

# A Delayed Black and Scholes Formula

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## Abstract

In this article we develop an explicit formula for pricing European options when the underlying stock price follows a non-linear stochastic functional differential equation. We believe that the proposed model is sufficiently flexible to fit real market data, and is yet simple enough to allow for a closed-form representation of the option price. Furthermore, the model maintains the completeness of the market. The derivation of the option-pricing formula is based on an equivalent martingale measure.

## 1 Introduction

The connection between the modeling of uncertainty and the description of financial variables was first established as early as 1900 with the doctoral dissertation of Louis Bachelier, a student of Henri Poincaré. Bachelier hypothesized that the movement of speculative prices in the market could be described as a random walk and gave the first mathematical description of Brownian motion, pointing out its Markovian nature ([Bac], [C]).

Subsequent advances in probability theory, particularly in stochastic analysis, were essential in the development of the theory of mathematical finance. In fact, what we know today as the modern theory of finance has evolved, from an essentially descriptive discipline to a rigorous theory. This is mainly because of the use of continuous-time models and stochastic analysis ([Me1],

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[Me<sub>2</sub>]). The *Black and Scholes Formula* has been one of the most important consequences of the study of continuous time models. On the other hand, the need for better ways of understanding the behavior of many natural processes has motivated the development of dynamic models of these processes that take into consideration the influence of past events on the current and future states of the system ([I.N], [Ku], [K.N], [Mo<sub>1</sub>], [Mo<sub>2</sub>], [M.T], [E.Ø.S]). This view is specially appropriate in the study of financial variables, since predictions about their evolution take strongly into account the knowledge of their past ([H.Ø], [S.K]). In this framework we will derive a formula for pricing options on stocks with hereditary structure (Theorem 4).

We recall that an *option* is a contract giving the owner the right to buy or sell an asset, in accordance with certain conditions and within a specified period of time. In particular, a *call option* gives its owner the right to buy a share of stock at the *maturity* or *expiration date* of the option, for a specified *exercise price*. The option is *exercised* when the exercise price is paid.

In this paper we consider the effect of the past in the determination of the fair price of a call option. In particular, we assume that the stock price satisfies a stochastic functional differential equation (sfde). We consider call options that can be exercised only at the maturity date, viz. *European call options*. We derive an *explicit* formula for the valuation of a European call option on a given stock (Theorem 4) (cf. ([B.S], [Me<sub>1</sub>], [H.R])). Note that based on the preprint version of the present paper, the logarithmic utility of an insider has been computed and the stability of the European call option has been proved in [S].

Tests of the classical Black and Scholes model against real market data suggest the existence of significant levels of randomness in the volatility of the stock price, as manifested in the observed phenomenon of *frowns* and *smiles* ([Bat]). One of the motivations behind our model for the stock price is to account for such volatility in a natural manner, while at the same time maintain an explicit formula for the option price. It is hoped that the parameters of the proposed model will allow enough flexibility for a better fit than that of the Black and Scholes model when tested against real market data.

Although options are a very particular class of financial securities, they show the characteristic properties of more general forms of investment such as *contingent claims* or *derivatives*. International markets for contingent claims have experienced remarkable growth in the last thirty years. This makes the study of option pricing of special interest in the present context, since this theory may lead to a general theory of pricing contingent claims with hereditary structure.

## 2 Stochastic delay models for the stock price

In this section we propose a stochastic delay model for the evolution of the stock price. We prove that the proposed model is feasible. In Section 3, we formulate and solve the option pricing problem for the model.

Consider a stock whose price at time  $t$  is given by a stochastic process  $S(t)$  satisfying the following stochastic functional differential equation (sfde):

$$\left. \begin{aligned} dS(t) &= f(t, S_t) dt + g(S(t-b))S(t) dW(t), & t \in [0, T] \\ S(t) &= \varphi(t), & t \in [-L, 0] \end{aligned} \right\} \quad (1)$$

on a probability space  $(\Omega, \mathcal{F}, P)$  with a filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  satisfying the usual conditions. In the above sfde,  $L$ ,  $b$  and  $T$  are *positive* constants with  $L \geq b$ . The space  $C([-L, 0], \mathbf{R})$  of all continuous functions  $\eta : [-L, 0] \rightarrow \mathbf{R}$  is a Banach space with the supremum norm

$$\|\eta\| := \sup_{s \in [-L, 0]} |\eta(s)|.$$

The drift coefficient  $f : [0, T] \times C([-L, 0], \mathbf{R}) \rightarrow \mathbf{R}$  is a given continuous functional, and  $g : \mathbf{R} \rightarrow \mathbf{R}$  is continuous. The initial process  $\varphi : \Omega \rightarrow C([-L, 0], \mathbf{R})$  is  $\mathcal{F}_0$ -measurable with respect to the Borel  $\sigma$ -algebra of  $C([-L, 0], \mathbf{R})$ . The process  $W$  is a one-dimensional standard Brownian motion adapted to the filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ ; and  $S_t \in C([-L, 0], \mathbf{R})$  stands for the segment  $S_t(s) := S(t+s)$ ,  $s \in [-L, 0]$ ,  $t \geq 0$ .

In the sequel, we will consider the following two candidates for the drift coefficient  $f$ .

Define the functionals  $f_i : [0, T] \times C([-L, 0], \mathbf{R}) \rightarrow \mathbf{R}$ ,  $i = 1, 2$ , by

$$f_1(t, \eta) := \mu\eta(-a)\eta(0), \quad f_2(t, \eta) = \mu\eta(-a),$$

for all  $(t, \eta) \in [0, T] \times C([-L, 0], \mathbf{R})$ , where  $\mu, a$  are positive constants with  $L = \max\{a, b\}$ . In other words, under these choices of  $f$ , the sfde (1) reduces to the following stochastic differential delay equations (sdde's):

$$\left. \begin{aligned} dS(t) &= \mu S(t-a)S(t) dt + g(S(t-b))S(t) dW(t), & t \in [0, T] \\ S(t) &= \varphi(t), & t \in [-L, 0]. \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} dS(t) &= \mu S(t-a) dt + g(S(t-b))S(t) dW(t), & t \in [0, T] \\ S(t) &= \varphi(t), & t \in [-L, 0]. \end{aligned} \right\} \quad (3)$$

In our next result, we will show that the above three models (1), (2) and (3) are feasible in the sense that they admit pathwise unique solutions such that  $S(t) > 0$  almost surely for all  $t \geq 0$  whenever the initial path  $\varphi(t) > 0$  for all  $t \in [-L, 0]$ .

**Hypotheses (E).**

(i) There is a positive constant  $L'$  such that

$$|f(t, \eta)| \leq L'(1 + \|\eta\|)$$

for all  $(t, \eta) \in [0, T] \times C([-L, 0], \mathbf{R})$ .

(ii) For each integer  $n > 0$ , there is a positive constant  $L_n$  such that

$$|f(t, \eta^1) - f(t, \eta^2)| \leq L_n \|\eta^1 - \eta^2\|$$

for all  $(t, \eta^i) \in [0, T] \times C([-L, 0], \mathbf{R})$  with  $\|\eta^i\| \leq n$ ,  $i=1,2$ .

(iii)  $f(t, \eta) > 0$  for all  $(t, \eta) \in [0, T] \times C([-L, 0], \mathbf{R})$ .

(iv)  $g : \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

(v)  $a$  and  $b$  are positive constants.

It is easy to see that  $f_2$  satisfies Hypotheses (E)(i),(ii) (but not (E)(iii)). Although  $f_1$  does not satisfy (E)(i), we will show in Theorem 1 below that the sdde (2) still admits a pathwise-unique positive solution.

**Theorem 1** *Assume Hypotheses (E). Then each of the sfde's (1), (2) and (3) has a pathwise unique solution  $S$  for a given  $\mathcal{F}_0$ -measurable initial process  $\varphi : \Omega \rightarrow C([-L, 0], \mathbf{R})$ . Furthermore, if  $\varphi(t) \geq 0$  for all  $t \in [-L, 0]$  a.s., then  $S(t) \geq 0$  for all  $t \geq 0$  a.s.. If in addition  $\varphi(0) > 0$  a.s., then  $S(t) > 0$  for all  $t \geq 0$  a.s..*

*Proof.*

*Case 1:  $f = f_1$ .*

Define  $l := \min\{a, b\} > 0$  and let  $t \in [0, l]$ . Then (2) gives

$$\left. \begin{aligned} dS(t) &= S(t)[\mu\varphi(t-a) dt + g(\varphi(t-b)) dW(t)], & t \in [0, l] \\ S(0) &= \varphi(0). \end{aligned} \right\} \quad (4)$$

Define the semimartingale

$$N(t) := \mu \int_0^t \varphi(u-a) du + \int_0^t g(\varphi(u-b)) dW(u), \quad t \in [0, l],$$

and denote by  $[N, N](t) = \int_0^t g(\varphi(u-b))^2 du$ ,  $t \in [0, l]$ , its quadratic variation. Then (4) becomes

$$dS(t) = S(t) dN(t), \quad t > 0, \quad S(0) = \varphi(0),$$

which has the unique solution

$$\begin{aligned} S(t) &= \varphi(0) \exp\{N(t) - \frac{1}{2}[N, N](t)\}, \\ &= \varphi(0) \exp\left\{\mu \int_0^t \varphi(u-a) du \right. \\ &\quad \left. + \int_0^t g(\varphi(u-b)) dW(u) - \frac{1}{2} \int_0^t g(\varphi(u-b))^2 du\right\}, \end{aligned}$$

for  $t \in [0, l]$ . This clearly implies that  $S(t) > 0$  for all  $t \in [0, l]$  almost surely, when  $\varphi(0) > 0$  a.s. By a similar argument, it follows that  $S(t) > 0$  for all  $t \in [l, 2l]$  a.s.. Therefore  $S(t) > 0$  for all  $t \geq 0$  a.s., by induction. Note that the above argument also gives existence and pathwise-uniqueness of the solution to (2).

*Case 2:  $f = f_2$ .*

First let  $t \in [0, l]$  and let  $\varphi(t) \geq 0$  a.s. for all  $t \in [-L, 0]$ . Then (3) becomes

$$\left. \begin{aligned} dS(t) &= \mu\varphi(t-a) dt + g(\varphi(t-b))S(t) dW(t), \quad t \in [0, l] \\ S(0) &= \varphi(0). \end{aligned} \right\} \quad (5)$$

Define the martingale

$$M(t) := \int_0^t g(\varphi(u-b)) dW(u), \quad t \in [0, l].$$

Then  $S$  solves the stochastic ordinary differential equation (sode)

$$\left. \begin{aligned} dS(t) &= \mu\varphi(t-a) dt + S(t) dM(t), \quad t \in [0, l] \\ S(0) &= \varphi(0). \end{aligned} \right\} \quad (6)$$

Let  $\psi$  be the solution of the sode

$$\left. \begin{aligned} d\psi(t) &= \psi(t) dM(t), \quad t \in [0, l] \\ \psi(0) &= 1, \end{aligned} \right\} \quad (7)$$

and define the random process  $y$  by

$$\left. \begin{aligned} y'(t) &= \mu\psi(t)^{-1}\varphi(t-a), \quad t \in [0, l] \\ y(0) &= \varphi(0). \end{aligned} \right\} \quad (8)$$

Denote by  $[M, M]$  the quadratic variation of  $M$ . Then, from (7), it follows that

$$\psi(t) = \exp\{M(t) - \frac{1}{2}[M, M](t)\} > 0$$

for all  $t \in [0, l]$ .

Define the process  $\tilde{S}$  by  $\tilde{S}(t) := \psi(t)y(t)$  for  $t \in [0, l]$ . Then by the product rule, it follows that

$$\left. \begin{aligned} d\tilde{S}(t) &= \mu\varphi(t-a) dt + \tilde{S}(t) dM(t), \quad t \in [0, l] \\ \tilde{S}(0) &= \varphi(0). \end{aligned} \right\} \quad (9)$$

Comparing (6) and (9), it follows by uniqueness that  $P$ -a.s.,  $S(t) = \tilde{S}(t)$  for all  $t \in [0, l]$ . Now using (8) and the fact that  $\varphi(t) \geq 0$  a.s. for all  $t \in [-L, 0]$ , it follows that  $y(t) \geq 0$  a.s. for all  $t \in [0, l]$ . Also it follows from (8) that  $y(t) > 0$  for all  $t \in [0, l]$  a.s. if  $\varphi(0) > 0$  a.s.. Hence  $S(t) = \tilde{S}(t) > 0$  for all  $t \in [0, l]$  a.s.. Using forward steps of length  $l$ , it is easy to see that  $S(t) > 0$  a.s. for all  $t \geq 0$ .

*Case 3:*  $f$  satisfies (E).

This is similar to Case 2. Details are left to the reader.  $\diamond$

### Remarks 1.

- (i) In Case 1 above, we need only require  $\phi(0) \geq 0$  (or  $\phi(0) > 0$ ) to conclude that the solution of (2) satisfies a.s.  $S(t) \geq 0$  for all  $t \geq 0$  (or  $S(t) > 0$  for all  $t \geq 0$ , resp.).
- (ii) A fourth feasible model for the stock price is obtained by taking  $f = f_3$  where

$$f_3(t, \xi) := h(t, \xi^{t-a})\xi(0), \quad (t, \xi) \in [0, T] \times C([-L, T], \mathbf{R}),$$

with  $\xi^t(s) := \xi(t \wedge s)$ ,  $t, s \in [-L, T]$ , and  $h : [0, T] \times C([-L, T], \mathbf{R}) \rightarrow \mathbf{R}$  is a continuous functional. Hence the stock price  $S$  satisfies the sfde

$$\begin{aligned} dS(t) &= h(t, S^{t-a})S(t) dt + g(S(t-b))S(t) dW(t), \quad t \in [0, T], \\ S(t) &= \varphi(t), \quad t \in [-L, 0]. \end{aligned}$$

Theorem 1 holds for the above model of the stock if Hypotheses (E) hold with E(iii) replaced by the following monotonicity condition:

(E)(iii)' For each  $\xi \in C([-L, T], \mathbf{R})$  with  $\xi(t) \geq 0$  for all  $t \in [-L, T]$ , one has

$$h(t, \xi) \geq 0 \text{ for all } t \in [0, T].$$

The proof is analogous to Case 2 in the proof of Theorem 1.

## 3 A delayed option pricing formula

Consider a market consisting of a riskless asset (e.g., a bond or bank account)  $B(t)$  with rate of return  $r \geq 0$  (i.e.,  $B(t) = e^{rt}$ ) and a single stock whose

price  $S(t)$  at time  $t$  satisfies the SDE (2) where  $\varphi(0) > 0$  a.s.. In the SDE (2), assume further that the delays  $a, b$  are positive and  $g$  is continuous. Consider an option, written on the stock, with maturity at some future time  $T > t$  and an exercise price  $K$ . Assume also that there are no transaction costs and that the underlying stock pays no dividends. Our main objective is to derive the fair price of the option at time  $t$ . In the following discussion, we will obtain an equivalent martingale measure with the help of Girsanov's theorem.

Let

$$\tilde{S}(t) := \frac{S(t)}{B(t)} = e^{-rt} S(t), \quad t \in [0, T],$$

be the discounted stock price process. Then by Itô's formula (the product rule), we obtain

$$\begin{aligned} d\tilde{S}(t) &= e^{-rt} dS(t) + S(t)(-re^{-rt}) dt \\ &= \tilde{S}(t) \left[ \{\mu S(t-a) - r\} dt + g(S(t-b)) dW(t) \right]. \end{aligned}$$

Let

$$\hat{S}(t) := \int_0^t \{\mu S(u-a) - r\} du + \int_0^t g(S(u-b)) dW(u), \quad t \in [0, T].$$

Then

$$d\tilde{S}(t) = \tilde{S}(t) d\hat{S}(t), \quad 0 < t < T. \quad (10)$$

Taking into account that  $\tilde{S}(0) = \varphi(0)$ , we have

$$\tilde{S}(t) = \varphi(0) + \int_0^t \tilde{S}(u) d\hat{S}(u), \quad t \in [0, T]. \quad (11)$$

We now recall Girsanov's theorem (see, e.g., Theorem 5.5 in [K.K]).

**Theorem 2 (Girsanov)** *Let  $W(t)$ ,  $t \in [0, T]$ , be a standard Wiener process on  $(\Omega, \mathcal{F}, P)$ . Let  $\Sigma$  be a predictable process such that  $\int_0^T |\Sigma(u)|^2 du < \infty$  a.s., and let*

$$\varrho_t := \exp \left\{ \int_0^t \Sigma(u) dW(u) - \frac{1}{2} \int_0^t |\Sigma(u)|^2 du \right\}, \quad t \in [0, T].$$

*Suppose that  $E_P(\varrho_T) = 1$ , where  $E_P$  denotes expectation with respect to the probability measure  $P$ . Define the probability measure  $Q$  on  $(\Omega, \mathcal{F})$  by  $dQ := \varrho_T dP$ . Then the process*

$$\widehat{W}(t) := W(t) - \int_0^t \Sigma(u) du, \quad t \in [0, T],$$

*is a standard Wiener process under the measure  $Q$ .*

From now on, we will assume that the function  $g : \mathbf{R} \rightarrow \mathbf{R}$  in the sdde (2) satisfies the following hypothesis:

**Hypothesis (B).**  $g(v) \neq 0$  whenever  $v \neq 0$ .

We want to apply Girsanov's theorem with the process

$$\Sigma(u) := -\frac{\{\mu S(u-a) - r\}}{g(S(u-b))}, \quad u \in [0, T].$$

Hypothesis (B) implies that  $\Sigma$  is well-defined, since by Theorem 1,  $S(t) > 0$  for all  $t \in [0, T]$  a.s.. Clearly  $\Sigma(t)$ ,  $t \in [0, T]$ , is a predictable process. Moreover,  $\int_0^T |\Sigma(u)|^2 du < \infty$  a.s., since almost sure continuity of the process  $S(t)$ ,  $t \in [0, T]$ , implies almost sure boundedness of  $S(t)$ ,  $t \in [0, T]$ , and Hypothesis (B) implies that  $1/g(v)$ ,  $v \in (0, \infty)$ , is bounded on bounded intervals. Now let  $l := \min(a, b)$ . Set  $\mathcal{F}_t := \mathcal{F}_0$  for  $t \leq 0$ . Then  $\Sigma(u)$ ,  $u \in [0, T]$ , is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_{T-l}$ . Hence, the stochastic integral  $\int_{T-l}^T \Sigma(u) dW(u)$  conditioned on  $\mathcal{F}_{T-l}$  has a normal distribution with mean zero and variance  $\int_{T-l}^T \Sigma(u)^2 du$ . Consequently, by the formula for the moment generating function of a normal distribution, we obtain

$$E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) \right\} \middle| \mathcal{F}_{T-l} \right) = \exp \left\{ \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\}$$

a.s.. Hence

$$E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) - \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_{T-l} \right) = 1$$

a.s.. Now the above relation easily implies that

$$\begin{aligned} & E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_{T-l} \right) \\ &= \exp \left\{ \int_0^{T-l} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-l} |\Sigma(u)|^2 du \right\} \end{aligned}$$

a.s.. Let  $k$  to be a positive integer such that  $0 \leq T - kl \leq l$ . Then by successive conditioning using backward steps of length  $l$ , an inductive argument gives

$$\begin{aligned} & E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_{T-kl} \right) \\ &= \exp \left\{ \int_0^{T-kl} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-kl} |\Sigma(u)|^2 du \right\} \end{aligned}$$



a.s.. Taking conditional expectation with respect to  $\mathcal{F}_0$  on both sides of the above equation, we obtain

$$\begin{aligned} & E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_0 \right) \\ &= E_P \left( \exp \left\{ \int_0^{T-kl} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-kl} |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_0 \right) = 1 \end{aligned}$$

a.s.. Taking the expectation of the above equation, we immediately obtain

$$E_P(\varrho_T) = 1.$$

Therefore, the Girsanov's theorem (Theorem 2) applies and the process

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{\mu S(u-a) - r\}}{g(S(u-b))} du, \quad t \in [0, T],$$

is a standard Wiener process under the measure  $Q$  defined by  $dQ := \varrho_T dP$  with

$$\varrho_T := \exp \left\{ - \int_0^T \frac{\{\mu S(u-a) - r\}}{g(S(u-b))} dW(u) - \frac{1}{2} \int_0^T \left| \frac{\mu S(u-a) - r}{g(S(u-b))} \right|^2 du \right\}$$

a.s.. Since the process  $\widehat{S}(t)$ ,  $t \in [0, T]$ , can be written in the form

$$\widehat{S}(t) = \int_0^t g(S(u-b)) d\widehat{W}(u), \quad t \in [0, T], \quad (12)$$

we conclude that  $\widehat{S}(t)$ ,  $t \in [0, T]$ , is a continuous  $Q$ -martingale (i.e., a continuous martingale under the measure  $Q$ ). Furthermore, by the representation (11), the discounted stock price process  $\widetilde{S}(t)$ ,  $t \in [0, T]$ , is also a continuous  $Q$ -martingale. In other words,  $Q$  is an equivalent martingale measure. By the well-known theorem on trading strategies (e.g., Theorem 7.1 in [K.K]), it follows that the market consisting of  $\{B(t), S(t) : t \in [0, T]\}$  satisfies the no-arbitrage property: There is no admissible self-financing strategy which gives an arbitrage opportunity.

We now establish the completeness of the market  $\{B(t), S(t) : t \in [0, T]\}$ .

From the proof of Theorem 1, it follows that the solution of the sdde (2) satisfies the relation

$$S(t) = \varphi(0) \exp \left\{ \int_0^t g(S(u-b)) dW(u) + \mu \int_0^t S(u-a) du - \frac{1}{2} \int_0^t g(S(u-b))^2 du \right\}$$

a.s. for  $t \in [0, T]$ . Hence we have

$$\widetilde{S}(t) = \varphi(0) \exp \left\{ \int_0^t g(S(u-b)) d\widehat{W}(u) - \frac{1}{2} \int_0^t g(S(u-b))^2 du \right\} \quad (13)$$

for  $t \in [0, T]$ . [ $S(t)$  and  $\tilde{S}(t)$  are the same except we consider  $\tilde{S}(t)$  as a functional of  $\tilde{W}$  and  $S(t)$  is considered as a functional of  $W$ ]. This implies that  $\mathcal{F}_t^S = \mathcal{F}_t^{\tilde{S}} = \mathcal{F}_t^{\widehat{W}}$ , the  $\sigma$ -algebras generated by  $\{S(u) : u \leq t\}$ ,  $\{\tilde{S}(u) : u \leq t\}$ ,  $\{\widehat{W}(u) : u \leq t\}$ , respectively. Let  $X$  be a contingent claim, viz. a bounded  $\mathcal{F}_T^S$ -measurable random variable. Consider the  $Q$ -martingale

$$M(t) := E_Q(e^{-rT}X | \mathcal{F}_t^S) = E_Q(e^{-rT}X | \mathcal{F}_t^{\widehat{W}}), \quad t \in [0, T].$$

By the martingale representation theorem (e.g., Theorem 9.4 in [K.K]), there exists an  $(\mathcal{F}_t^{\widehat{W}})$ -predictable process  $h(t)$ ,  $t \in [0, T]$ , such that

$$\int_0^t h(u)^2 du < \infty \quad a.s.,$$

and

$$M(t) = E_Q(e^{-rT}X) + \int_0^t h(u) d\widehat{W}(u), \quad t \in [0, T].$$

By (10) and (12) we obtain  $d\tilde{S}(u) = \tilde{S}(u)g(S(u-b))d\widehat{W}(u)$ . Define

$$\pi_S(t) := \frac{h(t)}{\tilde{S}(t)g(S(t-b))}, \quad \pi_B(t) := M(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0, T].$$

Consider the strategy  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$  which consists of holding  $\pi_S(t)$  units of the stock and  $\pi_B(t)$  units of the bond at time  $t$ . The value of the portfolio at time  $t$  is given by

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t),$$

hence

$$dV(t) = e^{rt}dM(t) + M(t)de^{rt} = \pi_B(t)de^{rt} + \pi_S(t)dS(t).$$

Consequently,  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$  is a self-financing strategy. Moreover,  $V(T) = e^{rT}M(T) = X$ , hence the contingent claim  $X$  is attainable. This shows that the market  $\{B(t), S(t) : t \in [0, T]\}$  is complete, since every contingent claim is attainable. Moreover, in order for the augmented market  $\{B(t), S(t), X : t \in [0, T]\}$  to satisfy the no-arbitrage property, the price  $V(t)$  of the claim  $X$  must be

$$V(t) = e^{-r(T-t)}E_Q(X | \mathcal{F}_t^S)$$

at each  $t \in [0, T]$  a.s.. See, e.g., [B.R] or Theorem 9.2 in [K.K].

The above discussion may be summarized in the following formula for the fair price  $V(t)$  of an option on the stock whose evolution is described by the sdde (2).

**Theorem 3** Suppose that the stock price  $S$  is given by the sdde (2), where  $\varphi(0) > 0$  and  $g$  satisfies Hypothesis (B). Let  $T$  be the maturity time of an option (contingent claim) on the stock with payoff function  $X$ , i.e.,  $X$  is an  $\mathcal{F}_T^S$ -measurable integrable random variable. Then at any time  $t \in [0, T]$ , the fair price  $V(t)$  of the option is given by the formula

$$V(t) = e^{-r(T-t)} E_Q(X | \mathcal{F}_t^S), \quad (14)$$

where  $Q$  denotes the probability measure on  $(\Omega, \mathcal{F})$  defined by  $dQ := \varrho_T dP$  with

$$\varrho_t := \exp \left\{ - \int_0^t \frac{\{\mu S(u-a) - r\}}{g(S(u-b))} dW(u) - \frac{1}{2} \int_0^t \left| \frac{\mu S(u-a) - r}{g(S(u-b))} \right|^2 du \right\}$$

for  $t \in [0, T]$ . The measure  $Q$  is a martingale measure and the market is complete.

Moreover, there is an adapted and square integrable process  $h(u)$ ,  $u \in [0, T]$  such that

$$E_Q(e^{-rT} X | \mathcal{F}_t^S) = E_Q(e^{-rT} X) + \int_0^t h(u) d\widehat{W}(u), \quad t \in [0, T]$$

and the hedging strategy is given by

$$\pi_S(t) := \frac{h(t)}{\widetilde{S}(t)g(S(t-b))}, \quad \pi_B(t) := M(t) - \pi_S(t)\widetilde{S}(t), \quad t \in [0, T]. \quad (15)$$

The following result is a consequence of Theorem 3. It gives a Black-Scholes-type formula for the value of a European option on the stock at any time prior to maturity.

**Theorem 4** Assume the conditions of Theorem 3. Let  $V(t)$  be the fair price of a European call option written on the stock  $S$  with exercise price  $K$  and maturity time  $T$ . Let  $\varphi$  denote the distribution function of the standard normal law, i.e.,

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad x \in \mathbf{R}.$$

Then for all  $t \in [T-l, T]$  (where  $l := \min\{a, b\}$ ),  $V(t)$  is given by

$$V(t) = S(t)\varphi(\beta_+(t)) - Ke^{-r(T-t)}\varphi(\beta_-(t)), \quad (16)$$

where

$$\beta_{\pm}(t) := \frac{\log \frac{S(t)}{K} + \int_t^T \left( r \pm \frac{1}{2} g(S(u-b))^2 \right) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

If  $T > l$  and  $t < T - l$ , then

$$V(t) = e^{rt} E_Q \left( H \left( \tilde{S}(T-l), -\frac{1}{2} \int_{T-l}^T g(S(u-b))^2 du, \int_{T-l}^T g(S(u-b))^2 du \right) \middle| \mathcal{F}_t \right) \quad (17)$$

where  $H$  is given by

$$H(x, m, \sigma^2) := x e^{m+\sigma^2/2} \varphi(\alpha_1(x, m, \sigma)) - K e^{-rT} \varphi(\alpha_2(x, m, \sigma)),$$

and

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + rT + m + \sigma^2 \right],$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + rT + m \right],$$

for  $\sigma, x \in \mathbf{R}^+$ ,  $m \in \mathbf{R}$ .

The hedging strategy is given by

$$\pi_S(t) = \varphi(\beta_+(t)), \quad \pi_B(t) = -K e^{-rT} \varphi(\beta_-(t)), \quad t \in [T-l, T].$$

**Remark 2.**

If  $g(x) = 1$  for all  $x \in \mathbf{R}^+$  then equation (16) reduces to the classical Black and Scholes formula. Note that, in contrast with the classical (non-delayed) Black and Scholes formula, the fair price  $V(t)$  in a general delayed model considered in Theorem 4 depends not only on the stock price  $S(t)$  at the present time  $t$ , but also on the whole segment  $\{S(v) : v \in [t-b, T-b]\}$ . (Of course  $[t-b, T-b] \subset [0, t]$  since  $t \geq T-l$  and  $l \leq b$ .)

*Proof of Theorem 4.*

Consider a European call option in the above market with exercise price  $K$  and maturity time  $T$ . Taking  $X = (S(T) - K)^+$  in Theorem 3, the fair price  $V(t)$  of the option is given by

$$V(t) = e^{-r(T-t)} E_Q((S(T) - K)^+ | \mathcal{F}_t) = e^{rt} E_Q((\tilde{S}(T) - K e^{-rT})^+ | \mathcal{F}_t),$$

at any time  $t \in [0, T]$ .

We now derive an explicit formula for the option price  $V(t)$  at any time  $t \in [T-l, T]$ . The representation (13) of  $\tilde{S}(t)$  implies that

$$\tilde{S}(T) = \tilde{S}(t) \exp \left\{ \int_t^T g(S(u-b)) d\widehat{W}(u) - \frac{1}{2} \int_t^T g(S(u-b))^2 du \right\}$$

for all  $t \in [0, T]$ . Clearly  $\tilde{S}(t)$  is  $\mathcal{F}_t$ -measurable. If  $t \in [T - l, T]$ , then  $-\frac{1}{2} \int_t^T g(S(u - b))^2 du$  is also  $\mathcal{F}_t$ -measurable. Furthermore, when conditioned on  $\mathcal{F}_t$ , the distribution of  $\int_t^T g(S(u - b)) d\widehat{W}(u)$  under  $Q$  is the same as that of  $\sigma\xi$ , where  $\xi$  is a Gaussian  $N(0, 1)$ -distributed random variable, and  $\sigma^2 = \int_t^T g(S(u - b))^2 du$ . Consequently, the fair price at time  $t$  is given by

$$V(t) = e^{rt} H \left( \tilde{S}(t), -\frac{1}{2} \int_t^T g(S(u - b))^2 du, \int_t^T g(S(u - b))^2 du \right),$$

where

$$H(x, m, \sigma^2) := E_Q(xe^{m+\sigma\xi} - Ke^{-rT})^+, \quad \sigma, x \in \mathbf{R}^+, m \in \mathbf{R}.$$

Now, an elementary computation yields the following:

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2} \varphi(\alpha_1(x, m, \sigma)) - Ke^{-rT} \varphi(\alpha_2(x, m, \sigma)).$$

Therefore,  $V(t)$  takes the form:

$$V(t) = S(t) \varphi(\beta_+) - Ke^{-r(T-t)} \varphi(\beta_-),$$

where

$$\beta_{\pm} = \frac{\log \frac{S(t)}{K} + \int_t^T \left( r \pm \frac{1}{2} g(S(u - b))^2 \right) du}{\sqrt{\int_t^T g(S(u - b))^2 du}}.$$

For  $T > l$  and  $t < T - l$ , from the representation (13) of  $\tilde{S}(t)$ , we have

$$\tilde{S}(T) = \tilde{S}(T - l) \exp \left\{ \int_{T-l}^T g(S(u - b)) d\widehat{W}(u) - \frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du \right\}.$$

Consequently, the option price at time  $t$  with  $t < T - l$  is given by

$$V(t) = e^{rt} E_Q \left( H \left( \tilde{S}(T - l), -\frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du, \int_{T-l}^T g(S(u - b))^2 du \right) \middle| \mathcal{F}_t \right).$$

To calculate the hedging strategy for  $t \in [T - l, T]$ , it suffices to use an idea from [B.R], pages 95–96. This completes the proof of the theorem.  $\diamond$

**Remark 3.**

For  $T > l$  and  $t < T - l$ , one can develop a recursive procedure to calculate (17) by taking backwards steps of length  $l$  from the maturity time  $T$  of the option. Coupled with numerical approximations, this recursive procedure can be used to compute the option price at any time  $t \in [0, T]$ . Obviously

$$V(t) = e^{rt} E_Q \left( E_Q((\tilde{S}(T) - Ke^{-rT})^+ | \mathcal{F}_{T-l}) \middle| \mathcal{F}_t \right).$$

The measurability arguments in the proof of Theorem 4 yield the following

$$\begin{aligned} & E_Q((\tilde{S}(T) - Ke^{-rT})^+ | \mathcal{F}_{T-l}) \\ &= H \left( \tilde{S}(T-l), -\frac{1}{2} \int_{T-l}^T g(S(u-b))^2 du, \int_{T-l}^T g(S(u-b))^2 du \right). \end{aligned}$$

In order to calculate the conditional expectation of  $E_Q((\tilde{S}(T) - Ke^{-rT})^+ | \mathcal{F}_{T-l})$  with respect to the  $\sigma$ -algebra  $\mathcal{F}_t$ , we will study the distribution of  $\tilde{S}(T-l)$  and  $\{S(u-b) : u \in [T-l, T]\}$  under the equivalent martingale measure  $Q$ . We will discuss only the conditional distribution of the solution process  $S(u-b)$  for  $u \in [T-l, T]$ . The corresponding distribution of  $\tilde{S}(T-l)$  can be handled in a similar way.

Now if  $T > kl$  for some positive integer  $k$  and  $t \in [T - (k+1)l, T - kl]$ , then for all  $u \in [T-l, T]$  the solution formula (13) implies that

$$S(u-b) = e^{r(u-b)} \tilde{S}(t) \prod_{j=1}^k \exp \left\{ \int_{\tau_{j-1}}^{\tau_j} g(S(v-b)) d\widehat{W}(v) - \frac{1}{2} \int_{\tau_{j-1}}^{\tau_j} g(S(v-b))^2 dv \right\}, \quad (18)$$

where

$$\tau_j := \begin{cases} t, & \text{if } j = 0, \\ T - (k-j+1)l, & \text{if } j = 1, \dots, k-1, \\ u-b, & \text{if } j = k. \end{cases}$$

Clearly  $\tilde{S}(t)$  is  $\mathcal{F}_t$ -measurable. The first factor in the product in (18) is

$$\exp \left\{ \int_t^{T-kl} g(S(v-b)) d\widehat{W}(v) - \frac{1}{2} \int_t^{T-kl} g(S(v-b))^2 dv \right\}. \quad (19)$$

This can be handled as before:  $-\frac{1}{2} \int_t^{T-kl} g(S(v-b))^2 dv$  is  $\mathcal{F}_t$ -measurable, the integrand  $\{g(S(v-b)) : v \in [t, T-kl]\}$  is also  $\mathcal{F}_t$ -measurable, and  $\widehat{W}$  is a standard Wiener process under the measure  $Q$ . Based on the conditional distribution under  $Q$ , one can construct an approximation of the integral  $\int_t^{T-kl} g(S(v-b)) d\widehat{W}(v)$  (see, e.g., [K.P] and [K.P.S]), which in turn yields an approximation of (19).

The second factor in the product in (18) has the form

$$\exp \left\{ \int_{T-kl}^{T-(k-1)l} g(S(v-b)) d\widehat{W}(v) - \frac{1}{2} \int_{T-kl}^{T-(k-1)l} g(S(v-b))^2 dv \right\}. \quad (20)$$

Using again (13), the integrand  $\{g(S(v-b)) : v \in [T-kl, T-(k-1)l]\}$  in the above expression can be rewritten using the relation:

$$S(v-b) = e^{r(v-b)} \tilde{S}(t) \exp \left\{ \int_t^{v-b} g(S(s-b)) d\widehat{W}(s) - \frac{1}{2} \int_t^{v-b} g(S(s-b))^2 ds \right\}.$$

In the same way as above, one can construct an approximation of the integrals in the required interval  $v \in [T - kl, T - (k - 1)l]$ , which will give an approximation of (20). Clearly one can treat the other factors in (18) in a similar fashion. Numerical approximation together with the rate of convergence may be discussed in the spirit of the work [H.M.Y] and will not be dealt with here.

**Remark 4.**

In the last delay period  $[T - l, T]$ , one can rewrite the option price  $V(t), t \in [T - l, T]$  in terms of the solution of a random Black-Scholes pde of the form

$$\left. \begin{aligned} \frac{\partial F(t, x)}{\partial t} &= -\frac{1}{2}g(S(t - b))^2 x^2 \frac{\partial^2 F(t, x)}{\partial x^2} - rx \frac{\partial F(t, x)}{\partial x} + rF(t, x), & 0 < t < T \\ F(T, x) &= (x - K)^+, & x > 0. \end{aligned} \right\} \quad (21)$$

The above time-dependent random final-value problem admits a unique  $(\mathcal{F}_t)_{t \geq 0}$ -adapted random field  $F(t, x)$ . Using the classical Itô-Ventzell formula ([Kun]) and (14) of Theorem 3, it can be shown that

$$V(t) = e^{-r(T-t)} F(t, S(t)), \quad t \in [T - b, T].$$

Note that the above representation is no longer valid if  $t \leq T - b$ , because in this range, the solution  $F$  of the final-value problem (17) is *anticipating* with respect to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ .

## 4 A stock price model with variable delay

In this section, we give an alternative model for the stock price dynamics with variable delay. In this case we are also able to develop a Black-Scholes formula for the option price.

Throughout this section, suppose  $h$  is a given fixed positive number. Denote  $[t] := kh$  if  $kh \leq t < (k + 1)h$ . Consider a market consisting of a riskless asset  $\xi$  with a variable (deterministic) continuous rate of return  $\lambda$ , and a stock  $S$  satisfying the following equations

$$\left. \begin{aligned} d\xi(t) &= \lambda(t)\xi(t) dt \\ dS(t) &= f(t, S([t]))S(t)dt + g(t, S([t]))S(t)dW(t) \end{aligned} \right\} \quad (22)$$

for  $t \in (0, T]$ , with initial conditions  $\xi(0) = 1$ ,  $S_0 \in C([-h, 0], \mathbf{R})$  and  $S(0) > 0$ . In the above model, let  $(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  be as in Section 2; and suppose  $f : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is a continuous function. Assume further that  $g : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous and  $g(t, v) \neq 0$  for all  $(t, v) \in [0, T] \times \mathbf{R}$ .

Under the above conditions, this model is feasible: That is  $S(t) > 0$  a.s. for all  $t > 0$ . This follows by an argument similar to the proof of Theorem 1, Section 2, and Remark 1(i).

Next, we will establish the completeness of the market  $\{\xi(t), S(t) : t \in [0, T]\}$  and the no-arbitrage property, following the approach in Section 3.

For  $t \in [kh, (k+1)h]$ , the solution of the second equation in (22) is given by

$$S(t) = S(kh) \exp \left( \int_{kh}^t g(s, S(kh)) dW(s) + \int_{kh}^t f(s, S(kh)) ds - \frac{1}{2} \int_{kh}^t g(s, S(kh))^2 ds \right). \quad (23)$$

As in Section 3, let

$$\tilde{S}(t) := \frac{S(t)}{\xi(t)} = S(t) e^{-\int_0^t \lambda(s) ds}, \quad t \in [0, T],$$

be the discounted stock price process. Again by Itô's formula, we obtain

$$\begin{aligned} d\tilde{S}(t) &= \frac{1}{\xi(t)} dS(t) + S(t) \left( -\frac{\lambda(t)}{\xi(t)} \right) dt \\ &= \tilde{S}(t) \left[ \{f(t, S(\lfloor t \rfloor)) - \lambda(t)\} dt + g(t, S(\lfloor t \rfloor)) dW(t) \right]. \end{aligned}$$

Let

$$\hat{S}(t) := \int_0^t \{f(u, S(\lfloor u \rfloor)) - \lambda(u)\} du + \int_0^t g(u, S(\lfloor u \rfloor)) dW(u), \quad t \in [0, T].$$

Then

$$d\tilde{S}(t) = \tilde{S}(t) d\hat{S}(t), \quad 0 < t < T, \quad (24)$$

and

$$\tilde{S}(t) = S(0) + \int_0^t \tilde{S}(u) d\hat{S}(u), \quad t \in [0, T]. \quad (25)$$

Define the stochastic process

$$\Sigma(u) := -\frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))}, \quad u \in [0, T].$$

It is clear that  $\Sigma(u)$  is  $\mathcal{F}_{[u]}^S$ -measurable for each  $u \in [0, T]$ . Furthermore, by backward conditioning using steps of length  $h$ , the reader may check that

$$E_P(\rho_T) = 1$$

where

$$\rho_T := \exp \left\{ -\int_0^T \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))} dW(u) - \frac{1}{2} \int_0^T \left| \frac{f(u, S(\lfloor u \rfloor)) - \lambda(u)}{g(u, S(\lfloor u \rfloor))} \right|^2 du \right\}.$$



(See the argument in Section 3 following the statement of Theorem 2.) Hence the Girsanov theorem (Theorem 2) applies, and it follows that the process

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))} du, \quad t \in [0, T],$$

is a standard Wiener process under the probability measure  $Q$  defined by  $dQ := \varrho_T dP$ .

We now establish the no-arbitrage property. To do so, let  $X$  be any contingent claim, viz. a bounded  $\mathcal{F}_T^S$ -measurable random variable. Define the process

$$M(t) := E_Q\left(\frac{X}{\xi(T)} \mid \mathcal{F}_t^S\right) = E_Q\left(\frac{X}{\xi(T)} \mid \mathcal{F}_t^{\widehat{W}}\right), \quad t \in [0, T].$$

Then  $M(t), t \in [0, T]$ , is an  $(\mathcal{F}_t^{\widehat{W}})$ -adapted  $Q$ -martingale. Hence, by the martingale representation theorem, there exists an  $(\mathcal{F}_t^{\widehat{W}})$ -predictable process  $h(t), t \in [0, T]$ , such that

$$\int_0^t h(u)^2 du < \infty \quad a.s.,$$

and

$$M(t) = E_Q\left(\frac{X}{\xi(T)}\right) + \int_0^t h(u) d\widehat{W}(u), \quad t \in [0, T].$$

Define

$$\pi_S(t) := \frac{h(t)}{\widetilde{S}(t)g(t, S(\lfloor t \rfloor))}, \quad \pi_\xi(t) := M(t) - \pi_S(t)\widetilde{S}(t), \quad t \in [0, T].$$

Consider the strategy  $\{(\pi_\xi(t), \pi_S(t)) : t \in [0, T]\}$  which consists of holding  $\pi_S(t)$  units of the stock and  $\pi_\xi(t)$  units of the bond at time  $t$ . The value of the portfolio at time  $t$  is given by

$$V(t) := \pi_\xi(t)\xi(t) + \pi_S(t)S(t) = \xi(t)M(t).$$

Moreover,

$$dV(t) = \xi(t)dM(t) + M(t)d\xi(t) = \pi_\xi(t)d\xi(t) + \pi_S(t)dS(t).$$

Consequently,  $\{(\pi_\xi(t), \pi_S(t)) : t \in [0, T]\}$  is a self-financing strategy. Clearly  $V(T) = \xi(T)M(T) = X$ , thus the contingent claim  $X$  is attainable. This shows that the market  $\{\xi(t), S(t) : t \in [0, T]\}$  is complete.

Moreover, in order for the augmented market  $\{\xi(t), S(t), X : t \in [0, T]\}$  to satisfy the no-arbitrage property, the price  $V(t)$  of the claim  $X$  must be

$$V(t) = \frac{\xi(t)}{\xi(T)} E_Q(X \mid \mathcal{F}_t^S)$$

at each  $t \in [0, T]$  a.s.. See, e.g., [B.R] or Theorem 9.2 in [K.K].

The above discussion may be summarized in the following formula for the fair price  $V(t)$  of an option on the stock whose evolution is described by the sdde (22).

**Theorem 5** *Suppose that the stock price  $S$  is given by the sdde (22), where  $S(0) > 0$  and  $g$  satisfies Hypothesis (B). Let  $T$  be the maturity time of an option (contingent claim) on the stock with payoff function  $X$ , i.e.,  $X$  is an  $\mathcal{F}_T^S$ -measurable integrable random variable. Then at any time  $t \in [0, T]$ , the fair price  $V(t)$  of the option is given by the formula*

$$V(t) = E_Q(X | \mathcal{F}_t^S) e^{-\int_t^T \lambda(s) ds}, \quad (26)$$

where  $Q$  denotes the probability measure on  $(\Omega, \mathcal{F})$  defined by  $dQ := \varrho_T dP$  with

$$\varrho_t := \exp \left\{ - \int_0^t \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))} dW(u) - \frac{1}{2} \int_0^t \left| \frac{f(u, S(\lfloor u \rfloor)) - \lambda(u)}{g(u, S(\lfloor u \rfloor))} \right|^2 du \right\}$$

for  $t \in [0, T]$ . The measure  $Q$  is a martingale measure and the market is complete.

Moreover, there is an adapted and square integrable process  $h(u)$ ,  $u \in [0, T]$  such that

$$E_Q \left( \frac{X}{\xi(T)} \middle| \mathcal{F}_t^S \right) = E_Q \left( \frac{X}{\xi(T)} \right) + \int_0^t h(u) d\widehat{W}(u), \quad t \in [0, T]$$

and the hedging strategy is given by

$$\pi_S(t) := \frac{h(t)}{\widetilde{S}(t)g(t, S(\lfloor t \rfloor))}, \quad \pi_\xi(t) := M(t) - \pi_S(t)\widetilde{S}(t), \quad t \in [0, T]. \quad (27)$$

The following result gives a Black-Scholes-type formula for the value of a European option on the stock at any time prior to maturity.

**Theorem 6** *Assume the conditions of Theorem 5. Let  $V(t)$  be the fair price of a European call option written on the stock  $S$  with exercise price  $K$  and maturity time  $T$ . Then for all  $t \in [T - \lfloor T \rfloor, T]$ ,  $V(t)$  is given by*

$$V(t) = S(t)\varphi(\beta_+(t)) - K\varphi(\beta_-(t))e^{-\int_t^T \lambda(s) ds}, \quad (28)$$

where

$$\beta_\pm(t) := \frac{\log \frac{S(t)}{K} + \int_t^T \left( \lambda(u) \pm \frac{1}{2}g(u, S(\lfloor u \rfloor))^2 \right) du}{\sqrt{\int_t^T g(u, S(\lfloor u \rfloor))^2 du}}.$$

If  $T > h$  and  $t < T - \lfloor T \rfloor$ , then

$$V(t) = e^{\int_0^t \lambda(s) ds} E_Q \left( H \left( \tilde{S}(T - \lfloor T \rfloor), -\frac{1}{2} \int_{T - \lfloor T \rfloor}^T g(u, S(\lfloor u \rfloor))^2 du, \int_{T - \lfloor T \rfloor}^T g(u, S(\lfloor u \rfloor))^2 du \right) \middle| \mathcal{F}_t \right) \quad (29)$$

where  $H$  is given by

$$H(x, m, \sigma^2) := x e^{m + \sigma^2/2} \varphi(\alpha_1(x, m, \sigma)) - K \varphi(\alpha_2(x, m, \sigma)) e^{-\int_0^T \lambda(s) ds},$$

and

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + \int_0^T \lambda(s) ds + m + \sigma^2 \right],$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + \int_0^T \lambda(s) ds + m \right],$$

for  $\sigma, x \in \mathbf{R}^+$ ,  $m \in \mathbf{R}$ .

The hedging strategy is given by

$$\pi_S(t) = \varphi(\beta_+(t)), \quad \pi_\xi(t) = -K \varphi(\beta_-(t)) e^{-\int_0^T \lambda(s) ds}, \quad t \in [T - \lfloor T \rfloor, T].$$

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