New combinatorial interpretations of r-Whitney and r-Whitney-Lah numbers

Eszter Gyimesi, Gábor Nyul[1,*]

Institute of Mathematics, University of Debrecen
H–4002 Debrecen P.O.Box 400, Hungary

Abstract

T. A. Dowling introduced Whitney numbers of the first and second kind concerning the so-called Dowling lattices of finite groups. It turned out that they are generalizations of Stirling numbers. Later, I. Mező defined r-Whitney numbers as common generalizations of Whitney numbers and r-Stirling numbers. Additionally, G.-S. Cheon and J.-H. Jung defined r-Whitney-Lah numbers.

In our paper, we give new combinatorial interpretations of r-Whitney and r-Whitney-Lah numbers, which correspond better with the combinatorial definitions of Stirling, r-Stirling, Lah and r-Lah numbers. These allow us to explain their properties in a purely combinatorial manner, as well as derive several new identities.

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1. Introduction

Stirling numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ are fundamental objects in enumerative combinatorics. A. Z. Broder [4], and later R. Merris [16] defined combinatorially a generalization of these numbers, the r-Stirling numbers, although they appeared in a previous work of L. Carlitz [5], who reached them in a different way. $\left[{n \atop k}\right]_r$ counts those permutations of $1, \ldots, n+r$ which are the product of $k+r$ disjoint cycles such that $1, \ldots, r$ belong to distinct cycles, while $\left\{{n \atop k}\right\}_r$ counts the partitions of $\{1, \ldots, n+r\}$ into $k+r$ nonempty subsets, where $1, \ldots, r$ are contained in distinct blocks ($0 \le k \le n$, $r \ge 0$). For $r = 0$ and $r = 1$, they give back the ordinary Stirling numbers directly and with shifted parameters, respectively. Recently, an alternative approach to r-Stirling numbers of the second kind was found by E. Gyimesi and G. Nyul [12] using combinatorial subspaces.

*Corresponding author
Email addresses: gyimesie@science.unideb.hu (Eszter Gyimesi), gnyul@science.unideb.hu (Gábor Nyul)

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Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ are close relatives of Stirling numbers. Their r-generalized variants, the r-Lah numbers were extensively studied by G. Nyul and G. Rácz [21]. $\left\lfloor {n \atop k} \right\rfloor_r$ is the number of partitions of $\{1, \ldots, n+r\}$ into $k+r$ nonempty ordered subsets such that $1, \ldots, r$ belong to distinct ordered blocks ($0 \leq k \leq n$, $r \geq 0$). As above, 0-Lah and 1-Lah numbers coincide with Lah numbers.

T. A. Dowling [11] constructed a certain lattice for a finite group of order m, now called Dowling lattice, and using the Möbius function, he introduced the corresponding Whitney numbers of the first kind $w_m(n,k)$ and Whitney numbers of the second kind $W_m(n,k)$ ($0 \leq k \leq n$, $m \geq 1$), which are independent of the group itself, but depend only on its order. Here and hereafter, we use these notations for unsigned Whitney numbers even in the first kind case. For the trivial group, we have $w_1(n,k) = \left[{n+1 \atop k+1} \right]$ and $W_1(n,k) = \left\{ {n+1 \atop k+1} \right\}$. M. Benoumhani [2] gave a detailed description of properties of these numbers.

It was the idea of I. Mező [17] to give a common generalization of r-Stirling numbers and Whitney numbers. He defined r-Whitney numbers of the first kind by equation

$$m^n x^{\overline{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)\,(mx-r)^k\,, \tag{1}$$

and r-Whitney numbers of the second kind by

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k) m^k x^{\underline{k}}. \tag{2}$$

However, r-Whitney numbers also appear from other directions under different names. As a special case of Stirling number pairs introduced by L. C. Hsu and P. J.-S. Shiue [13], R. B. Corcino, C. B. Corcino and R. Aldema [8], [7] studied the so-called (r,β)-Stirling numbers, which turn out to be equivalent to r-Whitney numbers. Moreover, B. Voigt [25] defined a generalization of Stirling numbers of the second kind with a sequence as a further parameter, and this variant gives back r-Whitney numbers of the second kind for arithmetic progressions, see A. Ruciński and B. Voigt [23].

G.-S. Cheon and J.-H. Jung [6] gave a detailed study of r-Whitney numbers of both kinds, based on an interpretation in connection with Dowling lattices. In addition, they introduced r-Whitney-Lah numbers by identity

$$WL_{m,r}(n,k) = \sum_{j=k}^{n} w_{m,r}(n,j)\,W_{m,r}(j,k)\,. \tag{3}$$

Ordinary Whitney-Lah numbers $WL_m(n,k)$ could be defined similarly.

We mention here that M. Merca [15] connected r-Whitney numbers to symmetric polynomials.

The above summarized results do not contain any purely combinatorial meaning of r-Whitney and r-Whitney-Lah numbers, not even in the ordinary case. There have been some attempts in this direction.

J. B. Remmel and M. L. Wachs [22] gave two combinatorial interpretations of Whitney numbers of the first kind and one for Whitney numbers of the second kind. I. Mező [18] derived some formulas for Whitney numbers of the second kind by the help of this latter interpretation.

M. Mihoubi and M. Rahmani [20] described r-Whitney and r-Whitney-Lah numbers using only set partitions for all of them. They obtained these interpretations through the partial r-Bell polynomials.

Finally, we remark that coloured set partitions of D. G. L. Wang [26] are actually counted by r-Whitney numbers of the second kind. H. Belbachir and I. E. Bousbaa [1] introduced a modification of these numbers, which they called translated r-Whitney and r-Whitney-Lah numbers, these are just equal to m^{n-k} times the r-Stirling and r-Lah numbers.

In the present paper we provide new combinatorial interpretations for all of these numbers.

For ordinary Whitney numbers of the second kind, this will coincide essentially with the interpretation of J. B. Remmel and M. L. Wachs [22], while for r-Whitney numbers of the second kind it is similar to that of M. Mihoubi and M. Rahmani [20]. But for r-Whitney numbers of the first kind and r-Whitney-Lah numbers, our interpretations are completely new, and really fit to the combinatorial definitions of r-Stirling and r-Lah numbers. We emphasize that these are new combinatorial interpretations in the ordinary case, as well.

In the rest of the paper, we shall choose the following strategy: We use our combinatorial interpretations as the definitions of r-Whitney and r-Whitney-Lah numbers. Thereafter, based on these definitions we proceed to derive some known properties by purely combinatorial proofs, as well as many new results. Among the identities, equations (1), (2) and (3) will appear, which shows that our definitions are equivalent to the original ones. In those theorems which state identities for each of these numbers, we give the proofs mainly in the first kind case, and point out the differences in the other cases if necessary.

2. Combinatorial definitions

First of all, we formulate our combinatorial definitions for r-Whitney and r-Whitney-Lah numbers.

Definition 2.1. *Let $0 \leq k \leq n$, $r \geq 0$, $n + r \geq 1$ and $m \geq 1$. Denote by $w_{m,r}(n, k)$ the number of coloured permutations in S_{n+r} which are the product of $k + r$ disjoint cycles such that*

- *the distinguished elements $1, \ldots, r$ belong to distinct cycles,*
- *the smallest elements of the cycles are not coloured,*
- *an element in a cycle containing a distinguished element is not coloured if there are no smaller numbers on the arc from the distinguished element to this element,*
- *the remaining elements are coloured with m colours.*

Moreover, let $w_{m,0}(0,0)=1$*. We call these numbers* r***-Whitney numbers of the first kind****, and such permutations* r***-Whitney coloured permutations****.*

Definition 2.2. *Let* $0 \leq k \leq n$*,* $r \geq 0$*,* $n+r \geq 1$ *and* $m \geq 1$*. Denote by* $W_{m,r}(n,k)$ *the number of coloured partitions of* $\{1,\ldots,n+r\}$ *into* $k+r$ *nonempty subsets such that*

- *the distinguished elements* $1,\ldots,r$ *belong to distinct blocks,*
- *the smallest elements of the blocks are not coloured,*
- *elements in blocks containing a distinguished element are not coloured,*
- *the remaining elements are coloured with* m *colours.*

Moreover, let $W_{m,0}(0,0)=1$*. We call these numbers* r***-Whitney numbers of the second kind****, and such partitions* r***-Whitney coloured partitions****.*

Definition 2.3. *Let* $0 \leq k \leq n$*,* $r \geq 0$*,* $n+r \geq 1$ *and* $m \geq 1$*. Denote by* $WL_{m,r}(n,k)$ *the number of coloured partitions of* $\{1,\ldots,n+r\}$ *into* $k+r$ *nonempty ordered subsets such that*

- *the distinguished elements* $1,\ldots,r$ *belong to distinct ordered blocks,*
- *the smallest elements of the ordered blocks are not coloured,*
- *an element in an ordered block containing a distinguished element is not coloured if there are no smaller numbers between the distinguished element and this element,*
- *the remaining elements are coloured with* m *colours.*

Moreover, let $WL_{m,0}(0,0)=1$*. We call these numbers* r***-Whitney-Lah numbers****, and such partitions* r***-Whitney-Lah coloured partitions****.*

In the cases of $r=1$; $m=1$; $m=1$, $r=0$ and $r=0$, the r-Whitney numbers give back Whitney numbers, r-Stirling numbers, classical Stirling numbers and multiples of Stirling numbers, respectively, and it holds similarly for r-Whitney-Lah numbers. More precisely, in the first kind case we have

$$w_{m,1}(n,k)=w_m(n,k)\,, w_{1,r}(n,k)=\begin{bmatrix} n \\ k \end{bmatrix}_r,$$

$$w_{1,0}(n,k)=\begin{bmatrix} n \\ k \end{bmatrix}, w_{m,0}(n,k)=m^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}.$$

In the proofs of the next section, we will often need to additionally colour with m colours the smallest elements of those cycles, blocks or ordered blocks which contain no distinguished element. We shall refer to them as *extended* r-Whitney coloured permutations, partitions or r-Whitney-Lah coloured partitions. If we have $k+r$ cycles, blocks or ordered blocks, then their numbers are $m^k w_{m,r}(n,k)$, $m^k W_{m,r}(n,k)$ and $m^k WL_{m,r}(n,k)$, respectively.

3. Identities and properties

In the sequel, we will need the notion of rising and falling factorials with difference m defined by

$$(x|m)^{\overline{0}} = 1, \quad (x|m)^{\overline{n}} = x\,(x+m)\cdots(x+(n-1)\,m)\,,$$

$$(x|m)^{\underline{0}} = 1, \quad (x|m)^{\underline{n}} = x\,(x-m)\cdots(x-(n-1)\,m)\,,$$

where $m, n \geq 1$. Especially, $(x|1)^{\overline{n}} = x^{\overline{n}}$ and $(x|1)^{\underline{n}} = x^{\underline{n}}$ are the simple rising and falling factorials. We will often need that $(mx|m)^{\overline{n}} = m^n x^{\overline{n}}$ and $(mx|m)^{\underline{n}} = m^n x^{\underline{n}}$.

For the smallest and largest possible values of k, we obtain by direct combinatorial arguments the following special values:

- $w_{m,r}(n,0) = (r|m)^{\overline{n}}$, $W_{m,r}(n,0) = r^n$, $WL_{m,r}(n,0) = (2r|m)^{\overline{n}}$
- $W_{m,r}(n,1) = \frac{1}{m}\left((m+r)^n - r^n\right)$,
  $WL_{m,r}(n,1) = \frac{1}{m}\left((m+2r|m)^{\overline{n}} - (2r|m)^{\overline{n}}\right)$ $(n \geq 1)$
- $w_{m,r}(n,n-1) = W_{m,r}(n,n-1) = m\binom{n}{2} + rn$,
  $WL_{m,r}(n,n-1) = mn\,(n-1) + 2rn$ $(n \geq 1)$
- $w_{m,r}(n,n) = W_{m,r}(n,n) = WL_{m,r}(n,n) = 1$

Different forms of the following polynomial identities can be found in several papers, in the first kind case see [11] (for $r = 1$), [6], [7], in the second kind case [11] (for $r = 1$), [6], [8], and for r-Whitney-Lah numbers [6]. We note that equivalent formulations (1), (2) of the first two equations serve as definitions by I. Mező [17].

Theorem 3.1. *If $n, r \geq 0$ and $m \geq 1$, then*

$$(x+r|m)^{\overline{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)\, x^k,$$

$$(x+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k)\,(x|m)^{\underline{k}},$$

$$(x+2r|m)^{\overline{n}} = \sum_{k=0}^{n} WL_{m,r}(n,k)\,(x|m)^{\underline{k}}.$$

Proof. In the proof we count r-Whitney coloured permutations of S_{n+r} with m colours such that we additionally colour with c colours the smallest elements of those cycles which contain no distinguished element.

If the number of cycles is $k+r$, then we have $w_{m,r}(n,k)$ r-Whitney coloured permutations $(k = 0, \ldots, n)$. Since there are k cycles containing no distinguished element, their smallest elements can be coloured with c colours in c^k ways. Summing up, we have $\sum\limits_{k=0}^{n} w_{m,r}(n,k)\, c^k$ possibilities.

On the other hand, first place the distinguished elements into distinct cycles. Then $r+i$ can open a new cycle, when it is coloured with c colours ($i=1,\ldots,n$). Furthermore, we can put $r+i$ after a distinguished element, when it gets no colour, or after a previously placed non-distinguished element, in which case it is coloured with m colours. This means that $r+i$ can be placed and coloured in $c+r+(i-1)\,m$ ways, hence the number of possibilities is $(c+r|m)^{\overline{n}}$, altogether.

For r-Whitney numbers of the second kind and r-Whitney-Lah numbers, the above approach does not work in the present form, it is not enough to colour the smallest elements of the blocks containing no distinguished element. We discuss the details in the second kind case.

We are interested in the number of extended r-Whitney coloured partitions of $\{1,\ldots,n+r\}$ with m colours such that we secondarily colour with c colours ($c \geq n$) the smallest elements of the blocks containing no distinguished element, where secondary colours have to be distinct.

On the one hand, we can partition the elements into $k+r$ blocks in extended r-Whitney sense in $m^k W_{m,r}(n,k)$ ways ($k=0,\ldots,n$), thereafter colour secondarily the non-distinguished smallest elements in $c^{\underline{k}}$ ways, which results in $\sum\limits_{k=0}^{n} W_{m,r}(n,k)\,(mc|m)^{\underline{k}}$ possibilities, since $m^k c^{\underline{k}} = (mc|m)^{\underline{k}}$.

Or, after placing the first r elements into distinct blocks, consider the element $r+i$ ($i=1,\ldots,n$). Suppose that the first $r+i-1$ elements are partitioned into $l+r$ blocks. If $r+i$ opens a new block, its primary and secondary colours come from m and $c-l$ colours, respectively. If we put $r+i$ into a block containing a distinguished element, it is uncoloured, while if it is placed into one of the l blocks without distinguished elements, it is coloured with m colours. Summarizing, $r+i$ can be placed and coloured in $m\,(c-l)+r+lm = mc+r$ ways, consequently the number of all possibilities is $(mc+r)^n$. □

Substituting $-x$ into these equations, we obtain the following important consequences immediately.

Corollary 3.1. *If $n,r \geq 0$ and $m \geq 1$, then*

$$(x-r|m)^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k}\, w_{m,r}(n,k)\, x^k,$$

$$(x-r)^n = \sum_{k=0}^{n} (-1)^{n-k}\, W_{m,r}(n,k)\,(x|m)^{\overline{k}},$$

$$(x-2r|m)^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k}\, WL_{m,r}(n,k)\,(x|m)^{\overline{k}}.$$

Remark 3.1. As a special case, for $x = 1$, Theorem 3.1 gives that the sum of r-Whitney numbers of the first kind with fixed n is

$$\sum_{k=0}^{n} w_{m,r}\left(n,k\right) = \left(r+1|m\right)^{\overline{n}}.$$

We notice that the similar sum of r-Whitney numbers of the second kind yields

$$\sum_{k=0}^{n} W_{m,r}\left(n,k\right) = D_{n,m,r},$$

a so-called r-Dowling number [6] (in case of $r = 1$, see also [2]).

The recurrences of r-Whitney numbers appear earlier in the literature, in the first kind case see [11] (for $r = 1$), [6], [7], [17], in the second kind case [2], [11] (for $r = 1$), [6], [8], [17], [26], while the recurrence for r-Whitney-Lah numbers can be found in [6]. Now, our combinatorial definitions allow us to prove them in a purely combinatorial way.

Theorem 3.2. *If $1 \leq k \leq n$, $r \geq 0$ and $m \geq 1$, then*

$$w_{m,r}\left(n+1,k\right) = w_{m,r}\left(n,k-1\right) + \left(mn+r\right) w_{m,r}\left(n,k\right),$$

$$W_{m,r}\left(n+1,k\right) = W_{m,r}\left(n,k-1\right) + \left(mk+r\right) W_{m,r}\left(n,k\right),$$

$$WL_{m,r}\left(n+1,k\right) = WL_{m,r}\left(n,k-1\right) + \left(m\left(n+k\right)+2r\right) WL_{m,r}\left(n,k\right).$$

Proof. We enumerate the r-Whitney coloured permutations of S_{n+r+1} with m colours which are the product of $k+r$ disjoint cycles.

If $n+r+1$ stands in a cycle alone, then the other elements constitute an r-Whitney coloured permutation of S_{n+r} with $k-1+r$ disjoint cycles, which gives us $w_{m,r}\left(n,k-1\right)$ possibilities. If it is contained in a cycle of length at least 2, then the other elements can be arranged into $k+r$ cycles in r-Whitney sense in $w_{m,r}\left(n,k\right)$ ways, and we can insert $n+r+1$ after one of the n non-distinguished elements, in which case it is coloured with m colours, or after a distinguished element getting no colour. $\square$

In the following theorem, we present new vertical recurrences for these numbers.

Theorem 3.3. *If $0 \leq k \leq n$, $r \geq 0$ and $m \geq 1$, then*

$$w_{m,r}\left(n+1,k+1\right) = \sum_{j=k}^{n} \left(mn+r|m\right)^{\underline{n-j}} w_{m,r}\left(j,k\right),$$

$$W_{m,r}\left(n+1,k+1\right) = \sum_{j=k}^{n} \left(m\left(k+1\right)+r\right)^{n-j} W_{m,r}\left(j,k\right),$$

$$WL_{m,r}\left(n+1,k+1\right) = \sum_{j=k}^{n} \left(m\left(n+k+1\right)+2r|m\right)^{\underline{n-j}} WL_{m,r}\left(j,k\right).$$

Proof. To prove this theorem, we need to count the r-Whitney coloured permutations of S_{n+r+1} with m colours which are the product of $k+r+1$ disjoint cycles.

Let $j+r+1$ be the largest one among the smallest elements of the cycles ($j=k,\ldots,n$). Start with an arrangement of the first $j+r$ elements into $k+r$ disjoint cycles in r-Whitney sense, which can be done in $w_{m,r}(j,k)$ ways. After that, in increasing order each remaining element can be inserted after a non-distinguished element, when it is coloured, or after one of the r distinguished elements. This gives $((j+1)m+r|m)^{\overline{n-j}} = (mn+r|m)^{\underline{n-j}}$ possibilities for the last $n-j$ elements. □

The explicit formula for r-Whitney numbers of the second kind is derived from their exponential generating function in [2] (for $r=1$), [6], [8], [17]. Now, based on our combinatorial interpretations, we can prove it applying the inclusion-exclusion principle, as well as a new formula for r-Whitney-Lah numbers.

Theorem 3.4. *If $0 \leq k \leq n$, $r \geq 0$, $m \geq 1$, then*

$$W_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(m(k-j)+r)^n,$$

$$WL_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(m(k-j)+2r|m)^{\overline{n}}.$$

Proof. In this proof we count the extended r-Whitney coloured partitions of $\{1,\ldots,n+r\}$ with m colours, where the smallest elements of the blocks are coloured secondarily with $k+r$ colours such that their colours are distinct and we use all of the secondary colours.

First, observe that the number of blocks is $k+r$ since we have to use all of the secondary colours exactly once. Therefore, the number of extended r-Whitney coloured partitions is $m^k W_{m,r}(n,k)$ and we can assign the secondary colours to the smallest elements of the blocks in $(k+r)!$ ways, which gives $(k+r)!m^k W_{m,r}(n,k)$ possibilities.

On the other hand, we can enumerate them by the inclusion-exclusion principle. Let X be the set of the extended r-Whitney coloured partitions together with the secondary colouring, where we not necessarily use all of the secondary colours, and let Y_h be the subset of X which contains those possibilities which do not use the hth secondary colour ($h=1,\ldots,k+r$).

To derive the cardinality of X, we can place the first r elements into distinct blocks and colour them with their secondary colours in $(k+r)^{\underline{r}}$ ways. Then, similarly to the last paragraph in the proof Theorem 3.1, we obtain that $r+i$ ($i=1,\ldots,n$) can be placed and coloured in $mk+r$ ways. Therefore, $|X| = (k+r)^{\underline{r}}(mk+r)^n$.

Since the distinguished elements belong to disctinct blocks, we have to use at least r secondary colours, hence the intersection of j sets of type Y_h has no elements if $j = k+1,\ldots,k+r$. On the other hand, the cardinality of the intersection of j sets of type Y_h is $(k+r-j)^{\underline{r}}(m(k-j)+r)^n$ for $j=1,\ldots,k$, which can be deduced similarly as by $|X|$ above. Then the inclusion-exclusion principle gives that the number of possibilities is

$$|X \setminus (Y_1 \cup \cdots \cup Y_{k+r})| = \sum_{j=0}^{k} (-1)^j \binom{k+r}{j} (k+r-j)^{\underline{r}} \left(m\left(k-j\right)+r\right)^n$$

altogether, and the assertion follows after some simplification. □

A simpler explicit formula appeared in [6] for r-Whitney-Lah numbers, which can be proved again combinatorially.

Theorem 3.5. *If $0 \le k \le n$, $r,m \ge 1$, then*

$$WL_{m,r}\left(n,k\right) = \binom{n}{k} \frac{(2r|m)^{\overline{n}}}{(2r|m)^{\overline{k}}}.$$

Proof. We need to enumerate the extended r-Whitney-Lah coloured partitions of $\{1,\ldots,n+r\}$ with m colours into $k+r$ nonempty ordered subsets. First, we place the distinguished elements into distinct ordered blocks. Then, we can choose and colour the first elements of the other ordered blocks in $\binom{n}{k}m^k$ ways. Finally, considering the remaining elements in increasing order, there are $2r+m(k+j-1)$ possibilities for the jth element ($j=1,\ldots,n-k$), since it is uncoloured if it is placed just before or after a distinguished element, and it is coloured at the other $k+j-1$ places.

This gives us $m^k WL_{m,r}\left(n,k\right) = \binom{n}{k}m^k\left(2r+mk|m\right)^{\overline{n-k}}$, from which the assertion follows by some simplification. (We notice that this form of the formula is also valid for $r=0$.) □

In the following two theorems we give two different expressions of r-Whitney and r-Whitney-Lah numbers with m colours by s-Whitney and s-Whitney-Lah numbers with l colours.

Theorem 3.6. *If $0 \le k \le n$, $r,s \ge 0$ and $m,l \ge 1$, then*

$$l^{n-k} w_{m,r}\left(n,k\right) = \sum_{j=k}^{n} \binom{n}{j} m^{j-k} w_{l,s}\left(j,k\right) \left(lr-ms|ml\right)^{\overline{n-j}},$$

$$l^{n-k} W_{m,r}\left(n,k\right) = \sum_{j=k}^{n} \binom{n}{j} m^{j-k} W_{l,s}\left(j,k\right) \left(lr-ms\right)^{n-j},$$

$$l^{n-k} WL_{m,r}\left(n,k\right) = \sum_{j=k}^{n} \binom{n}{j} m^{j-k} WL_{l,s}\left(j,k\right) \left(2lr-2ms|ml\right)^{\overline{n-j}}.$$

Proof. By Theorem 3.1, we can easily get

$$(mlx+lr|ml)^{\overline{n}} = l^n \, (mx+r|m)^{\overline{n}} = l^n \sum_{k=0}^{n} w_{m,r}\,(n,k)\, m^k x^k.$$

On the other hand, the binomial theorem for rising factorials and Theorem 3.1 give

$$\begin{aligned}
(mlx+lr|ml)^{\overline{n}} &= \sum_{j=0}^{n} \binom{n}{j} (mlx+ms|ml)^{\overline{j}}\,(lr-ms|ml)^{\overline{n-j}} \\
&= \sum_{j=0}^{n} \binom{n}{j} m^j \,(lx+s|l)^{\overline{j}}\,(lr-ms|ml)^{\overline{n-j}} \\
&= \sum_{j=0}^{n} \binom{n}{j} m^j \sum_{k=0}^{j} w_{l,s}\,(j,k)\, l^k x^k \,(lr-ms|ml)^{\overline{n-j}} \\
&= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} l^k m^j w_{l,s}\,(j,k)\,(lr-ms|ml)^{\overline{n-j}}\, x^k.
\end{aligned}$$

In the second kind case, when $r \geq s$ and $l \geq m$, we can give an interesting combinatorial background of this identity. (We notice that the same idea also works for r-Whitney numbers of the first kind and r-Whitney-Lah numbers if $r \geq s$ and $l = m$.) We enumerate extended r-Whitney coloured partitions of $\{1,\ldots,n+r\}$ with m colours into $k+r$ nonempty subsets, where we colour all but the smallest elements of the blocks with l secondary colours. Obviously, there are $l^{n-k} m^k W_{m,r}\,(n,k)$ possibilities.

Let j be the number of those non-distinguished elements which are in a block containing no distinguished element, or which belong to the blocks of $1,\ldots,s$ such that they are coloured by one of the first m colours among the secondary colours ($j = k,\ldots,n$). These j elements can be chosen in $\binom{n}{j}$ ways. There are $W_{l,s}\,(j,k)$ possibilities to partition these j elements together with $1,\ldots,s$ into $k+s$ blocks in s-Whitney sense, where the coloured elements are now coloured by their secondary colours. But all of these j elements still need to be coloured by m colours, by their primary colours if they belong to a block containing no distinguished element, and by their secondary colours if they share a block with one of the first s numbers.

The other $r-s$ distinguished elements are placed into distinct blocks without any colours. Then there are $s(l-m)+(r-s)l = lr-ms$ possibilities for each of the remaining $n-j$ non-distinguished elements, since they can be put into the blocks of $1,\ldots,s$ when they get colours from the last $l-m$ secondary colours, or they belong to a block of one of the other $r-s$ distinguished elements when they are simply coloured by their secondary colours.

Summarizing, the number of possibilities is $\sum\limits_{j=k}^{n} \binom{n}{j} m^j W_{l,s}\,(j,k)\,(lr-ms)^{n-j}$.

□

Theorem 3.7. *If $0 \le k \le n$, $r, s \ge 0$ and $m, l \ge 1$, then*

$$l^{n-k} w_{m,r}(n,k) = \sum_{j=k}^{n} m^{n-j} w_{l,s}(n,j) \binom{j}{k} (lr - ms)^{j-k},$$

$$l^{n-k} W_{m,r}(n,k) = \sum_{j=k}^{n} m^{n-j} W_{l,s}(n,j) \binom{j}{k} (lr - ms|ml)^{\underline{j-k}},$$

$$l^{n-k} WL_{m,r}(n,k) = \sum_{j=k}^{n} m^{n-j} WL_{l,s}(n,j) \binom{j}{k} (2lr - 2ms|ml)^{\underline{j-k}}.$$

Proof. As we have seen in the proof of Theorem 3.6, we have

$$(mlx + lr|ml)^{\overline{n}} = l^n \sum_{k=0}^{n} w_{m,r}(n,k)\, m^k x^k.$$

A non-straightforward application of Theorem 3.1 and the binomial theorem give

$$\begin{aligned}
(mlx + lr|ml)^{\overline{n}} &= m^n \left(l \left(x + \frac{lr - ms}{ml} \right) + s|l \right)^{\overline{n}} \\
&= m^n \sum_{j=0}^{n} w_{l,s}(n,j) \left(l \left(x + \frac{lr - ms}{ml} \right) \right)^{j} \\
&= \sum_{j=0}^{n} m^{n-j} w_{l,s}(n,j) (mlx + lr - ms)^j \\
&= \sum_{j=0}^{n} m^{n-j} w_{l,s}(n,j) \sum_{k=0}^{j} \binom{j}{k} (mlx)^k (lr - ms)^{j-k} \\
&= \sum_{k=0}^{n} \sum_{j=k}^{n} l^k m^{n+k-j} w_{l,s}(n,j) \binom{j}{k} (lr - ms)^{j-k} x^k.
\end{aligned}$$

□

If $l = m$, then the identities of the above two theorems become slightly simpler. A few of these special forms can be found in [6], [15].

Corollary 3.2. *If $0 \le k \le n$, $r, s \ge 0$ and $m \ge 1$, then*

$$w_{m,r}(n,k) = \sum_{j=k}^{n} \binom{n}{j} w_{m,s}(j,k) (r - s|m)^{\overline{n-j}},$$

$$W_{m,r}(n,k) = \sum_{j=k}^{n} \binom{n}{j} W_{m,s}(j,k) (r - s)^{n-j},$$

$$WL_{m,r}(n,k) = \sum_{j=k}^{n} \binom{n}{j} WL_{m,s}(j,k) (2r - 2s|m)^{\overline{n-j}}.$$

Corollary 3.3. *If $0 \le k \le n$, $r, s \ge 0$ and $m \ge 1$, then*

$$w_{m,r}(n,k) = \sum_{j=k}^{n} w_{m,s}(n,j) \binom{j}{k} (r-s)^{j-k},$$

$$W_{m,r}(n,k) = \sum_{j=k}^{n} W_{m,s}(n,j) \binom{j}{k} (r-s|m)^{\underline{j-k}},$$

$$WL_{m,r}(n,k) = \sum_{j=k}^{n} WL_{m,s}(n,j) \binom{j}{k} (2r-2s|m)^{\underline{j-k}}.$$

Remark 3.2. By more special choices of the parameters, we could obtain several interesting identities, but we shall not give them in full detail. We can express r-Whitney numbers with m colours by r-Whitney numbers with l colours (for $s = r$), Whitney numbers with l colours (for $s = 1$), Whitney numbers with m colours (for $s = 1$, $l = m$), s-Stirling numbers (for $l = 1$), r-Stirling numbers ($l = 1$, $s = r$), and ordinary Stirling numbers (for $l = 1$, $s = 0$), and it holds similarly for r-Whitney-Lah numbers. We notice that the expressions by the latter two types of numbers partly appear in [6], [15], [17].

The following proposition will be useful to derive consequences of some upcoming general theorems.

Theorem 3.8. *If $0 \le k \le n$, $r \ge 0$ and $m, l \ge 1$, then*

$$l^{n-k} w_{m,r}(n,k) = w_{ml,lr}(n,k),$$
$$l^{n-k} W_{m,r}(n,k) = W_{ml,lr}(n,k),$$
$$l^{n-k} WL_{m,r}(n,k) = WL_{ml,lr}(n,k).$$

Proof. Applying Theorem 3.1, we obtain

$$l^n \sum_{k=0}^{n} w_{m,r}(n,k)\, m^k x^k = (mlx + lr|ml)^{\overline{n}} = \sum_{k=0}^{n} w_{ml,lr}(n,k)\, m^k l^k x^k.$$

For r-Whitney numbers of the second kind, we can also give a direct combinatorial proof. Let L be an l-element set of secondary colours. We consider r-Whitney coloured partitions of $\{1,\dots,n+r\}$ with m colours into $k+r$ nonempty subsets, where we colour every element with a secondary colour from L, except the smallest elements of the blocks. The number these doubly coloured partitions is $l^{n-k} W_{m,r}(n,k)$.

We can assign a partition of $(\{1,\dots,r\} \times L) \cup \{r+1,\dots,n+r\}$ to a partition of the above type, as follows. Handle the elements of $\{1,\dots,r\} \times L$ as the distinguished ones. Let a further element belong to the block of (i,c) ($1 \le i \le r$, $c \in L$) if previously it shared a block with i and had secondary colour c, and leave the other blocks unchanged. This is an lr-Whitney coloured partition of $n + lr$ elements with ml colours into $k + lr$ blocks. This assignment gives a bijective correspondence between these two types of partitions. □

Now we prove that the binomial convolution of r-Whitney numbers with m colours and s-Whitney numbers with l colours is just equal to the product of a binomial coefficient and an $(lr+ms)$-Whitney number with ml colours.

Theorem 3.9. *If $n,k,h,r,s \geq 0,\ k+h \leq n$ and $m,l \geq 1$, then*

$$\binom{k+h}{k} w_{ml,lr+ms}\left(n,k+h\right) = \sum_{j=k}^{n-h} \binom{n}{j} l^{j-k} m^{n-j-h} w_{m,r}\left(j,k\right) w_{l,s}\left(n-j,h\right),$$

$$\binom{k+h}{k} W_{ml,lr+ms}\left(n,k+h\right) = \sum_{j=k}^{n-h} \binom{n}{j} l^{j-k} m^{n-j-h} W_{m,r}\left(j,k\right) W_{l,s}\left(n-j,h\right),$$

$$\binom{k+h}{k} WL_{ml,lr+ms}\left(n,k+h\right) = \sum_{j=k}^{n-h} \binom{n}{j} l^{j-k} m^{n-j-h} WL_{m,r}\left(j,k\right) WL_{l,s}\left(n-j,h\right).$$

Proof. Consider the $(lr+ms)$-Whitney coloured permutations of $S_{n+lr+ms}$ with ml colours which are the product of $k+h+lr+ms$ disjoint cycles, where we secondarily colour the smallest elements of the cycles by green or red such that the first lr numbers are green, the other ms distinguished elements are red, and the number of green elements is $k+lr$.

The number of these configurations is $\binom{k+h}{k} w_{ml,lr+ms}\left(n,k+h\right)$ since after arranging the elements into cycles with the colouring of Whitney sense, we have to choose those cycles which contain no distinguished element, but whose smallest element is green.

On the other hand, let j be the number of those non-distinguished elements which belong to a cycle having green minimal element ($j=k,\ldots,n-h$). They can be chosen in $\binom{n}{j}$ ways. By Theorem 3.8, these j elements together with the lr green distinguished elements, and the remaining $n-j+ms$ elements can be arranged into $k+lr$ and $h+ms$ disjoint cycles in $w_{ml,lr}\left(j,k\right) = l^{j-k} w_{m,r}\left(j,k\right)$ and $w_{ml,ms}\left(n-j,h\right) = m^{n-j-h} w_{l,s}\left(n-j,h\right)$, respectively. □

In the case of $l=m$, we have the following consequences by Theorem 3.8.

Corollary 3.4. *If $n,k,h,r,s \geq 0,\ k+h \leq n$ and $m \geq 1$, then*

$$\binom{k+h}{k} w_{m,r+s}\left(n,k+h\right) = \sum_{j=k}^{n-h} \binom{n}{j} w_{m,r}\left(j,k\right) w_{m,s}\left(n-j,h\right),$$

$$\binom{k+h}{k} W_{m,r+s}\left(n,k+h\right) = \sum_{j=k}^{n-h} \binom{n}{j} W_{m,r}\left(j,k\right) W_{m,s}\left(n-j,h\right),$$

$$\binom{k+h}{k} WL_{m,r+s}\left(n,k+h\right) = \sum_{j=k}^{n-h} \binom{n}{j} WL_{m,r}\left(j,k\right) WL_{m,s}\left(n-j,h\right).$$

In the following theorem and corollary, we present generalized orthogonality relations and some other connections of r-Whitney and r-Whitney-Lah numbers.

Theorem 3.10. *Let $0 \leq k \leq n$, $r, s \geq 0$ and $m, l \geq 1$. Then*

$$\sum_{j=k}^{n} (-1)^{j-k} l^{n-j} m^{j-k} w_{m,r}(n,j) W_{l,s}(j,k) = \binom{n}{k} (lr - ms|ml)^{\overline{n-k}},$$

$$\sum_{j=k}^{n} (-1)^{j-k} l^{n-j} m^{j-k} W_{m,r}(n,j) w_{l,s}(j,k) = \binom{n}{k} (lr - ms)^{n-k},$$

$$\sum_{j=k}^{n} (-1)^{j-k} l^{n-j} m^{j-k} WL_{m,r}(n,j) WL_{l,s}(j,k) = \binom{n}{k} (2lr - 2ms|ml)^{\overline{n-k}},$$

$$w_{ml,2lr-ms}(n,k) = \sum_{j=k}^{n} (-1)^{j-k} l^{n-j} m^{j-k} WL_{m,r}(n,j) w_{l,s}(j,k) \quad \text{if } 2lr \geq ms,$$

$$W_{ml,2ms-lr}(n,k) = \sum_{j=k}^{n} (-1)^{n-j} l^{n-j} m^{j-k} W_{m,r}(n,j) WL_{l,s}(j,k) \quad \text{if } 2ms \geq lr,$$

$$WL_{ml,\frac{lr+ms}{2}}(n,k) = \sum_{j=k}^{n} l^{n-j} m^{j-k} w_{m,r}(n,j) W_{l,s}(j,k)$$

if lr and ms have the same parity.

Proof. Applying Theorem 3.1 and Corollary 3.1, it follows that

$$\begin{aligned}
(mlx + lr - ms|ml)^{\overline{n}} &= l^n \left(m \left(x - \frac{s}{l} \right) + r|m \right)^{\overline{n}} \\
&= l^n \sum_{j=0}^{n} w_{m,r}(n,j) m^j \left(x - \frac{s}{l} \right)^j \\
&= \sum_{j=0}^{n} l^{n-j} m^j w_{m,r}(n,j) (lx - s)^j \\
&= \sum_{j=0}^{n} l^{n-j} m^j w_{m,r}(n,j) \sum_{k=0}^{j} (-1)^{j-k} W_{l,s}(j,k) l^k x^{\overline{k}} \\
&= \sum_{k=0}^{n} \sum_{j=k}^{n} (-1)^{j-k} l^{n+k-j} m^j w_{m,r}(n,j) W_{l,s}(j,k) x^{\overline{k}}.
\end{aligned}$$

The same expression can be expanded by the binomial theorem for rising factorials to obtain

$$\begin{aligned}
(mlx + lr - ms|ml)^{\overline{n}} &= \sum_{k=0}^{n} \binom{n}{k} (mlx|ml)^{\overline{k}} (lr - ms|ml)^{\overline{n-k}} \\
&= \sum_{k=0}^{n} \binom{n}{k} l^k m^k (lr - ms|ml)^{\overline{n-k}} x^{\overline{k}}.
\end{aligned}$$

□

Corollary 3.5. *Let $0 \leq k \leq n$, $r, s \geq 0$ and $m \geq 1$. Then*

$$\sum_{j=k}^{n} (-1)^{j-k} w_{m,r}(n,j) W_{m,s}(j,k) = \binom{n}{k} (r-s|m)^{\overline{n-k}},$$

$$\sum_{j=k}^{n} (-1)^{j-k} W_{m,r}(n,j) w_{m,s}(j,k) = \binom{n}{k} (r-s)^{n-k},$$

$$\sum_{j=k}^{n} (-1)^{j-k} WL_{m,r}(n,j) WL_{m,s}(j,k) = \binom{n}{k} (2r-2s|m)^{\overline{n-k}},$$

$$w_{m,2r-s}(n,k) = \sum_{j=k}^{n} (-1)^{j-k} WL_{m,r}(n,j) w_{m,s}(j,k) \quad \text{if } 2r \geq s,$$

$$W_{m,2s-r}(n,k) = \sum_{j=k}^{n} (-1)^{n-j} W_{m,r}(n,j) WL_{m,s}(j,k) \quad \text{if } 2s \geq r,$$

$$WL_{m,\frac{r+s}{2}}(n,k) = \sum_{j=k}^{n} w_{m,r}(n,j) W_{m,s}(j,k)$$

if r and s have the same parity.

Remark 3.3. We remark that if we additionally assume that $s = r$, then Kronecker's $\delta_{n,k}$ stands on the right-hand sides of the first three equations in this corollary. For this reason, we can call these identities generalized orthogonality. They partly appear in [11] (for $r = 1$) and [6], [17], [19].

Similarly, under the assumption $s = r$ the last equation becomes simply (3), which was used to originally define r-Whitney-Lah numbers by G.-S. Cheon and J.-H. Jung [6]. Now we give a combinatorial argument which directly results in this identity.

To prove it, we produce an r-Whitney-Lah coloured partition of the first $n + r$ positive integers into $k + r$ ordered blocks in the following way: First, we decompose them into $j + r$ disjoint cycles in r-Whitney sense ($j = k, \ldots, n$), and after identifying the cycles with their smallest elements, we partition them into $k + r$ blocks in r-Whitney sense. In each block rearrange the cycles into canonical form (list the least number first in the cycles, and sort the cycles in decreasing order of their first elements), finally shift the cycles containing a distinguished element to the front of their block in reverse order. Then, through the usual way, the blocks can be handled as ordered blocks of numbers. This construction ensures that the hereditary colours of the elements coincide with their colours by the r-Whitney-Lah rules.

In [2], [9] (for $r = 1$) and [6], [8], [15], r-Whitney numbers are expressed by the help of symmetric polynomials. We do the same for r-Whitney-Lah numbers, and we provide the combinatorial meanings of these identities.

Theorem 3.11. *If $0 \le k \le n$, $r \ge 0$ and $m \ge 1$, then*

$$w_{m,r}(n,k) = \sum_{0 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \prod_{j=1}^{n-k} (i_j m + r),$$

$$W_{m,r}(n,k) = \sum_{0 \le i_1 \le i_2 \le \cdots \le i_{n-k} \le k} \prod_{j=1}^{n-k} (i_j m + r),$$

$$WL_{m,r}(n,k) = \sum_{0 \le i_1 \le i_2 \le \cdots \le i_{n-k} \le k} \prod_{j=1}^{n-k} ((2i_j + j - 1)m + 2r),$$

in other words, $w_{m,r}(n,k)$ is the $(n-k)$th elementary symmetric polynomial of $r, m+r, \ldots, (n-1)m+r$, and $W_{m,r}(n,k)$ is the the $(n-k)$th complete symmetric polynomial of $r, m+r, \ldots, km+r$.

Proof. For an r-Whitney coloured permutation of $1, \ldots, n+r$ which is the product of $k+r$ disjoint cycles, denote by $r+1 \le j_1 < j_2 < \cdots < j_{n-k} \le n+r$ those elements which are not minimal in their cycles.

First, place the $k+r$ minimal elements into distinct cycles. Then we can put j_h after one of the distinguished elements, when it remains uncoloured, or after any of the other $j_h - r - 1$ previously placed smaller numbers, when it is coloured by m colours ($h = 1, \ldots, n-k$). Therefore, we have

$$w_{m,r}(n,k) = \sum_{r+1 \le j_1 < j_2 < \cdots < j_{n-k} \le n+r} \prod_{h=1}^{n-k} ((j_h - r - 1)m + r),$$

and the assertion follows by changing indices. □

It is known that the finite sequences of r-Whitney numbers of the first kind [9] (for $r=1$), [7] and r-Whitney numbers of the second kind [3], [9], [24] (for $r=1$), [6], [8], [26] are log-concave if we fix n. The explicit formula enables us to immediately deduce the same property for r-Whitney-Lah numbers.

Theorem 3.12. *Let $n, m \ge 1$ and $r \ge 0$. Then the sequence $(WL_{m,r}(n,k))_{k=0}^{n}$ is strictly log-concave, therefore it is unimodal.*

Proof. For $1 \le k \le n-1$, it follows from Theorem 3.5 that the inequality $WL_{m,r}^2(n,k) > WL_{m,r}(n,k-1)\, WL_{m,r}(n,k+1)$ to be proven is equivalent to $(k+1)(n-k+1)(2r+km) > k(n-k)(2r+(k-1)m)$. □

In the following theorem we describe the connection between r-Whitney transformations of the first and second kind, r-Whitney-Lah transformation and their inverses. Its second part appears in [11], but only for $r=1$.

Theorem 3.13. *Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be sequences of complex numbers and let $r \ge 0$, $m \ge 1$. Then*

- $b_n = \sum_{k=0}^{n} w_{m,r}(n,k)\, a_k$ $(n \geq 0)$ *if and only if* $a_n = \sum_{k=0}^{n} (-1)^{n-k} W_{m,r}(n,k)\, b_k$ $(n \geq 0)$,
- $b_n = \sum_{k=0}^{n} W_{m,r}(n,k)\, a_k$ $(n \geq 0)$ *if and only if* $a_n = \sum_{k=0}^{n} (-1)^{n-k} w_{m,r}(n,k)\, b_k$ $(n \geq 0)$,
- $b_n = \sum_{k=0}^{n} WL_{m,r}(n,k)\, a_k$ $(n \geq 0)$ *if and only if* $a_n = \sum_{k=0}^{n} (-1)^{n-k} WL_{m,r}(n,k)\, b_k$ $(n \geq 0)$.

Proof. Suppose that $b_n = \sum_{k=0}^{n} w_{m,r}(n,k)\, a_k$ $(n \geq 0)$. Then orthogonality gives

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^{n-k} W_{m,r}(n,k)\, b_k &= \sum_{k=0}^{n} (-1)^{n-k} W_{m,r}(n,k) \sum_{j=0}^{k} w_{m,r}(k,j)\, a_j \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{n-k} W_{m,r}(n,k)\, w_{m,r}(k,j)\, a_j \\
&= \sum_{j=0}^{n} \delta_{n,j} a_j = a_n.
\end{aligned}$$

The same direction of the second statement can be proved similarly. Applying these for the sequences $\left((-1)^n b_n\right)_{n=0}^{\infty}$ and $\left((-1)^n a_n\right)_{n=0}^{\infty}$, the opposite directions follow. □

L. L. Liu [14] proved that Whitney transformation of both kinds preserves log-convexity. Giving a direct proof, we extend this result to r-Whitney transformations, as well as, to r-Whitney-Lah transformation. To derive this, we need the following identities for 0-Whitney and 0-Whitney-Lah numbers, which can be proved similarly to the combinatorial proof of Theorem 3.6.

Lemma 3.1. *If $0 \leq k \leq n$ and $m \geq 1$, then*

$$w_{m,0}(n+1,k+1) = \sum_{j=k}^{n} \binom{n}{j} w_{m,0}(j,k)\,(n-j)!m^{n-j},$$

$$W_{m,0}(n+1,k+1) = \sum_{j=k}^{n} \binom{n}{j} W_{m,0}(j,k)\, m^{n-j},$$

$$WL_{m,0}(n+1,k+1) = \sum_{j=k}^{n} \binom{n}{j} WL_{m,0}(j,k)\,(n-j+1)!m^{n-j}.$$

Theorem 3.14. *The r-Whitney transform of both kinds and the r-Whitney-Lah transform of a log-convex sequence of nonnegative real numbers are also log-convex.*

Proof. Let $(a_n)_{n=0}^{\infty}$ be a sequence of nonnegative real numbers. In this proof, denote by $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ its 0-Whitney and r-Whitney transforms of the first kind, and let $(d_n)_{n=0}^{\infty}$ be the 0-Whitney transform of the first kind of $(a_{n+1})_{n=0}^{\infty}$, that is $b_n = \sum_{k=0}^{n} w_{m,0}(n,k)\, a_k$, $c_n = \sum_{k=0}^{n} w_{m,r}(n,k)\, a_k$, $d_n = \sum_{k=0}^{n} w_{m,0}(n,k)\, a_{k+1}$ $(n \geq 0)$.

First, we prove by induction on N that $(b_n)_{n=0}^{N}$ is log-convex if $(a_n)_{n=0}^{N}$ is log-convex. It can be easily checked for $N = 1$ or 2, and assume that it holds for some N. To prove this statement for $N+1$, suppose that $(a_n)_{n=0}^{N+1}$ is log-convex. Then $(b_n)_{n=0}^{N}$ and $(d_n)_{n=0}^{N}$ are log-convex by the induction hypothesis.

By Lemma 3.1, we can show that $(b_{n+1})_{n=0}^{N}$ is the binomial convolution of the log-convex sequences $(n!m^n)_{n=0}^{N}$ and $(d_n)_{n=0}^{N}$, since

$$
\begin{aligned}
b_{n+1} &= \sum_{k=0}^{n+1} w_{m,0}(n+1,k)\, a_k = \sum_{k=0}^{n} w_{m,0}(n+1,k+1)\, a_{k+1} \\
&= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} w_{m,0}(j,k)\, (n-j)! m^{n-j} a_{k+1} \\
&= \sum_{j=0}^{n} \binom{n}{j} (n-j)! m^{n-j} \sum_{k=0}^{j} w_{m,0}(j,k)\, a_{k+1} = \sum_{j=0}^{n} \binom{n}{j} (n-j)! m^{n-j} d_j.
\end{aligned}
$$

The Davenport-Pólya Theorem [10] gives that $(b_{n+1})_{n=0}^{N}$, hence $(b_n)_{n=0}^{N+1}$ are log-convex.

Finally, applying Corollary 3.2 we have

$$
\begin{aligned}
c_n &= \sum_{k=0}^{n} w_{m,r}(n,k)\, a_k = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} w_{m,0}(j,k)\, (r|m)^{\overline{n-j}}\, a_k \\
&= \sum_{j=0}^{n} \binom{n}{j} (r|m)^{\overline{n-j}} \sum_{k=0}^{j} w_{m,0}(j,k)\, a_k = \sum_{j=0}^{n} \binom{n}{j} (r|m)^{\overline{n-j}}\, b_j,
\end{aligned}
$$

and the log-convexity of $(c_n)_{n=0}^{\infty}$ follows again by the Davenport-Pólya Theorem, since $(b_n)_{n=0}^{\infty}$ and $((r|m)^{\overline{n}})_{n=0}^{\infty}$ are log-convex. $\square$

Since $(1)_{n=0}^{\infty}$ is trivially log-convex, Theorem 3.14 implies the following consequence.

Corollary 3.6. *If $r \geq 0$ and $m \geq 1$, then the sequence $(D_{n,m,r})_{n=0}^{\infty}$ is log-convex.*

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