On Shyr-Yu Theorem[1]

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Abstract

An alternative proof of Shyr-Yu Theorem is given. Some generalizations are also considered using fractional root decompositions and fractional exponents of words.

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1 Introduction

A word is primitive if it is not a power of another word. A well-known unsolved problem of theoretical computer science asks whether or not the language of all primitive words over a nontrivial alphabet is context-free or not [4, 5]. Among others, this (in)famous problem motivates the study of the combinatorial properties of primitive words. In addition, they have special importance in studies of automatic sequences [1, 8]. The Shyr-Yu Theorem [12] is a well-known classical result in this direction. The known proof of this result is rather involved [12]. The aim of this paper is to give a new simple proof of this well-known theorem, using some simple observations. In addition, we give the decomposition of words in the language p^+q^+, in particular we find fractional root decompositions and fractional exponents of the words appearing in the language.

2 Preliminaries

A *word* over Σ is a finite sequence of elements over some finite non-empty set Σ. If there is no danger of confusion, sometimes we omit the expression "over Σ". We call the set Σ an *alphabet,* the elements of Σ *letters.* Σ is called *trivial* if it is a singleton. Otherwise we also say that Σ is *nontrivial.* We also define the *empty word* λ consisting of zero letters. The *length* $|w|$ of a word w is the number of letters in w, where each letter is counted as many times as it occurs. Thus $|\lambda| = 0$. If $u = x_1 \cdots x_k$ and $v = x_{k+1} \cdots x_\ell$ are words over an alphabet Σ (with $x_1, \ldots, x_k, x_{k+1}, \ldots, x_\ell \in \Sigma$) then their *catenation* (which is also called their *product*) $uv = x_1 \cdots x_k x_{k+1} \cdots x_\ell$ is also a word over Σ. In addition, for every word $u = x_1 \cdots x_k$ over Σ (with $x_1, \ldots, x_k \in \Sigma$), $u\lambda = \lambda u = u$ $(= x_1 \cdots x_k)$. Moreover, $\lambda\lambda = \lambda$. Obviously, for every $u, v \in \Sigma^*$, $|uv| = |u| + |v|$. Clearly, then, for every words u, v, w (over Σ) $u(vw) = (uv)w$. In other words, $uu' = w'w$ whenever $u' = vw$ and $w' = uv$. Therefore, catenation is an associative operation and the empty word λ is the identity with respect to catenation. Let u, v, w be words with $u = vw$. Then we say that v is a *prefix* of u and w is a *suffix* of u. Two words u, v are said to be *conjugates* if there exists a word w with $uw = wv$. In particular, a word z is called *overlapping* (or *bordered*) if there are $u, v, w \in \Sigma^+$ with $z = uw = wv$. Otherwise we say that z is *non-overlapping* (or *unbordered*).

Proposition 2.1 [9] *u, v are conjugates if and only if there are words p, q with $u = pq$ and $v = qp$.* $\square$

Theorem 2.2 [9] *Let $u, v \in \Sigma^+$ with $uv = vu$. There exists $w \in \Sigma^+$ with $u, v \in w^+$.*

Lemma 2.3 [7] *If $uv = pq$ and $|u| \leq |p|$ for some $u, v, p, q \in \Sigma^+$, then $p = ur$ and $v = rq$ for some $r \in \Sigma^*$.* $\square$

Lemma 2.4 [9] *If $uv = vq, u \in \Sigma^+, v, q \in \Sigma^*$, then $u = wz, v = (wz)^k w, q = zw$ for some $w \in \Sigma^*, z \in \Sigma^+$ and $k \geq 0$.*

If u, v, w, z are words over Σ having $z = uvw$, then v is called a *subword* of z. The nonempty prefix, suffix, subword are also called a *proper* (or *nontrivial*) prefix, suffix, subword.

By the *free monoid* Σ^* *generated by* Σ we mean the set of all words (including the *empty word* λ) having catenation as multiplication. We set $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$, where the subsemigroup Σ^+ of Σ^* is said to be the *free semigroup generated by* Σ. Σ^* and Σ^+ have left and right cancellation, i.e. for every elements u, v, w of Σ^* or Σ^+, $uv = uw$ implies $v = w$ and $uv = wv$ implies $u = w$.

Subsets of Σ^* are referred to as *languages* over Σ. In particular, subsets of Σ^+ are referred to as λ-*free languages* over Σ.

Given a word u, we define $u^0 = \lambda,\ u^n = u^{n-1}u,\ n > 0, u^* = \{u^n : n \geq 0\}$ and $u^+ = u^* \setminus \{\lambda\}$. Thus u^n with $n \geq 0$ is the n-th *power* of u, while u^* is the *Kleene closure,* moreover, u^+ is the *semigroup closure* of u.

Given a list $c_1, \ldots, c_n$ of integers, let $gcd(c_1, \ldots, c_n)$ denote the greatest common divisor of $c_1, \ldots, c_n$.

Theorem 2.5 [6] *Let $u, v \in \Sigma^*$. $u, v \in w^+$ for some $w \in \Sigma^+$ if and only if there are $i, j \geq 0$ so that u^i and v^j have a common prefix (suffix) of length $|u| + |v| - gcd(|u|, |v|)$.* $\square$

A *primitive word* (over Σ, or actually over an arbitrary alphabet) is a nonempty word not of the form w^m for any nonempty word w and integer $m \geq 2$. Thus λ is a nonprimitive word because of $\lambda\lambda = \lambda$. The set of all primitive words over Σ will be denoted by $Q(\Sigma)$, or simply by Q if Σ is understood. Let $u \neq \lambda$ and let f be a primitive word with an integer $k \geq 1$ having $u = f^k$. We let $\sqrt{u} = f$ and call f the *primitive root* of the word u.

Theorem 2.6 [11] *Let $u, v \in \Sigma^*$. $w^i = uv$ for some $w \in \Sigma^*, i \geq 1$ if and only if there are $p, q \in \Sigma^*$ $w = pq, (qp)^i = vu$. Furthermore, $uv \in Q$ for some $u, v \in \Sigma^*$ if and only if $vu \in Q$.* $\square$

The next statement shows that, for every nonempty word, the primitive root is unambiguously determined.

Theorem 2.7 [9] *If* $u \neq \lambda$, *then there exists a unique primitive word* f *and a unique integer* $k \geq 1$ *such that* $u = f^k$. $\square$

Let $a^m b^n = c^k$ be an equation such that $a, b, c \in \Sigma^*$ and $m, n, k \geq 2$. We say that $a^m b^n = c^k$ *has only trivial solution* if $a^m b^n = c^k$ holds only if there exists a $w \in \Sigma^*$ with $a, b, c \in w^*$.

Theorem 2.8 (Lyndon-Schützenberger Theorem) [9] *The equation* $a^m b^n = c^k$ *with* $a, b, c \in \Sigma^*$ *has only trivial solutions.* $\square$

We have a direct consequence of Theorem 2.8 as below.

Theorem 2.9 *Let* $f, g \in Q, f \neq g$. *Then* $f^m g^n \in Q$ *for all* $m \geq 2, n \geq 2$. $\square$

Lemma 2.10 [3] *Let* $u, v \in Q$, *such that* $u^m = v^k w$ *for some* $k, m \geq 2$, *and* $w \in \Sigma^*$ *with* $|w| \leq |v|$.

Then exactly one of the following conditions holds:

(i) $u = v$ *and* $w \in \{u, \lambda\}$;

(ii) $m = k = 2$ *and there are* $p, q \in \Sigma^+, s \geq 1$ *with* $\sqrt{p} \neq \sqrt{q}$, $u = (pq)^{s+1} p^2 q$, $v = (pq)^{s+1} p, w = qp^2 q$. $\square$

Theorem 2.11 [3] *Let* $u, v \in Q$, *such that* $u^m = v^k w$ *for some prefix* w *of* v *and* $k, m \geq 2$. *Then* $u = v$ *and* $w \in \{u, \lambda\}$. $\square$

Theorem 2.12 (Shyr-Yu Theorem) [12] *Let* $f, g \in Q, f \neq g$. *Then* $|f^+ g^+ \cap \Sigma^+ \setminus Q| \leq 1$. *Moreover, if* f *and* g *are also non-overlapping, then* $f^+ g^+$ *contains only primitive words.*

3 Results

The next statement is an extended version of the first part of Theorem 2.9.

Theorem 3.1 *Let* $f, g \in Q, f \neq g$ *and* $n \geq 1$. *If* $fg^n \notin Q$ *then* $fg^{n+k} \in Q$ *for all* $k \geq 1$.

Proof: Suppose the contrary and let $u^i = fg^n, v^j = fg^{n+k}$ for some $u, v \in \Sigma^+, i, j > 1, k \geq 1$. We may assume without any restriction $u, v \in Q$. By our conditions, $v^j = u^i g^k$.

First we assume $k \geq 2$. Hence, by Theorem 2.8, $\sqrt{v} = \sqrt{u} = \sqrt{g}$, i.e. $u = v = g$. By $u^i = fg^n$, this results $f = g^{i-n}$ leading to $\sqrt{f} = \sqrt{g}$. Then, by $f, g \in Q$, we have $f = g$, a contradiction.

Now we suppose $k = 1$. By Theorem 2.6, there are $w, z \in Q$ with $w^i = g^n f, z^j = g^n fg$. Then $z^j = w^i w_1$, where w_1 is a prefix of w. Applying Theorem 2.11, $w = z$, which, by Theorem 2.6, implies $u = v$. But then $k = 0$, a contradiction. □

Proof of Theorem 2.12 : Let $f, g \in Q$ be distinct primitive words.

First we prove that the language f^+g^+ contains at most one non-primitive word. Suppose $f^m g^n \notin Q$. By Theorem 2.9, $m, n \geq 2$ is impossible. Therefore, we may assume either $f^m g \notin Q$ for some $m \geq 1$ or $fg^n \notin Q$ for some $n \geq 1$.

By Theorem 3.1, there exists at most one pair $m, n \geq 1$ of positive integers with $f^m g, fg^n \notin Q$. In addition, if $fg \notin Q$, then $f^m g, fg^n \in Q$, $m, n \geq 2$. Therefore, it is enough to prove that for every pair $m, n \geq 2$, $f^m g \notin Q$ implies $fg^n \in Q$.

Suppose the contrary and let $f^m g = u^i, fg^n = v^j$ for some $m, n, i, j \geq 2, u, v \in Q$ and let, say, $|g| \leq |f|$. Using Lemma 2.10, this is possible only if $m = i = 2$ and there are $p, q \in \Sigma^+, s \geq 1$ with $\sqrt{p} \neq \sqrt{q}$ and $u = (pq)^{s+1}p^2q$, $f = (pq)^{s+1}p, g = qp^2q$.

By Theorem 2.5 and $v^j = fg^n = (pq)^{s+1}p(qp^2q)^n$, this is impossible if either $|pq| + |v| \leq |(pq)^{s+1}p| + |qp|$, or $|qp^2q| + |v| \leq |(qp^2q)^n|$. On the other hand, if $|pq| + |v| > |qp(pq)^{s+1}p|$ and $|qp^2q| + |v| > |(qp^2q)^n|$ simultaneously hold, then using again $v^j = fg^n = (pq)^{s+1}p(qp^2q)^n$, we obtain $|v^2| + |pq| + |qp^2q| > |v^j| + |qp|$. Hence $|v^2| > |v^j| - |qp^2q|$ which leads since $|qppq| < |v|$ to $j = 2$. By $v^2 = fg^n$, this implies $v^2 = (pq)^{s+1}p(qp^2q)^n$. By the assumptions $|pq| + |v| > |qp(pq)^{s+1}p|$ and $|qp^2q| + |v| > |(qp^2q)^n|$, we can reach $|(qp^2q)^{n-2}| < |(pq)^{s+1}p| < |(qp^2q)^n|$. Thus $2n - 5 \leq s \leq 2n - 2$ which means $v = (pq)^{2n-4}r_1 = r_2(qp^2q)^{n-2}, |r_1| = |r_2|$ such that $r_1r_2 = (pq)^t p(qp^2q)^2, t \in \{0, 1, 2, 3\}$.

Note that, because of $v \in Q$, $\sqrt{p} = \sqrt{q}$ is impossible. We distinguish the following four cases.

Case 1. $t = 0, r_1 = pqpz_1, r_2 = z_2p^2q, z_1z_2 = pq^2, |z_1| = |z_2|$. Then $v = (pq)^{2n-4}pqpz_1 = z_2p^2q(qp^2q)^{n-2}$. Thus z_1 is a suffix and z_2 is a prefix of pq. Hence, by $|z_1| = |z_2| = \frac{1}{2}|p| + |q|$, we get $z_1 = p_2q, z_2 = p_1q'$ with $p = p_1p_2, |q'| = |q|, |p_1| = |p_2|$ for appropriate $q', p_1, p_2 \in \Sigma^+$. Obviously, then $z_1z_2 = p_2qp_1q' = pq^2$. This implies $p_2 = p_1$ and $q' = q$. This leads to $z_1 = z_2 = p_1q$. Thus $z_1z_2 = (p_1q)^2 = p_1^2q^2$ implying $qp_1 = p_1q$. Applying Theorem 2.2, $\sqrt{p_1} = \sqrt{q}$. By $p = p_1^2$, this implies $\sqrt{p} = \sqrt{q}$, a contradiction.

Case 2. $t = 1, r_1 = (pq)^2z_1, r_2 = z_2qp^2q, z_1z_2 = p^2q, |z_1| = |z_2|$. Then $v = (pq)^{2n-2}z_1 = z_2(qp^2q)^{n-1}$. Thus z_1 is a suffix and z_2 is a prefix of of pq. Hence, by $|z_1| = |z_2| = |p| + \frac{1}{2}|q|$, we get $z_1 = p'q_2, z_2 = pq_1$ with $q = q_1q_2, |p'| = |p|, |q_1| = |q_2|$ for appropriate $p', q_1, q_2 \in \Sigma^+$.

Obviously, then $z_1z_2 = p'q_2pq_1 = p^2q$. Hence we get $p' = p$ and $q_1 = q_2$. In other words, $z_1z_2 = (pq_1)^2 = p^2q_1^2$ leading to $q_1p = pq_1$. By Theorem 2.2, we get $\sqrt{q_1} = \sqrt{p}$ which is impossible because of $q = q_1^2$.

Case 3. $t = 2, r_1r_2 = (pq)^2p(qp^2q)^2$. Then $r_1 = (pq)^3p_1, r_2 = p_2pq^2p^2q$ with $p = p_1p_2, |p_1| = |p_2|$. But then, either $n = 2$ or $n > 2$, p_2 is simultaneously a prefix and a suffix of p. Therefore, $p_1 = p_2$ which implies $p = p_1^2$. Thus, we can write $v = (p_1^2q)^{2n-4}(p_1^2q)^3p_1 = p_1^3q^2p_1^4q(qp_1^4q)^{n-2}$ which implies $qp_1 = p_1q$ either $n = 2$ or $n > 2$. By Theorem 2.2, this means $\sqrt{q} = \sqrt{p_1}$ such that $p = p_1^2$. Therefore, $\sqrt{p} = \sqrt{q}$, a contradiction.

Case 4. $t = 3, r_1r_2 = (pq)^3p(qp^2q)^2$. In this case, $r_1 = (pq)^3pq_1, r_2 = q_2p^2q^2p^2q$ with $q = q_1q_2, |q_1| = |q_2|$. We observe that, either $n = 2$ or $n > 2$, q_1 is simultaneously a prefix and a suffix of q. Therefore, $q_1 = q_2$ with $q = q_1^2$. Thus, we can write $v = (pq_1^2)^{2n-4}(pq_1^2)^3pq_1 = q_1p^2q_1^4p^2q_1^2(q_1^2p^2q_1^2)^{n-2}$ which implies $pq_1 = q_1p$, if $n = 2$ and $pq_1^2pq_1 = q_1p^2q_1^2$, if $n > 2$. Both equalities lead to $pq_1 = q_1p$. By Theorem 2.2, this leads to $\sqrt{q_1} = \sqrt{p}$ such that $q = q_1^2$. Therefore, $\sqrt{p} = \sqrt{q}$, a contradiction.

It remains to show that, for every distinct pair $f, g \in Q$ of unbordered primitive words, $f^+g^+ \subseteq Q$. By Theorem 2.9, $m, n \geq 2$ implies $f^mg^n \in Q$. Thus it is enough to prove that for every $m, n \geq 2$, $f^mg, fg^n \in Q$.

Suppose that, contrary of our statement, there are $u \in Q, i > 1$ with $f^mg = u^i$. Consider $u_1, u_2 \in \Sigma^*, j \geq 0$ with $u = u_1u_2$ such that $f^m = u^ju_1$ and $g = u_2u^{i-j-1}$. By Theorem 2.11, $j \leq 1$. If $j = 0$, then $g = u_2u^{i-1}$ with $i - 1 > 0$. Then g is bordered, a contradiction. Thus we get $j = 1$. Therefore, $f^m = u_1u_2u_1, g = u_2(u_1u_2)^{i-2}$. If $u_1 \neq \lambda$, then f is bordered, a contradiction. Hence $f^m = u_2, g = u_2^{i-1}$. Clearly, then $f = \sqrt{f} = \sqrt{u_2} = \sqrt{g} = g$, a contradiction.

Now we assume that, contrary of our statement, $fg^n = v^k, v \in Q, k > 1$. By Theorem 2.6, there exists a word $z \in Q$ having $g^nf = z^k$. We have already proved that this is impossible. This completes the proof. □

Definition 3.2 *Let w be a finite word over the alphabet Σ. We define the fractional root of w as the shortest word noted ${}^f\sqrt{w}$ with the property that*

$(\sqrt[f]{w})^{(n+\alpha)} = w$ *where n is a positive integer and α is a positive rational number.*

Example for $w = abaabaab$, we find $(aba)^{(2+\frac{2}{3})} = abaabaab$, thus the fractional root of $w = abaabaab$ is $\sqrt[f]{w} = aba$.

Definition 3.3 *If $\sqrt[f]{w} = \sqrt{w} = w$ then we say that w is purely primitive (or aperiodic).*

For example $w = aababbb$ is purely primitive. The first part of Shyr-Yu Theorem could be refined in three cases according to the following statement:

Theorem 3.4 *Consider the language of p^+q^+. Then*

(i1) *If there exists k such that $p = (xq^k)^{i-1}x$ then the non-primitive factor is $W^i = pq^k = (xq^k)^i$ or $W^i = p^{k'}y(p^{k'}y)^{i-1}$. For this case, we could find finite classes of primitive words with fractional roots constructed as $pq = (xq^k)^{i-1}xq$, $pq^2 = (xq^k)^{i-1}xq^2, \cdots, pq^{k-1}$ or $pq = py(p^{k'}y)^{i-1}$, $p^2q = p^2y(p^{k'}y)^{i-1}, \cdots, p^{k'-1}q = p^{k'-1}y(p^{k'}y)^{i-1}$ which are primitive and not purely primitive. And infinite classes constructed by $p^m = p'y$ and $q^n = y'p'$ with $|p'|$ maximal and $m, n \geq 1$. Furthermore, if $|p'| = 0$ then the infinite classes have only purely primitive words.*

(i2) *There is a non-primitive word in the language and no finite class, thus we have infinite classes constructed by $p^m = p'y$ and $q^n = y'p'$ with $|p'|$ maximal and $m, n \geq 1$. Furthermore, if $|p'| = 0$ then the infinite classes have only purely primitive words.*

(ii) *There is no non-primitive word in the language, thus we have only infinite classes constructed by $p^m = p'y$ and $q^n = y'p'$ with $|p'|$ maximal and $m, n \geq 1$. Furthermore, if $|p'| = 0$ then the infinite classes are purely primitive.* □

In this variation of the first part of the Shyr-Yu Theorem, we find in each case an infinite class with words written in the form $w = (\sqrt[f]{w})^{(1+\frac{|p'|}{|\sqrt[f]{w}|})}$. In fact, if $|p'| = 0$ the form remains the one used in the Shyr-Yu theorem and we can deduce that $w = \sqrt[f]{w}$ thus each words are purely primitive words.

Examples :

1) $p = a$ and $q = b$ lead to only purely primitive words in the language p^+q^+.
2) $p = ab$ and $q = abb$ lead also to purely primitive words.
3) $p = aba$ and $q = abaab$ give a longest prefix of p^m which is also suffix of q^n namely $abaab$ for $m = 2$ and $n = 1$.

That is for $m \geq 2$ and $n \geq 1$ the word $w = p^m q^n$ could be written as $w = p^m q^n = (\sqrt[f]{w})^{(1+\frac{|p'|}{|\sqrt[f]{w}|})} = (w[1..|w| - |p'|])^{(1+\frac{|p'|}{|\sqrt[f]{w}|})} = (w[1..|w| - 5])^{(1+\frac{5}{|\sqrt[f]{w}|})}$. And for $m = n = 1$ we find $pq = abaabaab = (aba)^{2+\frac{2}{3}}$.

Proof of Theorem 3.4 : In case of (i1), we know that there exists a unique non-primitive word, namely W^i, in p^+q^+. We can write the general form of $W^i = (xq^k)^{i-1}xq^k$ or $W^i = p^{k'}y(p^{k'}y)^{i-1}$. Thus either $p = (xq^k)^{i-1}x$ or $q = y(p^k y)^{i-1}$. And then either $W^i = pq^k$ or $W^i = p^{k'}q$. In both cases we can find finite classes of words : $pq = (xq^k)^{i-1}xq$, $pq^2 = (xq^k)^{i-1}xq^2, \cdots, pq^{k-1} = (xq^k)^{i-1}xq^{i-1}$ or $pq = py(p^{k'}y)^{i-1}$, $p^2q = p^2y(p^{k'}y)^{i-1}, \cdots, p^{k'-1}q = p^{k'-1}y(p^{k'}y)^{i-1}$.

In each case, we can find an infinite class of words with the form $w = (\sqrt[f]{w})^{(1+\frac{|p'|}{|\sqrt[f]{w}|})}$ where $p^m = p'y$ and $q^n = y'p'$ with $|p'|$ maximal. We prove this fact by contradiction. Suppose that there is no p' which is maximal. That is we have an infinite sequence of words indexed by distinct couples (m_i, n_i) with increasing length of $p'_{(m_i,n_i)}$ such that $p^{m_i} = p'_{(m_i,n_i)}y$ and $q^{n_i} = y'p'_{(m_i,n_i)}$. As the sequence is infinite, we find a starting point in $q = ss'$ such that we could extend the rigth part of the word to an infinite by $p^\omega = (s's)^\omega$. By using Theorem 2.5, we have that $p = z^\ell$ and $s's = z^{\ell'}$. Thus p is not primitive. A contradiction. $\square$

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