Generalized universality in the massive sine-Gordon model

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A nontrivial interplay of the UV and IR scaling laws, a generalization of the universality is demonstrated in the framework of the massive sine-Gordon model, as a result of a detailed study of the global behavior of the renormalization group flow and the phase structure.

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I. INTRODUCTION

Our hope to cover the overpowering richness of physics by microscopic theories constructed on simple elementary principles is based on the concept of universality, the possibility of ignoring most of the short distance, microscopic parameters of the field theoretical models in describing the dynamics at finite scales [1]. Such an enormous simplification, obtained by inspecting the asymptotical UV scaling laws, is possible if the renormalized trajectory "spends" several orders of magnitude in the UV scaling regime and suppresses the sensitivity of the physics on the nonrenormalizable, i.e. irrelevant parameters.

But one tacitly assumes in this scenario that there are no other scaling laws in the theory or at least they do not influence the results obtained in the UV scaling regime. This is certainly a valid assumption for models with explicit mass gap in the Lagrangian which renders all non-Gaussian operators irrelevant at the IR fixed point. In fact, quantum fluctuations at longer length scales than the correlation length are suppressed in these models and the evolution of the running coupling constants slows down in the IR domain excluding other than the mass term as a relevant operator. Such a lack of nontrivial evolution in the IR scaling regime explains that these scaling laws are usually ignored and the UV part of the renormalized trajectory is alone sufficient to predict what happens with observables even if they correspond to the IR scaling regime.

But how about theories with massless bare particles? Spontaneous symmetry breaking, or dynamical mass generations in general, operates with soft long range modes, such as condensates, and may change the situation [2,3]. The piling up of infrared contributions during the evolution of the running coupling constants and appearing as an IR divergence in the naive vacuum opens the possibility of tunable parameters of the theories which make their impact on the dynamics due to a competition between the UV and the IR scaling regimes. Such a competition can be highly nontrivial because both of these regimes can cover unlimited orders of magnitude of evolution for the RG flow and may offer a new way of looking at complex systems.

One has to distinguish mass gap, installed in the theory by a mass term in the bare Lagrangian from particle masses generated dynamically in a theory without tree-level mass term. In fact, the dynamical mass generation mechanism, being an IR phenomenon, can strongly be influenced by the values of the running coupling constants at the UV-IR crossover, the initial conditions for the IR scaling laws. Once the IR scaling laws become nontrivial the resulting dynamics might well be more involved as in the UV scaling regime for no guarantee is found to exclude nonlocal operators from mixing.

The traditional way of "understanding" a large system starts with the identification of its components and their elementary behavior and then continues with the combination of these rules to come up with the whole picture. The competition between UV and IR fixed points may introduce microscopical parameters which influence the dynamics in the macroscopic range only. Such a renormalization microscope mechanism allows us to determine certain microscopical parameters by means of the long range dynamics and represents a microscopic, elementary feature of a model which remains undetected unless the whole system is considered.

We turn to a simple two-dimensional model in this work whose dynamics displays such a phenomenon. It has an explicit mass gap but this does not render the IR dynamics uninterestingly suppressed. This is because the natural strength of a coupling constant is its value when expressed in units of the scale inherent of the phenomenon to be described, e.g. the gliding cutoff in the framework of the renormalization group (RG) in momentum space. Since the non-Gaussian coupling constants of the local potential have positive mass dimension in two space-time dimensions, the usual slowing down of the evolution of dimensional coupling constants below the mass gap generates diverging dimensionless coupling constants, i.e. relevant parameters at the IR fixed point. But this is one part of the story only. The other part is that in order to have nontrivial competition between the UV and IR scaling regimes we need coupling constants which are nonrenormalizable, i.e. irrelevant at the UV fixed point and turn into relevant at the IR end point. In order to render the coupling constants of the local potential nonrenormalizable we need nonpolynomial coupling. The simplest, treatable case is the sine-Gordon (SG) model [4] whose Lagrangian contains a periodic local potential and which possesses two phases. All coupling constants of the local periodic potential are nonrenormalizable in one of the phases when the model is considered in the continuous space-time with nonperiodic space-time derivatives [5]. We are thus lead to the massive sine-Gordon (MSG) model [6–10] which has already been thoroughly investigated in the seventies as the bosonized version of the massive Schwinger model, 1 + 1 dimensional QED, the simplest model possessing confining vacuum [6]. The different UV and IR scaling laws have already been addressed in this model [7,8] but the careful identification of the full scaling regime, not only the asymptotical ends is needed to understand the global features of the RG flow diagram. This is the goal of the present work. One can only answer this question by solving the RG equations for the couplings and finding the effective potential. In this work we use the Wegner-Houghton (WH) functional RG method [11], where the renormalization procedure is defined by the blocked action with gliding sharp cutoff to determine the flow of the Fourier amplitudes of the model. The handicap of the WH-RG method is that one cannot go beyond (if it is even necessary) the local potential approximation (LPA), since the gradient expansion of the Lagrangian gives an indeterminate evolution equation due to the sharp cutoff used. There are several other methods in the literature which are based on the evolution of the effective action [12]. We show in this paper that the internal-space RG method [13], where the RG evolution is controlled by the mass, allows us to go beyond the LPA by taking wave-function renormalization into account and gives qualitatively the same results as the WH-RG method in the LPA.

Since the SG model has a condensed phase which arises at a finite cutoff k_{SI} one expects that if $M < k_{SI}$ the condensation also should appear in the MSG model. In our previous works [8,14] we showed in the WH-RG framework that the nontrivial scalings appear in the deep IR limit for the SG and the MSG models, indeed. Since the SG model has a trivial, constant effective potential in either phase [14,15], the RG methods based on the effective action [12,13] may fail in treating both the SG and the MSG models. Nevertheless, when the control parameter approaches the physical value of the mass, the internal-space RG with wave-function renormalization enables one to find an evolution of the parameters which is analogous to their WH-RG flow in the LPA. Also the sign of spinodal instability seems to appear as a singularity in the internal-space evolution. In the latter case the analogy mentioned above allows one to change from the internal-space RG analysis to the WH-RG framework and to determine the phase structure of the MSG model. The situation seems to be similar as that for the WH-RG framework where the IR limit $k \to 0$ of the blocked action is trivial but physics can be read off from its approaching this limit.

A side product of our RG analysis is that one recovers the well-known phase structure of the bosonized version of QED_2 in the LPA. This and the qualitatively insignificant role of the wave-function renormalization in the position of the boundary of the ionized phase found in our internal-space RG analysis are hints that the phase structure, as well as the spinodal instability should survive the inclusion of the wave-function renormalization in the evolution.

Another possibility might be the usage of the Polchinski RG method which uses the blocked action and a smooth cutoff. Unlike the WH-RG method based on summing up the loop corrections during the evolution, Polchinski's procedure sums up the perturbation expansion, therefore it misses the spinodal instability in an obvious manner. In fact, one can easily show that the local potential evolved by this scheme is not of parabolic shape, a feature is thought to be a crucial sign of spinodal instabilities in LPA.

The ionized (large β^2) phase of the MSG model can be an example where thorough numerical investigations show that the bare UV irrelevant coupling becomes relevant in the deep IR regime. This fact clearly shows that the treatments of the MSG model based on any perturbation expansion are doomed to fail.

The paper is organized as follows. In Sec. II we derive the evolution of the couplings of the MSG model in the framework of the WH-RG method in the LPA. In Sec. III the phase structure of the MSG model is presented. The RG microscope effect is discussed in Sec. IV. We show in Sec. V that our WH-RG results recover the well-known phase structure of QED_2. The evolution of the couplings in the internal-space RG framework is treated in Sec. VI and the role of the wave-function renormalization is discussed, and finally, in Sec. VII the conclusion is drawn up.

II. BLOCKING IN MOMENTUM SPACE

The MSG model is defined in 2-dimensional, infinite, Euclidean space-time by the Lagrangian

$$S_k = \int_x \left[\frac{1}{2}(\partial_\mu \phi_x)^2 + U_k(\phi_x)\right], \tag{1}$$

given in the leading order, local potential approximation (LPA) of the gradient expansion, k denotes the sharp momentum space cutoff and the potential is the sum $U_k(\phi) = \frac{1}{2}M_k^2\phi^2 + V_k(\phi)$, the second term being periodic,

$$V_k(\phi) = \sum_{n=1}^{\infty} u_n(k)\cos(n\beta_k\phi). \tag{2}$$

The blocking in momentum space [14], the lowering of the cutoff, $k \to k - \Delta k$, consists of the splitting of the field variable, $\phi = \tilde{\phi} + \phi'$ in such a manner that $\tilde{\phi}$ and ϕ' contain Fourier modes with $|p| < k - \Delta k$ and $k - \Delta k < |p| < k$, respectively, and the integration over ϕ' leads to the WH equation [11]

$$(2 + k\partial_k)\tilde{U}_k(\phi) = -\frac{1}{4\pi}\ln(1 + \tilde{U}_k''(\phi)) \quad (3)$$

for the dimensionless local potential $\tilde{U}_k = k^{-2}U_k$. This equation is obtained by assuming the absence of instabilities for the modes around the cutoff. Only the WH-RG scheme which uses a sharp gliding cutoff can account for the spinodal instability, which appears when the restoring force acting on the field fluctuations to be eliminated is vanishing and the resulting condensate generates tree-level contributions to the evolution equation [16]. The saddle point for the blocking step, ϕ_0', is obtained by minimizing the action, $S_{k-\Delta k}[\phi] = \min_{\phi'}(S_k[\phi + \phi'])$ [3,15]. The restriction of the minimization for plane waves gives

$$\tilde{U}_{k-\Delta k}(\phi) = \min_\rho\left[\rho^2 + \frac{1}{2}\int_{-1}^{1} du\tilde{U}_k(\phi + 2\rho\cos(\pi u))\right] \quad (4)$$

in LPA where the minimum is sought for the amplitude ρ only.

One can show that both evolution equations, Eqs. (3) and (4), preserve the period length of the potential $V_k(\phi)$ and the nonperiodic part of the potential, therefore $M_k^2 = M^2$ and $\beta_k = \beta$. Thus the mass is a relevant parameter of the LPA ansatz for all scales,

$$\tilde{M}_k^2 = \tilde{M}_\Lambda^2\left(\frac{k}{\Lambda}\right)^{-2}. \quad (5)$$

A. Asymptotic scaling

It is easy to find the asymptotic UV and IR scaling laws. One finds the evolution equation

$$k\partial_k\tilde{u}_n(k) = \left(\frac{\beta^2 n^2}{4\pi(1 + \frac{M^2}{k^2})} - 2\right)\tilde{u}_n(k) \quad (6)$$

in the UV regime after ignoring $\mathcal{O}(M^2/k^2)$ and $\mathcal{O}(|U_\Lambda''|^2/k^2)$ contributions with the solution

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda)\left(\frac{k}{\Lambda}\right)^{-2}\left(\frac{k^2 + M^2}{\Lambda^2 + M^2}\right)^{\beta^2 n^2/8\pi}. \quad (7)$$

The asymptotic IR scaling, well below the mass scale, is trivial because the mass gap freezes any scale dependence. The numerical solution of the complete evolution equation (3) is shown in Fig. 1.

B. Impact of the mass gap

It is instructive to compare the RG flow of the (massless) SG and the MSG models. The asymptotic UV evolution equations differ in $\mathcal{O}(M^2/k^2)$ terms only and the mass term gives small corrections to the scaling laws in this regime. But the mass gap freezes out the evolution for any values of β thus more significant differences should show up between the SG and the MSG models. The dimensionful potential approaches a constant in the SG model as a result of the loop-generated or instability driven evolution in the ionized or the molecular phase, respectively [14,15]. The evolution of the potential freezes out below the mass gap of the MSG model and a nontrivial potential is left over in the IR end point, reflecting the state of affairs at $k \approx M$. The IR scaling is trivial for $k < M$, $\tilde{u}_n(k) \sim k^{-2}$. In order to go beyond the asymptotic scaling analysis and to find out more precisely the changes brought by the nonperiodic mass term to the RG flow we have to rely on the numerical solutions of the evolution equations.

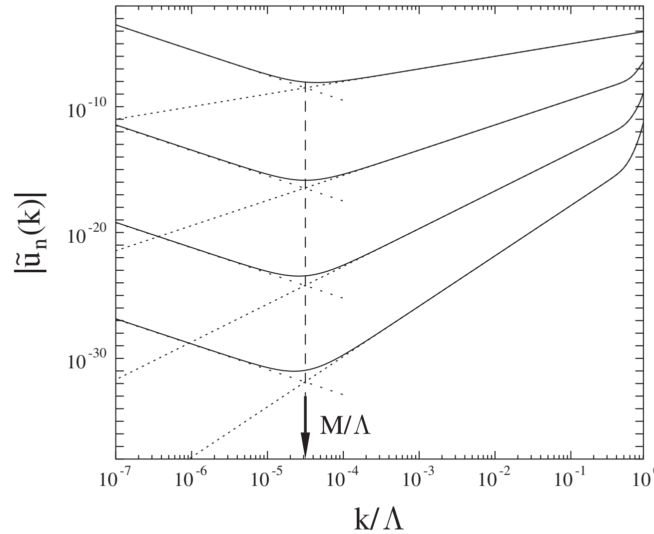

FIG. 1. RG flow of $\tilde{u}_1, \ldots, \tilde{u}_4$ for $\beta^2 = 12\pi$ and $M^2 = 10^{-9}\Lambda^2$ (solid line) or $M^2 = 0$ (dense dotted line). The plot also shows the asymptotic IR scaling $\sim k^{-2}$ (rare dotted line). The massive and the massless flows depart at the scale M, indicated by a dashed vertical line, placed at the intersection of the dotted lines.

Let us consider first the regime $\beta^2 > 8\pi$ which is free of spinodal instabilities in the massless case. The evolution of the first four coupling constants, $\tilde{u}_1, \ldots, \tilde{u}_4$ is shown in Fig. 1 for $\beta^2 > 8\pi$. The UV scaling regime is confined in this plot to the very beginning, around $k/\Lambda \approx 1$ [14], and what we see here is that the flows of the SG and the MSG models agree for $\beta^2 > 8\pi$ even in the IR scaling regime down to the mass gap. The freeze-out below the mass gap takes place naturally without generating spinodal instabilities.

The comparison of the massive and the massless cases is more involved for $\beta^2 < 8\pi$ due to the appearance of instabilities. If the scale k_{SI} where instabilities appear is higher than the mass gap, $k_{SI} > M$, then the RG flows of the MSG and SG models are similar down to M and both display instabilities, but they differ for $k < M$, as shown in Fig. 2. When the freeze-out scale is reached first during the evolution, i.e. the scale k_{SI} of the SG model with the same potential as $V(\phi)$ of the MSG model is smaller than M, then the instability does not occur in the MSG model.

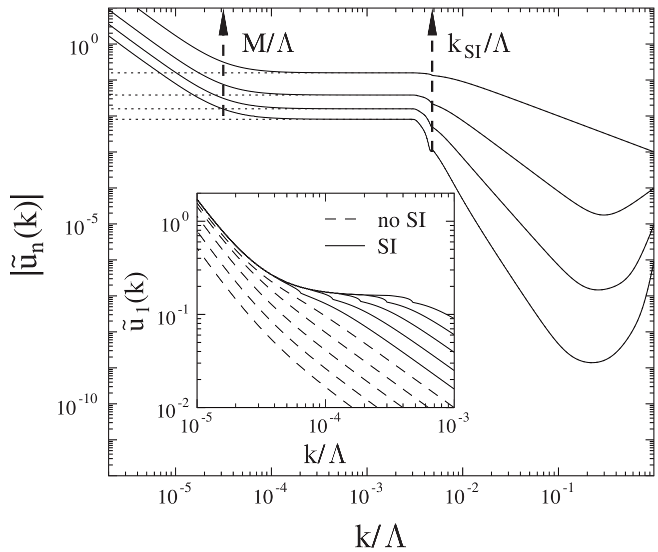

FIG. 2. RG flow of $\tilde{u}_1, \ldots, \tilde{u}_4$ for $\beta^2 = 4\pi$ and $M^2 = 10^{-9}\Lambda^2$ (solid line) or $M^2 = 0$ (dotted line). The scales of the spinodal instability $k_{\rm SI}/\Lambda$ and the mass M/Λ are indicated. The inset shows RG flows of $\tilde{u}_1$ for several different initial values at $M^2 = 10^{-9}\Lambda^2$.

It is instructive to determine the boundary of the region with spinodal instability in the coupling constant space when the single Fourier mode $\tilde{u}_n = \tilde{u}\delta_{1,n}$ is kept only. The condensate appears during the evolution at scale $k_{\rm SI}$, satisfying $k_{\rm SI}^2 + M^2 + U''_{k_{\rm SI}}(\phi) = 0$ for some ϕ. The approximate analytic expression, Eq. (7), gives

$$k_{\rm SI}^2 = (\Lambda^2 + M^2)\left(\frac{\Lambda^2 + M^2}{\beta^2 u(\Lambda)}\right)^{8\pi/(\beta^2 - 8\pi)} - M^2 \qquad (8)$$

for $\beta^2 < 8\pi$, suggesting that the coupling constant can be weak enough to allow the mass term to remove the condensate. The RG flow obtained numerically for $\tilde{u}_1(k)$ shown in the inset of Fig. 2 confirms, as well, that the mass can be strong enough to prevent the formation of instabilities. One can get an estimate of the critical value of the coupling constant by equating $k_{\rm SI}$ and M in Eq. (8),

$$u_{1c}(\Lambda) = \frac{\Lambda^2 + M^2}{\beta^2}\left(\frac{2M^2}{\Lambda^2 + M^2}\right)^{1-(\beta^2/8\pi)}. \qquad (9)$$

The boundary of the region with instability is shown in Fig. 3. In contrast to the SG model where the instability extends over the whole phase with $\beta^2 < 8\pi$ the mass term always wins at the IR end point of the flow of the MSG model and removes the condensate at some low but finite value of the scale k.

The disappearance of spinodal instability and the trivial scaling, $|\tilde{u}_n| \sim k^{-2}$ [7] can be made plausible also by the following, simple analytic consideration. Namely, Eq. (3) can be rewritten as

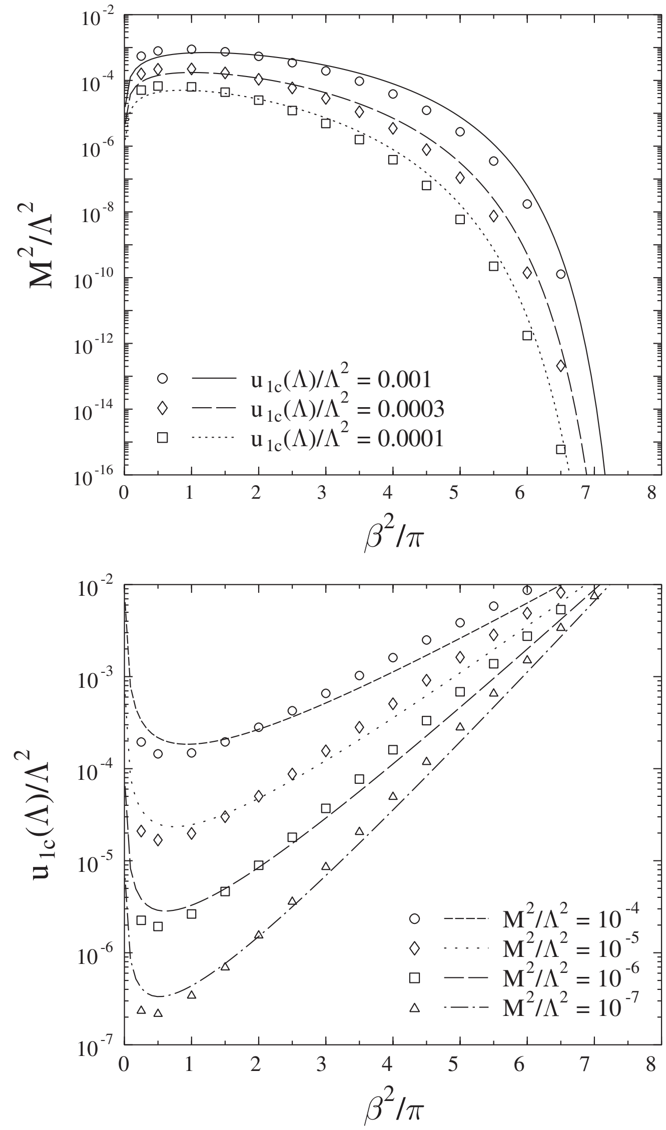

FIG. 3. The region with spinodal instability. Different symbols show the boundary obtained by the numerical computation with 10 coupling constants and the lines show the results obtained by taking into account a single Fourier mode only.

$$\sum_s \mathcal{B}_{n,s} k\partial_k \tilde{u}_s(k) = \sum_s \left[-2\mathcal{B}_{n,s} + \frac{\alpha_2 \beta^2 n^3 k^2}{k^2 + M^2}\right]\tilde{u}_s(k), \qquad (10)$$

with

$$\mathcal{B}_{n,s} = s\delta_{n,s} - \frac{s\beta^2 (n-s)^2 \tilde{u}_{|n-s|}(0) - (n+s)^2 \tilde{u}_{n+s}(0)}{2(k^2 + M^2)}, \qquad (11)$$

resulting in

$$\mathcal{B}_{n,s} \sim s\delta_{n,s} - \frac{s\beta^2 a_{n,s}}{2M^2}, \qquad (12)$$

a RG invariant quantity in the IR scaling region. Finally, the second term on the right-hand side of Eq. (10) can be neglected for $k \ll M$, yielding $\tilde{u}_n(k) \sim u_n(0)k^{-2}$ in the IR scaling region.

III. PHASE STRUCTURE

The mass term deforms the phase boundary of the SG model by extending the ionized phase. In this phase of the SG model the IR scaling law generates the scale dependence of the coupling constants $u_n(k)$ with $n \geq 2$ through $u_1(k)$, namely, renders the ratios $R_n^{\rm SG} = u_n(k)/u_1^n(k)$ RG invariant. It was checked numerically that $R_n^{\rm MSG} = |u_n(k)|/u_1^n(k)$ is RG invariant in the IR scaling region of the MSG model without condensate and the potential at $k = 0$ depends on the initial value of u_1 only.

The phase with condensate is similar to those of the SG model. The potential develops quickly into a superuniversal, initial condition independent shape [14] when $M < k \approx k_{\rm SI}$, cf. the inset of Fig. 2. But this scaling regime ends at $k \approx M$ where trivial scaling laws come into force down to $k = 0$. The matching of the IR scaling of the SG model [14] with the trivial scaling law gives $u_n(0) = (-1)^{n+1}2M^2/n^2\beta^2$.

The modification of the phase boundary induced by the mass can be seen by means of the sensitivity matrix [3,14], too. This matrix, defined as the derivatives of the running coupling constants with respect to the bare one,

$$S_{n,m} = \frac{\partial \tilde{u}_n(k)}{\partial \tilde{u}_m(\Lambda)}, \tag{13}$$

develops singularities when the UV and IR cutoffs are removed at the phase boundaries only. The typical behavior is depicted in Fig. 4, showing that the appearance of the condensate generates first singular turns and leads later to radically different scale dependence in this matrix.

The molecular phase of the two-dimensional SG model was compared with the four-dimensional Yang-Mills theory [14] because both support spinodal instabilities in their vacuum and the periodic symmetry of the SG model is formally similar to the center symmetry of the gauge theory. This analogy can be extended to the MSG model, where the breakdown of the periodicity which manifests itself in the scaling laws for $k < M$ is comparable with the spontaneous symmetry breaking in a four-dimensional Yang-Mills-Higgs system. The solitons cease to be stable and elementary plane-wave excitations appear in the scattering matrix of the MSG model due to the breakdown of the periodicity by the mass term. In a similar manner, the confining forces are lost by the spontaneous breakdown of the global symmetry and elementary excitations propagate in the vacuum of gauge theories. In other words, the competition of the confinement and the symmetry breaking scales in the gauge theory is similar to what happens between $k_{\rm SI}$ and M in the MSG model.

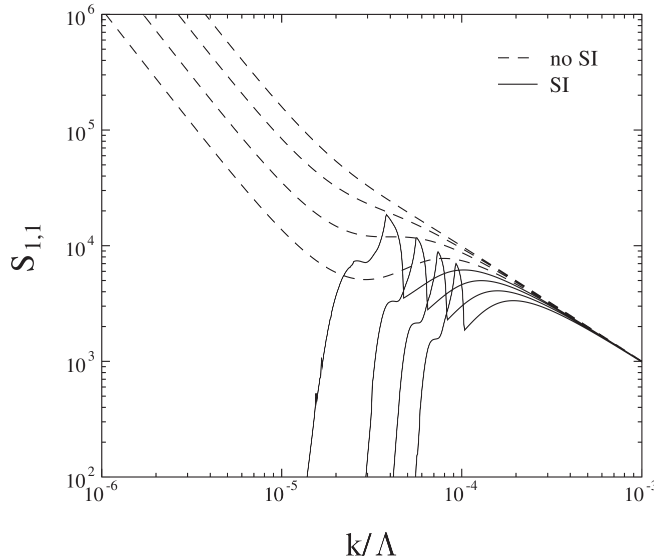

FIG. 4. The (1, 1) element sensitivity matrix is computed numerically for various initial conditions for $\beta^2 = 4\pi$ as the function of k/Λ. The trajectories of molecular and the ionized phases are shown by dashed and solid lines, respectively.

IV. RG MICROSCOPE

The nontriviality of the IR scaling regime opens the possibility of having relevant nonrenormalizable operators, being irrelevant around the UV fixed point, but becoming relevant in the IR scaling regime. To find out whether this can be the case let us take the running coupling constants, given by Eq. (7)

$$\tilde{u}_n(k) = (-1)^{n+1}\tilde{u}_1^n(\Lambda)R_n^{\rm MSG}k^{2n-2} \times \left(\frac{k}{\Lambda}\right)^{-2n} \times \left(\frac{k^2+M^2}{\Lambda^2+M^2}\right)^{n\beta^2/8\pi}, \tag{14}$$

and calculate the (1, 1) element of the sensitivity matrix in Eq. (13) in the limits $k \to 0$ and $\Lambda \to \infty$,

$$S_{1,1} \sim k^{-2}\Lambda^{(8\pi-\beta^2)/4\pi}. \tag{15}$$

We set $\beta^2 > 8\pi$, i.e. move deep in the ionized phase in the language of the SG model where all coupling constants are nonrenormalizable. But the IR scaling law makes the coupling constants relevant with its factor k^{-2}. More precisely, the dimension 2 coupling constants freeze out meaning that their values counted in the "natural" units of the running cutoff, k, diverge as the IR end point, $k = 0$, is approached. The result is the regain of the sensitivity of the long distance physics on the choice of the bare, microscopic, nonrenormalizable coupling constants which was suppressed in the UV scaling regime. As long as the UV cutoff Λ and the IR observational scale k are chosen according to $k = t^{-p_{\rm IR}}\mu$ and $\Lambda = t^{p_{\rm UV}}\mu$ where $p_{\rm IR} = p_{\rm UV}(\beta^2/8\pi - 1) > 0$ (with any $t > 1$ and $\mu > 0$), the observed IR dynamics depends on the choice of the bare coupling constant. This rearrangement is reminiscent of the traditional microscope in the sense that the amplification, i.e. divergence of the RG flow in the IR scaling regime balances the suppression, i.e. focusing of the flow in the UV scaling regime. As a result the large scale observations can fix the value of the microscopic parameter with good accuracy. Notice that this kind of dynamics must be put in the theory at microscopic scale even though it starts to influence the physics in the IR scaling regime. Such an interplay between the different scaling regimes represents a highly nontrivial, global extension of the simple universality idea which is based on the local analysis of the RG flow at a given fixed point and might be a key to phenomena like superconductivity and quark confinement [2,3].

V. TWO PHASES OF QED_2

The most remarkable consequence of the nontrivial phase structure of the MSG model is that the two-dimensional quantum electrodynamics (QED_2) also has two phases. The Lagrangian of the QED_2 is given as

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^{\mu}(\partial_{\mu} - \mathrm{i}eA_{\mu})\psi - m\bar{\psi}\psi, \quad (16)$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, m and e are the bare rest mass of the electron and the bare coupling constant, respectively. The bosonization rules give

$$\begin{aligned} :\bar{\psi}\psi: &\to -cmM\cos(2\sqrt{\pi}\phi), \\ :\bar{\psi}\gamma_5\psi: &\to -cmM\sin(2\sqrt{\pi}\phi), \\ j_{\mu} = :\bar{\psi}\gamma_{\mu}\psi: &\to \frac{1}{\sqrt{\pi}}\varepsilon_{\mu\nu}\partial^{\nu}\phi, \\ :\bar{\psi}\mathrm{i}\partial\psi: &\to \frac{1}{2}N_m(\partial_{\mu}\phi)^2, \end{aligned} \quad (17)$$

where N_m denotes normal ordering with respect to the fermion mass m, $c = \exp(\gamma)/2\pi$ with the Euler constant γ, and $M = e/\sqrt{\pi}$ the "meson" mass. The MSG Hamiltonian

$$\mathcal{H} = N_M \int_x \bigg[\frac{1}{2}\Pi_x^2 + \frac{1}{2}(\partial_1\phi_x)^2 + \frac{1}{2}M^2\phi_x^2 + u_1\cos(\beta\phi_x)\bigg] \quad (18)$$

for $\beta^2 = 4\pi$ and $u_1 = cmM$ is the bosonized version of the QED_2. According to Eq. (5) the dimensionful electric charge $e = M\sqrt{\pi}$ does not evolve. Using Eq. (7) the flow of the electron mass $m(k)$

$$m(k) = m(\Lambda)\bigg(\frac{k^2 + M^2}{\Lambda^2 + M^2}\bigg)^{1/2}, \qquad m(\Lambda) \equiv \frac{u_1(\Lambda)\sqrt{\pi}}{ce}, \quad (19)$$

inherits all the properties of the flow of the first Fourier amplitude $u_1(k)$. It implies that there exists a critical value

$$m_c(\Lambda) = \frac{u_c(\Lambda)\sqrt{\pi}}{ce} = \frac{\sqrt{2}}{4\pi c}(\Lambda^2 + e^2/\pi)^{1/2} \quad (20)$$

of the bare mass which determines whether the IR value of the mass depends on its UV value or not. For $m(\Lambda) > m_c(\Lambda)$ the evolution runs into the spinodal instability and the IR value of m becomes independent of its bare value:

$$m(k \to 0) = \frac{2e\sqrt{\pi}}{c4\pi^2}, \quad (21)$$

while for $m(\Lambda) < m_c(\Lambda)$ Eq. (19) implies the IR behavior

$$m(k \to 0) = m(\Lambda)\bigg(\frac{M^2}{\Lambda^2 + M^2}\bigg)^{1/2}. \quad (22)$$

Therefore the sensitivity matrix in the parameter space (m, e) taking the values $S_{1,1} = 0$ for $m(\Lambda) > m_c(\Lambda)$ and

$$S_{1,1} = \frac{\partial m(k)}{\partial m(\Lambda)} = \bigg(\frac{k^2 + e^2/\pi}{\Lambda^2 + e^2/\pi}\bigg)^{1/2}, \quad (23)$$

for $m(\Lambda) < m_c(\Lambda)$ indicates the existence of two different phases in QED_2. Lattice calculations also affirmed this result [17,18]. For large coupling ($e \gg m$), the model has a unique vacuum at $\varphi = 0$. For weak coupling ($e \ll m$), the reflection symmetry is spontaneously broken and the model has nontrivial vacua, located approximately at $\phi = \pm\sqrt{\pi}/2$. According to lattice simulations and density matrix RG studies of the MSG model the critical value which separates the two phases of the model is $m/e_c = 0.3335$. The analytical result $m/e_c = 0.3168$ for the critical point in Eq. (20) suggests that the RG methods using LPA enables us to determine the phase structure of the MSG model in a reliable manner.

VI. INTERNAL-SPACE RENORMALIZATION

One can go beyond the LPA by using the internal-space RG method. We define the generating functional of the connected Green functions as

$$W[j] = \log\int \mathcal{D}\phi e^{-S_B[\phi] + j\cdot\phi}, \quad (24)$$

with external source j_x. The shorthand notation $f \cdot g = \int_x f_x g_x$ is used. The effective action is defined as the Legendre-transform of $W[j]$,

$$\Gamma[\varphi] = j \cdot \varphi - W[j], \quad (25)$$

where the external source j_x is expressed in terms of φ_x according to the implicit equation

$$\varphi = \frac{\delta W[j]}{\delta j}. \quad (26)$$

The idea of internal-space RG is to eliminate quantum fluctuations successively ordering them according to their amplitudes. This can be achieved by introducing an additional mass term into the action,

$$S_\lambda[\phi] = S_B[\phi] + \frac{1}{2}\lambda^2\phi^2 \tag{27}$$

with the control parameter λ. For $\lambda = \lambda_0$ being of the order of the UV cutoff Λ the large-amplitude fluctuations are suppressed and decreasing the evolution parameter λ towards zero, they are continuously accounted for. Let us separate off the suppressing mass term from the evolving effective action,

$$\Gamma_\lambda[\varphi] = \bar{\Gamma}_\lambda[\varphi] + \frac{1}{2}\lambda^2\varphi^2, \tag{28}$$

and use the ansatz

$$\bar{\Gamma}_\lambda[\varphi] = \int_x \left[\frac{z_\lambda}{2}(\partial_\mu\varphi_x)^2 + U_\lambda(\varphi_x)\right] \tag{29}$$

with $U_\lambda(\varphi) = \frac{1}{2}M_\lambda^2\varphi^2 + V_\lambda(\varphi)$ and $V_\lambda(\varphi) = \sum_{n=1}^\infty u_n(\lambda)\cos(n\varphi)$. We introduced the function z_λ which stands for the wave-function renormalization. In order to get back the LPA approximation one has to choose $z_{\lambda=\Lambda} = 1/\beta^2$ and a simple scaling transformation of the internal space gives back the ansatz in Eq. (1) for the action. The functional evolution equation is

$$\partial_{\lambda^2}\bar{\Gamma}_\lambda = \frac{1}{2}\mathrm{Tr}[\lambda^2\delta_{x,y} + \bar{\Gamma}^{(2)}_{\lambda x,y}]^{-1}, \tag{30}$$

where $\bar{\Gamma}^{(2)}_{\lambda x,y} = \delta^2\bar{\Gamma}_\lambda/\delta\varphi_x\delta\varphi_y$, [13], reads

$$\partial_{\lambda^2}V_\lambda(\varphi) = \frac{1}{2}\int_{\mathbf{p}} \frac{1}{z_\lambda\mathbf{p}^2 + \lambda^2 + M_\lambda^2 + V_\lambda''(\varphi)},$$
$$\partial_{\lambda^2}z_\lambda = \frac{1}{2}\int_0^\Lambda \frac{d\mathbf{p}}{(2\pi)^2}\left[\frac{-2z_\lambda V'''^2}{(z_\lambda\mathbf{p}^2 + \lambda^2 + M_\lambda^2 + V_\lambda''(\varphi))^4} + \frac{4\mathbf{p}^2 z_\lambda^2 V'''^2}{(z_\lambda\mathbf{p}^2 + \lambda^2 + M_\lambda^2 + V_\lambda''(\varphi))^5}\right] \tag{31}$$

for homogeneous field configurations φ and the potential $V_\lambda(\varphi)$ of the ansatz Eq. (29), where $V_\lambda''(\varphi) = \partial^2 V_\lambda(\varphi)/\partial\varphi^2$ and $V_\lambda'''(\varphi) = \partial^3 V_\lambda(\varphi)/\partial\varphi^3$.

The two-dimensional momentum integrals can easily be performed, giving

$$(1 + \lambda^2\partial_{\lambda^2})\tilde{V}_\lambda = \frac{1}{8\pi z_\lambda}\log\left[\frac{z_\lambda(\Lambda/\lambda)^2 + 1 + \tilde{M}_\lambda^2 + \tilde{V}_\lambda''}{1 + \tilde{M}_\lambda^2 + \tilde{V}_\lambda''}\right],$$
$$\partial_{\lambda^2}z_\lambda = \frac{\tilde{V}'''^2}{8\pi}\left[\frac{1}{3(z_\lambda(\Lambda/\lambda)^2 + 1 + \tilde{M}_\lambda^2 + \tilde{V}_\lambda'')^3} - \frac{1}{3(1 + \tilde{M}_\lambda^2 + \tilde{V}_\lambda'')^3} - \frac{z_\lambda\Lambda^2}{(z_\lambda(\Lambda/\lambda)^2 + 1 + \tilde{M}_\lambda^2 + \tilde{V}_\lambda'')^4}\right], \tag{32}$$

where the notation of the dependences on the field variable φ is suppressed. By performing the Fourier expansion in both sides of Eqs. (32) one obtains a set of differential equations for the couplings $\tilde{u}_n(\lambda)$, z_λ, and $\tilde{M}_\lambda$. Since the left-hand side of the first one of Eqs. (31) does not contain polynomial terms, the mass parameter does not evolve, $M_\lambda^2 = M^2$. Thus the mass is a relevant parameter for all scales,

$$\tilde{M}_\lambda^2 = \lambda^{-2}M^2, \tag{33}$$

cf. Eq. (5). Note that the argument of the logarithm in the right-hand side of the first one of Eqs. (32) should be positive, otherwise the evolution fails. This criterion is reminiscent of the WH-RG method and it may generate spinodal instability like singularities on the internal-space RG renormalized trajectory, to be interpreted as quantum phase transition. In order to obtain the evolution equations for the couplings more easily we differentiate the first one of Eqs. (32) with respect to φ,

$$-\frac{1}{8\pi z_\lambda}\frac{\Lambda^2}{\lambda^2}\tilde{V}_\lambda''' = \left(z_\lambda\frac{\Lambda^2}{\lambda^2} + 1 + \tilde{M}^2 + \tilde{V}_\lambda''\right) \times (1 + \tilde{M}^2 + \tilde{V}_\lambda'')(1 + \lambda^2\partial_{\lambda^2})\tilde{V}_\lambda'. \tag{34}$$

The evolution should be started at $\lambda_0^2 \sim \mathcal{O}(\Lambda^2) \gg M^2 \gg |V_\lambda''|$. At these "UV" scales the large-amplitude field fluctuations with $|\varphi|^2 \gg |V_\lambda''|/\lambda^2$ are suppressed and the effective action can be calculated perturbatively.

A. Asymptotic scaling

The asymptotic scaling for λ^2 satisfying $\Lambda^2 \gg \lambda^2 \gg M^2 \gg |V_\lambda''|$ can be deduced by using the independent mode approximation. In this limit the wave-function renormalization z_λ does not evolve, $z_\lambda = z_\Lambda = 1/\beta^2$, so the evolution equation becomes

$$(1 + \lambda^2\partial_{\lambda^2})\tilde{V}_\lambda = -\frac{1}{8\pi z}\left[\frac{1}{1 + \tilde{M}^2} - \frac{1}{\Lambda^2/\lambda^2 + 1 + \tilde{M}^2}\right]\tilde{V}'', \tag{35}$$

where the field independent terms are neglected. The evolutions of the various couplings decouple and one finds

$$\tilde{u}_n(\lambda) = \tilde{u}_n(\lambda_0)\left(\frac{\lambda}{\Lambda}\right)^{-2}\left(\frac{\lambda^2 + M^2}{\Lambda^2 + \lambda^2 + M^2}\right)^{n^2\beta^2/8\pi} \tag{36}$$

for the dimensionless couplings, yielding

$$\tilde{u}_n \sim \lambda^{(n^2\beta^2/4\pi)-2}. \tag{37}$$

B. Ionized phase

In order to find the scaling laws in the nonasymptotic regime one has to solve a system of coupled evolution equations for the couplings $\tilde{u}_n(\lambda)$ and z_λ as the control parameter λ is decreased from $\lambda_0 = \Lambda$ down to $\lambda = 0$. The evolution equations have been derived and solved numerically. It was found that the increase of the number of the couplings $\tilde{u}_n$ beyond $n = 10$ does not influence the evolution of the first few couplings, similar to the WH-RG equations.

The evolution of the first four couplings, $\tilde{u}_1, \ldots, \tilde{u}_4$ is shown in Fig. 5 for $1/z_\Lambda = 12\pi$. The internal-space RG method gives qualitatively the same scaling laws both in the UV and in the IR regions just as the WH-RG method, which means that the evolution of z_λ is negligible, $z_\lambda \approx z_\Lambda$. The numerical value of the fundamental coupling constant $\tilde{u}_1(\lambda)$ was found to follow closely the analytic form of Eq. (36). The renormalized trajectory shares the feature, known from the WH-RG scheme that the SG and the MSG models with $1/z_\Lambda > 8\pi$ agree in the IR scaling regime down to the mass gap, therefore the evolution of z is trivial.

1. IR scaling

The decrease of the control parameter λ drives us out from the asymptotic region. According to Fig. 5 the IR scaling is $\tilde{u}_n \sim \lambda^{n(1/4\pi z_\Lambda - 2)}$. Such a scaling behavior can be obtained analytically from the functional RG equations (32) when the evolution of z_λ is neglected,

$$(1 + \lambda^2\partial_{\lambda^2})\tilde{V}'_\lambda + \tilde{V}''_\lambda(1 + \lambda^2\partial_{\lambda^2})\tilde{V}'_\lambda = -\frac{1}{8\pi z_\Lambda}\tilde{V}'''_\lambda \quad (38)$$

for $M^2 \ll \lambda^2 \ll \Lambda^2$. Making the ansatz

$$\tilde{u}_n(\lambda) = c_n\lambda^{n\eta} \quad (39)$$

with $\eta \geq 0$. For $n = 1$ one gets $\eta = 1/4\pi z_\Lambda - 2 > 0$. For $n > 1$ we obtain the recursion relation

$$c_n = \frac{\frac{1}{2}\beta^2\sum_{s=1}^{n-1}(2 + s\eta)s(n-s)^2 c_{n-s}c_s}{n(2 + n\eta - n^2\frac{\beta^2}{4\pi})}. \quad (40)$$

The coefficients c_n can be expressed in terms of c_1, since $c_1 = \tilde{u}_1(\Lambda)(\lambda/\Lambda)^\eta$ and therefore $c_n = (-1)^{n+1}\tilde{u}_1^n(\Lambda)R_n$, with $R_1 = 1$ and all R_n being independent of the bare couplings. These properties were confirmed numerically. They imply that the dimensionless effective action in the IR regime can be parametrized by the single bare parameter $\tilde{u}_1(\lambda)$. It is clear from Fig. 5 that the evolution of the potential freezes out below the mass gap and a nontrivial dimensionless potential is left over in the IR end point. The IR scaling is trivial for $\lambda < M$, $\tilde{u}_n(\lambda) \sim \lambda^{-2}$, just as in the case of the WH-RG method. Since the evolutions for the SG and MSG models are identical down to the scale $\lambda \sim M$, the evolving effective action of the MSG model inherits the properties of the SG model, namely, it depends on the initial value of the fundamental mode $\tilde{u}_1(\lambda_0)$ only. In fact, it was found numerically that $R_n^{\rm MSG} = |u_n(\lambda)|/u_1^n(\lambda)$ is RG invariant in the scaling region $\lambda < M$.

One sees also that for $1/z_\Lambda > 8\pi$ theories with various values of the mass M belong to the same phase. This conclusion arises because one detects the same scaling behavior of the dimensionless parameters $\tilde{u}_n(\lambda)$ for $\lambda/M < 1$ and the same qualitative behavior of the RG invariant constants $R_n^{\rm MSG}$ for all values of M.

C. Molecular phase

The typical evolution is depicted for several bare values $\tilde{u}_1(\Lambda)$ in Fig. 6 for $1/z_\Lambda < 8\pi$. Numerics shows immediately, that the evolution stops at a nonvanishing value of $\lambda = \lambda_c$ due to the appearing of a negative argument of the logarithm in the first one of Eqs. (32) for $\tilde{u}_1(\Lambda) > \tilde{u}_{1c}$, where $\tilde{u}_{1c}$ is a critical value of the first Fourier amplitude of the bare periodic potential. Then the evolution equation loses its validity and presumably an alternative RG equa-

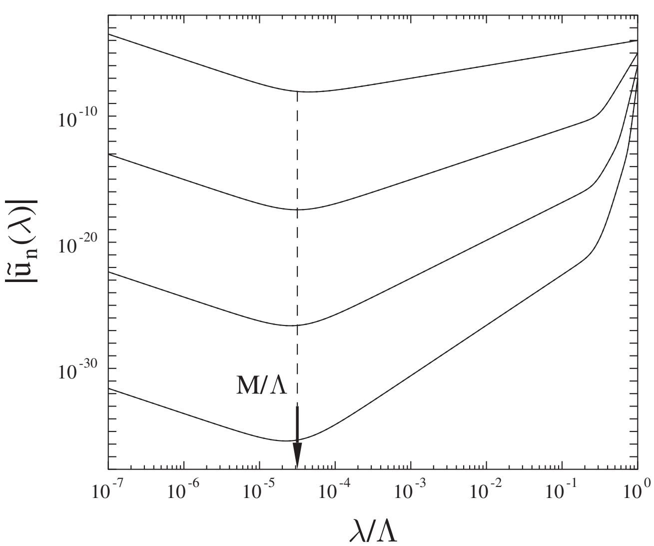

FIG. 5. RG flow of $\tilde{u}_1, \ldots, \tilde{u}_4$ for $1/z_\Lambda = 12\pi$ and $M^2 = 10^{-9}\Lambda^2$. The plot shows the flows obtained from the LPA ($z_\lambda = z_\Lambda$) and from beyond the LPA approximations but the flows coincide. The evolution is similar to the WH-RG evolution obtained in the previous section.

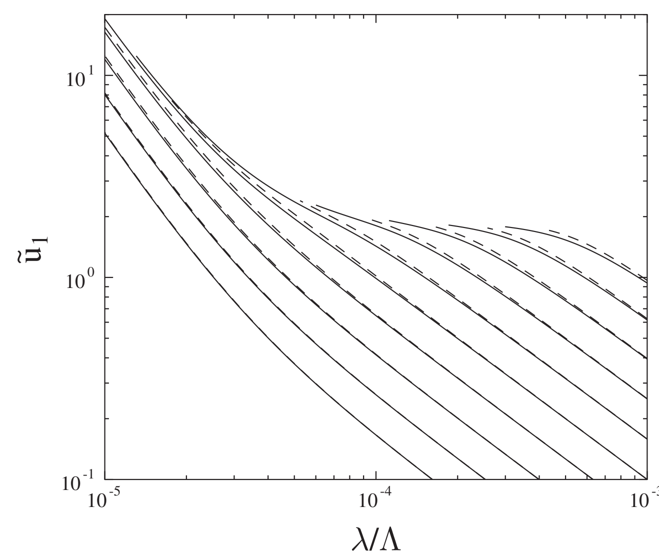

FIG. 6. Internal-space RG evolution of the coupling $\tilde{u}_1$ for several different initial values at $M^2 = 10^{-9}\Lambda^2$ for constant $z_\lambda = z_\Lambda$ (solid lines) and for running z_λ (dashed lines). In the vicinity of the phase boundary the flows depart slightly.

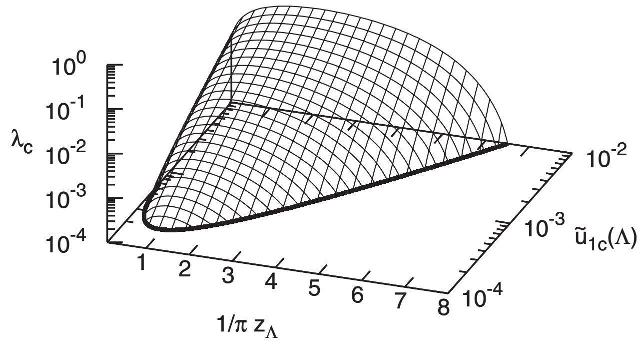

FIG. 7. The scale λ_c, where the singularity appears, as the function of the initial value of the first Fourier amplitude u_{1c} and $1/\pi z_\Lambda$ for $M^2 = 10^{-4}$.

tion is necessary as in the WH-RG framework, where the appearance of the spinodal instability implies a tree-level blocking relation [3,14,15]. Though plausible, it is not obvious that the singularity in the internal-space evolution is also rooted in the spinodal instability. The clarification of this point needs further efforts.

For bare values $\tilde{u}_1(\Lambda) < \tilde{u}_{1c}$ the evolution does not stop and goes below the mass scale. Then, as in the case of $1/z_\Lambda > 8\pi$, the trivial scaling $\tilde{u}_n(\lambda) \sim \lambda^{-2}$ is obtained in the limit $\lambda \to 0$. The running z_λ modifies the RG flow a bit, but only quantitatively changes the boundary of the phases.

Although one cannot follow the RG evolution for $\lambda < \lambda_c$ for $\tilde{u}_1(\Lambda) > \tilde{u}_{1c}$ by means of Eqs. (32), one can still determine the critical value $\tilde{u}_{1c}$ analytically, when the local periodic potential is restricted to its first Fourier mode and the LPA approximation is used. The singularity appears during the evolution at the scale λ_c satisfying $\lambda_C^2 + M^2 + V''_{\lambda_c}(\varphi) = 0$ for some φ. Using Eq. (36) as a good approximation of the scaling for $\lambda > \lambda_c$, one finds

$$\lambda_c^2 = \Lambda^2 \left(\frac{\tilde{u}_1(\Lambda)}{z_\Lambda}\right)^{(1-8\pi z_\Lambda)/8\pi z_\Lambda} - M^2 \qquad (41)$$

for $1/z_\Lambda < 8\pi$. The negativity of the right-hand side suggests that the coupling constant is sufficiently weak to allow the mass term to remove the singularity. For the opposite case $\lambda_c^2 > 0$ one can estimate the critical value of the coupling constant by equating λ_c to M in Eq. (41),

$$\tilde{u}_{1c} = z_\Lambda \left(\frac{2M^2}{\Lambda^2}\right)^{1-(1/8\pi z_\Lambda)}. \qquad (42)$$

The value of λ_c is plotted on the plane $(1/\pi z_\Lambda, \tilde{u}_1(\Lambda))$ in Fig. 7. In contrast to the SG model where the singularity appears for all $\tilde{u}_1(\Lambda)$ for $1/z_\Lambda < 8\pi$ the mass term always wins at the IR end point of the evolution for the MSG model and removes the singularity at some low but finite scale λ_c. This phase structure qualitatively remains unchanged when one considers many couplings and the running z_λ.

VII. SUMMARY

Some global features of the RG flow of the MSG model are discussed in this work. It is shown that the model possesses condensate of elementary excitations with nonvanishing momentum, spinodal instability, for weak enough mass in the remnant of the molecular phase of the SG model. This condensate, the sign of the periodicity of the local potential, generates nontrivial effective potential and phase structure despite of the explicit, stable mass for elementary excitations in the deep IR region. The sensitivity matrix allows us to study the way the ultraviolet parameters influence the IR physics. It was found that the suppression of the sensitivity on the nonrenormalizable bare coupling constants, generated in the UV scaling regime, can be overturned by the increasing sensitivity piled up in the IR scaling regime if the UV and the IR cutoffs are removed in a coordinated manner. As a result, a nontrivial, global extension of the universality is found which goes beyond the local studies of the RG flow around the UV fixed point only.

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