



Finite dimensional varieties on hypergroups

LÁSZLÓ SZÉKELYHIDI AND ŻYWILLA FECHNER 

Dedicated to the memory of Professor János Aczél.

Abstract. Let X be a hypergroup, K its compact subhypergroup and assume that (X, K) is a Gelfand pair. Connections between finite dimensional varieties and K -polynomials on X are discussed. It is shown that a K -variety on X is finite dimensional if and only if it is spanned by finitely many K -monomials. Next, finite dimensional varieties on affine groups over \mathbb{R}^d , where d is a positive integer are discussed. A complete description of those varieties using partial differential equations is given.

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1. Introduction

In this paper we investigate finite dimensional varieties and their applications in hypergroup settings. This study is motivated by results on spectral analysis and synthesis on vector modules discussed in Chapter 11 of [6] and also results in [7]. In the vector module settings a variety is a closed vector submodule. Spectral analysis for a variety means that there are nonzero finite dimensional subvarieties in every nonzero variety. On the other hand, spectral synthesis means that there are sufficiently many nonzero finite dimensional varieties in every nonzero subvariety. In this paper we are going to investigate finite dimensional varieties invariant with respect to a compact subhypergroup K of a hypergroup X such that (X, K) is a Gelfand pair, which means that the measure algebra of K -invariant measures is commutative.

2. Finite dimensional varieties

The terminology in this paper is in accordance with the monograph [1]. Let $X = (X, *, \check{\cdot}, e)$ be a hypergroup. Let $\mathcal{C}(X)$ denote the locally convex topological

vector space of all continuous complex-valued functions defined on X equipped with pointwise linear operations and the topology of compact convergence. The dual of $\mathcal{C}(X)$ can be identified with $\mathcal{M}_c(X)$, the space of all compactly supported complex measures on X and the pairing between $\mathcal{C}(X)$ and $\mathcal{M}_c(X)$ is given by

$$\langle \mu, f \rangle = \int_X f d\mu$$

for each μ in $\mathcal{M}_c(X)$ and f in $\mathcal{C}(X)$. For any function $f : X \rightarrow \mathbb{C}$ we define $\check{f}(x) := f(\check{x})$ for each x in X .

Convolution on $\mathcal{M}_c(X)$ is given by

$$\langle \mu * \nu, f \rangle = \int_X \int_X f(x * y) d\mu(x) d\nu(y)$$

for any μ, ν in $\mathcal{M}_c(X)$ and f in $\mathcal{C}(X)$. The explanation and the detailed discussion of the proper interpretation of the notation $f(x * y)$ can be found in Chapter 1 of [5]. The space $\mathcal{M}_c(X)$ with convolution is a unital algebra with the unit δ_e , where e denotes the unit of the hypergroup X . In general, for an arbitrary x in X the symbol δ_x denotes the point mass with support $\{x\}$.

Convolution of measures from $\mathcal{M}_c(X)$ and functions from $\mathcal{C}(X)$ is defined by

$$\mu * f(x) = \int_X f(x * \check{y}) d\mu(y)$$

for each μ in $\mathcal{M}_c(X)$, f in $\mathcal{C}(X)$ and $x \in X$. The convolution operator $\mu * f$ is continuous. With this action of $\mathcal{M}_c(X)$ on $\mathcal{C}(X)$ the space $\mathcal{C}(X)$ is a topological left module.

For any y in X and a continuous function $f : X \rightarrow \mathbb{C}$ we define the function $\tau_y f : X \rightarrow \mathbb{C}$ by the formula

$$\tau_y f(x) := f(x * y) := \int_X f(t) d(\delta_x * \delta_y)(t)$$

and call it the *left translation* of f by y . In a similar way one can define the right translation of f by y . A subset H of $\mathcal{C}(X)$ is called *left-translation invariant*, if for any f in H and any y in X the function $\tau_y f$ belongs to H . A closed, left invariant subspace of $\mathcal{C}(X)$ is called a *left variety*.

Let K be a compact subhypergroup of the hypergroup X . The function f in $\mathcal{C}(X)$ is called *K-invariant*, if it satisfies

$$f(k * x * l) = f(x)$$

for all x in X and k, l in K . The set of all K -invariant functions form a closed subspace of $\mathcal{C}(X)$ and it is denoted by $\mathcal{C}_K(X)$. Observe that f is K -invariant if and only if \check{f} is K -invariant.

For each f in $\mathcal{C}(X)$ the function defined by

$$f^\#(x) = \int_K \int_K f(k * x * l) d\omega(k) d\omega(l)$$

for each x in X is called the *projection* of f . The projection $f \mapsto f^\#$ is a continuous linear mapping on $\mathcal{C}(X)$ onto $\mathcal{C}_K(X)$. Moreover, $f^{\#\#} = f^\#$ and $(f^\#)^\vee = (\check{f})^\#$ for each f in $\mathcal{C}(X)$. Further, f is K -invariant if and only if $f^\# = f$.

The *projection* $\mu^\#$ of the measure μ in $\mathcal{M}_c(X)$ is defined by

$$\langle \mu^\#, f \rangle = \langle \mu, f^\# \rangle = \int_X \int_K \int_K f(k * x * l) d\omega(k) d\omega(l) d\mu(x)$$

for each f in $\mathcal{C}(X)$. Clearly $\mu^\#$ is a measure. A measure μ in $\mathcal{M}_c(X)$ is called *K-invariant* if $\mu^\# = \mu$. The projection $\mu \mapsto \mu^\#$ is the adjoint of the projection $f \mapsto f^\#$, hence it is a continuous linear mapping on $\mathcal{M}_c(X)$ onto the set $\mathcal{M}_{c,K}(X)$ of all K -invariant measures. Moreover, $\mu^{\#\#} = \mu^\#$ and $(\mu^\#)^\vee = (\check{\mu})^\#$ for each μ in $\mathcal{M}_c(X)$. Further, μ is K -invariant if and only if $\mu^\# = \mu$.

As a special case, the projection of the point mass δ_y is defined by

$$\langle \delta_y^\#, f \rangle = f^\#(y) = \int_K \int_K f(k * y * l) d\omega(k) d\omega(l).$$

We define the (left) K -translate of a function f by y in X in the following way:

$$\tau_y^\# f(x) = \delta_y^\# * f(x) = \int_K \int_K f(k * y * l * x) d\omega(k) d\omega(l)$$

for each x in X . In particular, for each K -invariant function f we have

$$\tau_y^\# f(x) = \int_K \int_K f(k * y * x) d\omega(k)$$

for each x and y in X . Similarly, for any μ in $\mathcal{M}_{c,K}(X)$ we define

$$\tau_y^\# \mu = \delta_y^\# * \mu.$$

From now on if we say that “Let (X, K) be a Gelfand pair”, then we mean that X is a hypergroup, $K \subseteq X$ is a compact subhypergroup, and (X, K) is a Gelfand pair, i.e. the algebra $\mathcal{M}_{c,K}(X)$ is commutative.

For every f in $\mathcal{C}_K(X)$ and for every y in X the K -invariant measure

$$D_{f;y} = \delta_y^\# - f(y)\delta_e$$

is called the *modified K-spherical difference*, or simply *modified K-difference* of f by increment y . The higher order modified differences are defined in the following way:

$$D_{f;y_1, \dots, y_{n+1}} := \prod_{j=1}^{n+1} D_{f;y_j}$$

for any natural number n and for each y_1, \dots, y_{n+1} in X . On the right hand side the product is meant as a convolution product.

The non-zero K -invariant function $s: X \rightarrow \mathbb{C}$ is called a K -spherical function, if it satisfies

$$\int_K s(x * k * y) d\omega(k) = s(x)s(y) \tag{2.1}$$

for each x and y in X . This is equivalent to the requirement that s satisfies (2.1) and $s(e) = 1$. K -spherical functions are exactly the common normalized eigenfunctions of all convolution operators corresponding to K -invariant measures, that is, $s(e) = 1$, and for each K -invariant measure μ there exists a complex number λ_μ such that

$$\mu * s = \lambda_\mu s$$

holds.

For a K -spherical function $s: X \rightarrow \mathbb{C}$ and a K -invariant measure μ for which the representation

$$\langle \mu, f \rangle = \int_X f(x) d\mu(x)$$

for f is in $\mathcal{C}_K(X)$, we define the generalized difference operator as follows:

$$D_{s;\tilde{\mu}} = \mu - \langle \mu, \check{s} \rangle \delta_0.$$

A subset H of $\mathcal{C}_K(X)$ is K -invariant, if for each f in H and y in K the function $\tau_y^\# f$ is in H . A closed K -invariant linear subspace of $\mathcal{C}_K(X)$ is a K -variety. The intersection of any family of K -varieties is a K -variety. The intersection of all K -varieties including the K -invariant function f is called the K -variety generated by f and is denoted by $\tau_K(f)$. This is the closure of the linear space spanned by all K -translates of f .

A function f in $\mathcal{C}_K(X)$ is called a generalized K -monomial, if there exists a spherical function s and a natural number d such that

$$D_{s;y_1, \dots, y_{n+1}} * f(x) = \left(\prod_{j=1}^{n+1} D_{s;y_j} \right) * f(x) = 0$$

for each x, y_1, \dots, y_{n+1} in X . If f is non-zero, then the spherical function s is unique and we call f a generalized spherical s -monomial, or simply generalized s -monomial, and the smallest number n with the above property we call the degree of f . For $f = 0$ we do not define the degree. A generalized s -monomial is simply called an s -monomial, if its K -variety is finite dimensional. A linear combination of (generalized) K -monomials are called (generalized) K -polynomials.

A K -variety in $\mathcal{C}_K(X)$ is called decomposable if it is the sum of two proper K -subvarieties. Otherwise the K -variety is called indecomposable. The dual concept is the following: the ideal I in $\mathcal{M}_{c,K}(X)$ is called decomposable, if it is

the intersection of two ideals which are different from I . Otherwise the ideal is said to be *indecomposable*.

In this paper we are interested in finite dimensional K -varieties on different hypergroups. We know that, by definition, every spherical monomial spans a finite dimensional K -variety.

Theorem 2.1. *Let (X, K) be a Gelfand pair. Every finite dimensional K -variety can be decomposed into a finite sum of indecomposable K -varieties.*

Proof. If V is a decomposable K -variety in $\mathcal{C}_K(X)$, then we have $V = V_1 + V_2$, where $V_1, V_2 \subsetneq V$, hence the dimension of V_1 and V_2 is smaller than the dimension of V . If both are indecomposable, then we are ready. If not, then we continue this process which must terminate as the dimensions are strictly decreasing. □

Let V be a finite dimensional K -variety in $\mathcal{C}_K(X)$ and let f_1, f_2, \dots, f_d form a basis in V . Then we have

$$\mu * f_k(x) = \sum_{j=1}^d \lambda_{k,j}(\mu) f_j(x) \tag{2.2}$$

for each μ in $M_{c,K}(X)$ and x, y in X , where $\lambda_{k,j} : M_{c,K}(X) \rightarrow \mathbb{C}$ are some functions ($j, k = 1, 2, \dots, d$). For ν in $M_{c,K}(X)$ we have

$$(\nu * (\mu * f_k))(x) = \sum_{j=1}^d \lambda_{k,j}(\mu) (\nu * f_j)(x) = \sum_{j=1}^d \sum_{i=1}^d \lambda_{k,j}(\mu) \lambda_{j,i}(\nu) f_i(x)$$

for each x in X . Obviously, the left hand side of this equation can be written as $((\nu * \mu) * f_k)(x)$, and we obtain

$$\sum_{i=1}^d \sum_{j=1}^d \lambda_{k,j}(\mu) \lambda_{j,i}(\nu) f_i(x) = ((\nu * \mu) * f_k)(x) = \sum_{i=1}^d \lambda_{k,i}(\nu * \mu) f_i(x).$$

By the linear independence of the f_i 's we infer

$$\lambda_{k,i}(\nu * \mu) = \sum_{j=1}^d \lambda_{k,j}(\mu) \lambda_{j,i}(\nu),$$

which can also be written, by the commutativity of $M_{c,K}(X)$, as

$$\lambda_{k,i}(\nu * \mu) = \sum_{j=1}^d \lambda_{k,j}(\mu) \lambda_{j,i}(\nu). \tag{2.3}$$

Let $M(\mathbb{C}^d)$ denote the algebra of complex $d \times d$ matrices. We define the mapping $\Lambda : M_{c,K}(X) \rightarrow M(\mathbb{C}^d)$ as

$$\Lambda(\mu) = (\Lambda_{i,j}(\mu))_{i,j=1}^d \text{ with } \Lambda_{i,j}(\mu) = \lambda_{i,j}(\mu),$$

then clearly $\Lambda(\delta_e) = I$, the $d \times d$ identity matrix, and Λ is an algebra homomorphism of $\mathcal{M}_{c,K}(X)$ into $M(\mathbb{C}^d)$:

$$\Lambda(\mu * \nu) = \Lambda(\mu)\Lambda(\nu) \tag{2.4}$$

holds for each μ, ν in $\mathcal{M}_{c,K}(X)$. We show that the matrix elements of Λ restricted to X , that is the functions $x \mapsto \lambda_{i,j}(\delta_x^\#)$ are K -polynomials. For the proof we shall need the following theorem (see [3,4]):

Theorem 2.2. *Let d be a positive integer and \mathcal{S} a family of commuting linear operators in $M(\mathbb{C}^d)$. Then \mathbb{C}^d decomposes into a direct sum of linear subspaces A_j such that each A_j is a minimal invariant subspace under the operators in \mathcal{S} . Further, \mathbb{C}^d has a basis in which every operator in \mathcal{S} is represented by an upper triangular matrix.*

In other words, there exist positive integers k, n_1, n_2, \dots, n_k with the property $n_1 + n_2 + \dots + n_k = n$, and there exists a regular matrix S such that every matrix L in \mathcal{S} has the form

$$L = S^{-1} \text{diag} \{L_1, L_2, \dots, L_k\} S$$

where L_j is upper triangular for $j = 1, 2, \dots, k$. Here $\text{diag} \{L_1, L_2, \dots, L_k\}$ denotes the block matrix with blocks L_1, L_2, \dots, L_k along the main diagonal, and all diagonal elements of the block L_j are the same. As a consequence the following theorem holds true.

Theorem 2.3. *Let (X, K) be a Gelfand pair, d a positive integer, and let $\Lambda : \mathcal{M}_{c,K}(X) \rightarrow M(\mathbb{C}^d)$ be a continuous mapping satisfying (2.4) for each μ, ν in $\mathcal{M}_{c,K}(X)$. Then there exist positive integers k, d_1, d_2, \dots, d_k with the property $d_1 + d_2 + \dots + d_k = d$, and there exists a regular matrix S such that*

$$\Lambda(\mu) = S^{-1} \text{diag} \{\Lambda_1(\mu), \Lambda_2(\mu), \dots, \Lambda_k(\mu)\} S \tag{2.5}$$

for each μ in $\mathcal{M}_{c,K}(X)$, where $\Lambda_j(\mu)$ is an upper triangular $d_j \times d_j$ matrix in which all diagonal elements are equal, and it satisfies (2.4) for each μ, ν in $\mathcal{M}_{c,K}(X)$ and for every $j = 1, 2, \dots, k$.

Theorem 2.4. *Let (X, K) be a Gelfand pair, d a positive integer. Suppose that $\Lambda : \mathcal{M}_{c,K}(X) \rightarrow M(\mathbb{C}^d)$ is an algebra homomorphism. Then the matrix elements $x \mapsto \lambda_{i,j}(\delta_x^\#)$ are K -polynomials of degree at most d .*

Proof. First we apply Theorem 2.3 to diagonalize L . For the sake of simplicity we suppose that $\Lambda(\mu)$ itself has the properties of the $\Lambda_j(\mu)$'s in Theorem 2.3, that is, $\Lambda(\mu) = (\lambda_{i,j}(\mu))_{i,j=1}^d$ is a $d \times d$ upper triangular matrix in which all diagonal elements are equal. We note that

$$\lambda_{i,j}^\#(x) = \lambda_{i,j}^\#(\delta_x) = \int_X \lambda_{i,j}^\#(t) d\delta_x(t) = \int_X \lambda_{i,j}(t) d\delta_x^\#(t) = \lambda_{i,j}(\delta_x^\#)$$

holds for each $i, j = 1, 2, \dots, d$ and x in X . This means that $\lambda_{i,j} = 0$ for $i > j$, it satisfies equation (2.3), and all diagonal elements in $\Lambda(\mu)$ are the same: $\lambda_{i,i} = \lambda_{j,j}$ for $i, j = 1, 2, \dots, d$. Then

$$\lambda_{i,j}(\delta_x^\# * \delta_y^\#) = \sum_{k=i}^j \lambda_{i,k}(\delta_x^\#) \cdot \lambda_{k,j}(\delta_y^\#) \tag{2.6}$$

holds for $i = 1, 2, \dots, j$ and for each x, y in X . We have

$$\begin{aligned} \lambda_{i,j}(\delta_x^\# * \delta_y^\#) &= \langle \delta_x^\# * \delta_y^\#, \lambda_{i,j} \rangle = \int_X \int_X \lambda_{i,j}(u * v) d\delta_x^\#(u) d\delta_y^\#(v) \\ &= \int_K \int_K \int_K \lambda_{i,j}(k_1 * x * k * y * l_1) d\omega_K(k_1) d\omega_K(k) d\omega_K(l_1) \\ &= \int_K \lambda_{i,j}^\#(x * k * y) d\omega_K(k). \end{aligned}$$

Substitution into (2.6) gives

$$\int_K \lambda_{i,j}^\#(x * k * y) d\omega_K(k) = \sum_{k=i}^j \lambda_{i,k}(\delta_x^\#) \cdot \lambda_{k,j}(\delta_y^\#) \tag{2.7}$$

for $i = 1, 2, \dots, j$ and for each x, y in X . If we put $j = i$ in (2.6) we get

$$\lambda_{i,i}(\delta_x^\# * \delta_y^\#) = \lambda_{i,i}(\delta_x^\#) \cdot \lambda_{i,i}(\delta_y^\#) \tag{2.8}$$

for $i = 1, 2, \dots, d$ and for each x, y in X . Hence we infer

$$\int_K \lambda_{i,i}^\#(x * k * y) d\omega_K(k) = \lambda_{i,i}(\delta_x^\#) \cdot \lambda_{i,i}(\delta_y^\#) = \lambda_{i,i}^\#(x) \cdot \lambda_{i,i}^\#(y)$$

which means that the functions $\lambda_{i,i}^\#$ ($i = 1, 2, \dots, d$) are K -spherical functions. By assumption, all $\lambda_{i,i}$'s ($i = 1, 2, \dots, d$) coincide, and we write $s = \lambda_{i,i}^\#$ for $i = 1, 2, \dots, d$. We show by induction on $j - i$ that $\lambda_{i,j}^\#$ is an s -monomial of degree at most $j - i$. First we show that

$$D_{s;y_1, y_2, \dots, y_{j-i+1}} \lambda_{i,j}^\#(x) = 0.$$

Clearly, the statement holds for $j - i = 0$. Suppose that we have proved it for $j - i \leq l$ and let $j = i + l + 1$. Then we have

$$\begin{aligned} &D_{s;y_1, y_2, \dots, y_{l+1}, y_{l+2}} \lambda_{i,i+l+1}^\#(x) \\ &= D_{s;y_1, y_2, \dots, y_{l+1}} \left[\int_K \lambda_{i,i+l+1}^\#(y_{l+2} * k * x) d\omega_K(k) - s(y_{l+2}) \lambda_{i,i+l+1}^\#(x) \right] \\ &= D_{s;y_1, \dots, y_{l+1}} \left[\sum_{k=i}^{i+l+1} \lambda_{i,k}^\#(x) \lambda_{k,i+l+1}^\#(y_{l+2}) \right] - s(y_{l+2}) D_{m;y_1, \dots, y_{l+1}} \lambda_{i,i+l+1}^\#(x) \\ &= D_{m;y_1, \dots, y_{l+1}} \left[\lambda_{i,i+l+1}^\#(x) s(y_{l+2}) \right] - s(y_{l+2}) D_{s;y_1, \dots, y_{l+1}} \lambda_{i,i+l+1}^\#(x) = 0. \end{aligned}$$

This shows that the functions $\lambda_{i,j}^\#$ are all generalized exponential monomials. By (2.6), the K -variety of $\lambda_{i,j}^\#$ is spanned by the functions $\lambda_{i,k}^\#$ for every $k = i, i+1, \dots, j$, hence it is finite dimensional. The proof is complete. \square

Corollary 2.5. *Let (X, K) be a Gelfand pair. An indecomposable K -variety on X consists of s -monomials for some K -spherical function s .*

Corollary 2.6. *Let (X, K) be a Gelfand pair. A K -variety on X is finite dimensional if and only if it is spanned by finitely many K -monomials.*

Proof. The statement follows from Theorem 2.1 and Corollary 2.5. \square

Now we have a characterization of K -monomials.

Corollary 2.7. *Let (X, K) be a Gelfand pair. Then the K -polynomials are exactly those continuous K -invariant functions on X whose K -variety is finite dimensional.*

3. Affine groups

In the previous sections we have seen that Gelfand pairs play an eminent role in our investigation. In fact, in the case of Gelfand pairs the commutativity of the basic structure, which may be a group, or hypergroup, can be relaxed to the commutativity of the measure algebra. It is obvious that if the hypergroup X is commutative, then so is its measure algebra $\mathcal{M}_c(X)$, and so are all of its subalgebras. The converse is also obvious: if the measure algebra $\mathcal{M}_c(X)$ is commutative, then the point masses commute with respect to convolution, but the commutativity of the convolution of point masses means exactly the commutativity of X . Nevertheless, in the case of Gelfand pairs we do not require the commutativity of the whole measure algebra $\mathcal{M}_c(X)$, but only its subalgebra $\mathcal{M}_{c,K}(X)$ of K -invariant measures. Typically, point masses are not K -invariant – apart from trivial cases. On the other hand, in general, semidirect products of groups are non-commutative. Still, a large class of examples for Gelfand pairs is served by semidirect product constructions – namely, by affine groups, as we shall see below. The point is that if we start with a commutative group X and a compact group K of automorphisms of X , then the pair $(X \rtimes K, K)$ is always a Gelfand pair. Here for X a commutative semigroup can be taken as well.

In this section we shall consider finite dimensional varieties on affine groups over \mathbb{R}^d , where d is a positive integer. We shall give a complete description of those varieties using partial differential equations.

Let $GL(\mathbb{R}^d)$ denote the *general linear group*, that is, the topological group of all linear bijections of the linear space \mathbb{R}^d . This is a locally compact group with the topology inherited from \mathbb{R}^{d^2} and with the group operation defined

as the composition of linear mappings. Given an arbitrary closed subgroup $K \subseteq GL(\mathbb{R}^d)$ the semidirect product $\mathbb{R}^d \rtimes K$ is called the *affine group* of K over \mathbb{R}^d , and it is denoted by $\text{Aff } K$. The group $\text{Aff } K$ can be identified with the group of all affine mappings of the form $x \mapsto kx + u$, where k is in K and u is in \mathbb{R}^d , and the group operation is the composition. We identify K with the closed subgroup of $\text{Aff } K$ formed by all elements $(0, k)$ with k in K , and \mathbb{R}^d with the closed normal subgroup of $\text{Aff } K$ formed by all elements (u, id) with u in \mathbb{R}^d , where id stands for the identity mapping.

If K is compact with normalized Haar measure ω_K , then we equip the orbit space $X = \text{Aff } K/K$ with the hypergroup structure given by

$$\int_X f d(\delta_{Kx} * \delta_{Ky}) = \int_K f(x + ky) d\omega_K(k)$$

for each x, y in \mathbb{R}^d and K -invariant continuous function $f : \mathbb{R}^d \rightarrow \mathbb{C}$, that is

$$f(kx) = f(x)$$

holds for each x in \mathbb{R}^d and k in K . We have:

$$\begin{aligned} \int_X f(x + ky) d\omega_K(k) &= \int_X f(k(k^{-1}x + y)) d\omega_K(k) = \\ \int_X f(k^{-1}x + y) d\omega_K(k) &= \int_X f(y + kx) d\omega_K(k), \end{aligned}$$

by the inversion invariance of the Haar measure and the K -invariance of f . It follows that the hypergroup X is commutative. The following theorem is fundamental.

Theorem 3.1. *Let $K \subseteq GL(\mathbb{R}^d)$ be a compact subgroup and $V \subseteq C_K(\mathbb{R}^d)$ a finite dimensional K -variety. Then $V \subseteq C^\infty(\mathbb{R}^d)$.*

Proof. By Corollary 2.6, it is enough to show that every K -monomial is infinitely many times differentiable. Let s be a K -spherical function. We choose a compactly supported continuous K -invariant function $g : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$\int_{\mathbb{R}^d} g(x)s(-x) dx \neq 0.$$

This is possible, as compactly supported functions form a dense set in $C_K(\mathbb{R}^d)$. We define the linear functional μ_g on $C_K(\mathbb{R}^d)$ by

$$\langle \mu_g, f \rangle = \int_{\mathbb{R}^d} f(x)g(x) dx$$

whenever f is in $C_K(\mathbb{R}^d)$. Clearly, μ_g is a K -invariant measure. Now let f be a generalized s -monomial of degree at most n . Then we have

$$\begin{aligned} 0 &= D_{s; \check{\mu}_g}^{n+1} * f = (\mu_g - \langle \mu_g, \check{s} \rangle \delta_0)^{n+1} * f \\ &= (\mu_g - [\int_{\mathbb{R}^d} s(-x)g(x) dx] \cdot \delta_0)^{n+1} * f, \end{aligned}$$

where the power on the left is a convolution product. Expanding the power we have an equation of the form

$$\psi * f = \left[\int_{\mathbb{R}^d} s(-x)g(x) dx \right]^{n+1} f,$$

where $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is infinitely differentiable. As the coefficient of f on the right is nonzero, f is infinitely differentiable. □

We shall use multi-index notation: if α, β are multi-indices in \mathbb{N}^r , then we write

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_r, \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_r^{\alpha_r}, \quad \alpha! = \alpha_1! \alpha_2! \dots \alpha_r!,$$

$$\alpha[j] = (\alpha_1, \alpha_2, \dots, \alpha_j - 1, \dots, \alpha_r), \quad \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \dots \binom{\alpha_r}{\beta_r}.$$

We consider $\mathcal{C}^\infty(\mathbb{R}^d)$ equipped with the Schwartz topology: a net (f_i) in $\mathcal{C}^\infty(\mathbb{R}^d)$ is convergent to f in $\mathcal{C}^\infty(\mathbb{R}^d)$ if and only if $\partial^\alpha f_i$ uniformly converges to $\partial^\alpha f$ on every compact subset of \mathbb{R}^d , for each multi-index α . A linear operator D on $\mathcal{C}^\infty(\mathbb{R}^d)$ is called a *differential operator*, if it is *support-decreasing*, that is

$$\text{supp } Df \subseteq \text{supp } f$$

holds for each f in $\mathcal{C}^\infty(\mathbb{R}^d)$. It is known (see [2, Theorem 1.4]) that every differential operator has the form

$$D = \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha,$$

where the a_α 's are infinitely differentiable functions. The effect of D on a function f in $\mathcal{C}^\infty(\mathbb{R}^d)$ is obvious. If $K \subseteq GL(\mathbb{R}^d)$ is a closed subgroup, then a differential operator D is called *K-invariant*, if

$$D(f \circ k) = Df \circ k$$

holds for each f in $\mathcal{C}^\infty(\mathbb{R}^d)$ and k in K . All K -invariant differential operators form a unital algebra over \mathbb{C} , which we denote by $\mathcal{D}_K(\mathbb{R}^d)$. The space $\mathcal{C}_K^\infty(\mathbb{R}^d)$ of all infinitely differentiable K -invariant functions is a left module over $\mathcal{D}_K(\mathbb{R}^d)$. From now on we assume that $K \subseteq GL(\mathbb{R}^d)$ is a compact subgroup with normalized Haar measure ω_K , then the K -invariant differential operators D_1, D_2, \dots, D_r form a generating set of the commutative algebra $\mathcal{D}_K(\mathbb{R}^d)$ of K -invariant differential operators (see [2, Chapter IV, §2]). Here $r \leq d$. For each λ in \mathbb{C}^r , let s_λ denote the unique K -spherical function such that

$$P(D_1, D_2, \dots, D_r) s_\lambda = P(\lambda) s_\lambda$$

whenever P is a complex polynomial in r variables. It means that every K -invariant differential operator has the form $P(D_1, D_2, \dots, D_r)$ with some complex polynomial P . For each vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ in \mathbb{C}^r we denote

by s_λ the unique K -spherical function satisfying $D_j s_\lambda = \lambda_j \cdot s_\lambda$ for each $j = 1, 2, \dots, r$. In other words, for every $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ in \mathbb{C}^r , s_λ is the unique solution of the system of partial differential equations

$$(D_j - \lambda_j)s_\lambda = 0 \quad \text{for } j = 1, 2, \dots, r \tag{3.1}$$

with $s_\lambda(0) = 1$. From the theory of partial differential equations it follows that $\lambda \mapsto s_\lambda$ is infinitely differentiable in each coordinate of the variable λ , hence it is infinitely differentiable; in fact, it is analytic. We denote by $\partial_i s_\lambda$ the partial derivative of s_λ with respect to λ_i . In this section we use the symbol ∂ **exclusively to denote differentiation with respect to λ** .

We show that for each multi-index α in \mathbb{N}^r the function $\partial^\alpha s_\lambda$ is an s_λ -monomial of degree at most $|\alpha|$. We keep our notation in the subsequent statements.

Lemma 3.2. *Let α be a multi-index in \mathbb{N}^r . Then for each $j = 1, 2, \dots, r$ we have*

$$(D_j - \lambda_j)\partial^\alpha s_\lambda = \alpha_j \partial^{\alpha[j]} s_\lambda.$$

Proof. We have $D_i s_\lambda = \lambda_i s_\lambda$ for $i = 1, 2, \dots, r$. Applying ∂_j on both sides we have

$$D_i \partial_j s_\lambda = \begin{cases} \lambda_i \partial_j s_\lambda & \text{if } i \neq j \\ s_\lambda + \lambda_j \partial_j s_\lambda & \text{if } i = j, \end{cases}$$

or

$$(D_i - \lambda_i)\partial_j s_\lambda = \begin{cases} 0 & \text{if } i \neq j \\ s_\lambda & \text{if } i = j. \end{cases}$$

Repeating this process we get

$$(D_i - \lambda_i)\partial_j^l s_\lambda = \begin{cases} 0 & \text{if } i \neq j \\ l \partial_i^{l-1} s_\lambda & \text{if } i = j, \end{cases}$$

and the statement follows by iteration. □

Lemma 3.3. *Let α, β be arbitrary multi-indices in \mathbb{N}^r . Then we have*

$$(D - \lambda)^\beta \partial^\alpha s_\lambda = \begin{cases} \beta! \binom{\alpha}{\beta} \partial^{\alpha - \beta} s_\lambda & \text{if } \beta \leq \alpha \\ 0 & \text{if } \beta \not\leq \alpha. \end{cases}$$

In particular, $(D - \lambda)^\alpha \partial^\alpha s_\lambda = \alpha! s_\lambda$.

Proof. Here we use the notation

$$(D - \lambda)^\alpha = (D_1 - \lambda_1)^{\alpha_1} (D_2 - \lambda_2)^{\alpha_2} \dots (D_r - \lambda_r)^{\alpha_r}.$$

The statement follows immediately from the previous lemma. □

Lemma 3.4. *The functions $\partial^\alpha s_\lambda$ are linearly independent for different multi-indices α in \mathbb{N}^r .*

Proof. Suppose that

$$\sum_{k=0}^N \sum_{|\alpha|=k} c_\alpha \partial^\alpha s_\lambda = 0$$

with some complex numbers c_α . Let β be arbitrary in \mathbb{N}^d with $|\beta| = N$, then applying the operator $(D - \lambda)^\beta$ on both sides of the equation we have, by the previous lemma

$$\sum_{|\alpha|=N} c_\alpha (D - \lambda)^\beta \partial^\alpha s_\lambda = 0.$$

If $\beta \neq \alpha$ but $|\beta| = |\alpha|$, then there exists an i with $1 \leq i \leq d$ such that $\beta_i > \alpha_i$. Hence the previous equation implies

$$0 = c_\beta (D - \lambda)^\beta \partial^\beta s_\lambda = c_\beta \beta! s_\lambda,$$

consequently $c_\beta = 0$. Repeating this argument we get the statement. □

Lemma 3.5. *Let P be a complex polynomial in r variables. If*

$$P(\partial_1, \partial_2, \dots, \partial_r) s_\lambda = 0,$$

then $P = 0$.

Proof. The statement follows from the previous lemma. □

Theorem 3.6. *For each multi-index α in \mathbb{N}^r , $\partial^\alpha s_\lambda$ is an s_λ -monomial of degree at most $|\alpha|$.*

Proof. We prove the statement by induction on $N = |\alpha|$, and it clearly holds for $N = 0$. Now we suppose that we have proved the statement for every $k = 0, 1, \dots, N$, and we prove it for $k = N + 1$. Let $D_{i_1}, D_{i_2}, \dots, D_{i_{N+1}}, D_{i_{N+2}}$ be given; then we have for $|\alpha| = N + 1$:

$$(D_{i_{N+2}} - \lambda_{i_{N+2}}) \partial^\alpha s_\lambda = \alpha_{i_{N+2}} \partial^{\alpha[i_{N+2}]} s_\lambda,$$

and, by assumption, the right hand side is an s_λ -monomial of degree at most

$$\alpha_1 + \dots + \alpha_{i_{N+2}} - 1 + \dots + \alpha_d = N + 1 - 1 = N.$$

Hence, applying $(D_{i_1} - \lambda_{i_1})(D_{i_2} - \lambda_{i_2}) \dots (D_{i_{N+1}} - \lambda_{i_{N+1}})$ on both sides, the statement follows. □

It turns out that all s_λ -monomials of the form $\partial^\alpha s$ with $|\alpha| \leq N$ span the space of K -monomials of degree at most N , as the following theorem shows.

Theorem 3.7. *Every s_λ -monomial of degree at most N is a linear combination of the functions $\partial^\alpha s_\lambda$ with $|\alpha| \leq N$.*

Proof. We prove the statement by induction on N , and it clearly holds for $N = 0$. We assume that we have proved it for N . Let $f \neq 0$ be an s_λ -monomial of degree at most $N + 1$; it follows that the functions $(D_i - \lambda_i)f$ are s_λ -monomials of degree at most N , hence, by our assumption, we have the representations

$$(D_i - \lambda_i)f = \sum_{k=0}^N \sum_{|\alpha|=k} a_{i,\alpha} \partial^\alpha s$$

with some complex numbers $a_{i,\alpha}$ for $i = 1, 2, \dots, r$. We define the polynomials

$$Q_i(z) = \sum_{k=0}^N \sum_{|\alpha|=k} a_{i,\alpha} z^\alpha$$

for z in \mathbb{C}^r and $i = 1, 2, \dots, r$. Then

$$(D_i - \lambda_i)f = Q_i(\partial)s_\lambda,$$

and, by Lemma 3.2

$$(D_j - \lambda_j)(D_i - \lambda_i)f = (\partial_j Q_i)(\partial)s_\lambda.$$

Similarly, we have

$$(D_i - \lambda_i)(D_j - \lambda_j)f = (\partial_i Q_j)(\partial)s_\lambda.$$

Consequently, we have

$$(\partial_j Q_i)(\partial)s_\lambda = (\partial_i Q_j)(\partial)s_\lambda.$$

By Lemma 3.5, we have $\partial_i Q_j = \partial_j Q_i$ for each $i, j = 1, 2, \dots, r$. We infer that there exists a complex polynomial P in r variables such that $\partial_i P = Q_i$ for $i = 1, 2, \dots, r$. Clearly, the degree of P is at most $N + 1$. We define

$$\varphi = P(\partial)s_\lambda.$$

Then we have

$$(D_i - \lambda_i)f = Q_i(\partial)s_\lambda,$$

and

$$(D_i - \lambda_i)\varphi = (D_i - \lambda_i)P(\partial)s_\lambda = (\partial_i P)(\partial)s_\lambda = Q_i(\partial)s_\lambda$$

for $i = 1, 2, \dots, r$. It follows that

$$(D_i - \lambda_i)(f - \varphi) = 0 \quad \text{for } i = 1, 2, \dots, r.$$

We conclude that $f - \varphi$ is a joint eigenfunction of the generators of $\mathcal{D}_K(\mathbb{R}^n)$ with the same eigenvalues as s_λ , hence, by the uniqueness of the K -spherical function s_λ , it is a constant multiple of s_λ : $f - \varphi = cs_\lambda$ with some complex number c . As φ is a linear combination of the partial derivatives $\partial^\alpha s_\lambda$ with $|\alpha| \leq N + 1$, our theorem is proved. □

Corollary 3.8. *Let $K \subseteq GL(\mathbb{R}^d)$ be a compact subgroup. Then the functions $\partial^\alpha s_\lambda$ with $|\alpha| \leq N$ form a basis of the linear space of s_λ -monomials of degree at most N .*

Corollary 3.9. *Let $K \subseteq GL(\mathbb{R}^d)$ be a compact subgroup. Then every K -monomial is analytic.*

The functions $\partial^\alpha s_\lambda$ are called *elementary s_λ -monomials*. Since $s_\lambda(0) = 1$ holds for each λ in \mathbb{C}^d , we have $\partial^\alpha s_\lambda(0) = 0$ for $|\alpha| \geq 1$.

Observe that in the case $d = 1$ the group $GL(\mathbb{R})$ is identified with the multiplicative group \mathbb{R}_o of nonzero real numbers. The only compact subgroups are the trivial one, and $\{-1, 1\}$. If $K = \{1\}$, then K -translation is the ordinary translation, hence K -varieties are exactly the translation invariant closed linear spaces, simply called varieties. Every function and measure is K -invariant, and the K -invariant differential operators are the differential operators with constant coefficients, that is the algebra $\mathcal{D}_K(\mathbb{R})$ is generated by $\frac{d}{dx}$. The K -spherical functions are exactly the exponential functions: $e_\lambda(x) = e^{\lambda x}$ with arbitrary complex numbers λ . The elementary e_λ -monomials are the functions

$$\frac{d^j \exp \lambda x}{d\lambda^j} = x^j e_\lambda(x), \quad j = 0, 1, \dots \tag{3.2}$$

These functions for $j = 0, 1, \dots, m_j - 1$ are exactly the basis of the solution space of the differential equation

$$\left(\frac{d}{dx} - \lambda_j\right)^{m_j} f = 0.$$

In fact, the solution space V_j of this differential equation is an indecomposable variety corresponding to the exponential e_{λ_j} , $j = 0, 1, \dots, m_j - 1$. If V is a nonzero finite dimensional variety, then we have $V = V_1 + V_2 + \dots + V_p$, which consists of exponential polynomials, that is, linear combinations of the functions in (3.2). We can apply a similar argument in the case $K = \{-1, 1\}$ to obtain the following theorem:

Theorem 3.10. *Let $K \subseteq GL(\mathbb{R})$ be a compact subgroup and $V \neq \{0\}$ a finite dimensional K -variety. Then there exist positive integers p , different complex numbers λ_j , positive numbers m_j and K -invariant differential operators D_j ($j = 1, 2, \dots, p$) such that V is the solution space of the differential equation*

$$(D_1 - \lambda_1)^{m_1} (D_2 - \lambda_2)^{m_2} \dots (D_p - \lambda_p)^{m_p} f = 0. \tag{3.3}$$

In the case $K = \{1\}$ we have $D_j = \frac{d}{dx}$, and in the case $K = \{-1, 1\}$ we have $D_j = \frac{d^2}{dx^2}$ for $j = 1, 2, \dots, p$. In both cases the dimension of V is $\sum_{j=1}^p m_j$.

Proof. The first statement about the D_j 's is obvious. Suppose that $K = \{-1, 1\}$. For each f in $C^\infty(\mathbb{R})$ we have

$$\left[\left(\frac{d^2}{dx^2} f\right) \circ (-id)\right](x) = f''(-x),$$

and

$$\left[\frac{d^2}{dx^2} (f \circ (-id)) \right] (x) = f''(-x)$$

for each x in \mathbb{R} , hence $\frac{d^2}{dx^2}$ is a K -invariant differential operator. As the rank of $\mathcal{D}_K(\mathbb{R})$ is obviously one, we have that $\mathcal{D}_K(\mathbb{R})$ is the polynomial algebra consisting of the polynomials $P(\frac{d^2}{dx^2})$ where P is an arbitrary complex polynomial. □

It follows that in the case $K = \{1\}$ every finite dimensional nonzero K -variety is the linear span of solutions of the differential equation

$$\left(\frac{d}{dx} - \lambda_1 \right)^{m_1} \left(\frac{d}{dx} - \lambda_2 \right)^{m_2} \dots \left(\frac{d}{dx} - \lambda_p \right)^{m_p} f = 0,$$

where the λ 's are different complex numbers and the m 's are positive integers, and in the case $K = \{-1, 1\}$ every finite dimensional nonzero K -variety is the linear span of the even solutions of the differential equation

$$\left(\frac{d^2}{dx^2} - \lambda_1 \right)^{m_1} \left(\frac{d^2}{dx^2} - \lambda_2 \right)^{m_2} \dots \left(\frac{d^2}{dx^2} - \lambda_p \right)^{m_p} f = 0,$$

where the λ 's are different complex numbers and the m 's are positive integers. In both cases $m_1 + m_2 + \dots + m_p = \dim V$.

4. Further examples

In this section we give more examples of finite dimensional varieties.

- (1) Our first example is the polynomial hypergroup X associated with the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$. Let $K = \{e\}$, the trivial subgroup, then the double coset space $X//K$ is identical to X , every function and measure is K -invariant. K -varieties are the solution spaces of linear homogeneous difference equation systems with constant coefficients. It follows easily that each proper K -variety is finite dimensional (see [5]).
- (2) Let K be a compact, and D a discrete hypergroup. We denote by X the hypergroup join $K \vee D$. The normalized Haar measure on K is denoted by ω_K . We recall that we identify the identity elements of K and D , and it will be the identity e of X . We write $K_e = K \setminus \{e\}$ and $D_e = D \setminus \{e\}$. The involution on X is defined as the extension of the involutions on K and D . The convolution on X is defined as follows: if x, y are in K , then $\delta_x * \delta_y = \delta_x *|_K \delta_y$, where $*|_K$ is the convolution in K . If x is in K and y is in D_e , then we have $\delta_x * \delta_y = \delta_y * \delta_x = \delta_x$. Finally, if x, y are in D and $x \neq \check{y}$, then we have $\delta_x * \delta_y = \delta_x *|_D \delta_y$, where $*|_D$ is the convolution in

D. Finally, if x is in D_e , then we have

$$\delta_x * |_D \delta_{\bar{x}} = \sum_{w \in D_e} c(w)\delta_w + c(e)\delta_e,$$

where $c : D \rightarrow \mathbb{C}$ is a finitely supported function with $c(e) \neq 0$. In this case the convolution of δ_x and $\delta_{\bar{x}}$ in X is defined as

$$\delta_x * \delta_{\bar{x}} = \sum_{w \in D_e} c(w)\delta_w + c(e)\omega_K.$$

It is known that with these definitions X is a hypergroup, and K is a compact subhypergroup of X . It is known that the double coset hypergroup $X//K$ is topologically isomorphic to D (see [1]). As (X, K) is a Gelfand pair if and only if the double coset hypergroup $X//K$ is commutative, in our case this is equivalent to the commutativity of the hypergroup D . It follows that K -spherical functions and K -monomials can be identified with the exponentials, resp. exponential monomials on D .

- (3) If $K = SO(\mathbb{R}^d)$, then $\mathcal{C}_K(\mathbb{R}^d)$ can be identified with the space of continuous radial functions, and K -spherical functions have the form

$$s_\lambda(x) = J_\lambda(\|x\|)$$

for each x in \mathbb{R}^d , where J_λ is the Bessel function defined by

$$J_\lambda(r) = \Gamma\left(\frac{d}{2}\right) \sum_{k=0}^{\infty} \frac{\lambda^k}{k! \Gamma(k + \frac{d}{2})} \left(\frac{r}{2}\right)^{2k}$$

for r in \mathbb{R} . Here s_λ is the unique K -spherical function corresponding to the eigenvalue λ of the Laplacian in radial form:

$$\frac{d^2 s_\lambda}{dr^2} + \frac{d-1}{r} \frac{ds_\lambda}{dr} = \lambda s_\lambda.$$

(see [7]). For every nonzero finite dimensional K -variety V there exists a positive integer r , different complex numbers $\lambda_1, \lambda_2, \dots, \lambda_r$ and nonnegative integers k_1, k_2, \dots, k_r such that V is spanned by the K -monomials $\frac{d^j s_{\lambda_i}}{d\lambda^j}$ for $i = 1, 2, \dots, r$ and $j = 0, 1, \dots, k_r - 1$, which can be obtained by termwise differentiation of the series of J_λ with respect to λ . In [7], it was proved that K -spectral synthesis holds for each K -variety, i.e. every K -variety is the topological sum of its finite dimensional K -varieties.

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László Székelyhidi
Institute of Mathematics
University of Debrecen
Debrecen
Hungary
e-mail: lszekelyhidi@gmail.com

Żywilla Fechner
Lodz University of Technology
ul. Wólczańska 215
90-924 Lodz
Poland
e-mail: zfechner@gmail.com

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