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# Limit theorems for the weights and the degrees in an $N$ -interactions random graph model

**Abstract:** A random graph evolution based on interactions of  $N$  vertices is studied. During the evolution both the preferential attachment rule and the uniform choice of vertices are allowed. The weight of an  $M$ -clique means the number of its interactions. The asymptotic behaviour of the weight of a fixed  $M$ -clique is studied. Asymptotic theorems for the weight and the degree of a fixed vertex are also presented. Moreover, the limits of the maximal weight and the maximal degree are described. The proofs are based on martingale methods.

**Keywords:** Random graph, Preferential attachment, Scale-free, Power law, Submartingale

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## 1 Introduction

Network theory is currently one of the most popular research topics. Random graphs are used to describe real-life networks. For overviews of random graph models and their properties see [1–3]. It is known that many real-life networks (e.g. the WWW, biological and social networks) are scale-free, that is their asymptotic degree distributions follow power laws. To describe the evolution of such networks, in [4], the preferential attachment model was suggested. However, in [4], the description of the evolution of the graph was informal. A rigorous definition of the preferential attachment model was given in [5], where a mathematical proof of the power law degree distribution was presented for  $d \leq n^{1/15}$  ( $d$  is the degree,  $n$  is the number of steps). For the recent development of the topic see [3,6].

Besides the degree distribution, other characteristics are also worth studying. The degree of a fixed vertex and the maximal degree in some preferential attachment models were investigated in [3,7,8]. In [9,10] the degree of a given vertex and the maximal degree were studied in a 2-parameter scale-free random graph model. A well-known technique to analyse the growth of the maximal degree is the martingale method (see [1,3,11]).

There are several versions of the preferential attachment model (see [3,12]). In [12] a general graph evolution procedure was introduced. In that procedure both the preferential attachment rule and the uniform choice of vertices are allowed, moreover, new links can be created between old vertices. The evolution method introduced in [13] in some sense resembles the one in [12]. However, the main feature of the model applied in [13] is the interaction of three vertices. Power law degree distribution in the

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three-interactions model was proved in [13,14]. The asymptotic behaviour of the weight and the degree of a fixed vertex, as well as the limits of the maximal weight and the maximal degree, were also described in [13,14]. Scale-free weight distributions both for the edges and the triangles in the three-interactions model were obtained in [15]. Instead of the three-interactions model, an evolution rule based on interactions of  $N$  vertices ( $N \geq 3$  fixed) was studied in [16].

Throughout this paper we shall study the following  $N$ -interactions model (defined in [16]). Here  $N \geq 3$  is a fixed integer. Usually, a complete graph with  $M$  vertices is called an  $M$ -clique. However, in this paper, we shall consider only that complete graph to be a clique, which is constructed by interactions of its vertices. We denote an  $M$ -clique by the symbol  $K_M$ . At each step the evolution of the graph is based on the interaction of  $N$  vertices. More precisely, at each step  $n = 1, 2, \dots$  we consider  $N$  vertices which will interact during that step. It means that we draw all non-existing edges between those vertices. So we obtain a clique  $K_N$ . Its complete subgraphs are also considered to be cliques. The weight of  $K_N$  and the weights of all sub-cliques in  $K_N$  are increased by 1. (That is we increase the weights of  $N$  vertices,  $\binom{N}{2}$  edges,  $\dots$ ,  $N$  different  $(N-1)$ -cliques and the  $N$ -clique  $K_N$  itself.)

The details of the evolution are the following. At time  $n = 0$  we start with  $N$  vertices, they interact, so they form an  $N$ -clique. Let the initial weight of this graph and the initial weights of its sub-cliques be one. After the initial step we start to increase the size of the graph. We select  $N$  vertices to interact. For the selection there are two possibilities at each step. On the one hand, with probability  $p$ , we add a new vertex that interacts with  $N-1$  old vertices (Step NEW). On the other hand, with probability  $(1-p)$ , we do not add any new vertex, but  $N$  old vertices interact (Step OLD). Here  $0 < p \leq 1$  is fixed.

Step NEW: When we add a new vertex, then we choose  $N-1$  old vertices. The new vertex and the  $N-1$  old vertices interact, so they together form an  $N$ -clique. However, to choose the  $N-1$  old vertices we have two possibilities: Choice PA and Choice UNI.

Choice PA: With probability  $r$  we choose an  $(N-1)$ -clique from the existing  $(N-1)$ -cliques according to the weights of the  $(N-1)$ -cliques. It means that an  $(N-1)$ -clique of weight  $w_t$  is chosen with probability  $w_t / \sum_h w_h$  (preferential attachment rule).

Choice UNI: On the other hand, with probability  $1-r$ , we choose  $N-1$  out of the existing vertices uniformly, that is, all groups of  $N-1$  vertices have the same chance to be chosen (uniform choice).

Step OLD: At a step when we do not add a new vertex, then  $N$  old vertices interact. As in the previous case, we have two options to choose the  $N$  old vertices: Choice PA and Choice UNI.

Choice PA: On the one hand, with probability  $q$ , we choose one  $N$ -clique  $K_N$  out of the existing  $N$ -cliques according to their weights. It means that the probability that we choose  $K_N$  is proportional to its weight (preferential attachment rule).

Choice UNI: On the other hand, with probability  $1-q$ , we choose from the existing vertices uniformly, that is all subsets consisting of  $N$  vertices have the same chance (uniform choice).

Power law degree and weight distributions for vertices in the general  $N$ -interactions model were obtained in [16,17]. The asymptotic behaviour of the weights of the  $N$ -cliques was examined and power law weight distribution for the  $N$ -cliques was obtained in [15].

In this paper we shall study the asymptotic behaviour of the weights and the degrees. Moreover, we shall also consider the limiting properties of the maximal weight and the maximal degree of vertices. In our proofs we follow some ideas of [13,14]. However, the combinatorial problems for general  $N$  are much more difficult than for  $N = 3$ .

The main results of the paper are the following. In Theorem 2.1, we describe the asymptotic behaviour of the weight of a fixed  $M$ -clique ( $1 \leq M \leq N$  fixed). The limit of the weight of a fixed vertex follows as a particular case with  $M = 1$  (Corollary 2.1). The asymptotic behaviour of the degree of a fixed vertex is also described (Theorem 2.2). Moreover, we find the limits of the maximal weight and the maximal degree (Theorems 2.3 and 2.4). Corollary 2.1, Theorems 2.2, 2.3 and 2.4 are extensions of the results in [13,14], where the 3-interactions model was studied.

The general structure of our results is  $\lim_{n \rightarrow \infty} n^{-\gamma} X_n = \nu$  almost surely, where  $\nu$  is a positive random variable. Therefore our results are in line with the corresponding results of [3,9–11]. The theorems are listed in Section 2. All the proofs and some auxiliary results are presented in Section 3.

## 2 Main results

Let  $1 \leq M \leq N$  be a fixed integer. We introduce the following notation.

$$\begin{aligned}\alpha_1 &= (1-p)q, & \alpha_2 &= \frac{N-1}{N}pr, & \alpha &= \alpha_1 + \alpha_2, \\ \beta_1 &= (N-1)(1-r), & \beta_2 &= \frac{N(1-p)(1-q)}{p}, & \beta &= \beta_1 + \beta_2, \\ \alpha' &= \alpha_1 + \frac{N-M}{N-1}\alpha_2, & \beta' &= \frac{N-M}{N-1}\beta_1 + \beta_2.\end{aligned}$$

We see that when  $M = 1$ , then  $\alpha' = \alpha$  and  $\beta' = \beta$ . First we list some results concerning the scale-free property.

**Remark 2.1.** *The scale-free property of our model means the following (see [16]). Let  $N \geq 3$  be fixed. Let  $X(n, w)$  denote the number of vertices of weight  $w$  after  $n$  steps. Let  $0 < p < 1$ ,  $q > 0$ ,  $r > 0$  and  $(1-r)(1-q) > 0$ . Then for all  $w = 1, 2, \dots$  we have*

$$\frac{X(n, w)}{n} \rightarrow x_w \quad (1)$$

almost surely, as  $n \rightarrow \infty$ , where  $x_w$ ,  $w = 1, 2, \dots$ , are positive numbers satisfying

$$x_w \sim C_0 w^{-(1+\frac{1}{\alpha})} \quad (2)$$

as  $w \rightarrow \infty$ , with  $C_0 = p\Gamma(1 + \frac{\beta+1}{\alpha}) / (\alpha\Gamma(1 + \frac{\beta}{\alpha}))$ . Let  $U(n, d)$  denote the number of vertices of degree  $d$  after  $n$  steps. Then, for any  $d \geq N-1$  we have

$$\frac{U(n, d)}{n} \rightarrow u_d \quad (3)$$

a.s. as  $n \rightarrow \infty$ , where  $u_d$ ,  $d = N-1, N, \dots$ , are positive numbers with

$$u_d \sim C_1 d^{-(1+\frac{1}{\alpha})} \quad (4)$$

as  $d \rightarrow \infty$ , where  $C_1 = p \left(\frac{\alpha}{\alpha_2}\right)^{-(1+\frac{1}{\alpha})} \Gamma(1 + \frac{\beta+1}{\alpha}) / (\alpha_2\Gamma(1 + \frac{\beta}{\alpha}))$ .

**Remark 2.2.** *In our model, besides the vertices, the cliques also have weights. It turns out that the weight distribution of the  $M$ -cliques is also power law. Let  $N \geq 3$  be fixed and let  $M$  be fixed with  $1 < M \leq N$  and denote by  $X_M(n, w)$  the number of  $M$ -cliques having weight  $w$  after  $n$  steps. If  $p > 0$  and either  $r > 0$  or  $(1-p)q > 0$ , then*

$$\frac{X_M(n, w)}{n} \rightarrow x_{M,w} \quad (5)$$

almost surely, as  $n \rightarrow \infty$ , where  $x_{M,w}$ ,  $w = 1, 2, \dots$ , are numbers satisfying

$$x_{M,w} \sim \frac{\mu}{\alpha'} \Gamma\left(1 + \frac{1}{\alpha'}\right) w^{-(1+\frac{1}{\alpha'})} \quad (6)$$

as  $w \rightarrow \infty$ , with  $\mu = p\binom{N-1}{M-1} + p(1-r)\binom{N-1}{M} + (1-p)(1-q)\binom{N}{M}$ . This result was presented in [15] for the case of  $M = 2$ ,  $N = 3$  and also for the case of  $M = N$  for arbitrary  $N > 2$ . The general case can be obtained by using the ideas of [18]. To this end one has to consider a slight modification of the general model of [18] and to prove appropriate versions of Theorems 2 and 3 in [18].

Now we turn to the asymptotic behaviour of a fixed clique and that of a fixed vertex. At time  $n = 0$ , the initial complete graph on  $N$  vertices contains  $\binom{N}{M}$   $M$ -cliques labelled by  $0, -1, \dots, -\left(\binom{N}{M} - 1\right)$ . When new  $M$ -cliques are born, they are labelled by  $1, 2, \dots$ . Let  $j \geq 0$  be a fixed integer. Let us denote by

$W[n, M, j]$  the weight of the  $j$ th  $M$ -clique after the  $n$ th step. Let us denote by  $D[n, j]$  the degree of the  $j$ th vertex after the  $n$ th step. (If  $n < j$ , then  $W[n, 1, j] = D[n, j] = 0$ .)

First, we study the weight of a fixed  $M$ -clique. At time  $n = 0$ , the initial complete graph on  $N$  vertices is symmetric. Therefore and by the evolution mechanism of our graph, it is enough to describe  $W[n, M, j]$  and  $D[n, j]$  for  $j = 0, 1, 2, \dots$ . The following theorem describes the asymptotic behaviour of the weight of a fixed  $M$ -clique.

**Theorem 2.1.** *Let  $j \geq 0$  and  $1 \leq M \leq N$  be fixed. Assume that  $\alpha > 0$ . Then*

$$W[n, M, j] \sim \frac{1}{\Gamma(1 + \alpha')} \gamma_{M,j} n^{\alpha'} \quad (7)$$

almost surely as  $n \rightarrow \infty$ , where  $\gamma_{M,j}$  is a positive random variable.

As a particular case with  $M = 1$ , the asymptotic behaviour of the weight of a fixed vertex is the following.

**Corollary 2.1.** *Let  $j \geq 0$  be fixed and let  $\alpha > 0$ . Then*

$$W[n, 1, j] \sim \frac{1}{\Gamma(1 + \alpha)} \gamma_{1,j} n^{\alpha} \quad (8)$$

almost surely as  $n \rightarrow \infty$ , where  $\gamma_{1,j}$  is a positive random variable.

We turn to the limit of the degree sequence of a fixed vertex.

**Theorem 2.2.** *Let  $j \geq 0$  be fixed and let  $\alpha > 0$ . Then*

$$D[n, j] \sim \frac{1}{\Gamma(1 + \alpha)} \frac{\alpha_2}{\alpha} \gamma_{1,j} n^{\alpha} \quad (9)$$

almost surely as  $n \rightarrow \infty$ , where the positive random variable  $\gamma_{1,j}$  is given in (8).

Now, we turn to the maximal weight and the maximal degree. Let us denote by  $\mathcal{W}_n$  the maximum of the weights of the vertices after  $n$  steps, that is

$$\mathcal{W}_n = \max\{W[n, 1, j] : -(N - 1) \leq j \leq n\}. \quad (10)$$

**Theorem 2.3.** *Let  $\alpha > 0$ . Then*

$$\mathcal{W}_n \sim \frac{1}{\Gamma(1 + \alpha)} \mu n^{\alpha} \quad \text{almost surely as } n \rightarrow \infty, \quad (11)$$

where  $\mu = \sup\{\gamma_{1,j} : j \geq -(N - 1)\}$  is a finite positive random variable with  $\gamma_{1,j}$  defined in (8).

Let us denote by  $\mathcal{D}_n$  the maximal degree after  $n$  steps, that is

$$\mathcal{D}_n = \max\{D[n, j] : -(N - 1) \leq j \leq n\}. \quad (12)$$

**Theorem 2.4.** *Let  $\alpha > 0$ . Then*

$$\mathcal{D}_n \sim \frac{1}{\Gamma(1 + \alpha)} \frac{\alpha_2}{\alpha} \mu n^{\alpha} \quad \text{almost surely as } n \rightarrow \infty, \quad (13)$$

where  $\mu = \sup\{\gamma_{1,j} : j \geq -(N - 1)\}$  is the positive random variable defined in Theorem 2.3.

Corollary 2.1 and Theorems 2.2, 2.3 and 2.4 are extensions of *Theorem 4.1* in [13] and *Theorems 5.2, 5.1, 5.3* in [14], respectively.

**Remark 2.3.** We see that the parameters  $\alpha_1$  and  $\alpha_2$  belong to the preferential attachment part of the model, while  $\beta_1$  and  $\beta_2$  are connected to the uniform choice part. Moreover, in the above Theorems 2.1-2.4 only  $\alpha_1$  and  $\alpha_2$  play role. Therefore the asymptotic behaviour in those theorems is not influenced by the uniform choice parameters. This phenomenon can be explained as follows. If  $j$  is fixed and  $n$  is large, then the  $j$ th vertex is 'old' among the  $V_n \sim np$  vertices. Therefore the degree and the weight of the  $j$ th vertex are relatively high compared to those of the 'young' vertices. As there are lot of 'young' vertices, therefore the uniform choice has minor influence on the degree and the weight of the  $j$ th vertex.

**Remark 2.4.** If we compare Corollary 2.1 and Theorem 2.2, we see that the asymptotic ratio of the weight and the degree of vertex  $j$  is

$$\frac{W[n, 1, j]}{D[n, j]} \sim \frac{\alpha}{\alpha_2} = \frac{(N-1)\mathbf{P}(\text{Step NEW and Choice PA}) + N\mathbf{P}(\text{Step OLD and Choice PA})}{(N-1)\mathbf{P}(\text{Step NEW and Choice PA})}. \quad (14)$$

And the last expression in (14) is nothing else but the ratio of the expected growth of the weights of the 'old' vertices to the expected growth of the degrees of the 'old' vertices during one step when the choice is PA. Similar observation is true for the asymptotic ratio of the maximal weight to the maximal degree. That is, by Theorems 2.3 and 2.4, we have  $\mathcal{W}_n/\mathcal{D}_n \sim \alpha/\alpha_2$ .

**Remark 2.5.** Let  $\tau_n$  denote the maximum of the labels of those vertices where the maximal weight is attained, that is let

$$\tau_n = \max\{j : W[n, 1, j] = \mathcal{W}_n\}.$$

Then the sequence  $\tau_n(\omega)$ ,  $n = 1, 2, \dots$  is bounded for almost all fixed elementary events  $\omega$ . It is a simple consequence of Theorem 2.3 and its proof. A similar statement is valid for the degrees.

### 3 Proofs and auxiliary lemmas

Introduce the following notation. Let  $\mathcal{F}_{n-1}$  denote the  $\sigma$ -algebra of observable events after the  $(n-1)$ th step. Let  $V_n$  denote the number of vertices after the  $n$ th step. Let  $j \geq 0$  and  $1 \leq M \leq N$  be fixed integers. Let  $I[n, M, j]$  be the indicator of the event that the  $j$ th  $M$ -clique exists after  $n$  steps, that is

$$I[n, M, j] = \begin{cases} 1, & \text{if } W[n, M, j] > 0, \\ 0, & \text{if } W[n, M, j] = 0. \end{cases}$$

Let  $J[n, M, j]$  be the indicator of the event that the  $j$ th  $M$ -clique is born at the  $n$ th step. Then  $J[n, M, j] = I[n, M, j] - I[n-1, M, j]$ . For all fixed positive integers  $j, k, l, M$ ,  $0 \leq j \leq l$ ,  $1 \leq k$ ,  $1 \leq M \leq N$ , we consider the following sequences:

$$b[n, M, k] = \prod_{i=1}^n \left(1 + \frac{\alpha'k}{i}\right)^{-1}, \quad (15)$$

$$d[n, M, k, j] = - \sum_{i=1}^{n-1} b[i+1, M, k] \frac{\binom{N}{M}}{N \binom{V_i}{M}} p\beta' \binom{W[i, M, j] + k - 1}{k-1}, \quad (16)$$

$$e_{n, M} = \prod_{i=1}^n \left(1 - \frac{\alpha'}{i}\right)^{-1}. \quad (17)$$

We can see that  $b[n, M, k]$  and  $e_{n, M}$  are deterministic, while  $d[n, M, k, j]$  is an  $\mathcal{F}_{n-1}$ -measurable random variable for any  $n, M, k$  and  $j$ . Using the definition of  $b[n, M, k]$  and the Stirling-formula for the Gamma function, we can show that

$$b[n, M, k] \sim b_{M, k} n^{-k\alpha'} \quad \text{as } n \rightarrow \infty, \quad (18)$$

where  $b_{M,k} = \Gamma(1 + \alpha'k) > 0$ ,  $k$  and  $M$  are fixed. Moreover, we can easily see that

$$e_{n,M} \sim \Gamma(1 - \alpha') n^{\alpha'} \quad \text{as } n \rightarrow \infty. \quad (19)$$

In the following lemma we introduce a martingale which will play an important role in the paper.

**Lemma 3.1.** *Let  $j, k, l, M, 0 \leq j \leq l, 1 \leq M \leq N$  be fixed nonnegative integers and let*

$$Z[n, M, k, j] = \left( b[n, M, k] \binom{W[n, M, j] + k - 1}{k} + d[n, M, k, j] \right) I[l, M, j]. \quad (20)$$

Then  $(Z[n, M, k, j], \mathcal{F}_n)$  is a martingale for  $n \geq l$ .

*Proof.* At each step, the weight of a fixed  $M$ -clique is increased by 1 if and only if it takes part in an interaction. The total weight of  $N$ -cliques after  $n$  steps is  $n + 1$ . The total weight of  $(N - 1)$ -cliques after  $n$  steps is  $N(n + 1)$ . When a new vertex is born and we choose  $N - 1$  vertices uniformly, the probability that the vertices of a given  $M$ -clique are selected is

$$\frac{\binom{V_n - M}{N - 1 - M}}{\binom{V_n}{N - 1}} = \frac{\binom{N - 1}{M}}{\binom{V_n}{M}}.$$

When we choose  $N$  vertices uniformly at random, the probability that the vertices of a given  $M$ -clique are chosen is

$$\frac{\binom{V_n - M}{N - M}}{\binom{V_n}{N}} = \frac{\binom{N}{M}}{\binom{V_n}{M}}.$$

Therefore, as in [15], it is easy to show that the probability that the  $j$ th  $M$ -clique takes part in interaction at step  $(n + 1)$  is

$$\begin{aligned} & pr \frac{\binom{N - M}{N} W[n, M, j]}{N(n + 1)} + p(1 - r) \frac{\binom{N - 1}{M}}{\binom{V_n}{M}} + (1 - p)q \frac{W[n, M, j]}{n + 1} + (1 - p)(1 - q) \frac{\binom{N}{M}}{\binom{V_n}{M}} = \\ & = \frac{W[n, M, j]}{n + 1} \alpha' + \frac{\binom{N}{M}}{N \binom{V_n}{M}} p\beta', \end{aligned} \quad (21)$$

provided that the  $j$ th  $M$ -clique exists at the  $l$ th step. Using this fact, we can see for  $n \geq l$

$$\begin{aligned} & \mathbb{E} \left\{ \binom{W[n + 1, M, j] + k - 1}{k} I[l, M, j] | \mathcal{F}_n \right\} = \\ & = I[l, M, j] \left( 1 - \left( \frac{W[n, M, j]}{n + 1} \alpha' + \frac{\binom{N}{M}}{N \binom{V_n}{M}} p\beta' \right) \right) \binom{W[n, M, j] + k - 1}{k} + \\ & \quad + I[l, M, j] \left( \frac{W[n, M, j]}{n + 1} \alpha' + \frac{\binom{N}{M}}{N \binom{V_n}{M}} p\beta' \right) \binom{W[n, M, j] + k}{k} = \\ & = I[l, M, j] \frac{\binom{N}{M}}{N \binom{V_n}{M}} p\beta' \binom{W[n, M, j] + k - 1}{k - 1} + I[l, M, j] \left( 1 + \alpha' \frac{k}{n + 1} \right) \binom{W[n, M, j] + k - 1}{k}. \end{aligned}$$

Multiplying both sides by  $b[n + 1, M, k]$ , we see that

$$\begin{aligned} & \mathbb{E} \left\{ b[n + 1, M, k] \binom{W[n + 1, M, j] + k - 1}{k} I[l, M, j] | \mathcal{F}_n \right\} = \\ & = I[l, M, j] \left( \binom{W[n, M, j] + k - 1}{k} b[n, M, k] + d[n, M, k, j] - d[n + 1, M, k, j] \right). \end{aligned} \quad (22)$$

Using that  $d[n + 1, M, k, j]$  is  $\mathcal{F}_n$ -measurable, we obtain the desired result.  $\square$

**Lemma 3.2.** *The following sequence is a nonnegative supermartingale*

$$\left( \frac{e_{n,M} I[k, M, j]}{W[n, M, j] - 1}, \mathcal{F}_n \right), \quad n = j, j+1, \dots \quad (23)$$

*Proof.* In a similar way as in the proof of Lemma 3.1, we have for  $n \geq k$

$$\begin{aligned} & \mathbb{E} \left\{ \frac{I[k, M, j]}{W[n+1, M, j] - 1} \middle| \mathcal{F}_n \right\} = \\ &= \left( \frac{W[n, M, j]}{n+1} \alpha' + \frac{\binom{N}{M}}{N \binom{V_n}{M}} p \beta' \right) \frac{I[k, M, j]}{W[n, M, j]} + \left( 1 - \left( \frac{W[n, M, j]}{n+1} \alpha' + \frac{\binom{N}{M}}{N \binom{V_n}{M}} p \beta' \right) \right) \frac{I[k, M, j]}{W[n, M, j] - 1} = \\ &= \left( \frac{W[n, M, j]}{n+1} \alpha' + \frac{\binom{N}{M}}{N \binom{V_n}{M}} p \beta' \right) \left( \frac{I[k, M, j]}{W[n, M, j]} - \frac{I[k, M, j]}{W[n, M, j] - 1} \right) + \frac{I[k, M, j]}{W[n, M, j] - 1} \leq \\ &\leq - \frac{\alpha' I[k, M, j]}{(n+1)(W[n, M, j] - 1)} + \frac{I[k, M, j]}{W[n, M, j] - 1} = \frac{I[k, M, j]}{W[n, M, j] - 1} \left( 1 - \frac{\alpha'}{n+1} \right). \end{aligned} \quad (24)$$

Multiplying both sides of (24) by  $e_{n+1,M}$ , we obtain the result.  $\square$

*Proof of Theorem 2.1.* The proof contains two parts. First, we will show that the result is valid with non-negative  $\gamma_{M,j}$ . Then we will show that  $\gamma_{M,j}$  is positive with probability 1.

Let  $B_{n+1} = \{W[n+1, M, j] = W[n, M, j] + 1\}$ . Consider the event that the  $j$ th  $M$ -clique exists after  $n$  steps. On this event, by (21),

$$\mathbb{P}(B_{n+1} | \mathcal{F}_n) \geq \frac{\alpha'}{n+1}. \quad (25)$$

The sequence  $(B_n, n \in \mathbb{N})$  is adapted to the sequence of  $\sigma$ -algebras  $(\mathcal{F}_n, n \in \mathbb{N})$ . Using *Corollary VII-2-6* of [19] and (25), we have

$$W[n, M, j] \rightarrow \infty \quad \text{a.s. as } n \rightarrow \infty. \quad (26)$$

Consider the martingale  $(Z[n, M, k, j], \mathcal{F}_n)$  introduced in Lemma 3.1 and let  $k = 1$ . Then

$$Z[n, M, 1, j] = (b[n, M, 1]W[n, M, j] + d[n, M, 1, j]) I[l, M, j]. \quad (27)$$

By the Marcinkiewicz strong law of large numbers, we have

$$V_n = pn + o\left(n^{1/2+\varepsilon}\right) \quad (28)$$

almost surely, for any  $\varepsilon > 0$ . By this fact and (18), we obtain that

$$d[n, M, 1, j] = - \sum_{i=1}^{n-1} b[i+1, M, 1] \frac{\binom{N}{M}}{N \binom{V_i}{M}} p \beta' \sim - \frac{1}{p^{M-1}} \binom{N}{M} \frac{M!}{N} \beta' \Gamma(1 + \alpha') \sum_{i=1}^{n-1} i^{-(\alpha'+M)} (1 + o(1)).$$

Using that  $\alpha' > 0$  and  $M > 0$ , we see that  $d[n, M, 1, j]$  converges as  $n \rightarrow \infty$ . Therefore the martingale  $Z[n, M, 1, j]$  is bounded from below. Now, we shall see that the martingale  $Z[n, M, 1, j]$  has bounded differences. The sequence  $b[n, M, 1]$  is decreasing, hence

$$Z[n+1, M, 1, j] - Z[n, M, 1, j] \leq b[n, M, 1] (W[n+1, M, j] - W[n, M, j]) \leq 1.$$

It is also easy to compute that

$$\begin{aligned} Z[n, M, 1, j] - Z[n+1, M, 1, j] &\leq (b[n, M, 1] - b[n+1, M, 1]) W[n, M, j] + (d[n, M, 1, j] - d[n+1, M, 1, j]) = \\ &= (b[n, M, 1] - b[n+1, M, 1]) W[n, M, j] + b[n+1, M, 1] \frac{\binom{N}{M}}{N \binom{V_n}{M}} p \beta' \leq \end{aligned}$$

$$\leq b[n+1, M, 1] \left( \alpha' + \frac{1}{N} p \beta' \right) \leq \alpha' + p \frac{\beta'}{N}.$$

So the martingale  $Z[n, M, 1, j]$  is bounded from below and it has bounded differences. Therefore, by *Proposition VII-3-9* of [19], it is convergent almost surely as  $n \rightarrow \infty$ . By the definition of  $Z[n, M, 1, j]$ , we see that  $b[n, M, 1]W[n, M, j]$  also converges almost surely on the event  $\{W[l, M, j] > 0\}$ . This fact, (26) and (18) imply that (8) is true with non-negative  $\gamma_{M,j}$ .

Now we will show that  $\gamma_{M,j}$  is positive with probability 1. Consider the supermartingale  $\left( \frac{e_{n,M} I[k, M, j]}{W[n, M, j]^{-1}}, \mathcal{F}_n \right)$ ,  $n \geq j$ , in Lemma 3.2. This supermartingale is nonnegative therefore, according to the submartingale convergence theorem, it converges almost surely.  $\lim_{l \rightarrow \infty} I[l, M, j] = 1$  almost surely, hence  $\frac{e_{n,M}}{W[n, M, j]^{-1}}$  also converges almost surely as  $n \rightarrow \infty$ . This and (19) imply that  $\gamma_{M,j}$  is positive almost surely.  $\square$

**Remark 3.1.** *Contrary to the weight, the degree of a fixed vertex can grow by  $0, 1, \dots, N-1$  at each step. Hence  $0 \leq D[n, j] - D[n-1, j] \leq N-1$  for all fixed  $j \geq 0$ . Moreover, the degree of a fixed vertex does not change at steps when we do not add a new vertex and the choice is done according to the preferential attachment rule.*

**Lemma 3.3.** *Let  $j \geq 0$  be fixed. For  $n \geq k$ , we have*

$$I[k, 1, j] \left( D[n, j] + \alpha_2 \frac{W[n, 1, j]}{n+1} \right) \leq \mathbf{E}\{I[k, 1, j]D[n+1, j] | \mathcal{F}_n\} = I[k, 1, j] \left( D[n, j] + \alpha_2 \frac{W[n, 1, j]}{n+1} + R_n \right),$$

where  $0 \leq R_n \leq (N-1) \frac{p\beta}{V_n}$ .

*Proof.* Consider the conditional expectation  $\mathbf{E}\{D[n+1, j] - D[n, j] | \mathcal{F}_n\}$  provided that the  $j$ th vertex with weight  $W[n, 1, j]$  and degree  $D[n, j]$  exists after  $k$  steps.

At each step when the model evolves according to the preferential attachment rule the degree of a fixed vertex can be increased by 0 or 1. Therefore, the expected growth of the degree of the  $j$ th vertex in the  $(n+1)$ th step when the choice is PA is  $\alpha_2 \frac{W[n, 1, j]}{n+1}$ . At steps when the choice is UNI the degree of a fixed vertex can be increased at most  $N-1$ . Moreover, using (21), we have the probability that the growth of the degree of the  $j$ th vertex is positive when the choice is UNI is bounded above by  $\frac{p\beta}{V_n}$ . Therefore, the expected growth of the degree of the  $j$ th vertex in the  $(n+1)$ th step when the choice is UNI is less than or equal to  $(N-1) \frac{p\beta}{V_n}$ .  $\square$

*Proof of Theorem 2.2.* Consider the following bounded random variable:  $\xi_n = \frac{I[k, 1, j]}{N-1} (D[n, j] - D[n-1, j])$ . By the Remark 3.1, we have  $0 \leq \xi_n \leq 1$ . Applying an appropriate version of *Corollary VII-2-6* of [19] (see *Proposition 2.4* of [20]), then using Lemma 3.3 and (8), we have

$$\begin{aligned} D[n, j] &= (N-1) \sum_{i=1}^n \xi_i \sim (N-1) \sum_{i=1}^n \mathbf{E}(\xi_i | \mathcal{F}_{i-1}) = \sum_{i=1}^n \left( \alpha_2 \frac{W[i-1, 1, j]}{i} + R_{i-1} \right) \sim \\ &\sim \frac{1}{\Gamma(1+\alpha)} \frac{\alpha_2}{\alpha} \gamma_{1,j} n^\alpha, \end{aligned} \quad (29)$$

provided that the  $j$ th vertex exists after  $k$  steps. As  $\lim_{k \rightarrow \infty} W[k, 1, j] = \infty$  a.s., we obtain the statement.  $\square$

The following lemma will be used to study the maximal weight. It is an extension of *Lemma 5.2* in [14].

**Lemma 3.4.** *For all fixed nonnegative integers  $k \geq 0$ ,  $1 \leq m \leq n$ , let*

$$S[m, n, k] = \sum_{j=m}^n \mathbf{E} \left( b[n, 1, k] \binom{W[n, 1, j] + k - 1}{k} I[n, 1, j] \right). \quad (30)$$

*Then there exists a positive constant  $C_k$  such that*

$$S[m, n, k] \leq C_k \sum_{j=m}^n j^{-\alpha k}. \quad (31)$$



*Proof.* We use induction on  $k$ . Let  $k = 0$ . Then

$$S[m, n, 0] = \sum_{j=m}^n \mathbb{E}(b[n, 1, 0]I[n, 1, j]) = \sum_{j=m}^n \mathbb{P}(W[n, 1, j] > 0) \leq n - m + 1.$$

Suppose that the statement is true for  $k - 1$ . By Lemma 3.1,  $Z[n, 1, k, j]$  is a martingale. The difference of two martingales is also a martingale. So, in the definition of  $Z[n, 1, k, j]$  changing  $I[l, 1, j]$  for  $J[l, 1, j]$ , we obtain again a martingale. Using the definitions of  $J[n, 1, j]$  and  $Z[n, 1, k, j]$ , we have

$$\begin{aligned} S[m, n, k] &= \sum_{j=m}^n \mathbb{E} \left( \sum_{l=j}^n (Z[l, 1, k, j] - d[n, 1, k, j]) J[l, 1, j] \right) = \\ &= \mathbb{E} \left( \sum_{j=m}^n \sum_{l=j}^n b[l, 1, k] J[l, 1, j] \right) + \mathbb{E} \left( \sum_{j=m}^n \sum_{l=j}^n (d[l, 1, k, j] - d[n, 1, k, j]) J[l, 1, j] \right). \end{aligned} \quad (32)$$

In the last step we used that  $W(l, 1, j) = 1$  if  $J[l, 1, j] = 1$ . Now, we give upper bounds for the two terms in (32) separately. We have already seen that, for a fixed  $k$ , the sequence  $b[n, 1, k]$  is decreasing. Therefore, applying also (18),

$$\mathbb{E} \left( \sum_{j=m}^n \sum_{l=j}^n b[l, 1, k] J[l, 1, j] \right) \leq \sum_{j=m}^n b[j, 1, k] \mathbb{E} \left( \sum_{l=j}^n J[l, 1, j] \right) \leq C_k^{(1)} \sum_{j=m}^n j^{-\alpha k}. \quad (33)$$

For the second term in (32), changing the order of summations and using that  $I[i, 1, j] = \sum_{l=j}^i J[l, 1, j]$ , we have

$$\begin{aligned} &\mathbb{E} \left( \sum_{j=m}^n \sum_{l=j}^n (d[l, 1, k, j] - d[n, 1, k, j]) J[l, 1, j] \right) = \\ &= \mathbb{E} \left( \sum_{i=m}^{n-1} \frac{b[i+1, 1, k]}{b[i, 1, k-1]} \frac{p}{V_i} \beta \sum_{j=m}^i b[i, 1, k-1] \binom{W[i, 1, j] + k - 2}{k-1} \frac{W[i, 1, j] + k - 1}{W[i, 1, j]} I[i, 1, j] \right) \leq \\ &\leq k \sum_{i=m}^{n-1} \frac{b[i+1, 1, k]}{b[i, 1, k-1]} \mathbb{E} \left( \frac{p}{V_i} \beta \sum_{j=m}^i b[i, 1, k-1] \binom{W[i, 1, j] + k - 2}{k-1} I[i, 1, j] \right). \end{aligned} \quad (34)$$

In the last step we applied that  $\frac{W[i, 1, j] + k - 1}{W[i, 1, j]} \leq k$  if  $I[i, 1, j] > 0$ .

Now, we give upper bounds on the events  $\{V_i < \frac{pi}{2}\}$  and  $\{V_i \geq \frac{pi}{2}\}$  separately. Using the induction hypothesis, we have

$$\mathbb{E} \left( \frac{p}{V_i} \beta \mathbb{I}_{\{V_i \geq \frac{pi}{2}\}} \sum_{j=m}^i b[i, 1, k-1] \binom{W[i, 1, j] + k - 2}{k-1} I[i, 1, j] \right) \leq \frac{2\beta}{i} C_{k-1} \sum_{j=m}^i j^{-\alpha(k-1)}. \quad (35)$$

(Here  $\mathbb{I}_A$  is the indicator of the set  $A$ .) On the other hand, by (18),

$$\begin{aligned} &\mathbb{E} \left( \frac{p}{V_i} \beta \mathbb{I}_{\{V_i < \frac{pi}{2}\}} \sum_{j=m}^i b[i, 1, k-1] \binom{W[i, 1, j] + k - 2}{k-1} I[i, 1, j] \right) \leq \\ &\leq \frac{p}{N} \beta \mathbb{P} \left\{ V_i < \frac{pi}{2} \right\} \sum_{j=m}^i b[i, 1, k-1] \binom{i+k-2}{k-1} = o \left( \frac{1}{i} \sum_{j=m}^i j^{-\alpha(k-1)} \right) \end{aligned} \quad (36)$$

as  $i \rightarrow \infty$ . In the above computation we used Hoeffding's exponential inequality (*Theorem 2* in [21]) to obtain the following upper bound:  $\mathbb{P} \left\{ V_i < \frac{pi}{2} \right\} \leq e^{-\varepsilon i}$ , where  $\varepsilon = \frac{p^2}{2}$ . Therefore, by (34), (36) and (35), we have

$$\mathbb{E} \left( \sum_{j=m}^n \sum_{l=j}^n (d[l, 1, k, j] - d[n, 1, k, j]) J[l, 1, j] \right) \leq$$

$$\leq kC_k^{(2)'} \sum_{i=m}^{n-1} i^{-\alpha} \sum_{j=m}^i \frac{1}{i} j^{-\alpha(k-1)} \leq C_k^{(3)} \sum_{j=m}^n j^{-\alpha k}. \quad (37)$$

Above we applied that, by (18),  $\frac{b[i+1,1,k]}{b[i,1,k-1]} = O(i^{-\alpha})$  as  $i \rightarrow \infty$ . Finally, (32), (33) and (37) give the result.  $\square$

*Proof of Theorem 2.3.* Let  $M[m, n] = \max\{W[n, 1, j] : -(N-1) \leq j < m\}$ , where  $1 \leq m \leq n$  fixed. From Corollary 2.1, we have

$$\Gamma(1+\alpha) \lim_{n \rightarrow \infty} n^{-\alpha} M[m, n] = \max\{\gamma_{1,j} : -(N-1) \leq j < m\} \quad (38)$$

almost surely. By (22), the following process is a submartingale:

$$b[n, 1, k] \binom{W[n, 1, j] + k - 1}{k} = b[n, 1, k] \binom{W[n, 1, j] + k - 1}{k} I[n, 1, j], \quad n \geq j.$$

Let  $Q[m, n] = \max_{m \leq j \leq n} W[n, 1, j]$ . Then  $0 \leq \mathcal{W}_n - M[m, n] \leq Q[m, n]$ . The maximum of increasing numbers of submartingales is also a submartingale, so

$$b[n, 1, k] \binom{Q[m, n] + k - 1}{k}, \quad n \geq m,$$

is a submartingale. For non-negative numbers the maximum is majorized by the sum. Therefore, and by Lemma 3.4, we obtain

$$\mathbb{E} \left( b[n, 1, k] \binom{Q[m, n] + k - 1}{k} \right) \leq S[m, n, k] \leq C_k \sum_{j=m}^n j^{-\alpha k}. \quad (39)$$

Since

$$0 \leq (b[n, 1, 1]Q[m, n])^k \leq \frac{b[n, 1, 1]^k}{b[n, 1, k]^k} k! b[n, 1, k] \binom{Q[m, n] + k - 1}{k}, \quad (40)$$

we see that the submartingale  $b[n, 1, 1]Q[m, n]$  is bounded in  $L^k$  for all  $k\alpha > 1$ . Hence, this submartingale converges almost surely and in  $L^k$  for every  $k > \frac{1}{\alpha}$ . Moreover, by (18), (39) and (40), we have

$$\mathbb{E} \left( \limsup_{n \rightarrow \infty} (n^{-\alpha} Q[m, n]) \right)^k \leq k! C_k \frac{1}{\Gamma(1+\alpha k)} \sum_{j=m}^{\infty} j^{-\alpha k}. \quad (41)$$

Now, using the monotone convergence theorem, we have

$$\mathbb{E} \left( \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} (n^{-\alpha} Q[m, n])^k \right) = 0,$$

for  $k > \frac{1}{\alpha}$ . As  $Q[m, n]$  is decreasing as  $m$  increases, so

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{-\alpha} Q[m, n] = 0 \quad \text{a.s.} \quad (42)$$

Therefore, as  $0 \leq \mathcal{W}_n - M[m, n] \leq Q[m, n]$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} (n^{-\alpha} (\mathcal{W}_n - M[m, n])) = 0 \quad \text{a.s.}$$

This relation and (38) imply (11). By relation (41),  $\mu$  is a.s. finite.  $\square$

*Proof of Theorem 2.4.* The evolution mechanism of the graph implies that  $D[n, j] \leq (N - 1)W[n, 1, j]$ . Therefore we have

$$\begin{aligned} & \max\{D[n, j] : -(N - 1) \leq j < m\} \leq \mathcal{D}_n \leq \\ & \leq \max\{D[n, j] : -(N - 1) \leq j < m\} + \max\{(N - 1)W[n, 1, j] : m \leq j \leq n\}. \end{aligned}$$

Multiplying both sides by  $n^{-\alpha}$  and then considering the limit as  $n \rightarrow \infty$ , Theorem 2.2 implies

$$\begin{aligned} & \max\left\{\frac{1}{\Gamma(1 + \alpha)} \frac{\alpha_2}{\alpha} \gamma_{1,j} : -(N - 1) \leq j < m\right\} \leq \\ & \leq \liminf_{n \rightarrow \infty} \mathcal{D}_n n^{-\alpha} \leq \limsup_{n \rightarrow \infty} \mathcal{D}_n n^{-\alpha} \leq \\ & \leq \max\left\{\frac{1}{\Gamma(1 + \alpha)} \frac{\alpha_2}{\alpha} \gamma_{1,j} : -(N - 1) \leq j < m\right\} + (N - 1) \limsup_{n \rightarrow \infty} n^{-\alpha} Q[m, n] \end{aligned}$$

as  $n \rightarrow \infty$ . As  $m \rightarrow \infty$ , by (42), we obtain the desired result.  $\square$

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