ is a nonvoid open interval, a two-variable function $M : I^2 \to I$ is called a *quasi-arithmetic mean* on I

if there exists a continuous, strictly monotone function $\varphi : I \to \mathbb{R}$ such that

$$M(x, y) = \mathcal{M}_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \tag{1.1}$$

for every $x, y \in I$. Here φ is called the *generating function* of the quasi-arithmetic mean.

In this definition by choosing $I = \mathbb{R}_+$ and $\varphi(x) = \ln x$ we get the geometric mean, while the power mean of exponent p can be obtained by taking $I = \mathbb{R}_+$ and $\varphi(x) = x^p$. Thus, the class of power means is contained in the class of quasi-arithmetic means, moreover, the only homogeneous quasi-arithmetic means are exactly the power means ([**28**]).

These classes of means as well as other possible generalizations of the power means provide a large field for research. Numerous researchers dealt with the equality and comparison problems, while others worked on Hölder and Minkowski-type inequalities (cf. [**6**], [**9**], [**10**], [**11**], [**14**], [**19**], [**20**], [**31**], [**32**], [**37**], [**38**], [**39**], [**40**], [**44**]). We concentrate on solving the *invariance equation* for these classes of means.

1.2. The general form of the invariance equation

Given two means $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $x, y \in \mathbb{R}_+$, the iteration sequence

$$\begin{aligned} x_1 &:= x, & y_1 &:= y, \\ x_{n+1} &:= M(x_n, y_n), & y_{n+1} &:= N(x_n, y_n) \qquad (n \in \mathbb{N}) \end{aligned} \tag{1.2}$$

is said to be the *Gauss-iteration* determined by the pair (M, N) with the initial values $(x, y) \in \mathbb{R}_+^2$. It is well-known (cf. [**8**], [**23**]) that if M and N are strict means then the sequences (x_n) and (y_n) are convergent and have equal limits $M \otimes N(x, y)$ which is a strict mean of the values x and y. The mean $M \otimes N$ defined by this procedure is called the *Gauss composition* of M and N.

A very important result ([**23**]) in characterizing the Gauss composition of means is the following: If $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ are two strict means, their Gauss composition $K = M \otimes N$ is the unique strict mean solution K of the functional equation

$$K(x, y) = K\big(M(x, y), N(x, y)\big) \qquad (x, y \in \mathbb{R}_+), \tag{1.3}$$

which is called the *invariance equation*. If (1.3) is valid then we say that K *is invariant with respect to the mean-type mapping* (M, N).

The simplest example when the invariance equation holds is the well-known identity

$$\sqrt{xy} = \sqrt{\frac{x+y}{2} \cdot \frac{2xy}{x+y}} \qquad (x, y \in \mathbb{R}_+),$$

that is,

$$\mathcal{G}(x, y) = \mathcal{G}(\mathcal{A}(x, y), \mathcal{H}(x, y)) \qquad (x, y \in \mathbb{R}_+).$$

Another invariance equation is the identity

$$\mathcal{A} \otimes \mathcal{G}(x, y) = \mathcal{A} \otimes \mathcal{G}(\mathcal{A}(x, y), \mathcal{G}(x, y)) \qquad (x, y \in \mathbb{R}_+),$$

where $\mathcal{A} \otimes \mathcal{G}$ denotes Gauss's *arithmetic-geometric mean*. This mean had an important role in the history of mathematics. In 1791, when he was only 14, Gauss played a game of choosing two numbers arbitrarily and creating the sequences defined in (1.2) with $M(x, y) = \mathcal{A}(x, y)$ and $N(x, y) = \mathcal{G}(x, y)$. He observed, by calculating the two sequences up to several digits, that the two values became indistinguishable very rapidly (i.e., they converged very rapidly to the common limit). On May 30, 1799 he discovered that the arc length of the famous (Bernoulli-) lemniscate with foci O_1 and O_2 can be expressed by $\frac{2\pi}{\mathcal{A}\otimes\mathcal{G}(\sqrt{2},2)} |O_1 O_2|$. He introduced the lemniscate functions and studied the theory of these functions. Later, he examined the elliptic functions (generalizations of the lemniscate functions) and the elliptic integrals and found the general form of $\mathcal{A} \otimes \mathcal{G}$, which is

$$\mathcal{A} \otimes \mathcal{G}(x, y) = \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{\sqrt{x^2 \cos^2 t + y^2 \sin^2 t}} \right)^{-1} \qquad (x, y \in \mathbb{R}_+).$$

The invariance equation in more general classes of means was studied extensively by many authors in various papers. The invariance of the arithmetic mean $\mathcal{A}$ (i.e., when in (1.3) K is equal to the arithmetic mean) with respect to two quasi-arithmetic means was first investigated by Sutô, and later by Matkowski (([**47**], [**48**]), [**33**]). This problem was completely solved by Daróczy and Páles ([**23**]), assuming only continuity of the unknown functions involved. The invariance equation involving three weighted quasi-arithmetic means was studied by Burai ([**13**], Jarczyk–Matkowski [**30**]) and Jarczyk ([**29**]). The final answer (where no additional regularity assumptions are required) was obtained in ([**29**]). The invariance of the arithmetic mean with respect to Lagrangian means was the subject of investigation of the paper [**35**] by Matkowski. The invariance of the arithmetic, geometric, and harmonic means with respect to the so-called Beckenbach–Gini means was studied by Matkowski ([**34**]). Pairs of Stolarsky means for which the geometric mean is invariant were determined by Błasińska-Lesk–Głazowska–Matkowski ([**7**]). The invariance of the arithmetic mean with respect to further means was studied by Głazowska–Jarczyk–Matkowski ([**27**]), Burai ([**12**]) and Domsta–Matkowski ([**25**]).

In recent years, under the supervision of Professor Zsolt Páles, I studied the invariance equation for different classes of means. The results the thesis is built upon appeared in the papers [**2**], [**3**], [**4**] and [**5**]. The exact references will always be given at the appropriate sections.

In the first part of the thesis we consider the invariance of the arithmetic mean with respect to the so-called *generalized quasi-arithmetic means*, and give the general solution under 4-times differentiability assumptions. In the second part we focus on the classes of Gini and Stolarsky means, and solve the invariance equation in several settings. We consider the cases when all the three means involved in the invariance equation come from the same class and also the cases when the three means are either Gini or Stolarsky means. By representing the Gini and the Stolarsky means as special cases of a common generalization, we will be able to determine the solutions of these equations. However, the procedure involves tedious computations, therefore we used the computer algebra package Maple V Release 9 to perform these calculations. In these cases, the exact code is always provided (and mostly the output as well), thus the interested reader can repeat and reproduce the calculations.

2

Generalized quasi-arithmetic means

In this chapter we consider the invariance equation involving a possible generalization of the quasi-arithmetic means. First, we recall the solution of the invariance equation for the class of quasi-arithmetic means defined in (1.1) and also for weighted quasi-arithmetic means. Then we state and prove the new results for generalized quasi-arithmetic means. These results appeared in the paper [**2**].

2.1. The invariance equation for quasi-arithmetic means

There is an extensive literature studying the invariance equation (1.3) in this class of two-variable means. The invariance of the arithmetic mean $\mathcal{A}$ with respect to two quasi-arithmetic means, i.e., the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y \qquad (x,y\in I) \tag{2.1}$$

(where I is a nonvoid open interval) was first investigated by Sutô, who found the analytic solutions ([**47**], [**48**]). Matkowski found the same solutions, but assuming only twice continuous differentiability concerning the generating functions of the quasi-arithmetic means ([**33**]). These regularity assumptions were weakened step-by-step by Daróczy, Maksa and Páles in the papers [**21**], [**22**], and finally in 2002 the following result was proved, which is the general solution of this - the so-called Matkowski–Sutô - problem:

Theorem 2.1. (Daróczy–Páles [**23**]) *The strictly monotone, continuous functions φ and ψ satisfy* (2.1) *if and only if*

(i) either there exist constants p, a, b, c, d with $acp \neq 0$ such that

$$\varphi(x)=a\,\mathrm{e}^{px}+b, \qquad \psi(x)=c\,\mathrm{e}^{-px}+d \qquad (x\in I);$$

(ii) or there exist non-zero constants a, c and constants b, d such that

$$\varphi(x) = ax + b, \qquad \psi(x) = cx + d \qquad (x \in I).$$

Furthermore, they solved the invariance equation for quasi-arithmetic means even in the general setting:

THEOREM 2.2. (Daróczy–Páles [**23**]) *If $\mathcal{M}_\varphi : I^2 \to I$, $\mathcal{M}_\psi : I^2 \to I$ and $\mathcal{M}_\kappa : I^2 \to I$ are quasi-arithmetic means on I, then the invariance equation*

$$\mathcal{M}_\varphi = \mathcal{M}_\psi \otimes \mathcal{M}_\kappa$$

holds on I^2 if and only if there exist a function f which is continuous and strictly monotone on I and a constant $p \in \mathbb{R}$ such that

$$\mathcal{M}_\varphi(x, y) = \mathcal{M}_f(x, y), \qquad \mathcal{M}_\psi(x, y) = \mathcal{M}_{\chi_p \circ f}(x, y)$$

and

$$\mathcal{M}_\kappa(x, y) = \mathcal{M}_{\chi_{-p} \circ f}(x, y)$$

hold for every $(x, y) \in I$, where

$$\chi_p(x) := \begin{cases} x, & \text{if } p = 0, \\ e^{px}, & \text{if } p \neq 0 \end{cases} \qquad (x \in I).$$

2.2. The invariance equation for weighted quasi-arithmetic means

If $I \subset \mathbb{R}$ is a nonvoid open interval, a two-variable function $M : I^2 \to I$ is called a *weighted quasi-arithmetic mean* on I if there exist a continuous, strictly monotone function $\varphi : I \to \mathbb{R}$ and a number $0 < \lambda < 1$ such that

$$M(x, y) = A_\varphi(x, y; \lambda) := \varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y))$$

for every $x, y \in I$. In this case the λ is said to be the *weight* and the function φ is the *generating function* of the weighted quasi-arithmetic mean.

If in this definition we choose $\varphi(x) := x$ for $x \in I$, then we have

$$M(x, y) = \lambda x + (1 - \lambda)y =: A(x, y; \lambda) \qquad (x, y \in I),$$

which is the well-known *weighted arithmetic mean* on I. In 2003, Daróczy and Páles solved the Matkowski-Sutô problem for weighted quasi-arithmetic means, which was the following: Let A_φ and A_ψ be two weighted quasi-arithmetic means on I with the same weight λ and the question is when the invariance equation

$$A\left(A_\varphi(x, y; \lambda), A_\psi(x, y; \lambda); \lambda\right) = A(x, y; \lambda) \tag{2.2}$$

holds for all $x, y \in I$. It is obvious that if $\lambda = \frac{1}{2}$ then this equation simplifies to equation (2.1), the original Matkowski–Sutô problem. They had the following result:

THEOREM 2.3. (Daróczy–Páles [**24**]) *Let φ, ψ be continuous, strictly monotone functions on I. If they are solutions of* (2.2) *with $\lambda \neq \frac{1}{2}$ and are continuously differentiable on I with nonvanishing derivatives on I, then there exist constants a, b, c, d with $ac \neq 0$ such that, for $x \in I$,*

$$\varphi(x) = ax + b, \qquad \psi(x) = cx + d.$$

In 2007, Burai investigated the Matkowski–Sutô-type equation

$$A_\varphi(x, y; \lambda) + A_\psi(x, y; 1 - \lambda) = x + y$$

and gave the continuously differentiable solutions φ and ψ ([**13**]).

The solution of the invariance equation for weighted quasi-arithmetic means in the general case was obtained by Jarczyk and Matkowski in 2006 under twice continuous differentiability conditions ([**30**]). In 2007, Jarczyk described the solution without these regularity assumptions.

THEOREM 2.4. (Jarczyk [**29**]) *Let I be an open interval. Continuous and strictly monotone functions $\varphi, \psi, \kappa : I \to \mathbb{R}$ and numbers $\lambda, \mu, \nu \in]0, 1[$ satisfy the functional equation*

$$\begin{aligned} \lambda\varphi\left(\psi^{-1}(\mu\psi(x) + (1 - \mu)\psi(y))\right) + \\ (1 - \lambda)\varphi\left(\kappa^{-1}(\nu\kappa(x) + (1 - \nu)\kappa(y))\right) = \lambda\varphi(x) + (1 - \lambda)\varphi(y) \end{aligned} \tag{2.3}$$

for all $x, y \in I$ if and only if the following two conditions are fulfilled:

(i) $\lambda = \dfrac{\nu}{1 - \mu + \nu}$,

(ii) there exist $a, c \in \mathbb{R}$, $ac \neq 0$ and $b, d \in \mathbb{R}$ such that

$$\psi(x) = a\varphi(x) + b \quad \text{and} \quad \kappa(x) = c\varphi(x) + d \quad (x \in I),$$

or $\lambda = \dfrac{1}{2}$ *and*

$$\psi(x) = a\,\mathrm{e}^{p\varphi(x)} + b \quad \text{and} \quad \kappa(x) = c\,\mathrm{e}^{-p\varphi(x)} + d \quad (x \in I),$$

with some $p \in \mathbb{R} \setminus \{0\}$.

2.3. The invariance equation for generalized quasi-arithmetic means

In this section we investigate the following generalization of the quasi-arithmetic mean: Given two continuous strictly monotone functions $\varphi_1, \varphi_2 : I \to \mathbb{R}$ such that φ_1 and φ_2 are strictly monotone in the same sense, the *generalized quasi-arithmetic mean* $\mathcal{M}_{\boldsymbol{\varphi}} : I^2 \to I$ is defined by

$$\mathcal{M}_{\boldsymbol{\varphi}}(x, y) := \varphi^{-1}(\varphi_1(x) + \varphi_2(y)) \qquad (x, y \in I),$$

where

$$\boldsymbol{\varphi} := (\varphi_1, \varphi_2), \qquad \varphi := \varphi_1 + \varphi_2.$$

From this generalization of the quasi-arithmetic mean, if $f : I \to \mathbb{R}$ is a strictly monotone function and $0 < \lambda < 1$, by choosing $\varphi_1(x) := \lambda f(x)$ and $\varphi_2(x) := (1 - \lambda) f(x)$ we can obtain the weighted quasi-arithmetic mean generated by f and with weight λ.

Our aim is to characterize the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means (to solve the Matkowski–Sutô problem for generalized quasi-arithmetic means), that is, to solve the equation

$$\mathcal{M}_{\boldsymbol{\varphi}}(x, y) + \mathcal{M}_{\boldsymbol{\psi}}(x, y) = x + y \qquad (x, y \in I),$$

which, in detailed form, is equivalent to the functional equation

$$(\varphi_1 + \varphi_2)^{-1} (\varphi_1(x) + \varphi_2(y)) + (\psi_1 + \psi_2)^{-1} (\psi_1(x) + \psi_2(y)) = x + y \tag{2.4}$$

for $x, y \in I$, where $\varphi_1, \varphi_2, \psi_1, \psi_2 : I \to \mathbb{R}$ are continuous, strictly monotone functions such that φ_1, φ_2 and ψ_1, ψ_2 are monotone in the same sense.

Clearly, if $\varphi_1 := \varphi_2 := \frac{\varphi}{2}$ and $\psi_1 := \psi_2 := \frac{\psi}{2}$, then (2.4) simplifies to (2.1). Functional equation (2.4) also generalizes equation (2.3) in the case when $\varphi(x) = x$ and $\lambda = \frac{1}{2}$, i.e., the outer mean is the arithmetic mean.

In order to formulate the solution of the invariance equation, we need the following definition:

Definition. Let $I \subset \mathbb{R}$ be a nonempty open interval. Let $\mathcal{D}^0(I)$ denote the class of all pairs (φ_1, φ_2) of continuous functions defined on I such that either φ_1 and φ_2 are strictly increasing or φ_1 and φ_2 are strictly decreasing. For $k \geqslant 1$, let $\mathcal{D}^k(I)$ denote the class of all those pairs (φ_1, φ_2) of k-times continuously differentiable functions defined on I such that $\varphi_1'(x)\varphi_2'(x) > 0$ for $x \in I$.

It easily follows from this definition that if the pair $(\varphi_1, \varphi_2) \in \mathcal{D}^k(I)$, then $(\varphi_1, \varphi_2) \in \mathcal{D}^0(I)$. Under the assumption $(\varphi_1, \varphi_2) \in \mathcal{D}^0(I)$, the left hand side of (2.4) is well-defined.

Now we can give the solution of the invariance equation (2.4).

THEOREM 2.5. (Baják–Páles [2]) *Let* $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \mathcal{D}^4(I)$. *Then, for every* x *and* y *in* I, *the functional equation* (2.4) *holds if and only if*

(i) *either there exist real constants* $p, a_1, a_2, c_1, c_2, b_1, b_2, d_1, d_2$ *with* $p \neq 0$, $a_1 a_2 > 0$, $c_1 c_2 > 0$ *and* $a_1 c_1 = a_2 c_2$ *such that, for* $x \in I$,

$$\varphi_1(x) = a_1 \, e^{px} + b_1 , \qquad \varphi_2(x) = a_2 \, e^{px} + b_2 \tag{2.5}$$

and

$$\psi_1(x) = c_1 \, e^{-px} + d_1 , \qquad \psi_2(x) = c_2 \, e^{-px} + d_2 ; \tag{2.6}$$

(ii) *or there exist real constants* a, b, c, d_1, d_2 *with* $ac \neq 0$ *such that, for* $x \in I$,

$$\varphi_1(x) + \varphi_2(x) = ax + b \tag{2.7}$$

and

$$\psi_1(x) = c\,\varphi_2(x) + d_1 , \qquad \psi_2(x) = c\,\varphi_1(x) + d_2 . \tag{2.8}$$

2.3.1. Proof of the sufficiency.

For the sufficiency part of Theorem 2.5, we need not require the 4-times continuous differentiability of the unknown functions, therefore we have the following stronger statement.

THEOREM 2.6. (Baják–Páles [2]) *Let* $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \mathcal{D}^0(I)$ *and assume that one of the alternatives (i)–(ii) of Theorem 2.5 holds. Then the functional equation* (2.4) *is satisfied for every* x *and* y *in* I.

PROOF. Assume that alternative (i) of Theorem 2.5 is valid, i.e., for $x \in I$,

$$\varphi_1(x) = a_1 \, e^{px} + b_1, \qquad \varphi_2(x) = a_2 \, e^{px} + b_2$$

and

$$\psi_1(x) = c_1 \, e^{-px} + d_1, \qquad \psi_2(x) = c_2 \, e^{-px} + d_2$$

for some real constants $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, p$ with $a_1 c_1 = a_2 c_2$ and $a_1 a_2 c_1 c_2 p \neq 0$.

Then $a_1 a_2 > 0$, $c_1 c_2 > 0$ and for $x \in I$,

$$(\varphi_1 + \varphi_2)(x) = (a_1 + a_2) \, e^{px} + b_1 + b_2 ,$$

which yields that

$$(\varphi_1 + \varphi_2)^{-1}(y) = \frac{1}{p} \ln \frac{y - b_1 - b_2}{a_1 + a_2} .$$

Thus, the means determined by φ_1, φ_2 and ψ_1, ψ_2 are of the form

$$\begin{aligned}\mathcal{M}_{\boldsymbol{\varphi}}(x, y) &= \frac{1}{p} \ln \frac{a_1\, e^{px} + b_1 + a_2\, e^{py} + b_2 - b_1 - b_2}{a_1 + a_2} \\ &= \frac{1}{p} \ln \frac{a_1\, e^{px} + a_2\, e^{py}}{a_1 + a_2}\end{aligned} \tag{2.9}$$

and

$$\mathcal{M}_{\boldsymbol{\psi}}(x, y) = -\frac{1}{p} \ln \frac{c_1\, e^{-px} + c_2\, e^{-py}}{c_1 + c_2}, \tag{2.10}$$

respectively. In view of $a_2\, c_2 = a_1\, c_1$, we have

$$\frac{a_1}{a_1 + a_2} = \frac{c_2}{c_1 + c_2}, \qquad \frac{a_2}{a_1 + a_2} = \frac{c_1}{c_1 + c_2}. \tag{2.11}$$

Substituting (2.9) and (2.10) into (2.4) and also using (2.11), for all $x, y \in I$, we get

$$\begin{aligned}\mathcal{M}_{\boldsymbol{\varphi}}(x, y) + \mathcal{M}_{\boldsymbol{\psi}}(x, y) &= \frac{1}{p}\left(\ln \frac{a_1\, e^{px} + a_2\, e^{py}}{a_1 + a_2} - \ln \frac{c_1\, e^{-px} + c_2\, e^{-py}}{c_1 + c_2}\right) \\ &= \frac{1}{p} \ln\left(\frac{\frac{a_1}{a_1+a_2}\, e^{px} + \frac{a_2}{a_1+a_2}\, e^{py}}{\frac{c_1}{c_1+c_2}\, e^{-px} + \frac{c_2}{c_1+c_2}\, e^{-py}}\right) \\ &= \frac{1}{p} \ln\left(\frac{\frac{a_1}{a_1+a_2}\, e^{px} + \frac{a_2}{a_1+a_2}\, e^{py}}{\frac{c_1}{c_1+c_2}\, e^{py} + \frac{c_2}{c_1+c_2}\, e^{px}}\, e^{px+py}\right) \\ &= \frac{1}{p} \ln e^{px+py} = x + y,\end{aligned}$$

which was to be proved.

Assume now that alternative (ii) of Theorem 2.5 is valid, i.e., for $x \in I$,

$$\varphi_1(x) + \varphi_2(x) = ax + b$$

and

$$\psi_1(x) = c\, \varphi_2(x) + d_1\,, \qquad \psi_2(x) = c\, \varphi_1(x) + d_2$$

for some real constants a, b, c, d_1, d_2 with $ac \neq 0$. Then

$$(\varphi_1 + \varphi_2)^{-1}(y) = \frac{y - b}{a}\,.$$

Thus, for $x, y \in I$,

$$\mathcal{M}_{\boldsymbol{\varphi}}(x, y) = \frac{\varphi_1(x) + \varphi_2(y) - b}{a}. \tag{2.12}$$

On the other hand, for $x \in I$,

$$\begin{aligned}\psi_1(x) + \psi_2(x) &= c\,\varphi_2(x) + d_1 + c\,\varphi_1(x) + d_2 \\ &= c\big(\varphi_1(x) + \varphi_2(x)\big) + d_1 + d_2 \\ &= c(ax + b) + d_1 + d_2,\end{aligned}$$

whence

$$(\psi_1 + \psi_2)^{-1}(y) = \frac{y - d_1 - d_2 - cb}{ca}.$$

Therefore, for $x, y \in I$,

$$\begin{aligned}\mathcal{M}_{\boldsymbol{\psi}}(x,\, y) &= \frac{\psi_1(x) + \psi_2(y) - d_1 - d_2 - cb}{ca} \\ &= \frac{c\,\varphi_2(x) + d_1 + c\,\varphi_1(y) + d_2 - d_1 - d_2 - cb}{ca} \\ &= \frac{\varphi_2(x) + \varphi_1(y) - b}{a}.\end{aligned} \tag{2.13}$$

Substituting the formulae obtained in (2.12) and (2.13) into (2.4), we get

$$\begin{aligned}\mathcal{M}_{\boldsymbol{\varphi}}(x,\, y) + \mathcal{M}_{\boldsymbol{\psi}}(x,\, y) &= \frac{\varphi_1(x) + \varphi_2(y) - b}{a} + \frac{\varphi_2(x) + \varphi_1(y) - b}{a} \\ &= \frac{\varphi_1(x) + \varphi_2(x) + \varphi_1(y) + \varphi_2(y) - 2b}{a} \\ &= \frac{ax + b + ay + b - 2b}{a} = x + y,\end{aligned}$$

hence the proof is complete. □

2.3.2. Partial derivatives of generalized quasi-arithmetic means.

To give the general solution of the invariance equation, we will need explicit formulae for the partial derivatives of the mean $\mathcal{M}_{\boldsymbol{\varphi}}$ along the diagonal of the Cartesian product $I \times I$.

THEOREM 2.7. (Baják–Páles [**2**]) *If $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{D}^k(I)$ then $\mathcal{M}_{\boldsymbol{\varphi}}$ is k-times continuously differentiable on $I \times I$ and, for all $x \in I$,*

(i) if $\boldsymbol{\varphi} \in \mathcal{D}^1(I)$, then

$$\partial_1 \mathcal{M}_{\boldsymbol{\varphi}}(x,\, x) = \frac{\varphi_1'}{\varphi'}(x), \qquad \partial_2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\, x) = \frac{\varphi_2'}{\varphi'}(x)\,; \tag{2.14}$$

(ii) if $\boldsymbol{\varphi} \in \mathcal{D}^2(I)$*, then*

$$\partial_1^2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \frac{\varphi'^2 \varphi_1'' - \varphi'' {\varphi_1'}^2}{\varphi'^3}(x)\,,$$

$$\partial_1 \partial_2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = -\frac{\varphi'' \varphi_1' \varphi_2'}{\varphi'^3}(x)\,, \tag{2.15}$$

$$\partial_2^2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \frac{\varphi'^2 \varphi_2'' - \varphi'' {\varphi_2'}^2}{\varphi'^3}(x)\,;$$

(iii) if $\boldsymbol{\varphi} \in \mathcal{D}^3(I)$*, then*

$$\partial_1^3 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \frac{3\varphi''^2 {\varphi_1'}^3 - \varphi''' \varphi' {\varphi_1'}^3 - 3\varphi'' \varphi'^2 \varphi_1'' \varphi_1' + \varphi'^4 \varphi_1'''}{\varphi'^5}(x)\,,$$

$$\partial_1^2 \partial_2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = -\frac{\varphi_2' (\varphi''' \varphi' {\varphi_1'}^2 - 3\,\varphi''^2 {\varphi_1'}^2 + \varphi'' \varphi'^2 \varphi_1'')}{\varphi'^5}(x)\,, \tag{2.16}$$

$$\partial_1 \partial_2^2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = -\frac{\varphi_1' (\varphi''' \varphi' {\varphi_2'}^2 - 3\,\varphi''^2 {\varphi_2'}^2 + \varphi'' \varphi'^2 \varphi_2'')}{\varphi'^5}(x)\,,$$

$$\partial_2^3 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \frac{3\varphi''^2 {\varphi_2'}^3 - \varphi''' \varphi' {\varphi_2'}^3 - 3\varphi'' \varphi'^2 \varphi_2'' \varphi_2' + \varphi'^4 \varphi_2'''}{\varphi'^5}(x)\,;$$

(iv) if $\boldsymbol{\varphi} \in \mathcal{D}^4(I)$*, then*

$$\begin{aligned}
\partial_1^4 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \Big(-\frac{1}{\varphi'^7} \Big(& \varphi'''' \varphi'^2 {\varphi_1'}^4 - 10\,\varphi''' \varphi'' \varphi' {\varphi_1'}^4 \\
& + 6\,\varphi''' \varphi'^3 \varphi_1'' {\varphi_1'}^2 + 15\,\varphi''^3 {\varphi_1'}^4 - 18\,\varphi''^2 \varphi'^2 \varphi_1'' {\varphi_1'}^2 \\
& + 3\,\varphi'' \varphi'^4 {\varphi_1''}^2 + 4\,\varphi'' \varphi'^4 \varphi_1''' \varphi_1' - \varphi_1'''' \varphi'^6 \Big) \Big)(x),
\end{aligned}$$

$$\begin{aligned}
\partial_1^3 \partial_2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \Big(-\frac{\varphi_2'}{\varphi'^7} \Big(& \varphi'''' \varphi'^2 {\varphi_1'}^3 - 10\,\varphi''' \varphi'' \varphi' {\varphi_1'}^3 + 3\,\varphi''' \varphi'^3 \varphi_1'' \varphi_1' \\
& + 15\,\varphi''^3 {\varphi_1'}^3 - 9\,\varphi''^2 \varphi'^2 \varphi_1'' \varphi_1' + \varphi'' \varphi'^4 \varphi_1''' \Big) \Big)(x)\,,
\end{aligned}$$

(2.17)

$$\begin{aligned}
\partial_1^2 \partial_2^2 \mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \Big(-\frac{1}{\varphi'^7} \Big(& \varphi'''' \varphi'^2 {\varphi_1'}^2 {\varphi_2'}^2 - 10\,\varphi''' \varphi'' \varphi' {\varphi_1'}^2 {\varphi_2'}^2 \\
& + \varphi''' \varphi'^3 \varphi_1'' {\varphi_2'}^2 + 15\,\varphi''^3 {\varphi_1'}^2 {\varphi_2'}^2 - 3\,\varphi''^2 \varphi'^2 \varphi_1'' {\varphi_2'}^2 \\
& + \varphi''' \varphi'^3 {\varphi_1'}^2 \varphi_2'' - 3\,\varphi''^2 \varphi'^2 {\varphi_1'}^2 \varphi_2'' + \varphi'' \varphi'^4 \varphi_1'' \varphi_2'' \Big) \Big)(x),
\end{aligned}$$

$$\partial_1\partial_2^3\mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \Big(-\frac{\varphi_1'}{\varphi'^7}\Big(\varphi''''\varphi'^2{\varphi_2'}^3 - 10\,\varphi'''\varphi''\varphi'{\varphi_2'}^3 + 3\,\varphi'''{\varphi'}^3\varphi_2''\varphi_2' + 15\,{\varphi''}^3{\varphi_2'}^3 - 9\,{\varphi'}^2{\varphi''}^2\varphi_2''\varphi_2' + \varphi''{\varphi'}^4\varphi_2'''\Big)\Big)(x)\,,$$

$$\partial_2^4\mathcal{M}_{\boldsymbol{\varphi}}(x,\ x) = \Big(-\frac{1}{\varphi'^7}\Big(\varphi''''{\varphi'}^2{\varphi_2'}^4 - 10\,\varphi'''\varphi''\varphi'{\varphi_2'}^4 + 6\,\varphi'''{\varphi'}^3\varphi_2''{\varphi_2'}^2 + 15\,{\varphi''}^3{\varphi_2'}^4 - 18\,{\varphi''}^2{\varphi'}^2\varphi_2''{\varphi_2'}^2 + 3\,\varphi''{\varphi'}^4{\varphi_2''}^2 + 4\,\varphi''{\varphi'}^4\varphi_2'''\varphi_2' - \varphi_2''''{\varphi'}^6\Big)\Big)(x)\,.$$

Proof. (i) Let $\boldsymbol{\varphi} \in \mathcal{D}^1(I)$. φ_1 and φ_2 are continuous, strictly monotone functions in the same sense, since $\varphi_1'\varphi_2' > 0$. Therefore $\varphi = \varphi_1 + \varphi_2$ is also continuous, strictly monotone, and hence invertible. Therefore,

$$\mathcal{M}_{\boldsymbol{\varphi}}(x, y) = (\varphi_1 + \varphi_2)^{-1}(\varphi_1(x) + \varphi_2(y)) \tag{2.18}$$

is well-defined for every $x,\ y \in I$. Having that φ' is continuous and does not vanish anywhere, it follows by the standard calculus rules that $\mathcal{M}_{\boldsymbol{\varphi}}$ is continuously differentiable on $I{\times}I$. If, in addition, $\boldsymbol{\varphi} \in \mathcal{D}^k(I)$ holds for some $k \geq 2$, then the k-times continuously differentiability of $\mathcal{M}_{\boldsymbol{\varphi}}$ also follows.

Using (2.18), we have

$$\varphi(\mathcal{M}_{\boldsymbol{\varphi}}(x, y)) = \varphi_1(x) + \varphi_2(y) \qquad (x, y \in I). \tag{2.19}$$

By differentiating this identity with respect to the first variable,

$$\varphi'(\mathcal{M}_{\boldsymbol{\varphi}}(x, y)) \cdot \partial_1\mathcal{M}_{\boldsymbol{\varphi}}(x, y) = \varphi_1'(x) \qquad (x, y \in I). \tag{2.20}$$

Thus, taking $y := x$, the first equality in (2.14) follows. The second equality in (2.14) can be obtained by differentiating (2.19) with respect to the second variable.

(ii) To prove the formulae for the second-order partial derivatives in (2.15), let $\boldsymbol{\varphi} \in \mathcal{D}^2(I)$. Then, by differentiating (2.20) with respect to the first variable, we get

$$\varphi''(\mathcal{M}_{\boldsymbol{\varphi}}(x, y)) \cdot (\partial_1\mathcal{M}_{\boldsymbol{\varphi}}(x, y))^2 + \varphi'(\mathcal{M}_{\boldsymbol{\varphi}}(x, y)) \cdot \partial_1^2\mathcal{M}_{\boldsymbol{\varphi}}(x, y) = \varphi_1''(x)\,.$$

Therefore, taking $y := x$ and using (2.14),

$$\varphi''(x) \cdot \left(\frac{\varphi_1'(x)}{\varphi'(x)}\right)^2 + \varphi'(x) \cdot \partial_1^2\mathcal{M}_{\boldsymbol{\varphi}}(x, x) = \varphi_1''(x)\,,$$

whence we get

$$\partial_1^2\mathcal{M}_{\boldsymbol{\varphi}}(x, x) = \frac{{\varphi'}^2\varphi_1'' - \varphi''{\varphi_1'}^2}{{\varphi'}^3}(x)\,.$$

By differentiating (2.20) with respect to the variable y and then substituting $y := x$, the second equality in (2.15) follows, while differentiating (2.19) twice with respect to the second variable, we obtain the last identity of (2.15).

In cases (iii) and (iv), the same argument provides the formulae for the third- and fourth-order partial derivatives. □

2.3.3. Necessary conditions.

In this subsection, we deduce various necessary conditions of the equality (2.4). Assuming first- and second-order differentiability properties of the unknown functions, ψ_1 and ψ_2 will be completely determined and described in terms of φ_1 and φ_2. The third- and fourth-order necessary conditions provide further differential equations for φ_1 and φ_2 which can finally be solved and thus the forms of the unknown functions can be determined.

First-order necessary condition.

Lemma 2.1. *Let* (φ_1, φ_2) *and* (ψ_1, ψ_2) *be pairs of class* $\mathcal{D}^1(I)$ *that satisfy the functional equation* (2.4). *Then*

$$\frac{\varphi_1'}{\varphi'} = \frac{\psi_2'}{\psi'} \quad \text{and} \quad \frac{\varphi_2'}{\varphi'} = \frac{\psi_1'}{\psi'}. \tag{2.21}$$

Proof. Differentiating (2.4) with respect to the first variable, we get

$$\partial_1\mathcal{M}_{\boldsymbol{\varphi}}(x, x) + \partial_1\mathcal{M}_{\boldsymbol{\psi}}(x, x) = 1,$$

whence, using (2.14),

$$\frac{\varphi_1'}{\varphi'} + \frac{\psi_1'}{\psi'} = 1.$$

Therefore

$$\frac{\varphi_1'}{\varphi'} = 1 - \frac{\psi_1'}{\psi'} = \frac{\psi' - \psi_1'}{\psi'} = \frac{\psi_2'}{\psi'}.$$

By differentiating (2.4) with respect to the second variable, the same calculation yields that

$$\frac{\varphi_2'}{\varphi'} = \frac{\psi_1'}{\psi'},$$

which makes our proof complete. □

Second-order necessary conditions.

LEMMA 2.2. (Baják–Páles [**2**]) *Let* (φ_1, φ_2) *and* (ψ_1, ψ_2) *be pairs of class* $\mathcal{D}^2(I)$ *that satisfy the functional equation* (2.4). *Then there exists a (nonzero) real constant c such that*

$$\varphi'\psi' = c. \tag{2.22}$$

PROOF. Differentiating (2.4) once with respect to both variables, we get

$$\partial_1\partial_2\mathcal{M}_{\boldsymbol{\varphi}}(x, x) + \partial_1\partial_2\mathcal{M}_{\boldsymbol{\psi}}(x, x) = 0,$$

which, by (2.15), yields that

$$\frac{\varphi''\varphi_1'\varphi_2'}{\varphi'^3} + \frac{\psi''\psi_1'\psi_2'}{\psi'^3} = 0.$$

Hence, by the first-order conditions in (2.21),

$$\frac{\varphi''}{\varphi'} + \frac{\psi''}{\psi'} = 0, \tag{2.23}$$

i.e.,

$$\varphi''\psi' + \varphi'\psi'' = 0,$$

which means that

$$(\varphi'\psi')' = 0.$$

Thus, by integrating, we get that there exists a constant c such that

$$\varphi'\psi' = c,$$

therefore (2.22) is valid. □

As a consequence of this and the previous lemma, we can formulate the following theorem, which describes the connection between the functions φ_1, φ_2 and ψ_1, ψ_2, respectively.

THEOREM 2.8. (Baják–Páles [**2**]) *Let* (φ_1, φ_2) *and* (ψ_1, ψ_2) *be pairs of class* $\mathcal{D}^2(I)$ *that satisfy the functional equation* (2.4). *Then there exists a (nonzero) real constant c such that*

$$\psi_1' = \frac{c\varphi_2'}{\varphi'^2} \qquad \textit{and} \qquad \psi_2' = \frac{c\varphi_1'}{\varphi'^2}.$$

PROOF. It immediately follows from (2.21) and (2.22). □

Third-order necessary conditions.

In order to formulate the higher-order necessary conditions, first we have to analyze the first- and second-order conditions. By differentiating (2.21), we get

$$\frac{\psi_1''}{\psi'} = \frac{\varphi_2''}{\varphi'} - 2\,\frac{\varphi''\varphi_2'}{\varphi'^2} \qquad \text{and} \qquad \frac{\psi_2''}{\psi'} = \frac{\varphi_1''}{\varphi'} - 2\,\frac{\varphi''\varphi_1'}{\varphi'^2}, \tag{2.24}$$

and, from (2.23), we get

$$\frac{\psi''}{\psi'} = -\frac{\varphi''}{\varphi'}. \tag{2.25}$$

Differentiating (2.25) once (assuming that (φ_1, φ_2) and (ψ_1, ψ_2) are pairs of class $\mathcal{D}^3(I)$),

$$\frac{\psi'''}{\psi'} = -\frac{\varphi'''}{\varphi'} + 2\left(\frac{\varphi''}{\varphi'}\right)^2 \tag{2.26}$$

and twice (assuming that (φ_1, φ_2) and (ψ_1, ψ_2) are pairs of class $\mathcal{D}^4(I)$),

$$\frac{\psi''''}{\psi'} = -\frac{\varphi''''}{\varphi'} + 6\,\frac{\varphi'''\varphi''}{\varphi'^2} - 6\left(\frac{\varphi''}{\varphi'}\right)^3. \tag{2.27}$$

Now we are able to state and prove the third-order necessary condition.

Lemma 2.3. (Baják–Páles [**2**]) *Let* (φ_1, φ_2) *and* (ψ_1, ψ_2) *be pairs of class* $\mathcal{D}^3(I)$ *that satisfy the functional equation* (2.4). *Then*

$$(\varphi'''\varphi' - \varphi''^2)(\varphi_1' - \varphi_2')\,\varphi_1'\,\varphi_2' + \varphi'^2\varphi''(\varphi_1''\varphi_2' - \varphi_1'\varphi_2'') = 0. \tag{2.28}$$

Proof. Differentiating (2.4) twice with respect to the first variable and once with respect to the second variable, we get

$$\partial_1^2\partial_2\mathcal{M}_{\boldsymbol{\varphi}}(x, x) + \partial_1^2\partial_2\mathcal{M}_{\boldsymbol{\psi}}(x, x) = 0\,.$$

Hence, applying (2.16) for the means $\mathcal{M}_{\boldsymbol{\varphi}}$ and $\mathcal{M}_{\boldsymbol{\psi}}$, we get

$$\frac{\varphi'''}{\varphi'}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'}{\varphi'} - 3\,\frac{\varphi''^2}{\varphi'^2}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'}{\varphi'} + \frac{\varphi''}{\varphi'}\frac{\varphi_1''}{\varphi'}\frac{\varphi_2'}{\varphi'} + \frac{\psi'''}{\psi'}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2'}{\psi'}$$
$$-3\,\frac{\psi''^2}{\psi'^2}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2'}{\psi'} + \frac{\psi''}{\psi'}\frac{\psi_1''}{\psi'}\frac{\psi_2'}{\psi'} = 0.$$

Now, using (2.21), (2.24), (2.25) and (2.26), it follows that

$$\frac{\varphi'''}{\varphi'}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'}{\varphi'} - 3\,\frac{\varphi''^2}{\varphi'^2}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'}{\varphi'} + \frac{\varphi''}{\varphi'}\frac{\varphi_1''}{\varphi'}\frac{\varphi_2'}{\varphi'} + \Big(-\frac{\varphi'''}{\varphi'} + 2\Big(\frac{\varphi''}{\varphi'}\Big)^2\Big)\frac{\varphi_2'^2}{\varphi'^2}\frac{\varphi_1'}{\varphi'}$$
$$-3\Big(-\frac{\varphi''}{\varphi'}\Big)^2\frac{\varphi_2'^2}{\varphi'^2}\frac{\varphi_1'}{\varphi'} + \Big(-\frac{\varphi''}{\varphi'}\Big)\Big(\frac{\varphi_2''}{\varphi'} - 2\,\frac{\varphi''\varphi_2'}{\varphi'^2}\Big)\frac{\varphi_1'}{\varphi'} = 0,$$

which simplifies to (2.28). □

Fourth-order necessary conditions.

Lemma 2.4. (Baják–Páles [**2**]) *Let* (φ_1, φ_2) *and* (ψ_1, ψ_2) *be pairs of class* $\mathcal{D}^4(I)$ *that satisfy the functional equation* (2.4)*. Then*

$$\varphi''(\varphi_1'\varphi_2'' - \varphi_2'\varphi_1'') = 0 \tag{2.29}$$

and

$$\varphi'''\varphi' - \varphi''^2 = 0. \tag{2.30}$$

Proof. Differentiating (2.4) twice with respect to both variables, we get

$$\partial_1^2\partial_2^2\mathcal{M}_{\boldsymbol{\varphi}}(x, x) + \partial_1^2\partial_2^2\mathcal{M}_{\boldsymbol{\psi}}(x, x) = 0.$$

By applying (2.17) for the means $\mathcal{M}_{\boldsymbol{\varphi}}$ and $\mathcal{M}_{\boldsymbol{\psi}}$, we get

$$\frac{\varphi''''}{\varphi'}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'^2}{\varphi'^2} - 10\,\frac{\varphi'''}{\varphi'}\frac{\varphi''}{\varphi'}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'^2}{\varphi'^2} + \frac{\varphi'''}{\varphi'}\frac{\varphi_1''}{\varphi'}\frac{\varphi_2'^2}{\varphi'^2} + 15\,\frac{\varphi''^3}{\varphi'^3}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2'^2}{\varphi'^2}$$
$$- 3\,\frac{\varphi''^2}{\varphi'^2}\frac{\varphi_1''}{\varphi'}\frac{\varphi_2'^2}{\varphi'^2} + \frac{\varphi'''}{\varphi'}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2''}{\varphi'} - 3\,\frac{\varphi''^2}{\varphi'^2}\frac{\varphi_1'^2}{\varphi'^2}\frac{\varphi_2''}{\varphi'} + \frac{\varphi''}{\varphi'}\frac{\varphi_1''}{\varphi'}\frac{\varphi_2''}{\varphi'}$$
$$+\frac{\psi''''}{\psi'}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2'^2}{\psi'^2} - 10\,\frac{\psi'''}{\psi'}\frac{\psi''}{\psi'}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2'^2}{\psi'^2} + \frac{\psi'''}{\psi'}\frac{\psi_1''}{\psi'}\frac{\psi_2'^2}{\psi'^2} + 15\,\frac{\psi''^3}{\psi'^3}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2'^2}{\psi'^2}$$
$$- 3\,\frac{\psi''^2}{\psi'^2}\frac{\psi_1''}{\psi'}\frac{\psi_2'^2}{\psi'^2} + \frac{\psi'''}{\psi'}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2''}{\psi'} - 3\,\frac{\psi''^2}{\psi'^2}\frac{\psi_1'^2}{\psi'^2}\frac{\psi_2''}{\psi'} + \frac{\psi''}{\psi'}\frac{\psi_1''}{\psi'}\frac{\psi_2''}{\psi'} = 0.$$

Similarly to the calculation in the third-order case, using (2.21), (2.24), (2.25), (2.26) and (2.27), we obtain

$$\begin{aligned}(\varphi''^2 - \varphi'''\varphi')\,\varphi''\varphi_1'\varphi_2'(7\,\varphi_1'\varphi_2' - \varphi'^2)&\\ + \varphi''^2\varphi'^2(\varphi_1' - \varphi_2')(\varphi_1''\varphi_2' - \varphi_1'\varphi_2'') &= 0.\end{aligned} \tag{2.31}$$

Multiplying (2.31) by $(\varphi_1' - \varphi_2')$ and (2.28) by $\varphi''(7\varphi_1'\varphi_2' - \varphi'^2)$ and adding these equations, we get

$$3\,\varphi''^2\varphi'^2\varphi_1'\varphi_2'(\varphi_1'\varphi_2'' - \varphi_2'\varphi_1'') = 0,$$

which, in view of $\varphi'^2\varphi_1'\varphi_2' \neq 0$, reduces to (2.29). Hence the last term on the left hand side in (2.28) is zero. Therefore, the first term is also equal to zero, i.e.,

$$(\varphi'''\varphi' - \varphi''^2)(\varphi_2' - \varphi_1') = 0 \tag{2.32}$$

holds.

By (2.29), the last term on the left hand side of (2.31) is zero, whence

$$\varphi''(\varphi''^2 - \varphi'''\varphi')(7\,\varphi_1'\varphi_2' - \varphi'^2) = 0$$

is obtained. Multiplying (2.32) by $\varphi''(\varphi_2' - \varphi_1')$ and adding it to the equation above,

$$\varphi''(\varphi'''\varphi' - \varphi''^2) = 0$$

follows.

Now assume that there exists a $q \in I$ such that $\varphi'''\varphi' - \varphi''^2 \neq 0$ at q. This means that there exists a neighborhood U of q such that $\varphi'''\varphi' - \varphi''^2 \neq 0$ for $u \in U$. Therefore $\varphi'' = 0$ for $u \in U$, which yields that $\varphi''' = 0$. Thus, for $u \in U$, we have $\varphi'''\varphi' - \varphi''^2 = 0$. This contradiction shows that $\varphi'''\varphi' - \varphi''^2$ vanishes on I, i.e., (2.30) follows. □

Now, having the statements of the previous lemmas, we are ready to determine the general solution of the invariance equation (2.4).

2.3.4. The proof of the main theorem.

By equality (2.30) of Lemma 2.4, it follows that

$$\left(\frac{\varphi''}{\varphi'}\right)' = 0, \tag{2.33}$$

which shows that there exists a constant p such that

$$\frac{\varphi_1'' + \varphi_2''}{\varphi_1' + \varphi_2'} = \frac{\varphi''}{\varphi'} = p\,.$$

According to the value of the constant p, we distinguish two cases.

Case 1: $p \neq 0$. Then φ'' does not vanish. By (2.29), we have that $\varphi_2'' = \frac{\varphi_1''\varphi_2'}{\varphi_1'}$. Therefore

$$p = \frac{\varphi_1'' + \varphi_2''}{\varphi_1' + \varphi_2'} = \frac{\varphi_1'' + \frac{\varphi_1''\varphi_2'}{\varphi_1'}}{\varphi_1' + \varphi_2'} = \frac{\varphi_1''}{\varphi_1'} .$$

Similarly, we can also obtain that $\frac{\varphi_2''}{\varphi_2'} = p$. Thus, φ_1 and φ_2 are solutions of the second-order linear differential equation $f'' - pf' = 0$. The general solution of this differential equation is of the form $f(x) = a\,\mathrm{e}^{px} + b$, hence (2.5) follows for some constants a_1, a_2, b_1, b_2 with $a_1a_2 > 0$.

By (2.25), we also have

$$\frac{\psi''}{\psi'} = -p .$$

An analogous argument shows that (2.6) holds. We note that ψ_1 and ψ_2 can also be obtained by using Theorem 2.8.

Case 2: $p = 0$. This means that $\varphi'' \equiv 0$ and, by (2.25), $\psi'' \equiv 0$ also holds. Therefore, there exist constants $a, b, \bar{c}, d$ with $a\bar{c} \neq 0$ such that

$$\varphi(x) = ax + b \qquad \text{and} \qquad \psi(x) = \bar{c}x + d.$$

From (2.21) we get

$$\psi_1' = \frac{\varphi_2'}{\varphi'}\,\psi' = \frac{\bar{c}}{a}\varphi_2' \qquad \text{and} \qquad \psi_2' = \frac{\varphi_1'}{\varphi'}\,\psi' = \frac{\bar{c}}{a}\varphi_1' ,$$

which means that there exist constants d_1, d_2 such that

$$\psi_1 = \frac{\bar{c}}{a}\,\varphi_2 + d_1 \qquad \text{and} \qquad \psi_2 = \frac{\bar{c}}{a}\,\varphi_1 + d_2.$$

Thus, with the notation $c := \bar{c}/a$, equations (2.7) and (2.8) hold. □

We remark that Matkowski and Volkmann, motivated by our investigations, considered a particular case of (2.4), namely when $\varphi_1 = \psi_2 =: \varphi$ and $\varphi_2 = \psi_1 =: \psi$. In this case, they could eliminate the unnatural 4-times continuous differentiability conditions (however, the solution of (1.3) in this class of means without regularity is still an open problem). They obtained:

THEOREM 2.9. (Matkowski–Volkmann [**36**]) *Let $\varphi, \psi : I \to \mathbb{R}$ be continuous and strictly increasing. Then the functional equation*

$$(\varphi + \psi)^{-1}\left(\varphi(x) + \psi(y)\right) + (\varphi + \psi)^{-1}\left(\varphi(y) + \psi(x)\right) = x + y \qquad (x, y \in I)$$

holds if and only if there exist $a, b \in \mathbb{R}$, $a > 0$, such that

$$\varphi(x) + \psi(x) = ax + b \qquad (x \in I).$$

3

Gini and Stolarsky means

In this chapter, our aim is to solve the invariance equations when the three means involved in the equation are either Gini or Stolarsky means. Due to the symmetry of these classes of means, this results six different types of invariance equations.

1. The equation when all three means are Gini means, i.e.,

$$G_{p,q}(G_{a,b}(x,y),G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+). \tag{3.1}$$

2. The equation when all three means are Stolarsky means, i.e.,

$$S_{p,q}(S_{a,b}(x,y),S_{c,d}(x,y)) = S_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+). \tag{3.2}$$

3. The equation when the outer mean is a Gini mean and one of the means inside is a Gini mean and the other is a Stolarsky mean, i.e.,

$$G_{p,q}(S_{a,b}(x,y),G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+). \tag{3.3}$$

4. The equation when the outer mean is a Gini mean and the means inside are both Stolarsky means, i.e.,

$$G_{p,q}(S_{a,b}(x,y),S_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+). \tag{3.4}$$

5. The equation when the outer mean is a Stolarsky mean and the means inside are both Gini means, i.e.,

$$S_{p,q}(G_{a,b}(x,y),G_{c,d}(x,y)) = S_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+). \tag{3.5}$$

6. The equation when the outer mean is a Stolarsky mean and one of the means inside is a Gini mean and the other is a Stolarsky mean, i.e.,

$$S_{p,q}(G_{a,b}(x,y),S_{c,d}(x,y)) = S_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+). \tag{3.6}$$

Our task in each case is to determine all 6-tuples (a,b,c,d,p,q) such that the corresponding identity holds.

Our idea was to follow a method similar to that used in the previous chapter, but in the case of Gini and Stolarsky means, the computation of the various higher-order partial derivatives of the means becomes very complicated. This motivated the usage of the computer algebra system Maple V Release 9 to perform these vast calculations, thus making the solving of these invariance equations more simple. However, a direct calculation would be too hard also for a computer algebra system, thus we need to reformulate the problem and find the appropriate setting the computer can work with.

We note here that invariance equations for different mean values, including some special cases of the above equations were investigated by G. Toader, S. Toader and I. Costin (cf. [**16**], [**17**], [**18**], [**49**], [**50**], [**51**], [**52**]). Some of these papers also make heavy use of computer algebra systems.

The solutions of the first two from the above equations can be found in the papers [**3**] and [**4**], whereas the other four equations are solved in [**5**]. However, the approach followed here will be based on [**5**], which uses a more general setting than the previous two papers. We consider a common generalization of both the Gini and the Stolarsky means, which enables us to deal with the six equations as special cases of a more general equation.

3.1. A common generalization

First we reformulate the invariance equation (1.3).

LEMMA 3.1. (Baják–Páles [**5**]) *If $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ are homogeneous (resp. symmetric) strict means, then their Gauss composition $M \otimes N$ is also homogeneous (resp. symmetric). Furthermore, if $K : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a homogeneous strict mean then $K = M \otimes N$, i.e.,* (1.3) *holds if and only if the single-variable function $F_{K,M,N} : \mathbb{R} \to \mathbb{R}$ defined by*

$$F_{K,M,N}(u) := \ln K\left(M(\mathrm{e}^u, \mathrm{e}^{-u}), N(\mathrm{e}^u, \mathrm{e}^{-u})\right) - \ln K(\mathrm{e}^u, \mathrm{e}^{-u}) \qquad (u \in \mathbb{R}), \tag{3.7}$$

vanishes everywhere on $\mathbb{R}$. In the case when K, M, N are analytic functions, $F_{K,M,N}$ is also analytic and vanishes on $\mathbb{R}$ if and only if

$$F^{(k)}_{K,M,N}(0) = 0 \tag{3.8}$$

for all $k \in \mathbb{N}$. If, additionally M, N and K are symmetric strict means, then $F_{K,M,N}$ is an even function and $F_{K,M,N}$ vanishes on $\mathbb{R}$ if and only if (3.8) *holds for all even $k \in \mathbb{N}$.*

Proof. Assume that M and N are homogeneous strict means and let $K := M \otimes N$. Replacing x and y by tx and ty in the invariance equation (1.3) and using the homogeneity of M and N, we get

$$\tfrac{1}{t}K\big(tM(x,y), tN(x,y)\big) = \tfrac{1}{t}K(tx,ty) \qquad (t,x,y \in \mathbb{R}_+).$$

Hence, with the notation $K_t(x,y) := \frac{1}{t}K(tx,ty)$, we obtain that, for every fixed $t \in \mathbb{R}_+$, the strict mean K_t is another solution of the invariance equation (1.3), i.e., it satisfies

$$K_t\big(M(x,y), N(x,y)\big) = K_t(x,y) \qquad (x,y \in \mathbb{R}_+).$$

By the unique solvability of (1.3), it follows that $K = K_t$ for all $t \in \mathbb{R}_+$. This results the homogeneity of K.

If the means M and N are symmetric, then applying the invariance equation (1.3) twice, for $x, y \in \mathbb{R}_+$ we get

$$K(x,y) = K\big(M(x,y), N(x,y)\big) = K\big(M(y,x), N(y,x)\big) = K(y,x),$$

which proves that K is also symmetric.

It is obvious that if (1.3) holds, the function F defined in (3.7) vanishes on $\mathbb{R}$. To prove the opposite implication, write $x = \sqrt{xy}\sqrt{\frac{x}{y}}$ and $y = \sqrt{xy}\sqrt{\frac{y}{x}}$. Then, utilizing the homogeneity of the means, the invariance equation (1.3) is equivalent to

$$K\left(M\left(\sqrt{\tfrac{x}{y}}, \sqrt{\tfrac{y}{x}}\right), N\left(\sqrt{\tfrac{x}{y}}, \sqrt{\tfrac{y}{x}}\right)\right) = K\left(\sqrt{\tfrac{x}{y}}, \sqrt{\tfrac{y}{x}}\right) \qquad (x,y \in \mathbb{R}_+).$$

With the notation $t := \sqrt{\frac{x}{y}}$, the above equality can be rewritten as

$$K\left(M\left(t,t^{-1}\right), N\left(t,t^{-1}\right)\right) = K\left(t,t^{-1}\right) \qquad (t \in \mathbb{R}_+).$$

Finally, by substituting $t = \mathrm{e}^u$ and taking the logarithm of the two sides, we get that the invariance equation is equivalent to

$$F_{K,M,N}(u) = \ln K\left(M(\mathrm{e}^u, \mathrm{e}^{-u}), N(\mathrm{e}^u, \mathrm{e}^{-u})\right) - \ln K(\mathrm{e}^u, \mathrm{e}^{-u}) = 0 \qquad (u \in \mathbb{R}).$$

Here we should observe that, by the mean value property, $F_{K,M,N}(0) = 0$. Thus, the proof of the remaining part of this theorem follows from known properties of analytic functions. □

Now we consider a common generalization of the Gini and Stolarsky means. For a given Borel probability measure μ on $[0,1]$ and parameters

$r, s \in \mathbb{R}$, the two variable mean $M_{r,s,\mu}$ is defined by
(3.9)

$$M_{r,s,\mu}(x,y) = \begin{cases} \left(\dfrac{\int_0^1 (x^t y^{1-t})^r \mathrm{d}\mu(t)}{\int_0^1 (x^t y^{1-t})^s \mathrm{d}\mu(t)} \right)^{\frac{1}{r-s}}, & \text{if } r \neq s \\ \exp \left(\dfrac{\int_0^1 \ln(x^t y^{1-t})(x^t y^{1-t})^r \mathrm{d}\mu(t)}{\int_0^1 (x^t y^{1-t})^r \mathrm{d}\mu(t)} \right), & \text{if } r = s \end{cases} \qquad (x, y \in \mathbb{R}_+).$$

For fixed $x, y \in \mathbb{R}_+$, the function $(r, s) \mapsto M_{r,s,\mu}(x, y)$ is continuous (moreover infinitely many times continuously differentiable) on $\mathbb{R} \times \mathbb{R}$.

The mean defined in (3.9) can be considered as a common generalization of both of the Gini and of the Stolarsky means, because if μ is equal to $\frac{\delta_0+\delta_1}{2}$ (where δ_x stands for the Dirac measure concentrated at x), we get the Gini mean $G_{r,s}$, and if μ is equal to the Lebesgue measure, we get the Stolarsky mean $S_{r,s}$. Therefore each of the six invariance equations which involve Gini or Stolarsky means are particular cases of the following equation

$$M_{p,q,\kappa}(M_{a,b,\mu}(x,y), M_{c,d,\nu}(x,y)) = M_{p,q,\kappa}(x,y) \qquad (x, y \in \mathbb{R}_+), \tag{3.10}$$

where each of μ, ν and κ is equal to the Lebesgue measure on $[0, 1]$ or to the measure $\frac{\delta_0+\delta_1}{2}$. In view of Lemma 3.1, the above invariance equation holds if and only if, for all $u \in \mathbb{R}$,

$$\begin{aligned} F_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(u) :=& \ln\big(M_{p,q,\kappa}(M_{a,b,\mu}(\mathrm{e}^u, \mathrm{e}^{-u}), M_{c,d,\nu}(\mathrm{e}^u, \mathrm{e}^{-u}))\big) \\ &- \ln\big(M_{p,q,\kappa}(\mathrm{e}^u, \mathrm{e}^{-u})\big) = 0, \end{aligned}$$

i.e., for all $k \in \mathbb{N}$,

$$F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(0) = 0. \tag{3.11}$$

In order to get a more useful representation of the means $M_{r,s,\mu}$, introduce the function $L_\mu : \mathbb{R} \to \mathbb{R}_+$ by

$$L_\mu(z) := \ln\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \mu_k \right),$$

where μ_k denotes the kth central moment of the measure μ defined as

$$\mu_k := \int_0^1 (t - \tfrac{1}{2})^k \mathrm{d}\mu(t) \qquad (k \in \mathbb{N} \cup \{0\}).$$

Assuming that μ is symmetric with respect to $\frac{1}{2}$ (i.e., $\mu(A) = \mu(1-A)$ for all Borel sets $A \subseteq [0,1]$) it follows that $\mu_{2k-1} = 0$ for all $k \in \mathbb{N}$.

Lemma 3.2. (Baják–Páles [**5**]) *Let μ be a Borel probability measure on $[0,1]$ and $r, s \in \mathbb{R}$. Then*

$$M_{r,s,\mu}(x,y) = \exp\left(M^*_{r,s,\mu}(\ln x, \ln y)\right) \qquad (x, y \in \mathbb{R}_+), \tag{3.12}$$

where $M^*_{r,s,\mu} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is defined by

$$M^*_{r,s,\mu}(u,v) := \begin{cases} \dfrac{u+v}{2} + \dfrac{L_\mu(r(u-v)) - L_\mu(s(u-v))}{r-s}, & \text{if } r \neq s, \\ \dfrac{u+v}{2} + (u-v)L'_\mu(r(u-v)), & \text{if } r = s. \end{cases} \tag{3.13}$$

Proof. We have the following computation:

$$\begin{aligned}
\int_0^1 (x^t y^{1-t})^r \mathrm{d}\mu(t) &= \int_0^1 (\sqrt{xy})^r (x^r)^{t-\frac{1}{2}} (y^r)^{\frac{1}{2}-t} \,\mathrm{d}\mu(t) \\
&= (\sqrt{xy})^r \int_0^1 \left(\mathrm{e}^{r(\ln x - \ln y)}\right)^{t-\frac{1}{2}} \mathrm{d}\mu(t) \\
&= (\sqrt{xy})^r \int_0^1 \sum_{k=0}^{\infty} \frac{(r(\ln x - \ln y))^k}{k!} (t - \tfrac{1}{2})^k \mathrm{d}\mu(t) \\
&= (\sqrt{xy})^r \sum_{k=0}^{\infty} \frac{(r(\ln x - \ln y))^k}{k!} \mu_k \\
&= (\sqrt{xy})^r \exp\left(L_\mu(r(\ln x - \ln y))\right).
\end{aligned}$$

From this formula, the statement of the lemma immediately follows if $r \neq s$. By passing the limit $s \to r$, the formula for the case $r = s$ can also be obtained. □

In order to obtain high-order approximation of the mean considered in (3.9), for a Borel probability measure μ and $m \in \mathbb{N}$, we define the following truncated function as

$$L_{\mu;m}(z) := \ln\left(\sum_{k=0}^{m} \frac{z^k}{k!}\mu_k\right) \qquad (z \in \mathbb{R}), \tag{3.14}$$

and, if $r, s \in \mathbb{R}$,

$$M_{r,s,\mu;m}(x, y) = \exp\left(M^*_{r,s,\mu;m}(\ln x, \ln y)\right) \qquad (x, y \in \mathbb{R}_+), \tag{3.15}$$

where $M^*_{r,s,\mu;m} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is defined by

$$M^*_{r,s,\mu;m}(u, v) := \begin{cases} \dfrac{u+v}{2} + \dfrac{L_{\mu;m}(r(u-v)) - L_{\mu;m}(s(u-v))}{r-s}, & \text{if } r \neq s, \\ \dfrac{u+v}{2} + (u-v)L'_{\mu;m}(r(u-v)), & \text{if } r = s. \end{cases} \tag{3.16}$$

The following lemma will play a very important role in solving the invariance equations.

Lemma 3.3. (Baják–Páles [**5**]) *Let μ be a Borel probability measure. Then, for all $m, i \in \mathbb{N}_0$ with $i \leqslant m$,*

$$(L^{(i)}_{\mu}(0)) = (L^{(i)}_{\mu;m}(0)). \tag{3.17}$$

Furthermore, for all $r, s \in \mathbb{R}$ and $m, i, j \in \mathbb{N}_0$ with $i + j \leqslant m$,

$$\partial_1^i \partial_2^j M_{r,s,\mu}(1, 1) = \partial_1^i \partial_2^j M_{r,s,\mu;m}(1, 1). \tag{3.18}$$

Proof. Define the functions $S_\mu : \mathbb{R} \to \mathbb{R}$ and $S_{\mu;m} : \mathbb{R} \to \mathbb{R}$ by

$$S_\mu(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}\mu_k \qquad \text{and} \qquad S_{\mu;m}(z) := \sum_{k=0}^{m} \frac{z^k}{k!}\mu_k.$$

Computing their ith derivatives, for all $0 \leqslant i \leqslant m$, we trivially have that

$$S^{(i)}_\mu(0) = S^{(i)}_{\mu;m}(0) = \mu_i.$$

On the other hand, the functions L_μ and $L_{\mu;m}$ are of the form $L_\mu = \ln \circ S_\mu$ and $L_{\mu;m} = \ln \circ S_{\mu;m}$, respectively. Thus, (3.17) follows by induction on i by using the smoothness of the logarithm function.

To prove (3.18), let $i, j, m \in \mathbb{N}$ such that $i + j \leqslant m$. By the identities (3.12) and (3.15), the partial derivatives

$$\partial_1^i \partial_2^j M_{r,s,\mu}(1, 1) \quad \text{and} \quad \partial_1^i \partial_2^j M_{r,s,\mu;m}(1, 1)$$

are expressed via the same expression in terms of the partial derivatives of

$$\partial_1^\alpha \partial_2^\beta M^*_{r,s,\mu}(0,0) \quad \text{and} \quad \partial_1^\alpha \partial_2^\beta M^*_{r,s,\mu;m}(0,0),$$

respectively, where $0 \leqslant \alpha \leqslant i$ and $0 \leqslant \beta \leqslant j$. Finally, observe that the partial derivatives

$$\partial_1^\alpha \partial_2^\beta M^*_{r,s,\mu}(0,0) \quad \text{and} \quad \partial_1^\alpha \partial_2^\beta M^*_{r,s,\mu;m}(0,0)$$

can be computed in the same form involving the derivatives

$$L_\mu^{(\gamma)}(0) \quad \text{and} \quad L_{\mu;m}^{(\gamma)}(0),$$

respectively, where $\gamma \leqslant \alpha+\beta \leqslant i+j \leqslant m$. Hence, the equality (3.18) follows from the first statement of this theorem. □

Using the following corollary, the computation of the higher order derivatives $F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}$ at 0 can be replaced by the computation of the derivatives $F^{(k)}_{M_{p,q,\kappa;m},M_{a,b,\mu;m},M_{c,d,\nu;m}}$ at 0 provided that $k \leqslant m$. Thus, we will be able to check condition (3.11) more easily in the sequel.

Corollary 3.1. (Baják–Páles [**5**]) *Let* $a,b,c,d,p,q \in \mathbb{R}$ *and* μ,ν,κ *be Borel probability measures on* $[0,1]$. *Then, for all* $k,m \in \mathbb{N}_0$ *with* $k \leqslant m$,

$$F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(0) = F^{(k)}_{M_{p,q,\kappa;m},M_{a,b,\mu;m},M_{c,d,\nu;m}}(0).$$

Proof. The statement easily follows from the second assertion of the previous lemma. □

3.2. The invariance equations

While solving the six invariance equations we will consider them as special cases of equation (3.10). The common sketch of finding the solutions of these equations is the following: By Corollary 3.1, we can consider each equation as the appropriate special case of the identity

(3.19)
$$F_{M_{p,q,\kappa;k},M_{a,b,\mu;k},M_{c,d,\nu;k}}(u) = M^*_{p,q,\kappa;k}(M^*_{a,b,\mu;k}(u,-u), M^*_{c,d,\nu;k}(u,-u)) - M^*_{p,q,\kappa;k}(u,-u) = 0.$$

We get solutions for the unknown parameters a,b,c,d,p,q by computing the Taylor coefficients of the function

$$F_{M_{p,q,\kappa;k},M_{a,b,\mu;k},M_{c,d,\nu;k}}$$

at $x = 0$ up to a sufficiently high order, and determining the conditions when all these coefficients vanish. This function is even due to the symmetry of the means, thus all coefficients of odd order are zero. Therefore in every case we have to differentiate up to 12th order to get sufficiently many conditions for the six parameters. Our task will be to determine the common roots of this system of six polynomial equations for six unknowns. The simple forms of the low-order coefficients immediately help us to reduce the number of parameters. Despite the fact that the computer can perform the calculations easily, the form of the higher-order coefficients is very complicated, as they involve high powers of the unknown parameters. These polynomials can be factorized, thus, by examining the factors, we can deduce solutions of the equations. However, after the factorization, besides the simple factors, these coefficients always contain higher-order polynomials of the unknowns. To determine if they have common roots, we will calculate the resultants of these polynomials with respect to one of the variables. Common roots can occur if and only if the resultants are zero (cf., e.g. [**15**], [**43**]). Thus we can get new conditions for the unknown parameters. In many cases, these resultants can also be factorized and by analyzing these factors, we can obtain new solutions of the equations. In the most complicated problems we have to calculate resultants of the resultants (with respect to another variable) to examine the cases when those resultants vanish. The exact forms of the resultants will mostly be suppressed as they contain very large constants and very high powers of the unknows, but the exact code to perform these calculations will always be given. The computation time on a dual-core processor does not take considerable time in any case.

The Gini and the Stolarsky means can be written as special cases of the mean defined in (3.9), with the help of the Dirac and the Lebesgue measure, respectively. To calculate the central moments of these measures, which will be needed for the calculations, let $\mathfrak{m}$ denote the measure $\frac{\delta_0+\delta_1}{2}$ and $\mathfrak{n}$ denote the Lebesgue measure on $[0,1]$. The ith central moments ($i \in \mathbb{N}_0$) of these measures are 0 whenever i is odd and

$$\mathfrak{m}_i = \frac{1}{2^i}, \qquad \mathfrak{n}_i = \frac{1}{(i+1)2^i},$$

when i is even.

In the syntax of the Maple language, we define these central moments as, for $0 \leqslant i \leqslant 6$,

```
> for i from 0 to 6 do m[2*i]:=1/2^(2*i) od:
> for i from 0 to 6 do n[2*i]:=1/((2*i+1)*2^(2*i)) od:
```

(When a Maple command is terminated by the semicolon sign “ : ”, the output of the command is suppressed, otherwise the commands are ended by

"$;$" to obtain a visible output.) Higher order central moments will not be needed in the calculations, because we do not have to use approximations of the means higher than 12th order.

The functions $L, K : \mathbb{R} \times \mathbb{N}_0 \to \mathbb{R}$ are defined by $L(u,k) := L_{\mathfrak{m};k}(u)$ and $K(u,k) := L_{\mathfrak{n};k}(u)$, which in the computer algebra language can be executed as

```
> L:=(u,k)->ln(add((u^(2*i))*m[2*i]/(2*i)!,i=0..k)):
> K:=(u,k)->ln(add((u^(2*i))*n[2*i]/(2*i)!,i=0..k)):
```

With the help of the functions L and K, we define $G, H : \mathbb{R}^4 \times \mathbb{N}_0 \to \mathbb{R}$ by

$$G(r,s,u,v,k) := M^*_{r,s,\mathfrak{m};k}(u,v)$$

and

$$H(r,s,u,v,k) := M^*_{r,s,\mathfrak{n};k}(u,v).$$

While using the computer algebra package, the functions G and H will play the roles of the approximations of the Gini and the Stolarsky means, respectively. We can give the order of the approximation as another variable, thus, to simplify the calculations, we can use as low-order approximation as possible in the computation of the higher-order derivatives. In the Maple language to define the functions G and H, we do

```
> G:=(r,s,u,v,k)->(u+v)/2+(L(r*(u-v),k)
     -L(s*(u-v),k))/(r-s):
> H:=(r,s,u,v,k)->(u+v)/2+(K(r*(u-v),k)
     -K(s*(u-v),k))/(r-s):
```

Using the functions G and H we can express the six special cases of (3.19) as follows

(3.20)

$$\begin{aligned}
&F_{M_{p,q,\mathfrak{m};k},M_{a,b,\mathfrak{m};k},M_{c,d,\mathfrak{m};k}}(u)\\
&\quad = G(p,q,G(a,b,u,-u,k),G(c,d,u,-u,k),k) - G(p,q,u,-u,k) = 0,\\
&F_{M_{p,q,\mathfrak{n};k},M_{a,b,\mathfrak{n};k},M_{c,d,\mathfrak{n};k}}(u)\\
&\quad = H(p,q,H(a,b,u,-u,k),H(c,d,u,-u,k),k) - H(p,q,u,-u,k) = 0,\\
&F_{M_{p,q,\mathfrak{m};k},M_{a,b,\mathfrak{n};k},M_{c,d,\mathfrak{m};k}}(u)\\
&\quad = G(p,q,H(a,b,u,-u,k),G(c,d,u,-u,k),k) - G(p,q,u,-u,k) = 0,\\
&F_{M_{p,q,\mathfrak{m};k},M_{a,b,\mathfrak{n};k},M_{c,d,\mathfrak{n};k}}(u)\\
&\quad = G(p,q,H(a,b,u,-u,k),H(c,d,u,-u,k),k) - G(p,q,u,-u,k) = 0,
\end{aligned}$$

$$
\begin{aligned}
&F_{M_{p,q,\mathfrak{n};k},M_{a,b,\mathfrak{m};k},M_{c,d,\mathfrak{m};k}}(u)\\
&\quad = H(p,q,G(a,b,u,-u,k),G(c,d,u,-u,k),k) - H(p,q,u,-u,k) = 0,\\
&F_{M_{p,q,\mathfrak{n};k},M_{a,b,\mathfrak{m};k},M_{c,d,\mathfrak{n};k}}(u)\\
&\quad = H(p,q,G(a,b,u,-u,k),H(c,d,u,-u,k),k) - H(p,q,u,-u,k) = 0.
\end{aligned}
$$

These equations correspond to equations (3.1)-(3.6). The first two involve only Gini and Stolarsky means, respectively, while the remaining four equations are mixed equations. We note here that in view of the symmetry of the Gini and the Stolarsky means in the parameters, we may always assume that $a \geqslant b$, $c \geqslant d$ and $p \geqslant q$ in the sequel.

3.2.1. The invariance equation for Gini means.

First we recall the characterization of the equality of two variable Gini means.

LEMMA 3.4. (Páles [**41**]) *Let $a, b, c, d \in \mathbb{R}$. Then the identity*

$$G_{a,b}(x,y) = G_{c,d}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if

- *(i) either $a + b = c + d = 0$ and, in this case, the two means are equal to the geometric mean;*
- *(ii) or $\{a, b\} = \{c, d\}$.*

We give the general solution of the invariance equation (3.1).

THEOREM 3.1. (Baják–Páles [**3**]) *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3.1)*, i.e.,*

$$G_{p,q}(G_{a,b}(x,y), G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if

- *(i) either $a + b = c + d = p + q = 0$, i.e., all the three means are equal to the geometric mean;*
- *(ii) or $\{a, b\} = \{c, d\} = \{p, q\}$, i.e., all the three means are equal to each other;*
- *(iii) or $\{a, b\} = \{-c, -d\}$ and $p + q = 0$, i.e., $G_{p,q}$ is the geometric mean and $G_{a,b} = G_{-c-d}$;*
- *(iv) or there exist $u, v \in \mathbb{R}$ such that $\{a, b\} = \{u + v, v\}$, $\{c, d\} = \{u - v, -v\}$, and $\{p, q\} = \{u, 0\}$ (in this case, $G_{p,q}$ is a power mean);*

(v) *or there exists* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{3w, w\}$, $c + d = 0$, *and* $\{p, q\} = \{2w, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{c,d}$ *is the geometric mean);*

(vi) *or there exists* $w \in \mathbb{R}$ *such that* $a + b = 0$, $\{c, d\} = \{3w, w\}$, *and* $\{p, q\} = \{2w, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{a,b}$ *is the geometric mean).*

As an immediate consequence, we obtain the following solution for the Matkowski–Sutô equation, i.e., when $G_{p,q}$ is equal to the arithmetic mean.

Corollary 3.2. (Baják–Páles [**3**]) *Let* $a, b, c, d \in \mathbb{R}$. *Then the Matkowski–Sutô-type equation*

$$G_{a,b}(x, y) + G_{c,d}(x, y) = x + y \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if

(i) *either* $\{a, b\} = \{c, d\} = \{1, 0\}$, *i.e., the two means are equal to the arithmetic mean;*

(ii) *or there exist* $v \in \mathbb{R}$ *such that* $\{a, b\} = \{1 + v, v\}$, $\{c, d\} = \{1 - v, -v\}$;

(iii) *or* $\{a, b\} = \left\{\frac{3}{2}, \frac{1}{2}\right\}$ *and* $c + d = 0$ *(in this case,* $G_{c,d}$ *is the geometric mean);*

(iv) *or* $a + b = 0$ *and* $\{c, d\} = \left\{\frac{3}{2}, \frac{1}{2}\right\}$ *(in this case,* $G_{a,b}$ *is the geometric mean).*

Proof of Theorem 3.1. The proof of the sufficiency of conditions (i)–(vi) is easy calculation, thus we have to prove the necessity of these conditions.

Using Lemma 3.1 and what we established previously, the invariance equation is equivalent to

$$F_{M_{p,q,\mathfrak{m};k}, M_{a,b,\mathfrak{m};k}, M_{c,d,\mathfrak{m};k}}(u) = G(p, q, G(a, b, u, -u, k), G(c, d, u, -u, k), k) - G(p, q, u, -u, k) = 0,$$

which, by Corollary 3.1, holds if and only if

$$F^{(k)}(0, k) = F^{(k)}_{M_{p,q,\mathfrak{m};k}, M_{a,b,\mathfrak{m};k}, M_{c,d,\mathfrak{m};k}}(0) = 0 \qquad (k \in \mathbb{N}), \tag{3.21}$$

where, in view of the first identity in (3.20), $F : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ and $k \in \mathbb{N}$ as

$$F(x, k) := G(p, q, G(a, b, x, -x, k), G(c, d, x, -x, k), k) - G(p, q, x, -x, k),$$

i.e., in the Maple language we set

```
> F:=(x,k)->G(p,q,G(a,b,x,-x,k),G(c,d,x,-x,k),k)
            -G(p,q,x,-x,k);
```

To derive the necessity of the conditions of Theorem 3.1, we check equation (3.21) only for the values $k = 2, 4, 6, 8, 10, 12$.

First we evaluate the second-order Taylor coefficient C_2 of $F(\cdot, 2)$ at $x = 0$:

```
> C[2]:=simplify(coeftayl(F(x,2),x=0,2));
```

which yields

$$C_2 := \frac{1}{4}a + \frac{1}{4}b + \frac{1}{4}c + \frac{1}{4}d - \frac{1}{2}p - \frac{1}{2}q$$

(We note here that the Maple-definition of the function G is valid only if $(p-q)(a-b)(c-d) \neq 0$, however, the Taylor coefficient C_2 and also the subsequent ones, are correct also in the singular case $(p-q)(a-b)(c-d) = 0$.) From (3.21) we have that all Taylor coefficients have to be zero, therefore we obtain our first necessary condition:

$$\frac{a+b+c+d}{4} = \frac{p+q}{2}.$$

If one tries to compute the higher-order coefficients C_4, $C_6, \ldots,$ then the expressions obtained are so complicated that it is hard to get further information. To show this, we calculate the 4th order Taylor coefficient by the command

```
> C[4]:=simplify(coeftayl(F(x,4),x=0,4));
```

Here we have that

$$\begin{aligned} C_4 :=& \frac{b^2\,p}{32} + \frac{a\,b\,p}{16} + \frac{a\,b\,q}{16} + \frac{a^2\,q}{32} + \frac{a^2\,p}{32} + \frac{c^2\,p}{32} + \frac{c^2\,q}{32} + \frac{d^2\,p}{32} + \frac{d^2\,q}{32} \\ &- \frac{a\,d\,q}{16} - \frac{a\,c\,p}{16} - \frac{a\,c\,q}{16} - \frac{a\,d\,p}{16} - \frac{b\,d\,q}{16} - \frac{b\,c\,p}{16} - \frac{b\,c\,q}{16} - \frac{b\,d\,p}{16} \\ &+ \frac{b^2\,q}{32} + \frac{c\,d\,p}{16} + \frac{c\,d\,q}{16} - \frac{b^3}{24} - \frac{d^3}{24} - \frac{d\,c^2}{24} - \frac{d^2\,c}{24} - \frac{b\,a^2}{24} - \frac{b^2\,a}{24} \\ &- \frac{a^3}{24} - \frac{c^3}{24} + \frac{q^3}{12} + \frac{p^3}{12} + \frac{p\,q^2}{12} + \frac{q\,p^2}{12} \end{aligned}$$

In order to simplify the evaluation of the higher-order Taylor coefficients C_k, we introduce the notations

$$w := \frac{a+b+c+d}{4} = \frac{p+q}{2},$$
$$v := \frac{a+b-(c+d)}{4},$$
$$t := \left(\frac{p-q}{2}\right)^2,$$
$$r := \frac{(a-b)^2+(c-d)^2}{8},$$
$$s := \frac{(a-b)^2-(c-d)^2}{8}.$$

(In these definitions we utilized the symmetry of the mean in the parameters, and in the case of w also the condition that $C_2 = 0$.) Then, provided that $a \geqslant b$, $c \geqslant d$ and $p \geqslant q$, we can express the parameters a, b, c, d, p, q in the following form:

```
> a:=w+v+sqrt(r+s); b:=w+v-sqrt(r+s);
  c:=w-v+sqrt(r-s); d:=w-v-sqrt(r-s);
  p:=w+sqrt(t); q:=w-sqrt(t);
```

$$a := w + v + \sqrt{r+s}$$
$$b := w + v - \sqrt{r+s}$$
$$c := w - v + \sqrt{r-s}$$
$$d := w - v - \sqrt{r-s}$$
$$p := w + \sqrt{t}$$
$$q := w - \sqrt{t}$$

These parameter transformations proved to be the most useful in the subsequent calculations, because they provide the most simple form for the higher-order coefficients.

Now we evaluate again the 4th order Taylor coefficient by inputting:

```
> C[4]:=simplify(coeftayl(F(x,4),x=0,4));
```

Then we obtain a much easier form than previously:

$$C_4 := \frac{1}{3}tw - \frac{1}{3}vs - \frac{1}{3}wr$$

The condition $C_4 = 0$ yields that $wt = wr + vs$. If $w = 0$, then $p + q = 0$ and hence (by Lemma 3.4) $G_{p,q}$ is equal to the geometric mean. Therefore the invariance equation (3.1) can be rewritten as

$$G_{a,b}(x, y)G_{c,d}(x, y) = xy \qquad (x, y \in \mathbb{R}_+).$$

This results

$$G_{a,b}(x, y) = \frac{1}{G_{c,d}(1/x, 1/y)} = G_{-c,-d}(x, y) \qquad (x, y \in \mathbb{R}_+).$$

Using Lemma 3.4 again, this identity yields that either $a + b = c + d = 0$ or $\{a, b\} = \{-c, -d\}$ must hold. Together with $p + q = 0$, these equations show that either condition (i) or condition (iii) of Theorem 3.1 must be satisfied.

In the rest of the proof, we assume that w is not zero. Then, we can express t in terms of w, v, r, s:

```
> t:=r+v*s/w;
```

$$t := r + \frac{vs}{w} \tag{3.22}$$

Next, we evaluate the 6th order Taylor coefficient:

```
> C[6]:=simplify(coeftayl(F(x,6),x=0,6));
```

$$C_6 := \frac{-2(-3w^2s^2 + 3v^2s^2 - 15w^2rv^2 - 5w^3sv - 10v^3sw + 15w^4v^2)}{45w}$$

If $v = 0$, then the condition $C_6 = 0$ simplifies to $w^2s^2 = 0$, whence $s = 0$ follows. Therefore (3.22) yields $t = r$ and we obtain that $a = c = p$ and $b = d = q$ which means that condition (ii) of Theorem 3.1 must be fulfilled.

In the rest of the proof, we assume that v is also not zero. We can observe that the 6th order coefficient C_6 does not involve higher-order powers of r. Therefore the equation $C_6 = 0$ can be solved for r.

```
> r:=(15*w^4*v^2-3*w^2*s^2+3*v^2*s^2-5*w^3*v*s
    -10*w*v^3*s)/(15*w^2*v^2);
```

$$r := \frac{15w^4v^2 - 3w^2s^2 + 3v^2s^2 - 5w^3vs - 10wv^3s}{15w^2v^2} \tag{3.23}$$

Finally, we evaluate the 8th order Taylor coefficient of $F(\cdot, 8)$, the 10th order Taylor coefficient of $F(\cdot, 10)$ and the 12th order Taylor coefficient of $F(\cdot, 12)$, respectively, at $x = 0$:

```
> C[8]:=factor(simplify(coeftayl(F(x,8),x=0,8)));
> C[10]:=factor(simplify(coeftayl(F(x,10),x=0,10)));
> C[12]:=factor(simplify(coeftayl(F(x,12),x=0,12)));
```

We have that

$$\begin{aligned}C_8 := \frac{(v-w)(v+w)s}{70875w^3v^2}&(2100w^3v^5 - 3850w^2v^4s + 4200w^5v^3 - 255wv^3s^2\\&+ 153v^2s^3 - 9245w^4v^2s - 7395w^3vs^2 - 153w^2s^3)\end{aligned}$$

$$\begin{aligned}C_{10} := \frac{2(v-w)(v+w)s}{1063125w^5v^4}&(28500w^5v^9 - 59675w^4v^8s + 34470w^3v^7s^2\\&+ 20100w^7v^7 - 299575w^6v^6s - 4260w^2v^6s^3 - 73200w^5v^5s^2\\&- 930wv^5s^4 + 66600w^9v^5 + 4805w^4v^4s^3 - 286500w^8v^4s\\&+ 279v^4s^5 - 169020w^7v^3s^2 - 16740w^3v^3s^4 - 558w^2v^2s^5\\&+ 45955w^6v^2s^3 + 17670w^5vs^4 + 279w^4s^5)\end{aligned}$$

$$\begin{aligned}C_{12} := \frac{2(v-w)(v+w)s}{2631234375w^7v^6}&(- 3272692500w^{10}v^8s + 22181100w^3v^9s^4\\&- 54365475w^6v^8s^3 - 25317375w^8v^6s^3 + 335826w^4v^2s^7\\&- 559710wv^7s^6 - 215221875w^6v^{12}s + 22875570w^6v^4s^5\\&+ 16977870w^5v^3s^6 - 7649370w^3v^5s^6 - 1246797750w^8v^{10}s\\&- 34684335w^8v^2s^5 - 777170000w^{11}v^5s^2 - 159926550w^7v^5s^4\\&- 335826w^2v^4s^7 + 641072375w^{10}v^4s^3 + 270963000w^5v^{11}s^2\\&- 1046615000w^9v^7s^2 + 11659365w^4v^6s^5 - 133190250w^5v^7s^4\\&- 1967022000w^{12}v^6s + 177650700w^9v^3s^4 - 8768790w^7vs^6\\&+ 385915750w^7v^9s^2 + 149400w^2v^8s^5 - 98002025w^4v^{10}s^3\\&+ 76725000w^7v^{13} - 57172500w^9v^{11} - 478665000w^{11}v^9\\&+ 188100000w^{13}v^7 - 111942w^6s^7 + 111942v^6s^7)\end{aligned}$$

The coefficients in C_8, C_{10}, C_{12} are obviously zero if $s(v-w)(v+w) = 0$. Thus, we have to consider the sub-cases $s = 0$, $v = w$ and $v = -w$.

In the case $s = 0$, (3.22) and (3.23) imply that $t = r = w^2$. Then we get that

$$\{a, b\} = \{2w + v, v\}, \qquad \{c, d\} = \{2w - v, -v\}, \qquad \{p, q\} = \{2w, 0\},$$

i.e., condition (iv) holds with $u := 2w$. Conversely, if condition (iv) holds and $u \neq 0$, then we have

(3.24)

$$G_{p,q}(G_{a,b}(x,y), G_{c,d}(x,y)) = G_{u,0}(G_{u+v,v}(x,y), G_{u-v,-v}(x,y)) =$$

$$\left(\frac{x^{u+v}+y^{u+v}}{2(x^v+y^v)} + \frac{x^{u-v}+y^{u-v}}{2(x^{-v}+y^{-v})}\right)^{\frac{1}{u}} = \left(\frac{x^{u+v}+y^{u+v}}{2(x^v+y^v)} + \frac{x^u y^v + y^u x^v}{2(x^v+y^v)}\right)^{\frac{1}{u}} =$$

$$\left(\frac{x^u+y^u}{2}\right)^{\frac{1}{u}} = G_{u,0}(x,y) = G_{p,q}(x,y).$$

Thus (3.1) is satisfied if $u \neq 0$. If $u = 0$, then the parameters also fulfill condition (iii), hence (3.1) holds in this case, too.

If $v = w$, then, by (3.22) and (3.23), $r = w^2 - s$ and $t = r + s = w^2$, respectively. Hence,

$$\{a,b\} = \{3w, w\}, \qquad c + d = 0, \qquad \{p,q\} = \{2w, 0\},$$

i.e., condition (v) holds. Conversely, if condition (v) holds and $w \neq 0$, then, using the identity (3.24) with $u := 2w$, $v := w$, we have

$$G_{p,q}(G_{a,b}(x,y), G_{c,d}(x,y)) =$$
$$G_{2w,0}(G_{2w+w,w}(x,y), G_{2w-w,-w}(x,y)) = G_{2w,0}(x,y) = G_{p,q}(x,y),$$

which shows that (3.1) is fulfilled. If $w = 0$, then all the three means are geometric means, and hence (3.1) holds trivially.

The last case when $v = -w$ holds, similarly to the case $v = w$, implies that condition (vi) is valid. If condition (vi) holds, then (3.1) can also be verified.

In the rest of the proof, we can assume that $s(v+w)(v-w)$ is not zero. The Taylor coefficients C_8, C_{10} and C_{12} are of the form

$$C_8 = \frac{(v-w)(v+w)s}{70875 w^3 v^2} P_8, \qquad C_{10} = \frac{2(v-w)(v+w)s}{1063125 w^5 v^4} P_{10}$$

and

$$C_{12} = \frac{2(v-w)(v+w)s}{2631234375 w^7 v^6} P_{12},$$

where P_8, P_{10} and P_{12} are polynomials of the variables v, w and s. They can be obtained by the following Maple commands (whose output is suppressed):

```
> P[8]:=op(5,C[8]): P[10]:=op(5,C[10]):
  P[12]:=op(5,C[12]):
```

The equalities $C_8 = C_{10} = C_{12} = 0$ and $s(v + w)(v - w) \neq 0$ imply that $P_8 = P_{10} = P_{12} = 0$. In what follows, we show that there is no solution v, w, s to this system of equations.

The variable s is a common root of the polynomials P_8 and P_{10}. Therefore the resultant $R_{8,10}$ of these two polynomials (with respect to s) is zero:

```
R[8,10]:=factor(resultant(op(5,C[8]),op(5,C[10]),s));
```

$$R_{8,10} := 136687500 w^{15} v^{15} (v - w)^2 (v + w)^2$$
$$(1178440166794705680\, v^{18} - 3484948813233498140\,0 w^2 v^{16}$$
$$+ 27095657773476976150\, w^4 v^{14} + 21571639531850248315\,39 w^6 v^{12}$$
$$+ 19335728720363587723895\, w^8 v^{10} + 7709834076285490475883\,8 w^{10} v^8$$
$$+ 1355417160647340535502\,90 w^{12} v^6 + 444986124077664744\,66 w^{18}$$
$$+ 2100034048587009260985\, w^{16} v^2 + 5297452851848849749955\,7 w^{14} v^4)$$

After the computation of $R_{8,10}$, we determine the resultant $R_{8,12}$ of P_8 and P_{12}, and the resultant $R_{10,12}$ of P_{10} and P_{12} respectively, by performing the commands

```
> R[8,12]:=factor(resultant(op(5,C[8]),op(5,C[12]),s)):
  R[10,12]:=factor(resultant(op(5,C[10]),op(5,C[12]),s)):
```

We do not output the explicit results of these computations, but we get that

$$R_{8,10} = K_{8,10} w^{15} v^{15} (w + v)^2 (w - v)^2 P_{8,10}(v, w),$$
$$R_{8,12} = K_{8,12} w^{21} v^{21} (w + v)^3 (w - v)^3 P_{8,12}(v, w)$$

and

$$R_{10,12} = K_{10,12} w^{35} v^{35} (w + v)^6 (w - v)^6 P_{10,12}(v, w),$$

where $K_{8,10}$, $K_{8,12}$ and $K_{10,12}$ are nonzero real constants and $P_{8,10}$, $P_{8,12}$ and $P_{10,12}$ are polynomials of v and w. Since the cases $w = 0$, $v = 0$, $v = w$ and $v = -w$ have already been discussed, the only remaining possible solution can occur as a common root (v, w) of the following system of equations:

$$P_{8,10}(v, w) = 0, \quad P_{8,12}(v, w) = 0, \quad P_{10,12} = 0.$$

However, by calculating the resultant Q_1 of polynomials $P_{8,10}$ and $P_{8,12}$ with respect to v

```
Q[1]:=factor(resultant(op(4,C[8,10]),op(4,C[8,12]),v));
```

we have that the resultant Q_1 is of the form

$$Q_1 = q_1 w^{468},$$

while calculating the resultant Q_2 of polynomials $P_{8,12}$ and $P_{10,12}$ with respect to v

```
Q[2]:=factor(resultant(op(4,C[8,12]),op(4,C[10,12]),v));
```

we have that the resultant Q_2 is of the form

$$Q_2 = q_2 w^{1196},$$

where q_1 and q_2 are nonzero (large) constants. This yields that $P_{8,10} = P_{8,12} = P_{10,12} = 0$ is not solvable for v and w since we already have that $w \neq 0$. Thus the solution set of (3.1) is described completely. □

As an application of Theorem 3.1, consider the following problem. If $x, y \in \mathbb{R}_+$, determine the Gauss composition of the Gini means $G_{4,1}(x, y)$ and $G_{2,-1}(x, y)$, i.e., determine the common limit of the sequences defined as follows:

$$x_1 := x, \qquad y_1 := y,$$

$$x_{n+1} := G_{4,1}(x_n, y_n) = \sqrt[3]{\frac{x_n^4 + y_n^4}{x_n + y_n}}, \qquad y_{n+1} := G_{2,-1}(x_n, y_n) = \sqrt[3]{\frac{x_n^2 + y_n^2}{\frac{1}{x_n} + \frac{1}{y_n}}}$$

where $n \in \mathbb{N}$.

Using case (iv) of Theorem 3.1,

$$G_{u+v,v} \otimes G_{u-v,-v} = G_{u,0},$$

with $u = 3$ and $v = 1$, it is very easy to determine that the Gauss composition of the two means is the power mean of exponent 3, i.e., the common limit of the two sequences is

$$G_{3,0}(x, y) = M_3(x, y) = \sqrt[3]{\frac{x^3 + y^3}{2}}.$$

3.2.2. The invariance equation for Stolarsky means.

We start with the characterization of the equality of two variable Stolarsky means, which will be needed later.

LEMMA 3.5. (Páles [**42**]) *Let $a, b, c, d \in \mathbb{R}$. Then the identity*

$$S_{a,b}(x, y) = S_{c,d}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if one of the following possibilities is valid:

(i) $a + b = c + d = 0$ *and, in this case, the two means are equal to the geometric mean;*

(ii) $\{a, b\} = \{c, d\}$.

We give the general solution of the invariance equation (3.2).

THEOREM 3.2. (Baják–Páles [**4**]) *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3.2)*, i.e.,*

$$S_{p,q}(S_{a,b}(x, y), S_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a + b = c + d = p + q = 0$*, i.e., all the three means are equal to the geometric mean;*

(ii) $\{a, b\} = \{c, d\} = \{p, q\}$*, i.e., all the three means are equal to each other;*

(iii) $\{a, b\} = \{-c, -d\}$ *and* $p + q = 0$*, i.e.,* $S_{p,q}$ *is the geometric mean and* $S_{a,b} = S_{-c,-d}$.

It is interesting to observe here that the parameter sets when (3.1) holds is much bigger then the corresponding set for (3.2).

As a consequence also in this case, we obtain the following solution for the Matkowski–Sutô equation, i.e., when $S_{p,q}$ is equal to the arithmetic mean in (3.2).

COROLLARY 3.3. (Baják–Páles [**4**]) *Let $a, b, c, d \in \mathbb{R}$. Then the Matkowski–Sutô-type equation*

$$S_{a,b}(x, y) + S_{c,d}(x, y) = x + y \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if $\{a, b\} = \{c, d\} = \{2, 1\}$*, i.e., both means are equal to the arithmetic mean.*

PROOF OF THEOREM 3.2. Using Lemma 3.1 and the previous results of this chapter, the invariance equation in this case is equivalent to

$$F_{M_{p,q,\text{n};k}, M_{a,b,\text{n};k}, M_{c,d,\text{n};k}}(u) = \\ H(p, q, H(a, b, u, -u, k), H(c, d, u, -u, k), k) - H(p, q, u, -u, k) = 0,$$

which, by Corollary 3.1, holds if and only if

$$F^{(k)}(0, k) = F^{(k)}_{M_{p,q,\text{n};k}, M_{a,b,\text{n};k}, M_{c,d,\text{n};k}}(0) = 0 \qquad (k \in \mathbb{N}), \tag{3.25}$$

where, in view of the second identity in (3.20), $F : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ and $k \in \mathbb{N}$ as

$$F(x,k) := H(p,q,H(a,b,x,-x,k),H(c,d,x,-x,k),k) - H(p,q,x,-x,k),$$

i.e., in the Maple language we set

```
> F:=(x,k)->H(p,q,H(a,b,x,-x,k),H(c,d,x,-x,k),k)
          -H(p,q,x,-x,k);
```

To derive the necessity of the conditions of Theorem 3.2, we have to check equation (3.25) for the values $k = 2, 4, 6, 8, 10, 12$.

First we evaluate the second-order Taylor coefficient C_2 of $F(\cdot, 2)$ at $x = 0$:

```
> C[2]:=simplify(coeftayl(F(x,2),x=0,2));
```

which yields

$$C_2 := \frac{1}{12}d - \frac{1}{6}p + \frac{1}{12}b - \frac{1}{6}q + \frac{1}{12}a + \frac{1}{12}c$$

(We have to note here that the Maple-definition of the function H is valid only if $(p-q)(a-b)(c-d) \neq 0$, but C_2 and all the subsequent Taylor coefficients are correct also in the singular case $(p-q)(a-b)(c-d) = 0$.)

From (3.25) we have that all Taylor coefficients have to be zero, therefore we obtain the first necessary condition, which is the same as in the case for Gini means:

$$\frac{a+b+c+d}{4} = \frac{p+q}{2}.$$

In order to simplify the evaluation of the higher-order Taylor coefficients, we introduce the same notations as previously, utilizing $C_2 = 0$:

$$\begin{aligned}
w &:= \frac{a+b+c+d}{4} = \frac{p+q}{2}, \\
v &:= \frac{a+b-(c+d)}{4}, \\
t &:= \left(\frac{p-q}{2}\right)^2, \\
r &:= \frac{(a-b)^2+(c-d)^2}{8}, \\
s &:= \frac{(a-b)^2-(c-d)^2}{8}.
\end{aligned} \tag{3.26}$$

Then, provided that $a \geqslant b$, $c \geqslant d$ and $p \geqslant q$, we can express the parameters a, b, c, d, p, q in the following form:

```
> a:=w+v+sqrt(r+s); b:=w+v-sqrt(r+s);
  c:=w-v+sqrt(r-s); d:=w-v-sqrt(r-s);
  p:=w+sqrt(t); q:=w-sqrt(t);
```

$$a := w + v + \sqrt{r+s}$$
$$b := w + v - \sqrt{r+s}$$
$$c := w - v + \sqrt{r-s}$$
$$d := w - v - \sqrt{r-s}$$
$$p := w + \sqrt{t}$$
$$q := w - \sqrt{t}$$

Now we evaluate the 4th order Taylor coefficient by performing the command

```
> C[4]:=simplify(coeftayl(F(x,4),x=0,4));
```

Then we obtain

$$C_4 := -\frac{1}{45}vs - \frac{4}{135}wv^2 + \frac{1}{45}wt - \frac{1}{45}wr$$

The condition $C_4 = 0$ yields that $wt = wr + vs + \frac{4}{3}wv^2$.

If $w = 0$, then $p + q = 0$, which means that $S_{p,q}$ is the geometric mean. Therefore the invariance equation can be written as

$$S_{a,b}(x, y)S_{c,d}(x, y) = xy \qquad (x, y \in \mathbb{R}_+).$$

This results

$$S_{a,b}(x, y) = \frac{1}{S_{c,d}(1/x, 1/y)} = S_{-c,-d}(x, y) \qquad (x, y \in \mathbb{R}_+).$$

Using Lemma 3.5, this identity yields that either $a + b = c + d = 0$ or $\{a, b\} = \{-c, -d\}$ must hold. In this case we get that one of the conditions (i) or (iii) of our theorem is valid. Conversely, if condition (i) or (iii) holds then (3.2) can easily be seen.

In the rest of the proof, we may assume that w is not zero. Then, from condition $C_4 = 0$, we can express t in terms of w, v, r, s:

```
> t:=r+v*s/w+(4/3)*v^2;
```

$$t := r + \frac{vs}{w} + \frac{4}{3}v^2 \tag{3.27}$$

Next, we evaluate the 6th order Taylor coefficient C_6 of $F(\cdot, 6)$ at $x = 0$:

```
> C[6]:=factor(simplify(coeftayl(F(x,6),x=0,6)));
```

We get that

$$C_6 := \frac{2(9w^2s^2 + 8w^2v^4 + 45w^2rv^2 + 39w^3sv + 6wv^3s - 13w^4v^2 - 9v^2s^2)}{8505w}$$

Here, if $v = 0$ then $C_6 = 0$ implies that $s = 0$. Hence, from (3.27) it follows that $t = r$ and we get that $a = c = p$ and $b = d = q$, i.e., condition (ii) of our theorem holds. Conversely, if condition (ii) holds then the invariance equation (2.8) is trivially valid.

In the rest of the proof, we may assume that v is also not zero. The 6th order coefficient C_6 does not involve higher-order powers of r, thus the equation $C_6 = 0$ can be solved for r.

```
> r:=(13*w^4*v^2-9*w^2*s^2-8*w^2v^4+9*v^2*s^2-39*w^3*v*s
     -6*w*v^3*s)/(45*w^2*v^2);
```

$$r := \frac{13w^4v^2 - 9w^2s^2 - 8w^2v^4 + 9v^2s^2 - 39w^3vs - 6wv^3s}{45w^2v^2} \tag{3.28}$$

Now, we evaluate the 8th order Taylor coefficient of $F(\cdot, 8)$, the 10th order Taylor coefficient of $F(\cdot, 10)$ and the 12th order Taylor coefficient of $F(\cdot, 12)$, respectively, at $x = 0$:

```
> C[8]:=factor(simplify(coeftayl(F(x,8),x=0,8)));
> C[10]:=factor(simplify(coeftayl(F(x,10),x=0,10)));
> C[12]:=factor(simplify(coeftayl(F(x,12),x=0,12)));
```

We have that

$$\begin{aligned} C_8 := \frac{1}{9568125w^3v^2}(&4347w^5vs^3 - 3616w^8v^4 + 1242w^2v^6s^2 + 16137w^6v^2s^2 \\ &+ 224w^4v^8 - 4644w^3v^3s^3 + 756w^3v^7s + 81w^4s^4 + 11976w^7v^3s + 81v^4s^4 \\ &+ 4992w^6v^6 - 162w^2v^2s^4 - 17379w^4v^4s^2 - 12732w^5v^5s + 297wv^5s^3) \end{aligned}$$

$$
\begin{aligned}
C_{10} := \frac{2}{2841733125w^5v^4}&(141632w^{10}v^8 - 2187w^2v^4s^6 - 58806w^3v^5s^5 \\
&+ 107406w^5v^3s^5 - 338432w^{12}v^6 + +34928w^6v^{12} - 523908w^9v^3s^3 \\
&+ 2187w^4v^2s^6 + 126459w^4v^{10}s^2 + 316272w^8v^{10} + 143541w^8v^6s^2 \\
&+ 92508w^5v^{11}s + 24948w^2v^8s^4 + 1319016w^{11}v^5s - 847044w^9v^7s \\
&+ 927324w^{10}v^4s^2 - 1197324w^6v^8s^2 - 564480w^7v^9s - 729w^6s^6 \\
&+ 55026w^3v^9s^3 - 794610w^5v^7s^3 + 672786w^6v^4s^4 + 3402wv^7s^5 \\
&- 52002w^7vs^5 - 313065w^4v^6s^4 + 1263492w^7v^5s^3 - 384669w^8v^2s^4 \\
&+ 729v^6s^6)
\end{aligned}
$$

$$
\begin{aligned}
C_{12} := \frac{2}{872767286015625w^7v^6}&(54403812w^4v^4s^8 - 36269208w^6v^2s^8 \\
&- 2472351012w^7v^3s^7 - 36269208w^2v^6s^8 - 6728400999w^4v^8s^6 \\
&- 23518313469w^8v^4s^6 + 28808296272w^{10}v^{14} - 43502512128w^{16}v^8 \\
&- 32563674368w^{14}v^{10} + 55532868672w^{12}v^{12} + 4082581552w^8v^{16} \\
&+ 51381378wv^9s^7 + 1616450580w^3v^{11}s^5 + 66139190820w^7v^7s^5 \\
&+ 5633126127w^4v^{12}s^4 - 27686800350w^5v^9s^5 + 23171026860w^{11}v^3s^5 \\
&- 198268186320w^{13}v^5s^3 + 9067302w^8s^8 - 50400222003w^{12}v^4s^4 \\
&+ 8922828051w^{10}v^2s^6 + 527871816w^2v^{10}s^6 + 2575113768w^5v^5s^7 \\
&+ 33853773582w^{10}v^6s^4 + 8610065640w^5v^{13}s^3 + 78734957472w^8v^8s^4 \\
&+ 806989878w^9vs^7 - 56327456871w^8v^{12}s^2 - 297179050266w^{10}v^{10}s^2 \\
&- 14684723544w^{13}v^9s + 203758408416w^{15}v^7s + 2387944944w^{14}v^6s^2 \\
&- 67821635178w^6v^{10}s^4 + 10150547496w^7v^{15}s + 13660568109w^6v^{14}s^2 \\
&- 84844566450w^7v^{11}s^3 + 337457994084w^{12}v^8s^2 + 9067302v^8s^8 \\
&- 3441528204w^9v^{13}s - 67735201590w^9v^9s^3 + 342237888720w^{11}v^7s^3 \\
&- 961134012w^3v^7s^7 + 20796014601w^6v^6s^6 - 63239867910w^9v^5s^5 \\
&- 195782704164w^{11}v^{11}s)
\end{aligned}
$$

The Taylor coefficients C_8, C_{10} and C_{12} are of the form

$$
C_8 = \frac{1}{9568125w^3v^2}P_8, \quad C_{10} = \frac{2}{2841733125w^5v^4}P_{10}
$$

and

$$C_{12} = \frac{2}{872767286015625 w^7 v^6} P_{12},$$

where P_8, P_{10} and P_{12} are polynomials of the variables v, w and s. The equalities $C_8 = C_{10} = C_{12} = 0$ imply that $P_8 = P_{10} = P_{12} = 0$.

The variable s is a common root of the polynomials P_8 and P_{10}. Therefore the resultant $R_{8,10}$ of these two polynomials (with respect to s) is zero:

```
> R[8,10]:=factor(resultant(op(4,C[8]),op(4,C[10]),s));
```

$$R_{8,10} := 282429536481000000000 w^{24} v^{24} (v-w)^8 (v+w)^8$$
$$(395726752304\, v^{32} - 28019198519832\, w^2 v^{30} + 1192972799035666\, w^4 v^{28}$$
$$- 36617671790074251\, w^6 v^{26} + 601554420387156651\, w^8 v^{24}$$
$$- 3652037976710860175\, w^{10} v^{22} - 1101310194408221307\, w^{12} v^{20}$$
$$+ 62048533824813847173\, w^{14} v^{18} - 175575191501013599783\, w^{16} v^{16}$$
$$+ 52614376847529172973\, w^{18} v^{14} + 435211540238087039223\, w^{20} v^{12}$$
$$- 793895884964266327270\, w^{22} v^{10} - 773252618095825970136\, w^{24} v^8$$
$$- 492199682627262911866\, w^{26} v^6 + 183522699320559043726\, w^{28} v^4$$
$$- 43030934088053846752\, w^{30} v^2 + 912066926976343384\, w^{32})$$

The resultant is zero if either $vw(v-w)(v+w) = 0$ holds or v and w are solutions of a homogeneous two variable polynomial equation of degree 32.

First, consider the case when $vw(v-w)(v+w) = 0$. We have that $vw \neq 0$, hence $(v-w)(v+w) = 0$ must hold, i.e, $v = \pm w$. Thus, from (3.27) and (3.28), we get that $r = \frac{w^2}{9} \mp s$ and $t = \frac{13}{9} w^2$, respectively.

In the case when $v = w$, the equations in (3.26) yield that

$$c = -d, \quad a = \frac{7}{3} w, \quad b = \frac{5}{3} w, \quad p = \left(1 + \frac{\sqrt{13}}{3}\right) w, \quad q = \left(1 - \frac{\sqrt{13}}{3}\right) w. \tag{3.29}$$

The first equality yields that $S_{c,d}$ is the geometric mean. Hence, we may assume that $c = -d = 1$. To simplify the computations, we can also assume that $w = 3$. We show that these parameters are not solutions of the invariance equation. For $k \in \mathbb{N}$, we now have that

$$\begin{aligned} F(x,k) :=& H\Big(3 + \sqrt{13}, 3 - \sqrt{13}, H(7,5,x,-x,k), H(1,-1,x,-x,k), k\Big) \\ & - H\Big(3 + \sqrt{13}, 3 - \sqrt{13}, x, -x, k\Big). \end{aligned}$$

In Maple, we input

```
> F:=(x,k)->H(3+sqrt(13),3-sqrt(13),H(7,5,x,-x,k),
    H(1,-1,x,-x,k),k)-H(3+sqrt(13),3-sqrt(13),x,-x,k);
```

We compute the 10th order Taylor coefficient of this function by

```
> simplify(coeftayl(F(x,10),x=0,10));
```

whence we get that this coefficient is $-\frac{12352}{5775}$, i.e., it is not zero, which means that the parameters in (3.29) do not provide solution to the invariance equation.

In the case when $v = -w$, we have that

$$a = -b, \quad c = \frac{7}{3}\,w, \quad d = \frac{5}{3}\,w, \quad p = \left(1 + \frac{\sqrt{13}}{3}\right)w, \quad q = \left(1 - \frac{\sqrt{13}}{3}\right)w,$$

whence a similar calculation as in the previous case shows that we again do not get an additional solution to the invariance equation. Thus, we have to return to the case when v and w are solutions of a homogeneous two variable polynomial equation.

The variable s is also a common root of the two polynomials P_8 and P_{12}. Therefore the resultant $R_{8,12}$ of these polynomials (with respect to s), and the resultant $R_{10,12}$ of P_{10} and P_{12} (with respect to s), is again zero:

```
> R[8,12]:=factor(resultant(op(4,C[8]),op(4,C[12]),s)):
  R[10,12]:=factor(resultant(op(4,C[10]),op(4,C[12]),s)):
```

We do not output the explicit results, but we get that

$$\begin{aligned} R_{8,10} &= K_{8,10} w^{24} v^{24} (w+v)^8 (w-v)^8 P_{8,10}(v,w), \\ R_{8,12} &= K_{8,12} w^{32} v^{32} (w+v)^{10} (w-v)^{10} P_{8,12}(v,w) \end{aligned}$$

and

$$R_{10,12} = K_{10,12} w^{48} v^{48} (w+v)^{15} (w-v)^{15} P_{10,12}(v,w),$$

where $K_{8,10}$, $K_{8,12}$ and $K_{10,12}$ are nonzero real constants and $P_{8,10}$, $P_{8,12}$ and $P_{10,12}$ are polynomials of v and w. Since the cases $w = 0$, $v = 0$, $v = w$ and $v = -w$ have already been discussed, the only remaining possible solution can occur as a common root (v, w) of the following system of equations:

$$P_{8,10}(v,w) = 0, \quad P_{8,12}(v,w) = 0, \quad P_{10,12} = 0.$$

However, by calculating the resultant Q_1 of polynomials $P_{8,10}$ and $P_{8,12}$ with respect to v

```
Q[1]:=factor(resultant(op(4,C[8,10]),op(4,C[8,12]),v));
```

we have that the resultant Q_1 is of the form

$$Q_1 = q_1 w^{1408},$$

while calculating the resultant Q_2 of polynomials $P_{8,12}$ and $P_{10,12}$ with respect to v

```
Q[2]:=factor(resultant(op(4,C[8,12]),op(4,C[10,12]),v));
```

we have that the resultant Q_2 is of the form

$$Q_2 = q_2 w^{2904},$$

where q_1 and q_2 are nonzero (large) constants. This yields that $P_{8,10} = P_{8,12} = P_{10,12} = 0$ is not solvable for v and w since we already have that $w \neq 0$. Thus we have all solutions of (3.2). □

3.2.3. The mixed invariance equations.

Now we are going to solve the remaining four mixed invariance equations. The following theorems completely describe the solution sets of these equations.

Theorem 3.3. (Baják–Páles [**5**]) *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3.3)*, i.e.,*

$$G_{p,q}(S_{a,b}(x,y), G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a + b = c + d = p + q = 0$, i.e., all the three means are equal to the geometric mean;

(ii) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{w, 2w\}$ and $\{c, d\} = \{p, q\} = \{0, w\}$, i.e., all the three means are equal to each other, and they are also equal to the power mean of exponent w;

(iii) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{w, 2w\}$, $\{c, d\} = \{0, -w\}$ and $p + q = 0$, i.e., $G_{p,q}$ is the geometric mean and the two means $S_{a,b}$ and $G_{-c,-d}$ are equal to each other, and are equal to the power mean of exponent w;

(iv) there exists $w \in \mathbb{R}$ such that $a + b = 0$, $\{c, d\} = \{3w, w\}$, and $\{p, q\} = \{2w, 0\}$ (in this case, $G_{p,q}$ is a power mean and $S_{a,b}$ is the geometric mean);

(v) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{w, 2w\}$, $\{c, d\} = \{-w, -2w\}$ and $\{p, q\} = \{0, -w\}$, i.e., $S_{a,b}$ is the power mean of exponent w and $G_{p,q}$ is the power mean of exponent $-w$.

Proof. The sufficiency of the conditions (i)–(v) can easily be checked. Thus, we have to prove the necessity of these conditions.

The invariance equation in this case is equivalent to

$$F^{(k)}_{M_{p,q,\mathfrak{m}},M_{a,b,\mathfrak{n}},M_{c,d,\mathfrak{m}}}(0) = 0 \qquad (k \in \mathbb{N}),$$

which, by Lemma 3.1 and Corollary 3.1, holds if and only if

$$F^{(k)}(0,k) = F^{(k)}_{M_{p,q,\mathfrak{m};k},M_{a,b,\mathfrak{n};k},M_{c,d,\mathfrak{m};k}}(0) = 0 \qquad (k \in \mathbb{N}), \tag{3.30}$$

where, in view of the third identity in (3.20), $F : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ and $k \in \mathbb{N}$ as

$$F(x,k) := G(p,q,H(a,b,x,-x,k),G(c,d,x,-x,k),k) - G(p,q,x,-x,k),$$

i.e., in Maple we set

```
> F:=(x,k)->G(p,q,H(a,b,x,-x,k),G(c,d,x,-x,k),k)
            -G(p,q,x,-x,k);
```

Now we derive the necessity of the conditions of Theorem 3.3 by checking (3.30) for the values $k = 2, 4, 6, 8, 10, 12$.

The second-order Taylor coefficient C_2 of $F(\cdot, 2)$ at $x = 0$:

```
> C[2]:=simplify(coeftayl(F(x,2),x=0,2));
```

which yields

$$C_2 := -\frac{1}{2}\,p + \frac{1}{4}\,c + \frac{1}{12}\,a + \frac{1}{12}\,b + \frac{1}{4}\,d - \frac{1}{2}\,q$$

Since $C_2 = 0$, we have that

$$\frac{a+b}{12} + \frac{c+d}{4} = \frac{p+q}{2}.$$

Now we introduce the following notations (which will be different than that in the solution of the previous two invariance equations, but by the same motivation):

$$w := \frac{a+b}{12} + \frac{c+d}{4} = \frac{p+q}{2}, \qquad v := \frac{a+b}{12} - \frac{c+d}{4}, \qquad t := \Big(\frac{p-q}{2}\Big)^2,$$
$$r := \frac{(a-b)^2 + 9(c-d)^2}{72}, \qquad s := \frac{(a-b)^2 - 9(c-d)^2}{72}. \tag{3.31}$$

Then, provided that $a \geqslant b$, $c \geqslant d$ and $p \geqslant q$, we can express the parameters a, b, c, d, p, q in the following form:

```
> a:=3*w+3*v+3*sqrt(r+s); b:=3*w+3*v-3*sqrt(r+s);
  c:=w-v+sqrt(r-s); d:=w-v-sqrt(r-s);
  p:=w+sqrt(t); q:=w-sqrt(t);
```

$$a := 3w + 3v + 3\sqrt{r+s}, \qquad c := w - v + \sqrt{r-s}, \qquad p := w + \sqrt{t},$$

$$b := 3w + 3v - 3\sqrt{r+s}, \qquad d := w - v - \sqrt{r-s}, \qquad q := w - \sqrt{t}.$$

Now we evaluate the fourth-order Taylor coefficient C_4 of $F(\cdot, 4)$ at $x = 0$:

```
> C[4]:=simplify(coeftayl(F(x,4),x=0,4));
```

which yields

$$C_4 := -\frac{2}{5}wv^2 - \frac{2}{5}w^2v - \frac{2}{15}w^3 - \frac{7}{15}wr - \frac{2}{15}ws - \frac{2}{15}v^3 - \frac{2}{15}vr - \frac{7}{15}vs + \frac{1}{3}wt$$

First we consider the case when $w = 0$. Then $p + q = 0$, which means that $G_{p,q}$ is the geometric mean. Therefore the invariance equation (3.3) can be rewritten as

$$S_{a,b}(x, y)G_{c,d}(x, y) = xy \qquad (x, y \in \mathbb{R}_+).$$

This yields

$$S_{a,b}(x, y) = \frac{1}{G_{c,d}(1/x, 1/y)} = G_{-c,-d}(x, y) \qquad (x, y \in \mathbb{R}_+).$$

Using the result of Alzer and Ruscheweyh ([**1**]), this identity yields that both means are power means, i.e., either $a + b = c + d = 0$ or there exists a real number u such that $\{a, b\} = \{u, 2u\}$ and $\{c, d\} = \{0, -u\}$ must hold. In this case we get that one of the conditions (i) or (iii) of our theorem is valid.

In the rest of the proof, we may assume that w is not zero. Hence, equation $C_4 = 0$ can be solved for t:

```
> t:=1/5*(6*w*v^2+6*w^2*v+2*w^3+7*w*r+2*w*s+2*v^3+
     2*v*r+7*v*s)/w;
```

$$t := \frac{6\,wv^2 + 6\,w^2v + 2\,w^3 + 7\,wr + 2\,ws + 2\,v^3 + 2\,vr + 7\,vs}{5w}$$

The next commands compute the Taylor coefficients C_6, C_8, C_{10} and C_{12}

```
> C[6]:=simplify(coeftayl(F(x,6),x=0,6));
  C[8]:=simplify(coeftayl(F(x,8),x=0,8));
  C[10]:=simplify(coeftayl(F(x,10),x=0,10));
  C[12]:=simplify(coeftayl(F(x,12),x=0,12));
```

which yields:

$$
\begin{aligned}
C_6 := -\frac{2}{7875w}\Big(&-1440\,w^2v^4 - 3818\,w^3vr - 1212\,w^4r - 246\,w^2r^2 - 1191\,w^2s^2 + 168\,v^4r \\
&+ 588\,v^4s + 84\,v^2r^2 + 1029\,v^2s^2 - 2318\,wv^3s + 588\,v^2rs - 6807\,w^2v^2r - 912\,w^2rs \\
&- 4182\,w^2v^2s - 5533\,w^3vs - 1408\,wv^3r - 1632\,w^4s - 96\,w^5v + 1635\,w^4v^2 + 34\,w^6 \\
&+ 84\,v^6 - 324\,wvrs - 1970\,w^3v^3 - 246\,wv^5 - 162\,wvr^2 - 162\,wvs^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_8 := \frac{1}{39375w^2}\Big(&-40332\,w^7s + 1428\,wr^3v^2 - 19152\,w^3r^2s - 328280\,w^4v^3r - 8747\,w^2vs^3 \\
&- 351580\,w^4v^3s - 65732\,w^2v^5s + 2856\,v^5rs - 32380\,w^2v^3s^2 - 41560\,w^2v^3r^2 - 29097\,w^3rs^2 \\
&+ 1428\,v^7s + 408\,v^5r^2 + 4998\,v^5s^2 + 136\,v^3r^3 - 8404\,v^7w^2 - 4344\,w^3r^3 - 39096\,w^3v^6 \\
&- 17776\,w^6v^3 + 53646\,w^7v^2 - 63684\,w^4v^5 + 6704\,w^8v - 8914\,w^5v^4 + 5831\,v^3s^3 + 1224\,wv^8 \\
&- 7914\,w^3s^3 - 36372\,w^5r^2 - 59862\,w^5s^2 + 136\,v^9 + 408\,v^7r + 1116\,w^9 - 30912\,w^7r \\
&+ 10404\,wr^2v^2s - 88284\,w^5rs + 17850\,wv^4s^2 - 305152\,w^5v^2s - 238010\,w^3v^4r - 46912\,w^2v^5r \\
&- 180760\,w^3v^4s - 142484\,w^6vr + 4998\,v^3rs^2 - 162634\,w^6vs - 313472\,w^5v^2r + 8976\,wv^6s \\
&- 153250\,w^3v^2r^2 - 143530\,w^3v^2s^2 + 1428\,v^3r^2s + 4998\,ws^3v^2 - 114640\,w^4vr^2 + 3876\,wv^6r \\
&- 275350\,w^4vrs + 20349\,wrv^2s^2 - 3052\,w^2vr^3 - 128410\,w^4vs^2 - 98590\,w^2v^3rs - 242680\,w^3v^2rs \\
&- 10176\,w^2vr^2s - 13746\,w^2vrs^2 + 4080\,wv^4r^2 + 19380\,wv^4rs\Big),
\end{aligned}
$$

We do not output the explicit forms of C_{10} and C_{12} as they contain very large polynomials of the variables v, w, r and s. Unfortunately, these polynomials involve higher-order powers of all variables, thus we cannot express any of these variables. To summarize the forms of these coefficients, we have that

$$C_{2k} = \frac{c_{2k}}{w^{k-2}} P_{2k}(r, s, v, w) \qquad (k = 3, 4, 5, 6),$$

where c_{2k} is a nonzero real constant and P_{2k} is a four variable polynomial. By condition (3.30), we have that $C_6 = C_8 = C_{10} = C_{12} = 0$, which yields the following system of four equations for the unknowns r, s, v and w:

$$P_{2k}(r, s, v, w) = 0 \qquad (k = 3, 4, 5, 6). \tag{3.32}$$

To eliminate the variable r first, observe that if (r, s, v, w) is a solution of (3.32), then r is the common root of the polynomials $P_{2k}(\cdot, s, v, w)$ for $k = 3, 4, 5, 6$. Therefore the resultant of $P_6(\cdot, s, v, w)$ and $P_{2k}(\cdot, s, v, w)$ with respect to r is zero for $k = 4, 5, 6$. Thus, we need to compute these resultants:

```
> R[6,8]:=factor(resultant(op(2,C[6]),op(2,C[8]),r)):
  R[6,10]:=factor(resultant(op(2,C[6]),op(2,C[10]),r)):
  R[6,12]:=factor(resultant(op(2,C[6]),op(2,C[12]),r)):
```

Again, we do not output the explicit results of these computations. Here we get that

$$R_{6,2k} = c_{6,2k}w^{k-1}(w+v)^2 P_{6,2k}(s,v,w) \qquad (k = 4,5,6),$$

where $c_{6,2k}$ is a nonzero real constant and $P_{6,2k}$ is a three variable polynomial. Thus, either $w+v=0$ or (s,v,w) is a solution of the following system of three polynomial equations:

$$P_{6,2k}(s,v,w) = 0 \qquad (k = 4,5,6). \tag{3.33}$$

We consider first the case when $w+v=0$ (but $w \neq 0$). Then $a+b=0$ and hence $S_{a,b}(x,y) = \sqrt{xy} = G_{0,0}(x,y)$ holds. This shows that the invariance equation (3.3) is now equivalent to (3.1), where $a = b = 0$. Observe that, by $p+q = 2w \neq 0$, $G_{p,q}$ cannot be the geometric mean. Therefore, in Theorem 3.1, the first five cases cannot be valid. Thus, only the sixth case of Theorem 3.1 can hold, whence we get that there exists $w \in \mathbb{R}$ such that $\{c,d\} = \{3w,w\}$, and $\{p,q\} = \{2w,0\}$ (i.e., the assertion (iv) of Theorem 3.3 holds). (In this, we can also get that $t = r - s = w^2$.)

In the rest of the proof, we assume that $w(w+v) \neq 0$. Then (3.33) must be satisfied by (s,v,w). Repeating the above procedure, we compute the resultants of $P_{6,8}(\cdot,v,w)$ and $P_{6,2k}(\cdot,v,w)$ with respect to s for $k = 5,6$:

```
> R[6,8,10]:=factor(resultant(op(4,R[6,8]),
             op(4,R[6,10]),s)):
  R[6,8,12]:=factor(resultant(op(4,R[6,8]),
             op(4,R[6,12]),s)):
```

The explicit result of these computations is suppressed again. Here we obtain that, for $k = 5,6$

$$R_{6,8,2k} = c_{6,8,2k}(2w+v)v^5(v-w)^{12}w^{4(k-1)}(w+v)^{3(k-1)}P_{6,8,2k}(v,w),$$

where $c_{6,8,2k}$ is a nonzero real constant and $P_{6,8,2k}$ is a two variable homogeneous polynomial. Thus either $v \in \{0,w,-2w\}$ or (v,w) is a solution of the following system of two polynomial equations:

$$P_{6,8,2k}(v,w) = 0 \qquad (k = 5,6). \tag{3.34}$$

Assume that $v \in \{0,w,-2w\}$. We distinguish three subcases according to the inclusion $v \in \{0,w,-2w\}$.

Subcase 1: $v = w$. In this case $c + d = 0$, thus $G_{c,d}$ is the geometric mean. However, by simplifying the resultants $C_{6,8}$, $C_{6,10}$ and $C_{6,12}$, we get that these resultants are of the form $C_{6,2k} = c_{6,2k}w^{8(k-1)}$, hence they can only be zero if $w = 0$. This means that we do not get new solutions in this subcase, and only assertion (i) of Theorem 3.3 can hold.

Subcase 2: $v = -2w$. First we simplify the resultants $C_{6,8}$, $C_{6,10}$ and $C_{6,12}$ and denote them by $K_{6,8}$, $K_{6,10}$ and $K_{6,12}$, respectively, by performing the commands

```
> v:=-2*w;
  K[6,8]:=factor(simplify(R[6,8])):
  K[6,10]:=factor(simplify(R[6,10])):
  K[6,12]:=factor(simplify(R[6,12])):
```

We have that in this case the resultants are of the form

$$K_{6,2k} = d_{6,2k} w^{2k-4}(9s + 4w^2) Q_{6,2k}(s, w) \qquad (k = 4, 5, 6),$$

where $d_{6,2k}$ is a nonzero real constant and $Q_{6,2k}$ is a two variable polynomial. Thus, either the system of the polynomial equations $Q_{6,8}(s, w) = Q_{6,10}(s, w) = Q_{6,12}(s, w) = 0$ has a solution or $9s + 4w^2 = 0$. Computing the resultants of these polynomials by

```
> resultant(op(4,K[6,8]),op(4,K[6,10]),s):
  resultant(op(4,K[6,8]),op(4,K[6,12]),s):
```

we have that the resultants can only be zero if $w = 0$ in which case we do not get a new solution.

If $s = -4w^2/9$, first we simplify the Taylor coefficients C_6, C_8, C_{10} and C_{12} and denote them by K_6, K_8, K_{10} and K_{12}, respectively, by the commands

```
> s:=-4*w^2/9:
  K[6]:=factor(simplify(C[6])):
  K[8]:=factor(simplify(C[8])):
  K[10]:=factor(simplify(C[10])):
  K[12]:=factor(simplify(C[12])):
```

We have that these coefficients are of the form

$$K_{2k} = d_{2k} w(9r - 5w^2) Q_{2k}(r, w) \qquad (k = 3, 4, 5, 6),$$

where d_{2k} is a nonzero real constant and Q_{2k} is a two variable polynomial. Thus, either the system of polynomial equations $Q_6(r, w) = Q_8(r, w) = Q_{10}(r, w) = Q_{12}(r, w) = 0$ has a solution or $9r - 5w^2 = 0$. Computing the resultants of these polynomials by

```
> resultant(op(4,K[6]),op(4,K[8]),r):
  resultant(op(4,K[6]),op(4,K[10]),r):
  resultant(op(4,K[6]),op(4,K[12]),r):
```

we have that the resultants can only be zero if $w = 0$. Therefore we again do not have new solutions.

In the other case, when $r = 5w^2/9$, we get that $t = w^2$. Thus $\{a, b\} = \{-2w, -4w\}$, $\{c, d\} = \{2w, 4w\}$ and $\{p, q\} = \{0, 2w\}$, i.e., assertion (v) of Theorem 3.3 holds.

Subcase 3: $v = 0$. We simplify the resultants $C_{6,8}$, $C_{6,10}$ and $C_{6,12}$ and denote them by $K_{6,8}$, $K_{6,10}$ and $K_{6,12}$, respectively, by performing the commands

```
> v:=0;
  K[6,8]:=factor(simplify(R[6,8])):
  K[6,10]:=factor(simplify(R[6,10])):
  K[6,12]:=factor(simplify(R[6,12])):
```

We have that in this case the resultants are of the form

$$K_{6,2k} = d_{6,2k} w^{2k-4} (9s + 4w^2)^2 Q_{6,2k}(s, w) \qquad (k = 4, 5, 6),$$

where $d_{6,2k}$ is a nonzero real constant and $Q_{6,2k}$ is a two variable polynomial. Thus, either the system of the polynomial equations $Q_{6,8}(s, w) = Q_{6,10}(s, w) = Q_{6,12}(s, w) = 0$ has a solution or $9s + 4w^2 = 0$. Computing the resultants of these polynomials by

```
> resultant(op(3,K[6,8]),op(3,K[6,10]),s):
  resultant(op(3,K[6,8]),op(3,K[6,12]),s):
```

we have that the resultants can only be zero if $w = 0$. Therefore solving the system of equations does not give new solutions.

If $s = -4w^2/9$, first we simplify the Taylor coefficients C_6, C_8, C_{10} and C_{12} and denote them by K_6, K_8, K_{10} and K_{12}, respectively, by the commands

```
> s:=-4*w^2/9:
  K[6]:=factor(simplify(C[6])):
  K[8]:=factor(simplify(C[8])):
  K[10]:=factor(simplify(C[10])):
  K[12]:=factor(simplify(C[12])):
```

We have that these coefficients are of the form

$$K_{2k} = d_{2k}w(9r - 5w^2)Q_{2k}(r, w) \qquad (k = 3, 4, 5, 6),$$

where d_{2k} is a nonzero real constant and Q_{2k} is a two variable polynomial. Thus either the system of polynomial equations $Q_6(r, w) = Q_8(r, w) = Q_{10}(r, w) = Q_{12} = 0$ has a solution or $9r-5w^2 = 0$. Computing the resultants of these polynomials by

```
> resultant(op(3,K[6]),op(4,K[8]),r):
  resultant(op(3,K[6]),op(4,K[10]),r):
  resultant(op(3,K[6]),op(4,K[12]),r):
```

we have that the resultants can only be zero if $w = 0$. Therefore we again do not have new solutions.

Otherwise, if $r = 5w^2/9$, we get that $t = w^2$. Therefore $\{a, b\} = \{2w, 4w\}$, $\{c, d\} = \{0, 2w\}$ and $\{p, q\} = \{0, 2w\}$, i.e., assertion (ii) of Theorem 3.3 holds.

Finally, we may assume that $v \notin \{0, w, -2w\}$. Then (3.34) must be satisfied by (v, w). We compute the resultant of $P_{6,8,10}(\cdot, w)$ and $P_{6,8,12}(\cdot, w)$:

```
> R[6,8,10,12]:=factor(resultant(
                  op(5,R[6,8,10])*op(6,R[6,8,10]),
                  op(5,R[6,8,12])*op(6,R[6,8,12]),v)):
```

Here we obtain that

$$R_{6,8,10,12} = c_{6,8,10,12}w^{35690},$$

where $c_{6,8,10,12}$ is a nonzero real constant, showing that the polynomials $P_{6,8,10}$ and $P_{6,8,12}$ do not have common root if $w \neq 0$. □

THEOREM 3.4. (Baják–Páles [**5**]) *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3.4)*, i.e.,*

$$G_{p,q}(S_{a,b}(x, y), S_{c,d}(x, y)) = G_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a + b = c + d = p + q = 0$*, i.e., all the three means are equal to the geometric mean;*

(ii) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{c, d\} = \{w, 2w\}$ and $\{p, q\} = \{0, w\}$, i.e., all the three means are equal to each other, and they are equal to the power mean of exponent w;

(iii) $\{a, b\} = \{-c, -d\}$ *and* $p + q = 0$*, i.e., $G_{p,q}$ is the geometric mean and* $S_{a,b} = S_{-c,-d}$.

Proof. We use a similar argument as in the previous proof.

The sufficiency of the conditions (i)–(iii) can easily be checked. To derive the necessity of the conditions of Theorem 3.4, we are going to check equation

$$F^{(k)}(0,k) = F^{(k)}_{M_{p,q,\mathfrak{m};k},M_{a,b,\mathfrak{n};k},M_{c,d,\mathfrak{n};k}}(0) = 0 \qquad (k \in \mathbb{N}), \tag{3.35}$$

only for the values $k = 2, 4, 6, 8, 10, 12$, where, in view of the fourth identity in (3.20), $F : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ and $k \in \mathbb{N}$ as

$$F(x,k) := G(p,q,H(a,b,x,-x,k),H(c,d,x,-x,k),k) - G(p,q,x,-x,k).$$

In the description of the proof, we will only provide the mathematical details. The Maple commands that produce the results will be omitted since they are completely analogous to those in the previous proof.

First, we evaluate the second-order Taylor coefficient of $F(\cdot,2)$ at $x = 0$, which yields

$$C_2 := -\frac{1}{2}p + \frac{1}{12}c + \frac{1}{12}a + \frac{1}{12}b + \frac{1}{12}d - \frac{1}{2}q$$

By condition (3.35), $C_2 = 0$, thus

$$\frac{a+b+c+d}{12} = \frac{p+q}{2}.$$

Now we introduce the following notations

$$w := \frac{a+b+c+d}{12} = \frac{p+q}{2}, \qquad v := \frac{a+b-(c+d)}{12}, \qquad t := \Big(\frac{p-q}{2}\Big)^2,$$

$$r := \frac{(a-b)^2+(c-d)^2}{72}, \qquad s := \frac{(a-b)^2-(c-d)^2}{72}.$$

Then, provided that $a \geqslant b$, $c \geqslant d$ and $p \geqslant q$, we can express the parameters a, b, c, d, p, q in the following form:

$$a := 3w + 3v + 3\sqrt{r+s}, \qquad c := 3w - 3v + 3\sqrt{r-s}, \qquad p := w + \sqrt{t},$$

$$b := 3w + 3v - 3\sqrt{r+s}, \qquad d := 3w - 3v - 3\sqrt{r-s}, \qquad q := w - \sqrt{t}.$$

Now we evaluate the fourth-order Taylor coefficient C_4 of $F(\cdot,4)$ at $x = 0$, which yields

$$C_4 := -\frac{4}{5}wv^2 - \frac{4}{15}w^3 - \frac{3}{5}wr - \frac{3}{5}vs + \frac{1}{3}wt$$

First we have to consider the case when $w = 0$. Then $p + q = 0$, thus $G_{p,q}$ is the geometric mean. The invariance equation (3.4) now can be rewritten as

$$S_{a,b}(x,y)S_{c,d}(x,y) = xy \qquad (x,y \in \mathbb{R}_+).$$

This yields

$$S_{a,b}(x,y) = \frac{1}{S_{c,d}(1/x, 1/y)} = S_{-c,-d}(x,y) \qquad (x, y \in \mathbb{R}_+),$$

whence we have that in this case one of the conditions (i) or (iii) of our theorem is valid. We note that this result also follows from Theorem 3.2, since the geometric mean is also a Stolarsky mean.

In the rest of the proof we assume that $w \neq 0$. Hence we can solve $C_4 = 0$ for t:

$$t := \frac{4w^3 + 9wr + 9vs + 12wv^2}{5w}$$

We also have to compute the Taylor coefficients C_6, C_8, C_{10} and C_{12} in the same way as in the proof of Theorem 3.3. Here we have that

$$C_6 := \frac{2}{7875w}\big(2214\, wv^3 s + 10863\, w^3 vs + 648\, wvrs + 324\, w^2 r^2 + 2025\, w^2 s^2 + 2088\, w^4 r - 141\, w^4 v^2 + 2376\, w^2 v^4 - 236\, w^6 - 1701\, v^2 s^2 + 10989\, w^2 v^2 r\big)$$

$$C_8 := -\frac{1}{13125w^2}\big(-2136\, w^9 + 25008\, w^3 v^6 - 16654\, w^7 v^2 + 5832\, w^2 vr^2 s + 20952\, w^5 r^2 + 37800\, w^5 s^2 - 12393\, wrv^2 s^2 + 52002\, w^2 rv^3 s + 204372\, w^5 v^2 r + 16872\, w^7 r + 144018\, w^3 v^4 r - 16524\, v^4 s^2 w + 18225\, w^3 rs^2 + 6075\, w^2 vs^3 + 223272\, w^4 v^3 s + 1944\, w^3 r^3 + 92826\, w^3 v^2 r^2 + 18798\, w^5 v^4 + 175554\, w^4 vrs + 113622\, w^6 vs + 92502\, w^3 v^2 s^2 - 4131\, v^3 s^3 + 28368\, w^2 v^5 s\big)$$

$$C_{10} := \frac{2}{19490625w^3}\big(385977069\, w^4 v^4 s^2 + 737675424\, w^7 v^3 s + 266772447\, w^4 v^6 r + 132586875\, vw^5 s^3 + 611501967\, w^5 v^5 s + 135594000\, w^6 rs^2 - 11932272\, v^5 s^3 w + 725783409\, w^6 v^2 s^2 + 42027822\, v^7 sw^3 + 690152319\, w^6 v^2 r^2 + 400604049\, w^8 v^2 r + 120654603\, w^4 v^2 r^3 + 417636081\, w^4 v^4 r^2 + 865710342\, w^6 v^4 r - 23864544\, v^6 s^2 w^2 + 30314736\, w^3 v^3 s^3 + 24603750\, w^4 r^2 s^2 - 2992516\, w^{12} - 45335067\, w^{10} v^2 + 102201777\, w^6 v^6 + 1458489888\, w^5 v^3 rs + 353110833\, w^4 v^2 rs^2 + 172975662\, w^3 v^5 sr + 341082333\, w^5 vr^2 s + 710902818\, w^7 vrs + 7453296\, w^3 r^3 vs - 13423806\, w^2 r^2 v^2 s^2 + 111825684\, w^3 r^2 v^3 s - 35796816\, w^2 rv^4 s^2 - 8949204\, wrv^3 s^3 + 16402500\, w^3 vrs^3 - 38472759\, w^8 v^4 + 33510024\, w^4 v^8 + 63395784\, w^8 r^2 + 19511856\, w^{10} r + 30314736\, w^6 r^3 + 1863324\, w^4 r^4 - 2237301\, v^4 s^4 + 83288250\, w^8 s^2 + 4100625\, w^4 s^4 + 161393481\, w^9 vs\big)$$

$$
\begin{aligned}
C_{12} := -\frac{2}{14780390625w^4}\big(&403741013760\, w^5v^8r + 1533816436995\, w^6v^3s^3 + 147670340625\, w^5v^2s^4 \\
&+ 477511402500\, w^9rs^2 + 42575378760\, v^9w^4s + 522785591820\, w^{11}v^2r - 22003319520\, v^7s^3w^2 \\
&+ 2393789829120\, w^9v^2s^2 + 14167659375\, w^5rs^4 + 2341418351265\, w^9v^4r - 80822008720\, w^{13}v^2 \\
&+ 1426729627620\, w^7v^2r^3 + 114233471370\, w^4v^5s^3 + 1302340759380\, w^6v^7s + 2409383453130\, w^7v^6r \\
&- 8251244820\, v^6s^4w + 811261449855\, w^5v^4r^3 + 1111235137515\, w^5v^6r^2 + 28335318750\, w^5r^3s^2 \\
&+ 748889853750\, w^8vs^3 + 1595845152\, w^5r^5 + 136561899375\, w^{11}s^2 + 123363069120\, w^{11}r^2 \\
&+ 2833531875\, w^4vs^5 - 1237686723\, v^5s^5 + 262774218960\, w^7v^8 + 2817190076265\, w^8v^5s \\
&+ 789387491385\, w^5r^2v^2s^2 + 174787358220\, w^4v^3r^3s + 7979225760\, w^4vr^4s + 28335318750\, w^4vr^2s^3 \\
&+ 574262460000\, w^6vrs^3 + 513728658315\, w^4v^5r^2s + 358526321280\, w^4v^7rs - 12376867230\, w^3r^3v^2s^2 \\
&- 49507468920\, w^3r^2v^4s^2 - 12376867230\, w^2r^2v^3s^3 - 33004979280\, w^2rv^5s^3 - 66009958560\, w^3rv^6s^2 \\
&- 6188433615\, wrv^4s^4 + 1078454668365\, w^5v^6s^2 + 4352648210865\, w^7v^4r^2 + 1354355588070\, w^{10}v^3s \\
&+ 135314370180\, w^5r^4v^2 - 32530323735\, w^3v^8s^2 + 2326102237980\, w^9v^2r^2 + 291555804375\, w^7r^2s^2 \\
&+ 4385267094735\, w^7v^2rs^2 + 5785680885480\, w^6v^5sr + 2159275162860\, w^8vr^2s + 17727300960\, w^{13}r \\
&+ 8019256823460\, w^8v^3rs + 512836526340\, w^6r^3vs + 4451805506610\, w^6r^2v^3s - 3686158848\, w^{15} \\
&+ 146366403840\, w^4rv^3s^3 + 136945365120\, w^9r^3 + 47843358750\, w^7s^4 - 15356483415\, w^3v^4s^4 \\
&+ 178593908460\, w^{12}vs - 223894817745\, w^{11}v^4 + 109692392715\, w^9v^6 + 32931348336\, w^5v^{10} \\
&+ 1663233280740\, w^{10}vrs + 2328040789110\, w^5rv^4s^2 + 4337072582355\, w^7v^4s^2 + 36591600960\, w^7r^4\big)
\end{aligned}
$$

This means that the coefficients are of the form

$$C_{2k} = \frac{c_{2k}}{w^{k-2}} P_{2k}(r, s, v, w) \qquad (k = 3, 4, 5, 6),$$

where c_{2k} is a nonzero real constant and P_{2k} is a four variable polynomial. By condition (3.35), we have that $C_6 = C_8 = C_{10} = C_{12} = 0$, which yields now the following system of four equations for the unknowns r, s, v and w:

$$P_{2k}(r, s, v, w) = 0 \qquad (k = 3, 4, 5, 6). \tag{3.36}$$

We use the same method as in the proof of Theorem 3.3 to determine if these polynomials have common roots. First we eliminate the variable r by calculating resultant $R_{6,2k}$ of $P_6(\cdot, s, v, w)$ and $P_{2k}(\cdot, s, v, w)$ for $k = 4, 5, 6$. We get that

$$R_{6,2k} = c_{6,2k} w^{2k-2} P_{6,2k}(s, v, w) \qquad (k = 4, 5, 6),$$

where $c_{6,2k}$ is a nonzero real constant and $P_{6,2k}$ is a three variable polynomial. In the next step we eliminate the variable s by calculating the resultants $R_{6,8,10}$ and $R_{6,8,12}$ of $P_{6,8}(\cdot, v, w)$ and $P_{6,2k}(\cdot, v, w)$, respectively. We obtain that, for $k = 5, 6$

$$R_{6,8,2k} = c_{6,8,2k} v^4 w^{12(k-1)} (3v - 5w)(3v + 5w)(v - w)^{12}(v + w)^{12} P_{6,8,2k}(v, w),$$

where $c_{6,8,2k}$ is a nonzero real constant and $P_{6,8,2k}$ is a two variable homogeneous polynomial. Thus either $v \in \{0, w, -w, -5w/3, 5w/3\}$ or (v, w) is a solution of the following system of two polynomial equations:

$$P_{6,8,2k}(v, w) = 0 \qquad (k = 5, 6). \tag{3.37}$$

Assume that $v \in \{0, w, -w, -5w/3, 5w/3\}$. We have to distinguish the following cases:

Subcase 1: $v = 0$. In this case, by simplifying the resultants $R_{6,8}$, $R_{6,10}$ and $R_{6,12}$ and denoting them by $K_{6,8}$, $K_{6,10}$ and $K_{6,12}$, respectively, we get that

$$K_{6,2k} = d_{6,2k} w^{2k-4} s^2 Q_{6,2k}(s, w) \qquad (k = 4, 5, 6),$$

where $d_{6,2k}$ is a nonzero real constant and $Q_{6,2k}$ is a two variable polynomial. Thus either the system of the polynomial equations $Q_{6,8}(s, w) = Q_{6,10}(s, w) = Q_{6,12}(s, w) = 0$ has a solution or $s = 0$. However, by computing the resultants of $Q_{6,8}(\cdot, w)$ and $Q_{6,10}(\cdot, w)$, we get that this resultant can only be zero if $w = 0$, therefore we do not have new solutions here.

On the other hand, if $s = 0$, then simplifying the Taylor coefficients C_6, C_8, C_{10} and C_{12}, denoting them by K_6, K_8, K_{10} and K_{12}, respectively, and solving the system of equations $K_6(r, w) = K_8(r, w) = K_{10}(r, w) = K_{12}(r, w) = 0$, we have that there may exist a solution if either $w = 0$ or $9r - w^2 = 0$. If $r = w^2/9$, then we obtain that $t = w^2$. Hence $\{a, b\} = \{c, d\} = \{4w, 2w\}$ and $\{p, q\} = \{2w, 0\}$, i.e., all the three means are equal to the power mean of exponent $2w$, thus assertion (ii) of Theorem 3.4 holds.

Subcase 2: $v = \pm w$. In both cases, by simplifying $R_{6,8}, R_{6,10}$ and $R_{6,12}$, we have that these resultants can only be simultaneously zero if $w = 0$, thus we do not get new solutions.

Subcase 3: $v = \pm 5w/3$. By simplifying the resultants $R_{6,8}, R_{6,10}$ and $R_{6,12}$ and denoting them by $K_{6,8}, K_{6,10}$ and $K_{6,12}$, respectively, we have that

$$K_{6,2k} = d_{6,2k} w^{2k-2} Q_{6,2k}(s, w) \qquad (k = 4, 5, 6),$$

where $d_{6,2k}$ is a nonzero real constant and $Q_{6,2k}$ is a two variable polynomial. However, the resultants of the polynomials $Q_{6,8}$ and $Q_{6,10}$ is zero if and only if $w = 0$, thus in this case we do not get new solutions either.

Finally, we may assume that $v \notin \{0, w, -w, -5w/3, 5w/3\}$. Then (3.37) must be satisfied by (v, w). Computing the resultant $R_{6,8,10,12}$ of $P_{6,8,10}$ and $P_{6,8,12}$ we obtain that

$$R_{6,8,10,12} = c_{6,8,10,12} w^{17100},$$

where $c_{6,8,10,12}$ is a nonzero real constant, showing that the polynomials $P_{6,8,10}$ and $P_{6,8,12}$ do not have common root.

□

THEOREM 3.5. (Baják–Páles [**5**]) *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3.5)*, i.e.,*

$$S_{p,q}(G_{a,b}(x, y), G_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a + b = c + d = p + q = 0$, i.e., all the three means are equal to the geometric mean;

(ii) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{c, d\} = \{0, w\}$ and $\{p, q\} = \{w, 2w\}$, i.e., all the three means are equal to each other, and they are equal to the power mean of exponent w;

(iii) $\{a, b\} = \{-c, -d\}$ and $p + q = 0$, i.e., $S_{p,q}$ is the geometric mean and $G_{a,b} = G_{-c,-d}$;

(iv) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ and $c + d = 0$, i.e., $G_{c,d}$ is the geometric mean and $S_{p,q}$ is the power mean of exponent $2w$;

(v) there exists a $w \in \mathbb{R}$ such that $\{c, d\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ and $a + b = 0$, i.e., $G_{a,b}$ is the geometric mean and $S_{p,q}$ is the power mean of exponent $2w$.

PROOF. We again use a similar argument as in the previous proofs.

The sufficiency of the conditions (i)–(v) can easily be checked. To derive the necessity of the conditions of Theorem 3.5, we check equation

$$F^{(k)}(0, k) = F^{(k)}_{M_{p,q,\mathfrak{n};k}, M_{a,b,\mathfrak{m};k}, M_{c,d,\mathfrak{m};k}}(0) = 0 \qquad (k \in \mathbb{N}) \tag{3.38}$$

only for the values $k = 2, 4, 6, 8, 10, 12$. Here, in view of the fifth identity in (3.20), $F : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ and $k \in \mathbb{N}$ as

$$F(x, k) := H(p, q, G(a, b, x, -x, k), G(c, d, x, -x, k), k) - H(p, q, x, -x, k).$$

We evaluate the second-order Taylor coefficient of $F(\cdot, 2)$ at $x = 0$, which in this case yields

$$C_2 := -\frac{1}{6}\, p + \frac{1}{4}\, c + \frac{1}{4}\, a + \frac{1}{4}\, b + \frac{1}{4}\, d - \frac{1}{6}\, q$$

By condition (3.38), $C_2 = 0$, thus

$$\frac{a + b + c + d}{4} = \frac{p + q}{6}.$$

This identity motivates to introduce the following notations

$$w := \frac{a+b+c+d}{4} = \frac{p+q}{6}, \qquad v := \frac{a+b-(c+d)}{4}, \qquad t := \Big(\frac{p-q}{6}\Big)^2,$$
$$r := \frac{(a-b)^2+(c-d)^2}{8}, \qquad s := \frac{(a-b)^2-(c-d)^2}{8}.$$

Then, provided that $a \geqslant b, c \geqslant d, p \geqslant q$, we can express the parameters a, b, c, d, p, q in the following form:

$$a := w + v + \sqrt{r+s}, \qquad c := w - v + \sqrt{r-s}, \qquad p := 3w + 3\sqrt{t},$$
$$b := w + v - \sqrt{r+s}, \qquad d := w - v - \sqrt{r-s}, \qquad q := 3w - 3\sqrt{t}.$$

Now we evaluate the fourth-order Taylor coefficient C_4 of $F(\cdot, 4)$ at $x = 0$, which yields

$$C_4 := \frac{4}{15}w^3 - \frac{1}{3}wr - \frac{1}{3}vs + \frac{3}{5}wt$$

If $w = 0$, then $p + q = 0$ and hence $S_{p,q}$ is the geometric mean. The same argument as in the previous proofs shows that assertion (i) or (iii) of Theorem 3.5 must hold. We note that the same consequence immediately follows also from Theorem 3.1, since the geometric mean is also a Gini mean. Therefore in the rest of the proof we assume that $w \neq 0$. Then $C_4 = 0$ is solvable for t:

$$t := -\frac{4w^3 - 5wr - 5vs}{9w}$$

Computing the Taylor coefficients C_6, C_8, C_{10} and C_{12}, we obtain that

$$C_6 := \frac{2}{315w}\big(44\,w^6 - 105\,w^4v^2 - 8\,wvrs - 5\,w^3vs + 105\,w^2v^2r + 70\,wv^3s - 40\,w^4r - 4\,w^2r^2$$
$$+ 21\,w^2s^2 - 25\,v^2s^2\big)$$

$$C_8 := -\frac{1}{315w^2}\big(-350\,w^5v^4 - 378\,w^7v^2 - 136\,w^5r^2 - 8\,w^3r^3 - 25\,v^3s^3 + 136\,w^9 + 224\,w^2v^5s$$
$$+ 17\,w^2vs^3 + 140\,w^5v^2r + 280\,w^4v^3s + 18\,w^3v^2s^2 + 51\,w^3rs^2 + 350\,w^3v^4r + 238\,w^3v^2r^2$$
$$+ 84\,w^5s^2 + 238\,w^2v^3sr - 24\,w^2vsr^2 - 34\,w^4vsr - 75\,wrv^2s^2 + 8\,w^7r - 6\,w^6vs\big)$$

$$C_{10} := \frac{2}{155925w^3}\big(57750\,w^6v^4r + 47520\,w^6rs^2 + 162855\,w^6v^2r^2 + 19866\,w^8s^2 + 13844\,w^{12}$$
$$+ 183645\,w^4v^4s^2 + 162855\,w^5v^5s - 138600\,w^7v^3s - 37040\,w^3v^3s^3 - 107415\,w^8v^2r$$
$$+ 88935\,w^4v^6r + 160545\,w^4v^4r^2 + 10230\,w^4r^2s^2 + 46035\,w^5vs^3 + 57705\,w^6v^2s^2 - 3125\,v^4s^4$$
$$+ 61380\,w^3v^3r^2s + 46035\,w^4v^2r^3 + 46705\,w^9vs + 55110\,w^3v^7s - 218295\,w^8v^4 + 86800\,w^{10}r$$
$$- 37040\,w^6r^3 - 101475\,w^{10}v^2 - 88935\,w^6v^6 + 1705\,w^4s^4 - 62184\,w^8r^2 - 34395\,w^5vsr^2$$
$$- 5680\,w^3r^3vs - 18750\,w^2r^2v^2s^2 - 12500\,wrv^3s^3 + 120582\,w^3v^5sr - 32430\,w^7vsr - 1420\,w^4r^4$$
$$- 20535\,w^4rv^2s^2 + 434280\,w^5v^3rs + 6820\,w^3vrs^3\big)$$

$$
\begin{aligned}
C_{12} := -\frac{2}{42567525w^4}\big(&157342185\, w^7 v^4 r^2 + 141576435\, w^7 v^4 s^2 + 104873769\, w^5 v^6 s^2 - 431875\, v^5 s^5 \\
&+ 27495468\, w^9 v^2 r^2 + 9066057\, w^5 v^2 s^4 + 8300292\, w^5 v^2 r^4 - 159128970\, w^{10} v^3 s - 243232\, w^5 r^5 \\
&+ 30180150\, w^8 v s^3 - 9065920\, w^7 r^4 + 37302944\, w^{13} r + 62522460\, w^6 v^7 s + 65044980\, w^7 v^2 r^3 \\
&- 130225095\, w^9 v^4 r + 943215\, w^5 r s^4 + 19745404\, w^{12} v s - 77297220\, w^{11} v^2 r + 16939923\, w^7 r^2 s^2 \\
&+ 16234218\, w^4 v^5 s^3 + 2692690\, w^7 s^4 - 74729655\, w^8 v^5 s + 66705639\, w^5 v^6 r^2 - 37059712\, w^9 r^3 \\
&+ 2885883\, w^{11} s^2 + 15345728\, w^{11} r^2 + 50945895\, w^5 v^4 r^3 - 23543520\, w^{13} v^2 + 11971960\, v^9 w^4 s \\
&+ 10780770\, w^7 v^6 r + 1886430\, w^5 r^3 s^2 - 3378375\, w^3 v^8 s^2 + 21585564\, w^9 r s^2 + 188643\, w^4 v s^5 \\
&+ 32516795\, w^6 v^3 s^3 + 20931308\, w^9 v^2 s^2 - 16896332\, w^6 r^3 v s - 21533691\, w^5 r^2 v^2 s^2 \\
&- 36263680\, w^4 r v^3 s^3 + 36060820\, w^{10} v s r - 46638660\, w^8 v s r^2 + 51334215\, w^7 r v^2 s^2 \\
&- 1216160\, w^4 r^4 v s - 4318750\, w^3 r^3 v^2 s^2 - 4318750\, w^2 r^2 v^3 s^3 - 2159375\, w r v^4 s^4 \\
&+ 237567330\, w^6 v^3 r^2 s + 163873710\, w^5 v^4 r s^2 + 45864819\, w^4 v^5 s r^2 + 310774464\, w^6 v^5 r s \\
&+ 146726580\, w^8 v^3 r s + 40216176\, w^4 v^7 s r + 13833820\, v^3 w^4 r^3 s + 1886430\, w^4 v r^2 s^3 \\
&+ 33201168\, w^6 v r s^3 - 77486409\, w^9 v^6 + 19819800\, w^5 v^8 r - 12524375\, w^3 v^4 s^4 \\
&- 19819800\, w^7 v^8 - 78062985\, w^{11} v^4 - 6279808\, w^{15}\big)
\end{aligned}
$$

This means that the coefficients are of the form

$$C_{2k} = \frac{c_{2k}}{w^{k-2}} P_{2k}(r, s, v, w) \qquad (k = 3, 4, 5, 6),$$

where c_{2k} is a nonzero real constant and P_{2k} is a four variable polynomial. Thus we obtain the following system of four equations for the unknowns r, s, v and w:

$$P_{2k}(r, s, v, w) = 0 \qquad (k = 3, 4, 5, 6).$$

The resultants $R_{6,2k}$ of the polynomials $P_6(\cdot, s, v, w)$ and $P_{2k}(\cdot, s, v, w)$ are

$$R_{6,2k} = c_{6,2k} w^{2k-2} s(v - w)(v + w) P_{6,2k}(s, v, w) \qquad (k = 4, 5, 6),$$

where $c_{6,2k}$ is a nonzero real constant and $P_{6,2k}$ is a three variable polynomial. Thus either $s = 0, v + w = 0, v - w = 0$ or (s, v, w) is a solution of the following system of three polynomial equations:

$$P_{6,2k}(s, v, w) = 0 \qquad (k = 4, 5, 6).$$

First we consider the solution of the system of equations. We eliminate the variable s by calculating the resultants $R_{6,8,10}$ and $R_{6,8,12}$ of $P_{6,8}(\cdot, v, w)$ and $P_{6,2k}(\cdot, v, w)$. We obtain that

$$R_{6,8,2k} = c_{6,8,2k} v w^{10k-15} (v - w)^3 (v + w)^3 P_{6,8,2k}(v, w) \qquad (k = 5, 6),$$

where $c_{6,8,2k}$ is a nonzero real constant and $P_{6,8,2k}$ is a two variable homogeneous polynomial. Then, by calculating the resultant of $P_{6,8,10}$ and $P_{6,8,12}$, we get that this resultant is zero if and only if $w = 0$, thus we do not get new solutions.

Now we have to distinguish the following subcases:

Subcase 1: $v = 0$. By simplifying the Taylor coefficients and calculating the resultants, we obtain that either $w = 0$ or $r = w^2$. If $r = w^2$, then again by simplifying the Taylor coefficients, we get that $s = 0$ must hold, and $t = w^2/9$. Thus $\{a, b\} = \{c, d\} = \{2w, 0\}$ and $\{p, q\} = \{4w, 2w\}$, i.e, all the three means are equal to the power mean of exponent $2w$, therefore condition (ii) of Theorem 3.5 is valid.

Subcase 2: $v = w$. Then $c + d = 0$, therefore $G_{c,d}$ is the geometric mean. By simplifying the Taylor coefficients and calculating their resultants, we get that either $w = 0$ or $r + s = w^2$. If $r + s = w^2$, then $t = w^2/9$, thus $\{a, b\} = \{3w, w\}$ and $\{p, q\} = \{4w, 2w\}$, i.e., $S_{p,q}$ is the power mean of exponent $2w$, therefore condition (iv) of Theorem 3.5 holds.

Subcase 3: $v = -w$. Then $a+b = 0$, therefore $G_{a,b}$ is the geometric mean. The same argument as in the previous subcase shows that either $w = 0$ or $r - s = w^2$ must hold. If $r - s = w^2$, then $t = w^2/9$, thus $\{c, d\} = \{3w, w\}$ and $\{p, q\} = \{4w, 2w\}$, i.e., $S_{p,q}$ is the power mean of exponent $2w$, therefore condition (v) of Theorem 3.5 is valid.

Subcase 4: $s = 0$. Then by simplifying the resultants $R_{6,8,10}$ and $R_{6,8,12}$, we have that these resultants can only be zero if $v = 0, v = w, v = -w$ or $w = 0$. Thus we do not get new solutions in this subcase. □

THEOREM 3.6. (Baják–Páles [**5**]) *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3.6)*, i.e.,*

$$S_{p,q}(G_{a,b}(x, y), S_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

- *(i) $a + b = c + d = p + q = 0$, i.e., all the three means are equal to the geometric mean;*
- *(ii) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{0, w\}$ and $\{c, d\} = \{p, q\} = \{w, 2w\}$, i.e., all the three means are equal to each other, and they are equal to the power mean of exponent w;*
- *(iii) there exists a $w \in \mathbb{R}$ such that $\{a, b\} = \{0, w\}$, $\{-c, -d\} = \{w, 2w\}$ and $p + q = 0$, i.e., $S_{p,q}$ is the geometric mean and the two means $G_{a,b}$ and $S_{-c,-d}$ are equal to each other, and are equal to the power mean of exponent w;*

(iv) *there exists a* $w \in \mathbb{R}$ *such that* $\{a,b\} = \{w,3w\}$, $\{p,q\} = \{2w,4w\}$ *and* $c+d=0$, *i.e.,* $S_{c,d}$ *is the geometric mean and* $S_{p,q}$ *is the power mean of exponent* $2w$;

(v) *there exists a* $w \in \mathbb{R}$ *such that* $\{a,b\} = \{w,2w\}$, $\{c,d\} = \{-w,-2w\}$ *and* $\{p,q\} = \{w,2w\}$, *i.e.,* $S_{c,d}$ *is the power mean of exponent* $-w$ *and* $S_{p,q}$ *is the power mean of exponent* w.

Proof. We again use a similar argument as before.

The sufficiency of the conditions (i)–(v) can easily be checked. To derive the necessity of the conditions of Theorem 3.6, we check equation

$$F^{(k)}(0,k) = F^{(k)}_{M_{p,q,\mathfrak{n};k},M_{a,b,\mathfrak{m};k},M_{c,d,\mathfrak{n};k}}(0) = 0 \qquad (k \in \mathbb{N}), \tag{3.39}$$

for the values $k = 2,4,6,8,10,12$. Here, in view of the sixth identity in (3.20), $F : \mathbb{R}\times\mathbb{N} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ and $k \in \mathbb{N}$ as

$$F(x,k) := H(p,q,G(a,b,x,-x,k),H(c,d,x,-x,k),k) - H(p,q,x,-x,k).$$

The second-order Taylor coefficient of $F(\cdot,2)$ at $x=0$ in this case yields

$$C_2 := -\frac{1}{6}p + \frac{1}{12}c + \frac{1}{4}a + \frac{1}{4}b + \frac{1}{12}d - \frac{1}{6}q$$

By condition (3.39) $C_2 = 0$, thus

$$\frac{a+b}{4} + \frac{c+d}{12} = \frac{p+q}{6}.$$

Now we introduce the following notations

$$w := \frac{a+b}{4} + \frac{c+d}{12} = \frac{p+q}{6}, \qquad v := \frac{a+b}{4} - \frac{c+d}{12}, \qquad t := \Big(\frac{p-q}{6}\Big)^2,$$
$$r := \frac{9(a-b)^2 + (c-d)^2}{72}, \qquad s := \frac{9(a-b)^2 - (c-d)^2}{72}.$$

Then we can express the parameters a,b,c,d,p,q in the following form:

$$a := w + v + \sqrt{r+s}, \qquad c := 3w - 3v + 3\sqrt{r-s}, \qquad p := 3w + 3\sqrt{t},$$
$$b := w + v - \sqrt{r+s}, \qquad d := 3w - 3v - 3\sqrt{r-s}, \qquad q := 3w - 3\sqrt{t}.$$

The fourth-order Taylor coefficient C_4 of $F(\cdot,4)$ at $x=0$ yields

$$C_4 := \frac{2}{5}w^2v - \frac{2}{5}wv^2 + \frac{2}{15}w^3 - \frac{7}{15}wr + \frac{2}{15}ws + \frac{2}{15}v^3 + \frac{2}{15}vr - \frac{7}{15}vs + \frac{3}{5}wt$$

If $w=0$, then $p+q=0$, and hence $S_{p,q}$ is the geometric mean. The same argument as in previous proofs shows that assertion (i) or (iii) of Theorem

3.6 must hold. Thus, in what follows we may assume that $w \neq 0$. We can solve $C_4 = 0$ for t:

$$t := -\frac{2w^3 + 6w^2v - 7wr - 7vs - 6wv^2 + 2ws + 2v^3 + 2vr}{9w}$$

We compute the Taylor coefficients C_6, C_8, C_{10} and C_{12}. Here we have that

$$\begin{aligned} C_6 := -\frac{2}{315w}\big(& -255\,w^2v^2r + 122\,w^3vr - 28\,v^2sr - 78\,wv^3s + 48\,wv^3r + 2\,wvs^2 + 2\,wvr^2 + 32\,w^2rs \\ & - 157\,w^3vs + 48\,w^4s - 2\,w^2r^2 - 47\,w^2s^2 + 49\,v^2s^2 - 28\,v^4s + 8\,v^4r + 4\,v^2r^2 + 150\,w^2v^2s + 4\,v^6 \\ & - 26\,w^6 + 4\,wvrs + 50\,w^3v^3 - 48\,w^2v^4 + 12\,w^4r + 123\,w^4v^2 - 48\,w^5v + 6\,wv^5\big) \end{aligned}$$

$$\begin{aligned} C_8 := -\frac{1}{1575w^2}\big(& -240\,wr^2v^4 - 1600\,w^2v^5r + 5074\,w^6vs + 323\,vs^3w^2 - 28\,w^2r^3v - 1384\,w^2v^3r^2 \\ & - 168\,v^5sr + 11004\,w^4v^3s + 8746\,w^3v^4r + 528\,v^6sw - 84\,wr^3v^2 + 294\,v^2s^3w - 1050\,v^4s^2w \\ & + 5570\,w^3v^2r^2 + 2180\,w^2v^5s + 1137\,w^3rs^2 - 84\,v^3sr^2 - 3044\,w^6vr + 9152\,w^5v^2r + 294\,v^3s^2r \\ & - 4298\,w^4vs^2 + 540\,w^9 + 8\,v^9 + 64\,w^3r^3 - 11064\,w^4v^3r + 4130\,w^3v^2s^2 - 8992\,w^5v^2s - 228\,wrv^6 \\ & - 84\,v^7s + 504\,w^7r - 1404\,w^7s + 28\,w^5r^2 - 72\,wv^8 - 314\,w^3s^3 + 294\,v^5s^2 - 1598\,w^5v^4 \\ & - 3822\,w^7v^2 + 1664\,w^6v^3 + 896\,w^8v - 1572\,w^4v^5 + 1240\,w^3v^6 - 244\,v^7w^2 + 24\,v^7r + 24\,v^5r^2 \\ & + 8\,v^3r^3 + 3118\,w^2v^3sr + 24\,w^2r^2vs - 354\,w^2rvs^2 + 612\,wr^2v^2s - 1197\,wrv^2s^2 + 1140\,wrv^4s \\ & + 7766\,w^4vrs - 8120\,w^3v^2sr - 5816\,w^3v^4s - 3488\,w^4vr^2 - 672\,w^3r^2s - 2716\,w^5rs - 844\,w^2v^3s^2 \\ & + 2278\,w^5s^2 - 343\,v^3s^3\big) \end{aligned}$$

Here we suppress the exact forms of C_{10} and C_{12}, but we have that the coefficients are of the form

$$C_{2k} = \frac{c_{2k}}{w^{k-2}} P_{2k}(r, s, v, w) \qquad (k = 3, 4, 5, 6),$$

where c_{2k} is a nonzero real constant and P_{2k} is a four variable polynomial. Thus, we obtain the following system of four equations for the unknowns r, s, v and w:

$$P_{2k}(r, s, v, w) = 0 \qquad (k = 3, 4, 5, 6). \tag{3.40}$$

The resultants $R_{6,2k}$ of the polynomials $P_6(\cdot, s, v, w)$ and $P_{2k}(\cdot, s, v, w)$ are

$$R_{6,2k} = c_{6,2k} w^{k-1} (v - w)(v + w) P_{6,2k}(s, v, w) \qquad (k = 4, 5, 6),$$

where $c_{6,2k}$ is a nonzero real constant and $P_{6,2k}$ is a three variable polynomial. Thus either $v - w = 0$, $v + w = 0$ or (s, v, w) is a solution of the following system of three polynomial equations:

$$P_{6,2k}(s, v, w) = 0 \qquad (k = 4, 5, 6). \tag{3.41}$$

If $v - w = 0$, then $c + d = 0$, hence $S_{c,d}$ is the geometric mean. By simplifying the coefficients C_6, C_8, C_{10} and C_{12} and computing their resultants, we have that these coefficients can only be zero if $w = 0$ or $r + s = w^2$. If $r+s = w^2$, we get that $t = w^2/9$, thus $\{a, b\} = \{3w, w\}$ and $\{p, q\} = \{4w, 2w\}$, i.e., $S_{p,q}$ is the power mean of exponent $2w$, therefore assertion (iv) of Theorem 3.6 holds.

On the other hand, if $v + w = 0$, then $a + b = 0$, hence $G_{a,b}$ is the geometric mean. The same method (or applying Theorem 3.2) shows that in this case all the three means are equal to the geometric mean, thus we do not get new solutions.

Now we consider the system of equations in (3.41). We eliminate the variable s by calculating the resultants $R_{6,8,10}$ and $R_{6,8,12}$ of $P_{6,8}(\cdot, v, w)$ and $P_{6,10}(\cdot, v, w)$, and $P_{6,8}(\cdot, v, w)$ and $P_{6,12}(\cdot, v, w)$, respectively. We obtain that, for $k = 5, 6$

$$R_{6,8,2k} = c_{6,8,2k} v^4 w^{2k-4} (2v - w)^2 (v - 2w)(v + 7w) \\ (v - w)^3 (v + w)^{4k+2} P_{6,8,2k}(v, w),$$

where $c_{6,8,2k}$ is a nonzero real constant and $P_{6,8,2k}$ is a two variable homogeneous polynomial. Then, by calculating the resultant of $P_{6,8,10}(\cdot, w)$ and $P_{6,8,12}(\cdot, w)$, we get that this resultant is zero if and only if $w = 0$, thus we do not get new solutions.

Finally, we consider the remaining cases:

Subcase 1: $v = 0$. In this case, by simplifying the coefficients C_6, C_8, C_{10} and C_{12}, and calculating their resultants with respect to r, we have that either $w = 0$ or $9s - 4w^2 = 0$. If $s = 4w^2/9$, then by again simplifying the Taylor coefficients, we obtain that $r = 5w^2/9$ and $t = w^2/9$. Thus $\{a, b\} = \{2w, 0\}$ and $\{c, d\} = \{p, q\} = \{4w, 2w\}$, i.e., all the three means are equal to the power mean of exponent $2w$, hence assertion (ii) of Theorem 3.6 holds.

Subcase 2: $v = 2w$. The same argument as in the previous subcase shows that here we also have that either $w = 0$ or $9s - 4w^2 = 0$. If $s = 4w^2/9$, then we obtain that $r = 5w^2/9$ and $t = w^2/9$. Thus $\{a, b\} = \{4w, 2w\}$, $\{c, d\} = \{-2w, -4w\}$ and $\{p, q\} = \{4w, 2w\}$, i.e., $S_{c,d}$ is the power mean of exponent $-2w$ and $S_{p,q}$ is the power mean of exponent $2w$, hence assertion (v) of Theorem 3.6 holds.

Subcase 3: $2v = w$. In this case, by simplifying the resultants $R_{6,8}, R_{6,10}$ and $R_{6,12}$, we get that these resultants are zero if and only if $w = 0$, thus we do not get new solutions.

Subcase 4: $v = -7w$. The same argument as in the previous subcase shows that we do not get new solutions in this case either. □

It is worth noticing that in Theorem 3.3, Theorem 3.4, Theorem 3.5 and Theorem 3.6, the means $G_{p,q}$ and all the Stolarsky means involved (except in case (iii) of Theorem 3.4) are equal to power means.

In the proofs of the six theorems for the Gini and the Stolarsky means, after transforming the problem, we use the computer algebra package Maple to help us in performing the vast calculations. However, during the procedure we always had to analyze and interpret the outputs of the calculations that could not have been carried out without using the high performance of the computer algebra package.

Summary

The Gini and the Stolarsky means form a very important area of research in the theory of mean values. In the most general case (i.e., when $p \neq q$), the Gini mean of two positive real numbers x and y is defined by

$$G_{p,q}(x, y) = \left(\frac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}},$$

and if $(p - q)pq(x - y) \neq 0$ the Stolarsky mean of two positive real numbers x and y is

$$S_{p,q}(x, y) := \left(\frac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{\frac{1}{p-q}}.$$

For the complete definition covering all exceptional cases, see the definitions in Chapter 1.

Another important class of mean values is the class of quasi-arithmetic means. The quasi-arithmetic mean (generated by the strictly monotone continuous function φ) of x and y from a nonvoid open interval I is

$$\mathcal{M}_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right).$$

A possible generalization of the quasi-arithmetic means is the following: If the continuous, strictly monotone functions φ_1 and φ_2 are strictly monotone in the same sense on an interval I, the generalized quasi-arithmetic mean is defined by

$$\mathcal{M}_{\boldsymbol{\varphi}}(x, y) := \varphi^{-1}(\varphi_1(x) + \varphi_2(y)) \qquad (x,\ y \in I),$$

where

$$\boldsymbol{\varphi} := (\varphi_1, \varphi_2), \qquad \varphi := \varphi_1 + \varphi_2.$$

If $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ are two strict means, their Gauss composition $K = M \otimes N$ is the unique strict mean solution K of the functional equation

$$K(x, y) = K\big(M(x, y), N(x, y)\big) \qquad (x, y \in \mathbb{R}_+),$$

which is the invariance equation. For the detailed definition of the Gauss composition and the characterization of the solution of the invariance equation as well as for some examples when the invariance equation holds, see Chapter 1.

In the thesis we examined and solved the invariance equation for the above generalization of the quasi-arithmetic means as well as for the classes of Gini and Stolarsky means.

The invariance of the arithmetic mean with respect to two quasi-arithmetic means (the Matkowski–Sutô problem) was first investigated by Matkowski. The general solution of this problem without any regularity assumption was described by Daróczy and Páles. They also solved the invariance equation for quasi-arithmetic means in the general case. The complete solution of the invariance equation in the class of weighted quasi-arithmetic means was given by Jarczyk. These preliminary results in detailed form can be found in the first two sections of Chapter 2. In the third section we deal with the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means, which is also a Matkowski–Sutô-type problem. Our theorem generalizes the solution of Daróczy and Páles and the result of Jarczyk in the case when the outer mean in the invariance equation is the arithmetic mean, but under some higher-order regularity conditions for the generating functions.

Theorem. *Let φ_1, φ_2 and ψ_1, ψ_2 be 4-times continuously differentiable functions defined on a nonempty open interval I such that $\varphi_1'(x)\varphi_2'(x) > 0$ and $\psi_1'(x)\psi_2'(x) > 0$ (i.e., φ_1, φ_2 and ψ_1, ψ_2 are strictly monotone in the same sense, respectively) for $x \in I$. Then, for every x and y in I, the functional equation*

$$(\varphi_1 + \varphi_2)^{-1}(\varphi_1(x) + \varphi_2(y)) + (\psi_1 + \psi_2)^{-1}(\psi_1(x) + \psi_2(y)) = x + y$$

holds if and only if

(i) either there exist real constants $p, a_1, a_2, c_1, c_2, b_1, b_2, d_1, d_2$ with $p \neq 0$, $a_1 a_2 > 0$, $c_1 c_2 > 0$ and $a_1 c_1 = a_2 c_2$ such that, for $x \in I$,

$$\varphi_1(x) = a_1 e^{px} + b_1, \qquad \varphi_2(x) = a_2 e^{px} + b_2,$$

and

$$\psi_1(x) = c_1 e^{-px} + d_1, \qquad \psi_2(x) = c_2 e^{-px} + d_2;$$

(ii) or there exist real constants a, b, c, d_1, d_2 with $ac \neq 0$ such that, for $x \in I$,

$$\varphi_1(x) + \varphi_2(x) = ax + b,$$

and

$$\psi_1(x) = c\,\varphi_2(x) + d_1\,, \qquad \psi_2(x) = c\,\varphi_1(x) + d_2.$$

The method used in the proof of the above theorem is based on computing the partial derivatives of the means along the diagonal of $I \times I$ up to fourth order and thus getting conditions to prove the necessity part of the solutions. For the sufficiency part the regularity conditions are not needed.

In Chapter 3 we concentrate on solving the invariance equation when the three means involved are either Gini or Stolarsky means, which results six equations. With the help of a common generalization of both the Gini and the Stolarsky means, we are able to deal with these equations as special cases of a more general equation.

First we reformulate the general invariance equation:

LEMMA. *If $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ are homogeneous (resp. symmetric) strict means, then their Gauss composition $M \otimes N$ is also homogeneous (resp. symmetric). Furthermore, if $K : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a homogeneous strict mean then $K = M \otimes N$, i.e., the invariance equation*

$$K(x,y) = K\big(M(x,y), N(x,y)\big) \qquad (x, y \in \mathbb{R}_+),$$

holds if and only if the single-variable function $F_{K,M,N} : \mathbb{R} \to \mathbb{R}$ defined by

$$F_{K,M,N}(u) := \ln K\left(M(\mathrm{e}^u, \mathrm{e}^{-u}), N(\mathrm{e}^u, \mathrm{e}^{-u})\right) - \ln K(\mathrm{e}^u, \mathrm{e}^{-u}) \qquad (u \in \mathbb{R}),$$

vanishes everywhere on $\mathbb{R}$. In the case when K, M, N are analytic functions, $F_{K,M,N}$ is also analytic and vanishes on $\mathbb{R}$ if and only if

$$F^{(k)}_{K,M,N}(0) = 0$$

for all $k \in \mathbb{N}$. If, additionally M, N and K are symmetric strict means, then $F_{K,M,N}$ is an even function and $F_{K,M,N}$ vanishes on $\mathbb{R}$ if and only if the derivatives above vanish for all even $k \in \mathbb{N}$.

If r and s are two different real parameters and μ is a Borel probability measure on $[0, 1]$, the two-variable mean

$$M_{r,s,\mu}(x,y) = \left(\frac{\int_0^1 (x^t y^{1-t})^r \,\mathrm{d}\mu(t)}{\int_0^1 (x^t y^{1-t})^s \,\mathrm{d}\mu(t)} \right)^{\frac{1}{r-s}}$$

is a common generalization of both the Gini and the Stolarsky means (for the complete definition, see Chapter 3). If μ is equal to $\frac{\delta_0+\delta_1}{2}$ (where δ_x

stands for the Dirac measure concentrated at x), we get the Gini mean $G_{r,s}$, and if μ is equal to the Lebesgue measure, we get the Stolarsky mean $S_{r,s}$. This means that each of the six invariance equations can be considered as a particular case of the equation

$$M_{p,q,\kappa}(M_{a,b,\mu}(x,y), M_{c,d,\nu}(x,y)) = M_{p,q,\kappa}(x,y) \qquad (x,y \in \mathbb{R}_+),$$

where each of μ, ν and κ is equal to the Lebesgue measure on $[0,1]$ or to the measure $\frac{\delta_0+\delta_1}{2}$. In view of the lemma, the above invariance equation holds if and only if, for all $u \in \mathbb{R}$,

$$\begin{aligned} F_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(u) &:= \ln\left(M_{p,q,\kappa}(M_{a,b,\mu}(e^u, e^{-u}), M_{c,d,\nu}(e^u, e^{-u}))\right) \\ &\quad - \ln\left(M_{p,q,\kappa}(e^u, e^{-u})\right) = 0, \end{aligned}$$

i.e., for all $k \in \mathbb{N}$,

$$F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(0) = 0.$$

To get a more useful representation of the means $M_{r,s,\mu}$, we introduce the function $L_\mu : \mathbb{R} \to \mathbb{R}_+$ by

$$L_\mu(z) := \ln\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\mu_k\right),$$

where μ_k denotes the kth central moment of the measure μ. Assuming that μ is symmetric with respect to $\frac{1}{2}$ it follows that $\mu_{2k-1} = 0$ for all $k \in \mathbb{N}$. With the help of funcion L_μ, we can express the main expression of the mean $M_{r,s,\mu}$ in the following form:

LEMMA. *If μ be a Borel probability measure on $[0,1]$ and $r, s \in \mathbb{R}$, then*

$$M_{r,s,\mu}(x,y) = \exp\left(M^*_{r,s,\mu}(\ln x, \ln y)\right) \qquad (x,y \in \mathbb{R}_+),$$

where $M^*_{r,s,\mu} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is defined by

$$M^*_{r,s,\mu}(u,v) := \frac{u+v}{2} + \frac{L_\mu(r(u-v)) - L_\mu(s(u-v))}{r-s}.$$

To simplify the calculations, we consider an approximation of the mean $M_{r,s,\mu}$. If μ is a Borel probability measure and $m \in \mathbb{N}$, define the functions

$$L_{\mu;m}(z) := \ln\left(\sum_{k=0}^{m} \frac{z^k}{k!}\mu_k\right) \qquad (z \in \mathbb{R}),$$

and, if $r, s \in \mathbb{R}$,

$$M_{r,s,\mu;m}(x,y) = \exp\left(M^*_{r,s,\mu;m}(\ln x, \ln y)\right) \qquad (x,y \in \mathbb{R}_+),$$

where $M^*_{r,s,\mu;m} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is defined by (suppressing the exceptional case $r = s$)

$$M^*_{r,s,\mu;m}(u, v) := \frac{u + v}{2} + \frac{L_{\mu;m}(r(u - v)) - L_{\mu;m}(s(u - v))}{r - s}.$$

The following lemma states that instead of the functions $M_{r,s,\mu}$ and L_μ, we can use the truncated functions $M_{r,s,\mu;m}$ and $L_{\mu;m}$ to compute the higher-order derivatives needed in the calculations.

Lemma. *Let μ be a Borel probability measure. Then, for all $m, i \in \mathbb{N}_0$ with $i \leqslant m$,*

$$(L^{(i)}_\mu(0)) = (L^{(i)}_{\mu;m}(0)).$$

Furthermore, for all $r, s \in \mathbb{R}$ and $m, i, j \in \mathbb{N}_0$ with $i + j \leqslant m$,

$$\partial_1^i \partial_2^j M_{r,s,\mu}(1, 1) = \partial_1^i \partial_2^j M_{r,s,\mu;m}(1, 1).$$

As an immeadiate consequence of the lemma, the computation of the higher order derivatives $F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}$ at 0 can be replaced by the computation of the derivatives $F^{(k)}_{M_{p,q,\kappa;m},M_{a,b,\mu;m},M_{c,d,\nu;m}}$ at 0 provided that $k \leqslant m$.

Corollary. *Let $a, b, c, d, p, q \in \mathbb{R}$ and μ, ν, κ be Borel probability measures on* $[0, 1]$. *Then, for all $k, m \in \mathbb{N}_0$ with $k \leqslant m$,*

$$F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(0) = F^{(k)}_{M_{p,q,\kappa;m},M_{a,b,\mu;m},M_{c,d,\nu;m}}(0).$$

This means that it is sufficient to check these easier conditions while solving the invariance equations. We can consider each equation as the suitable special case of the identity

$$F_{M_{p,q,\kappa;k},M_{a,b,\mu;k},M_{c,d,\nu;k}}(u) = M^*_{p,q,\kappa;k}(M^*_{a,b,\mu;k}(u, -u), M^*_{c,d,\nu;k}(u, -u)) - M^*_{p,q,\kappa;k}(u, -u) = 0.$$

We get solutions for the unknown parameters a, b, c, d, p, q by computing the Taylor coefficients of the function

$$F_{M_{p,q,\kappa;k},M_{a,b,\mu;k},M_{c,d,\nu;k}}$$

at $x = 0$ up to a sufficiently high order, and determining the conditions when all these coefficients vanish. This function is even due to the symmetry of the means, thus all coefficients of odd order are zero. Therefore we have to differentiate up to 12th order to get sufficiently many conditions for the six parameters. Our task is to determine the common roots of this system of six polynomial equations. The Taylor coefficients can be factorized and by analyzing these factors, we get solutions to the invariance equations. In several cases, among these factors we get high-order multivariable polynomials of the unknowns. We calculate the resultants of these polynomials to check if they have common roots. We used the computer algebra package Maple V Release 9 to perform the tedious computations.

In Chapter 3, using the above method, we give the general solutions of the six invariance equations involving Gini and Stolarsky means. In the following theorem we describe the solution when all three means are Gini means with possibly different parameters.

Theorem. *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation*

$$G_{p,q}(G_{a,b}(x,y), G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

holds if and only if one of the following possibilities holds:

(i) $a+b=c+d=p+q=0$*, i.e., all the three means are equal to the geometric mean;*
(ii) $\{a,b\}=\{c,d\}=\{p,q\}$*, i.e., all the three means are equal to each other;*
(iii) $\{a,b\}=\{-c,-d\}$ *and* $p+q=0$*, i.e.,* $G_{p,q}$ *is the geometric mean and* $G_{a,b}=G_{-c-d}$*;*
(iv) there exist $u, v \in \mathbb{R}$ *such that* $\{a,b\}=\{u+v,v\}$*,* $\{c,d\}=\{u-v,-v\}$*, and* $\{p,q\}=\{u,0\}$ *(in this case,* $G_{p,q}$ *is a power mean);*
(v) there exists $w \in \mathbb{R}$ *such that* $\{a,b\}=\{3w,w\}$*,* $c+d=0$*, and* $\{p,q\}=\{2w,0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{c,d}$ *is the geometric mean);*
(vi) there exists $w \in \mathbb{R}$ *such that* $a+b=0$*,* $\{c,d\}=\{3w,w\}$*, and* $\{p,q\}=\{2w,0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{a,b}$ *is the geometric mean).*

The exact Maple code to define the appropriate functions, compute the Taylor coefficients and the resultants as well as an application of this theorem is given in Chapter 3.

In the next theorem, we give the general solution of the invariance equation for Stolarsky means.

Theorem. *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation*

$$S_{p,q}(S_{a,b}(x,y), S_{c,d}(x,y)) = S_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a+b=c+d=p+q=0$*, i.e., all the three means are equal to the geometric mean;*
(ii) $\{a,b\}=\{c,d\}=\{p,q\}$*, i.e., all the three means are equal to each other;*

(iii) $\{a,b\} = \{-c,-d\}$ *and* $p+q=0$*, i.e.,* $S_{p,q}$ *is the geometric mean and* $S_{a,b} = S_{-c,-d}$.

The following theorems completely describe the solution sets of the mixed equations, i.e., when the three means involved are either Gini or Stolarsky means.

Theorem. *Let* $a,b,c,d,p,q \in \mathbb{R}$. *Then the invariance equation*

$$G_{p,q}(S_{a,b}(x,y), G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a+b=c+d=p+q=0$*, i.e., all the three means are equal to the geometric mean;*

(ii) there exists a $w \in \mathbb{R}$ *such that* $\{a,b\} = \{w,2w\}$ *and* $\{c,d\} = \{p,q\} = \{0,w\}$*, i.e., all the three means are equal to each other, and they are also equal to the power mean of exponent* w*;*

(iii) there exists a $w \in \mathbb{R}$ *such that* $\{a,b\} = \{w,2w\}$, $\{c,d\} = \{0,-w\}$ *and* $p+q=0$*, i.e.,* $G_{p,q}$ *is the geometric mean and the two means* $S_{a,b}$ *and* $G_{-c,-d}$ *are equal to each other, and are equal to the power mean of exponent* w*;*

(iv) there exists $w \in \mathbb{R}$ *such that* $a+b=0$, $\{c,d\} = \{3w,w\}$*, and* $\{p,q\} = \{2w,0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $S_{a,b}$ *is the geometric mean);*

(v) there exists a $w \in \mathbb{R}$ *such that* $\{a,b\} = \{w,2w\}$, $\{c,d\} = \{-w,-2w\}$ *and* $\{p,q\} = \{0,-w\}$*, i.e.,* $S_{a,b}$ *is the power mean of exponent* w *and* $G_{p,q}$ *is the power mean of exponent* $-w$.

Theorem. *Let* $a,b,c,d,p,q \in \mathbb{R}$. *Then the invariance equation*

$$G_{p,q}(S_{a,b}(x,y), S_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a+b=c+d=p+q=0$*, i.e., all the three means are equal to the geometric mean;*

(ii) there exists a $w \in \mathbb{R}$ *such that* $\{a,b\} = \{c,d\} = \{w,2w\}$ *and* $\{p,q\} = \{0,w\}$*, i.e., all the three means are equal to each other, and they are equal to the power mean of exponent* w*;*

(iii) $\{a,b\} = \{-c,-d\}$ *and* $p+q=0$*, i.e.,* $G_{p,q}$ *is the geometric mean and* $S_{a,b} = S_{-c,-d}$.

Theorem. *Let* $a,b,c,d,p,q \in \mathbb{R}$. *Then the invariance equation*

$$S_{p,q}(G_{a,b}(x,y), G_{c,d}(x,y)) = S_{p,q}(x,y) \qquad (x,y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a + b = c + d = p + q = 0$, *i.e., all the three means are equal to the geometric mean;*

(ii) *there exists a* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{c, d\} = \{0, w\}$ *and* $\{p, q\} = \{w, 2w\}$, *i.e., all the three means are equal to each other, and they are equal to the power mean of exponent* w;

(iii) $\{a, b\} = \{-c, -d\}$ *and* $p + q = 0$, *i.e.,* $S_{p,q}$ *is the geometric mean and* $G_{a,b} = G_{-c,-d}$;

(iv) *there exists a* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ *and* $c + d = 0$, *i.e.,* $G_{c,d}$ *is the geometric mean and* $S_{p,q}$ *is the power mean of exponent* $2w$;

(v) *there exists a* $w \in \mathbb{R}$ *such that* $\{c, d\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ *and* $a + b = 0$, *i.e.,* $G_{a,b}$ *is the geometric mean and* $S_{p,q}$ *is the power mean of exponent* $2w$.

Theorem. *Let* $a, b, c, d, p, q \in \mathbb{R}$. *Then the invariance equation*

$$S_{p,q}(G_{a,b}(x, y), S_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

is valid if and only if one of the following possibilities holds:

(i) $a + b = c + d = p + q = 0$, *i.e., all the three means are equal to the geometric mean;*

(ii) *there exists a* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{0, w\}$ *and* $\{c, d\} = \{p, q\} = \{w, 2w\}$, *i.e., all the three means are equal to each other, and they are equal to the power mean of exponent* w;

(iii) *there exists a* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{0, w\}$, $\{-c, -d\} = \{w, 2w\}$ *and* $p + q = 0$, *i.e.,* $S_{p,q}$ *is the geometric mean and the two means* $G_{a,b}$ *and* $S_{-c,-d}$ *are equal to each other, and are equal to the power mean of exponent* w;

(iv) *there exists a* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ *and* $c + d = 0$, *i.e.,* $S_{c,d}$ *is the geometric mean and* $S_{p,q}$ *is the power mean of exponent* $2w$;

(v) *there exists a* $w \in \mathbb{R}$ *such that* $\{a, b\} = \{w, 2w\}$, $\{c, d\} = \{-w, -2w\}$ *and* $\{p, q\} = \{w, 2w\}$, *i.e.,* $S_{c,d}$ *is the power mean of exponent* $-w$ *and* $S_{p,q}$ *is the power mean of exponent* w.

Összefoglalás

A Gini és a Stolarsky közepek egy igen intenzíven kutatott területet alkotnak a középértékek elméletén belül. A legáltalánosabb esetben (azaz $p \neq q$ esetén) az x és y pozitív valós számok Gini közepe a

$$G_{p,q}(x, y) = \left(\frac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}},$$

valamint $(p-q)pq(x-y) \neq 0$ esetén az x és y pozitív valós számok Stolarsky közepe az

$$S_{p,q}(x, y) := \left(\frac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{\frac{1}{p-q}}$$

formulával értelmezett. A Gini és a Stolarsky közepek kivételes esetekre vonatkozó alakjai megtalálhatók az 1. Fejezetben.

A kvázi-aritmetikai közepek szintén nagyon fontos középosztályt alkotnak. Ha x és y egy nemüres, nyílt intervallum elemei és φ egy szigorúan monoton, folytonos függvény, akkor

$$\mathcal{M}_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right)$$

az x és y (φ által generált) kvázi-aritmetikai közepe.

Ezen középosztály egy lehetséges általánosítása a következő: Legyenek φ_1 és φ_2 szigorúan monoton, folytonos függvények úgy, hogy az I intervallumon a két függvény azonos értelemben szigorúan monoton, ekkor

$$\mathcal{M}_{\boldsymbol{\varphi}}(x, y) := \varphi^{-1}(\varphi_1(x) + \varphi_2(y)) \qquad (x,\ y \in I),$$

az x és y általánosított kvázi-aritmetikai közepe, ahol

$$\boldsymbol{\varphi} := (\varphi_1, \varphi_2), \qquad \varphi := \varphi_1 + \varphi_2.$$

Ha $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ két szigorú közép, akkor a két közép $K = M \otimes N$ Gauss kompozíciója a

$$K(x, y) = K\big(M(x, y), N(x, y)\big) \qquad (x, y \in \mathbb{R}_+)$$

függvényegyenlet, az ún. invariancia egyenlet egyértelmű olyan K megoldása, amely maga is szigorú közép. Az 1. Fejezet tartalmazza a Gauss kompozíció részletes definícióját, az invariancia egyenlet megoldásának jellemzését, valamint néhány példát közepek invarianciájára.

A dolgozatban a kvázi-aritmetikai közepek fenti általánosítására, illetve a Gini és a Stolarsky közepekre vonatkozó invariancia egyenleteket tanulmányoztuk és oldottuk meg.

A számtani közép két kvázi-aritmetikai középre vonatkozó invarianciáját (az ún. Matkowski–Sutô problémát) először Sutô oldotta meg, majd Matkowski ugyanazokat a megoldásokat találta gyengébb regularitás mellett. Ennek a problémának az általános megoldását regularitási feltételek nélkül Daróczy és Páles adták meg. Ugyancsak megoldották a kvázi-aritmetikai közepekre vonatkozó általános invariancia egyenletet. A súlyozott kvázi-aritmetikai közepekkel felírt invariancia egyenlet megoldása Jarczyk nevéhez fűződik. Mindezen korábbi eredmények részletes ismertetése megtalálható a 2. Fejezet első két szakaszában. A harmadik szakaszban a számtani középnek az általánosított kvázi-aritmetikai közepekre vonatkozó invarianciájával foglalkozunk, ami szintén egy Matkowski–Sutô típusú probléma. A megfogalmazott tétel általánosítja Daróczy és Páles, valamint Jarczyk eredményét abban az esetben, amikor az invariancia egyenletben szereplő külső közép a számtani közép, azonban az eredmény igazolásához szükséges volt a generáló függvények magasabbrendű regularitását feltételezni.

TÉTEL. *Legyenek φ_1, φ_2 és ψ_1, ψ_2 négyszer folytonosan differenciálható függvények egy I nemüres, nyílt intervallumon úgy, hogy $\varphi_1'(x)\varphi_2'(x) > 0$ és $\psi_1'(x)\psi_2'(x) > 0$ (azaz φ_1, φ_2 illetve ψ_1, ψ_2 azonos értelemben szigorúan monoton) minden $x \in I$ esetén. Ekkor a*

$$(\varphi_1 + \varphi_2)^{-1} (\varphi_1(x) + \varphi_2(y)) + (\psi_1 + \psi_2)^{-1} (\psi_1(x) + \psi_2(y)) = x + y$$

függvényegyenlet pontosan akkor teljesül minden x, $y \in I$ esetén, ha

(i) vagy léteznek p, a_1, a_2, c_1, c_2, b_1, b_2, d_1, d_2 valós konstansok, melyekre $p \neq 0$, $a_1 a_2 > 0$, $c_1 c_2 > 0$ és $a_1 c_1 = a_2 c_2$ teljesül úgy, hogy bármely $x \in I$ esetén

$$\varphi_1(x) = a_1\, e^{px} + b_1\,, \qquad \varphi_2(x) = a_2\, e^{px} + b_2\,,$$

és

$$\psi_1(x) = c_1\, e^{-px} + d_1\,, \qquad \psi_2(x) = c_2\, e^{-px} + d_2\,;$$

(ii) vagy léteznek a, b, c, d_1, d_2 valós konstansok, melyekre $ac \neq 0$ teljesül úgy, hogy bármely $x \in I$ esetén

$$\varphi_1(x) + \varphi_2(x) = ax + b\,,$$

és

$$\psi_1(x) = c\,\varphi_2(x) + d_1\,, \qquad \psi_2(x) = c\,\varphi_1(x) + d_2.$$

A fenti tétel bizonyításában a szükségesség igazolásához azon a módon nyerünk feltételeket, hogy kiszámoljuk a közepek parciális deriváltjait $I \times I$ átlóján negyedrendig bezárólag. Az elégségesség bizonyítása során ez a regularitási feltétel elhagyható, így egy erősebb állítás is megfogalmazható.

A 3. Fejezetben az invariancia egyenlet azon eseteit tárgyaljuk, melyekben az előforduló közepek mindegyike Gini vagy Stolarsky közép. Ez összesen hat egyenletet eredményez. Azáltal, hogy a Gini és a Stolarsky közepek egy közös általánosításával dolgozunk, ezen hat egyenlet mindegyikét egy általánosabb egyenlet speciális eseteként oldhatjuk meg.

Először átfogalmazzuk az invariancia egyenlet általános alakját:

Lemma. *Legyenek $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ homogén (szimmetrikus) szigorú közepek. Ekkor ezen két közép $M \otimes N$ Gauss kompozíciója szintén homogén (szimmetrikus). Továbbá, ha $K : \mathbb{R}_+^2 \to \mathbb{R}_+$ homogén szigorú közép, akkor $K = M \otimes N$ - azaz teljesül a*

$$K(x, y) = K\big(M(x, y), N(x, y)\big) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet - pontosan akkor, ha az

$$F_{K,M,N}(u) := \ln K\left(M(\mathrm{e}^u, \mathrm{e}^{-u}), N(\mathrm{e}^u, \mathrm{e}^{-u})\right) - \ln K(\mathrm{e}^u, \mathrm{e}^{-u}) \qquad (u \in \mathbb{R})$$

módon értelmezett $F_{K,M,N} : \mathbb{R} \to \mathbb{R}$ egyváltozós függvény azonosan eltűnik az egész számegyenesen. Abban az esetben, ha K, M, N analitikus függvények, akkor $F_{K,M,N}$ szintén analitikus és pontosan akkor tűnik el az egész számegyenesen, ha minden $k \in \mathbb{N}$ esetén

$$F^{(k)}_{K,M,N}(0) = 0.$$

Ha még ezen felül M, N és K szimmetrikus szigorú közepek, akkor az $F_{K,M,N}$ függvény páros, és pontosan akkor tűnik el az egész számegyenesen, ha a fenti deriváltak eltűnnek minden páros k esetén.

Ha r és s két különböző valós paraméter és μ egy Borel valószínűségi mérték a $[0, 1]$ intervallumon, akkor az

$$M_{r,s,\mu}(x, y) = \left(\frac{\int_0^1 (x^t y^{1-t})^r \mathrm{d}\mu(t)}{\int_0^1 (x^t y^{1-t})^s \mathrm{d}\mu(t)} \right)^{\frac{1}{r-s}}$$

módon értelmezett kétváltozós közép a Gini és a Stolarsky közepek közös általánosítása (a közép teljes definícióját a 3. Fejezet tartalmazza). Amennyiben a μ mérték megegyezik a $\frac{\delta_0+\delta_1}{2}$ mértékkel (ahol δ_x az x pontba koncentrált Dirac mértéket jelöli), akkor a fenti közép a $G_{r,s}$ Gini közepet adja, illetve ha μ a Lebesgue-mérték, akkor pedig az $S_{r,s}$ Stolarsky közepet kapjuk. Ez azt jelenti, hogy az általunk vizsgált hat invariancia egyenlet tekinthető az

$$M_{p,q,\kappa}(M_{a,b,\mu}(x,y), M_{c,d,\nu}(x,y)) = M_{p,q,\kappa}(x,y) \qquad (x,y \in \mathbb{R}_+)$$

egyenlet hat különböző speciális esetének, ahol μ, ν és κ vagy a $[0,1]$ intervallumon értelmezett Lebesgue mértékkel, vagy a $\frac{\delta_0+\delta_1}{2}$ mértékkel egyenlők. A fenti lemma alapján az invariancia egyenlet pontosan akkor teljesül, ha minden $u \in \mathbb{R}$ esetén

$$F_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(u) := \ln\left(M_{p,q,\kappa}(M_{a,b,\mu}(e^u, e^{-u}), M_{c,d,\nu}(e^u, e^{-u}))\right) - \ln\left(M_{p,q,\kappa}(e^u, e^{-u})\right) = 0,$$

azaz

$$F^{(k)}_{M_{p,q,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(0) = 0$$

minden $k \in \mathbb{N}$-re.

Ahhoz, hogy az $M_{r,s,\mu}$ közép egy, a számolások során használhatóbb alakját megkapjuk, értelmezzük az

$$L_\mu(z) := \ln\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\mu_k\right)$$

függvényt, ahol μ_k a μ mérték k-adik centrált momentumát jelöli. Feltéve, hogy μ szimmetrikus az $\frac{1}{2}$-re, kapjuk, hogy $\mu_{2k-1} = 0$ minden $k \in \mathbb{N}$ esetén. Az L_μ függvény segítségével az $M_{r,s,\mu}$ közép fenti alakja a következő formában is írható:

Lemma. *Legyenek $r, s \in \mathbb{R}$ és μ egy Borel valószínűségi mérték a* $[0,1]$ *intervallumon. Ekkor*

$$M_{r,s,\mu}(x,y) = \exp\left(M^*_{r,s,\mu}(\ln x, \ln y)\right) \qquad (x,y \in \mathbb{R}_+),$$

ahol $M^*_{r,s,\mu} : \mathbb{R}^2_+ \to \mathbb{R}_+$ a következő módon értelmezett:

$$M^*_{r,s,\mu}(u,v) := \frac{u+v}{2} + \frac{L_\mu(r(u-v)) - L_\mu(s(u-v))}{r-s}.$$

Hogy a számolásokat tovább egyszerűsítsük, megadjuk az $M_{r,s,\mu}$ közép egy approximációját. Ha $m \in \mathbb{N}$ és μ egy Borel valószínűségi mérték, akkor legyen

$$L_{\mu;m}(z) := \ln\left(\sum_{k=0}^{m} \frac{z^k}{k!}\mu_k\right) \qquad (z \in \mathbb{R}),$$

valamint $r, s \in \mathbb{R}$ esetén legyen

$$M_{r,s,\mu;m}(x, y) = \exp\left(M^*_{r,s,\mu;m}(\ln x, \ln y)\right) \qquad (x, y \in \mathbb{R}_+),$$

ahol $M^*_{r,s,\mu;m} : \mathbb{R}^2_+ \to \mathbb{R}_+$ a következő módon értelmezett:

$$M^*_{r,s,\mu;m}(u, v) := \frac{u + v}{2} + \frac{L_{\mu;m}(r(u - v)) - L_{\mu;m}(s(u - v))}{r - s}.$$

Az alábbi lemma azt állítja, hogy a magasabbrendű deriváltak számolása során az $M_{r,s,\mu}$ és L_μ függvények helyett dolgozhatunk az $M_{r,s,\mu;m}$ és $L_{\mu;m}$ csonkított függvényekkel.

Lemma. *Legyen μ egy Borel valószínűségi mérték. Ekkor minden olyan $m, i \in \mathbb{N}_0$ esetén, melyekre $i \leqslant m$, teljesül, hogy*

$$(L^{(i)}_\mu(0)) = (L^{(i)}_{\mu;m}(0)).$$

Továbbá, minden $r, s \in \mathbb{R}$ és $m, i, j \in \mathbb{N}_0$ esetén, melyekre $i + j \leqslant m$, igaz, hogy

$$\partial_1^i \partial_2^j M_{r,s,\mu}(1, 1) = \partial_1^i \partial_2^j M_{r,s,\mu;m}(1, 1).$$

A lemma azonnali következményeként kapjuk, hogy a számításainkban $k \leqslant m$ esetén az $F^{(k)}_{M_{p,q,\kappa}, M_{a,b,\mu}, M_{c,d,\nu}}$ deriváltak 0-beli értéke helyett használhatjuk az $F^{(k)}_{M_{p,q,\kappa;m}, M_{a,b,\mu;m}, M_{c,d,\nu;m}}$ deriváltak 0-beli értékét.

Következmény. *Legyenek $a, b, c, d, p, q \in \mathbb{R}$ és μ, ν, κ a $[0, 1]$ intervallumon értelmezett Borel valószínűségi mértékek. Ekkor minden $k, m \in \mathbb{N}_0$ esetén, melyekre $k \leqslant m$, teljesül, hogy*

$$F^{(k)}_{M_{p,q,\kappa}, M_{a,b,\mu}, M_{c,d,\nu}}(0) = F^{(k)}_{M_{p,q,\kappa;m}, M_{a,b,\mu;m}, M_{c,d,\nu;m}}(0).$$

Tehát az invariancia egyenletek megoldása során elegendő a fenti feltételeket vizsgálni. Mindegyik egyenlet tekinthető az

$$F_{M_{p,q,\kappa;k}, M_{a,b,\mu;k}, M_{c,d,\nu;k}}(u) = \\ M^*_{p,q,\kappa;k}(M^*_{a,b,\mu;k}(u, -u), M^*_{c,d,\nu;k}(u, -u)) - M^*_{p,q,\kappa;k}(u, -u) = 0$$

egyenlet megfelelő speciális esetének. Az ismeretlen a, b, c, d, p, q paraméterek meghatározásához először az

$$F_{M_{p,q,\kappa;k}, M_{a,b,\mu;k}, M_{c,d,\nu;k}}$$

függvény $x = 0$ pontbeli, elegendően nagy rendű Taylor együtthatóit számoljuk ki, majd meghatározzuk azokat a feltételeket, melyek esetén ezen együtthatók mindegyike 0. Ez a függvény a közepek szimmetriája miatt páros, ezért minden páratlan rendű együttható zérus. Ezért ahhoz, hogy elegendő számú feltételt kapjunk a hat ismeretlen paraméterre, egészen 12. rendig kell a deriváltakat kiszámolnunk. A kapott, hat polinomiális egyenletből álló egyenletrendszernek kell meghatároznunk az összes közös gyökét. A Taylor együtthatók faktorizálhatók, és ezen tényezők vizsgálatával kapjuk az invariancia egyenletek megoldásait. A tényezők között azonban a legtöbb esetben rendre előfordulnak az ismeretlen paraméterek magas fokú, többváltozós polinomjai. Ezen polinomok közös gyökeit rezultánsok számolásával keressük. A nagy és hosszadalmas számítások elvégzéséhez a Maple V Release 9 komputeralgebra rendszert használtuk.

A 3. Fejezetben a fenti módszer segítségével megadjuk a Gini illetve Stolarsky közepekre vonatkozó hat invariancia egyenlet megoldását. A következő tétel azt az esetet tárgyalja, amikor az egyenletben előforduló három közép mindegyike Gini közép, melyeknek paraméterei különbözőek is lehetnek.

Tétel. *Legyenek $a, b, c, d, p, q \in \mathbb{R}$. Ekkor a*

$$G_{p,q}(G_{a,b}(x,y), G_{c,d}(x,y)) = G_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet pontosan akkor teljesül, ha a következő esetek valamelyike fennáll:

(i) $a + b = c + d = p + q = 0$, *azaz mindhárom közép megegyezik a geometriai középpel;*

(ii) $\{a, b\} = \{c, d\} = \{p, q\}$, *azaz a három közép egyenlő;*

(iii) $\{a, b\} = \{-c, -d\}$ *és* $p + q = 0$, *azaz* $G_{p,q}$ *a geometriai közép és* $G_{a,b} = G_{-c-d}$*;*

(iv) léteznek $u, v \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{u + v, v\}$, $\{c, d\} = \{u - v, -v\}$ *és* $\{p, q\} = \{u, 0\}$ *(ebben az esetben* $G_{p,q}$ *hatványközép);*

(v) létezik $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{3w, w\}$, $c + d = 0$ *és* $\{p, q\} = \{2w, 0\}$ *(ekkor* $G_{p,q}$ *hatványközép és* $G_{c,d}$ *a geometriai közép);*

(vi) létezik $w \in \mathbb{R}$ *úgy, hogy* $a + b = 0$, $\{c, d\} = \{3w, w\}$ *és* $\{p, q\} = \{2w, 0\}$ *(ekkor* $G_{p,q}$ *hatványközép és* $G_{a,b}$ *a geometriai közép).*

A megfelelő függvények definiálásához és a Taylor együtthatók illetve a rezultánsok kiszámolásához szükséges Maple parancsokat, valamint a tétel egy alkalmazását a 3. Fejezet tartalmazza.

Az alábbi tételben megadjuk a Stolarsky közepekre vonatkozó invariancia egyenlet megoldását.

TÉTEL. *Legyenek* $a, b, c, d, p, q \in \mathbb{R}$. *Ekkor az*

$$S_{p,q}(S_{a,b}(x, y), S_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet pontosan akkor teljesül, ha a következő esetek valamelyike fennáll:

(i) $a + b = c + d = p + q = 0$, *azaz mindhárom közép megegyezik a geometriai középpel;*
(ii) $\{a, b\} = \{c, d\} = \{p, q\}$, *azaz a három közép egyenlő;*
(iii) $\{a, b\} = \{-c, -d\}$ *és* $p + q = 0$, *azaz* $S_{p,q}$ *a geometriai közép és* $S_{a,b} = S_{-c,-d}$.

A következő tételekben jellemezzük a vegyes egyenleteket, azaz azokat az eseteket, amikor a közepek Gini és Stolarsky közepek is lehetnek.

TÉTEL. *Legyenek* $a, b, c, d, p, q \in \mathbb{R}$. *Ekkor a*

$$G_{p,q}(S_{a,b}(x, y), G_{c,d}(x, y)) = G_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet pontosan akkor teljesül, ha a következő esetek valamelyike fennáll:

(i) $a + b = c + d = p + q = 0$, *azaz mindhárom közép megegyezik a geometriai középpel;*
(ii) létezik $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{w, 2w\}$ *és* $\{c, d\} = \{p, q\} = \{0, w\}$, *azaz a három közép egyenlő, és megegyeznek a* w *paraméterű hatványközéppel;*
(iii) létezik $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{w, 2w\}$, $\{c, d\} = \{0, -w\}$ *és* $p + q = 0$, *azaz* $G_{p,q}$ *a geometriai közép és az* $S_{a,b}$ *és* $G_{-c,-d}$ *közepek egyenlők, és megegyeznek a* w *paraméterű hatványközéppel;*
(iv) létezik $w \in \mathbb{R}$ *úgy, hogy* $a + b = 0$, $\{c, d\} = \{3w, w\}$ *és* $\{p, q\} = \{2w, 0\}$ *(ekkor* $G_{p,q}$ *hatványközép és* $S_{a,b}$ *a geometriai közép);*
(v) létezik $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{w, 2w\}$, $\{c, d\} = \{-w, -2w\}$ *és* $\{p, q\} = \{0, -w\}$, *azaz* $S_{a,b}$ *a* w *paraméterű hatványközép és* $G_{p,q}$ *a* $-w$ *paraméterű hatványközép.*

TÉTEL. *Legyenek* $a, b, c, d, p, q \in \mathbb{R}$. *Ekkor a*

$$G_{p,q}(S_{a,b}(x, y), S_{c,d}(x, y)) = G_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet pontosan akkor teljesül, ha a következő esetke valamelyike fennáll:

(i) $a + b = c + d = p + q = 0$, *azaz mindhárom közép megegyezik a geometriai középpel;*
(ii) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{c, d\} = \{w, 2w\}$ *és* $\{p, q\} = \{0, w\}$, *azaz a három közép egyenlő, és megegyeznek a* w *paraméterű hatványközéppel;*
(iii) $\{a, b\} = \{-c, -d\}$ *és* $p + q = 0$, *azaz* $G_{p,q}$ *a geometriai közép és* $S_{a,b} = S_{-c,-d}$.

TÉTEL. *Legyenek* $a, b, c, d, p, q \in \mathbb{R}$. *Ekkor az*

$$S_{p,q}(G_{a,b}(x, y), G_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet pontosan akkor teljesül, ha a következő esetek valamelyike fennáll:

(i) $a + b = c + d = p + q = 0$, *azaz mindhárom közép megegyezik a geometriai középpel;*
(ii) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{c, d\} = \{0, w\}$ *és* $\{p, q\} = \{w, 2w\}$, *azaz a három közép egyenlő, és megegyeznek a* w *paraméterű hatványközéppel;*
(iii) $\{a, b\} = \{-c, -d\}$ *és* $p + q = 0$, *azaz* $S_{p,q}$ *a geometriai közép és* $G_{a,b} = G_{-c,-d}$;
(iv) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ *és* $c + d = 0$, *azaz* $G_{c,d}$ *a geometriai közép és* $S_{p,q}$ *a* $2w$ *paraméterű hatványközép;*
(v) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{c, d\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ *és* $a + b = 0$, *azaz* $G_{a,b}$ *a geometriai közép és* $S_{p,q}$ *a* $2w$ *paraméterű hatványközép.*

TÉTEL. *Legyenek* $a, b, c, d, p, q \in \mathbb{R}$. *Ekkor az*

$$S_{p,q}(G_{a,b}(x, y), S_{c,d}(x, y)) = S_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

invariancia egyenlet pontosan akkor teljesül, ha a következő esetek valamelyike fennáll:

(i) $a + b = c + d = p + q = 0$, *azaz mindhárom közép megegyezik a geometriai középpel;*
(ii) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{0, w\}$ *és* $\{c, d\} = \{p, q\} = \{w, 2w\}$, *azaz a három közép egyenlő, és megegyeznek a* w *paraméterű hatványközéppel;*
(iii) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{0, w\}$, $\{-c, -d\} = \{w, 2w\}$ *és* $p + q = 0$, *azaz* $S_{p,q}$ *a geometriai közép és a* $G_{a,b}$ *és* $S_{-c,-d}$ *közepek egyenlők és megegyeznek a* w *paraméterű hatványközéppel;*

(iv) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{w, 3w\}$, $\{p, q\} = \{2w, 4w\}$ *és* $c + d = 0$, *azaz* $S_{c,d}$ *a geometriai közép és* $S_{p,q}$ *a* $2w$ *paraméterű hatványközép;*

(v) *létezik* $w \in \mathbb{R}$ *úgy, hogy* $\{a, b\} = \{w, 2w\}$, $\{c, d\} = \{-w, -2w\}$ *és* $\{p, q\} = \{w, 2w\}$, *azaz* $S_{c,d}$ *a* $-w$ *paraméterű hatványközép és* $S_{p,q}$ *a* w *paraméterű hatványközép.*

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