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AKADÉMIAI KIADÓ

INTERVAL CHAINS AND COMPLETENESS IN ULTRAPOWERS OF ORDERED SETS

Zoltán Boros[1,*] and Péter Tóth[1]

[1] Institute of Mathematics, University of Debrecen, Pf. 400, 4002 Debrecen, Hungary

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ABSTRACT

The ultrapower T^* of an arbitrary ordered set T is introduced as an infinitesimal extension of T. It is obtained as the set of equivalence classes of the sequences in T, where the corresponding relation is generated by a free ultrafilter on the set of natural numbers. It is established that T^* always satisfies Cantor's property, while one can give the necessary and sufficient conditions for T so that T^* would be complete or it would fulfill the open completeness property, respectively. Namely, the density of the original set determines the open completeness of the extension, while independently, the completeness of T^* is determined by the cardinality of T.

KEYWORDS

ordered sets, interval chains, Cantor's property, completeness, ultrafilter, ultrapower

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1. INTRODUCTION

A well known statement from the theory of ordered fields is that an ordered field is complete if and only if it simultaneously fulfills the Archimedean property and Cantor's property. To demonstrate the independence of these properties, one needs to construct an ordered field which fulfills Cantor's property but is not complete. This question is usually treated in the framework of non-standard analysis, for instance in the works of Stroyan and Luxemburg [7], [5].

However in the above cited publications one can also find an idea for a construction that does not require the development of the theory of non-standard analysis or any other specific method of formal logic. This idea can be easily described as follows. Let us choose an adequate family of subsets of the set of natural numbers called a free ultrafilter, and then use it to define an equivalence relation on the set of all sequences of the elements of a given set R. This provides a partition, and the set of the equivalence classes is called the ultrapower of R. Clearly, the result of this method is an extension of the original set, since the classes of the constant sequences can be considered as representatives of the original elements.

* Corresponding author. E-mail: zboros@science.unideb.hu

The aforementioned works contain only the main ideas without the technical parts, but in the recent publication [2] of Corazza, a detailed construction of the ultrapower of $\mathbb{Q}$ (denoted by $\mathbb{Q}^{\mathbb{N}}/U$) is displayed. One of the main objectives of his work is to construct a non-Archimedean ordered field that fulfills Cantor's property (in [2] it is referred as Nested Intervals Property). As mentioned above, such a construction provides an ordered field with the Cantor property which is not complete.

Not surprisingly, only the ordering of the ultrapower plays an important role during the investigation of these two order-related properties: completeness and the Cantor property, while the field operations are irrelevant at that point. Actually this fact motivates us to generalize these constructions, introducing the ultrapower of an arbitrary ordered set T, and investigating Cantor's property and completeness in its extension T^*.

We may note that while free ultrafilters play central role in all of the constructions cited before, the concepts of filters and ultrafilters are more widely investigated than the corresponding free (or nonprincipal) ones. In fact the ultrafilter used in the construction of the ultrapower has to fulfill some conditions which ensure that the extension is proper (not trivial), but these conditions do not appear in the classical definition of the ultrafilter, e.g. in the monograph of Jech [3]. Hence in [5] and [7] a so-called free ultrafilter is used, while in [2] a nonprincipal ultrafilter is used. We avoid the elaboration of the details of the proof of the existence of a free ultrafilter by appropriately using Tarski's classical existence theorem for ultrafilters (which may be found in [3, Theorem 7.5]).

Our process of showing Cantor's property for the ultrapower sometimes resembles Corazza's methods, although at one point the fact that we start from an arbitrary ordered set makes a significant difference. Namely, in [2] it is shown that in $\mathbb{Q}^{\mathbb{N}}/U$ the intersection of a chain of countably many open intervals is nonempty. This property is referred to as open completeness, and it clearly implies both Cantor's property and the lack of completeness (due to the non-Archimedean feature). In our more abstract setting it is reasonable to investigate these properties separately. Namely, starting from an ordered set T, we shall prove that T^* always satisfies Cantor's property (as well as its extension to nested non-empty open intervals), while we can give the necessary and sufficient conditions for T so that T^* would be complete or it would fulfill the open completeness property (in the sense of [2, Definition 3.2]), respectively. Namely, the density of the original set determines the open completeness of the extension, while independently, the completeness of T^* is determined by the cardinality of T.

2. PARTICULAR PROPERTIES OF ORDERED SETS

In this section we collect the basic concepts for ordered sets that are in the focus of this paper.

As usual, we call a nonempty set X equipped with a relation $\leq$ (on X) an ordered set if the relation $\leq$ is reflexive, anti-symmetric, transitive, and linear (i.e., $x \leq y$ or $y \leq x$ for all $x, y \in X$).

Once the relation $\leq$ on X is given, we shall also use the relations $\geq$, $<$ and $>$ in the usual sense.

We shall use the concepts of *lower/upper bound*, *minimum/maximum* (denoted by min and max, respectively), a set being *bounded from below/above*, *least upper bound* (sup) and *greatest lower bound* (inf) in the usual sense as well [3, Definition 2.2].

DEFINITION 2.1. An ordered set $(X, \leq)$ is called *complete* if every nonempty subset of X, that is bounded from above, has a least upper bound.

It is well known that the ordered set $(X, \leq)$ is complete if, and only if, every nonempty subset of X, that is bounded from below, has a greatest lower bound (a proof in an abstract setting can be found, for instance, in [1, Theorem 4.6]).

We will define *intervals* as particular subsets of an ordered set $(X, \leq)$ in the usual way. Namely, if $a, b \in X$ such that $a \leq b$, let $[a, b] = \{x \in X \mid a \leq x \leq b\}$. Similarly, if $a, b \in X$ such that $a < b$, let $]a, b[= \{x \in X \mid a < x < b\}$.

We call a sequence $\langle I_n \rangle$ of non-empty intervals an *interval chain* if $I_{n+1} \subseteq I_n$ for every $n \in \mathbb{N}$. We can describe Cantor's property and the open completeness of an ordered set X by the phenomena that the intersection of an arbitrary interval chain of closed, respectively, open intervals is non-empty.

DEFINITION 2.2. We say that an ordered set X satisfies *Cantor's property* if

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \emptyset$$

for any sequences $\langle a_n\rangle, \langle b_n\rangle : \mathbb{N} \longrightarrow X$ fulfilling

$$a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$$

for every $k \in \mathbb{N}$.

DEFINITION 2.3. We say that an ordered set X is *open complete* if

$$\bigcap_{n\in\mathbb{N}}]a_n, b_n[\neq \emptyset$$

for any sequences $\langle a_n\rangle, \langle b_n\rangle : \mathbb{N} \longrightarrow X$ fulfilling

$$a_k \leq a_{k+1} < b_{k+1} \leq b_k$$

for every $k \in \mathbb{N}$.

Finally, we introduce the concept of density in ordered sets.

DEFINITION 2.4. We say that an ordered set X is *dense everywhere* if, for any $a, b \in X$ fulfilling $a < b$, there exists $c \in X$ such that $a < c < b$.

3. AN EXTENSION OF ORDERED SETS

3.1. Ultrafilter

We introduce the concept of a free ultrafilter. For the power set of an arbitrary set X we will use the notation $\mathcal{P}(X)$, i.e. the elements of $\mathcal{P}(X)$ are the subsets of X.

DEFINITION 3.1. Let $\mathbf{J}$ be an infinite set. The nonempty family of sets $\mathcal{U} \subseteq \mathcal{P}(\mathbf{J})$ is called a *free filter* on $\mathbf{J}$, if

(1) $K \in \mathcal{U}$ and $K \subseteq L \subseteq \mathbf{J}$ implies $L \in \mathcal{U}$,
(2) $K, L \in \mathcal{U}$ implies $K \cap L \in \mathcal{U}$,
(3) $K \in \mathcal{U}$ implies that K is infinite.

Moreover, $\mathcal{U}$ is called a *free ultrafilter* if it is a free filter and

(4) if $K \subseteq \mathbf{J}$, then $K \in \mathcal{U}$ or $\mathbf{J} \setminus K \in \mathcal{U}$ holds.

REMARK 3.2. In many works (such as [3]) filters and ultrafilters are defined on arbitrary sets (and not particularly infinite ones) such that assumption *(3)* is replaced by the weaker condition

(3') $\emptyset \notin \mathcal{U}$.

It is known that any filter can be extended to an ultrafilter (see [3, Theorem 7.5]). Using this result we prove the following statement.

THEOREM 3.3. Let $\mathbf{J}$ be an infinite set and let $K \subseteq \mathbf{J}$ be also infinite. Then there exists a free ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\mathbf{J})$ such that $K \in \mathcal{U}$.

Proof. Let us define the so-called Fréchet filter:

$$\mathcal{F} = \{S \subseteq \mathbf{J} \mid \mathbf{J} \setminus S \text{ is finite.}\}$$

It is easy to see that $\mathcal{F}$ is indeed a filter. Let us define another subset of $\mathcal{P}(\mathbf{J})$:

$$\mathcal{M} = \{M \subseteq \mathbf{J} \mid \exists L \in \mathcal{F} : K \cap L \subseteq M\}.$$

Now we show that $\mathcal{M}$ is a filter. Let M, N be arbitrary sets in $\mathcal{M}$, thus there exist $L_M, L_N \in \mathcal{F}$ such that $K \cap L_M \subseteq M$ and $K \cap L_N \subseteq N$.

(1) If $M \subseteq S$ then $K \cap L_M \subseteq M \subseteq S$, so $S \in \mathcal{M}$.
(2) $K \cap (L_M \cap L_N) = (K \cap L_M) \cap (K \cap L_N) \subseteq M \cap N$, and as $L_M \cap L_N \in \mathcal{F}$, it also holds that $M \cap N \in \mathcal{M}$.

(3') Assume $\emptyset \in \mathcal{M}$, which means $K \cap L = \emptyset$ for some $L \in \mathcal{F}$. But this would imply $K \subseteq \mathbf{J} \setminus L$ and that is impossible, since K is infinite and $\mathbf{J} \setminus L$ is finite.

Notice that $\mathcal{F} \subseteq \mathcal{M}$ trivially holds. Now $\mathcal{M}$ can be extended to an ultrafilter $\mathcal{U}$. However $\mathcal{U}$ is, in particular, a free ultrafilter. Indeed, if $F \in \mathcal{U}$ for some finite subset F, then $\mathbf{J} \setminus F \notin \mathcal{U}$ which contradicts $\mathbf{J} \setminus F \in \mathcal{F}$. □

As a corollary of this statement, we get that there exists a free ultrafilter on the set of natural numbers.

3.2. Ultrapower of an ordered set

In the next step we construct a so called *ultrapower* of any ordered set T. The existence of a free ultrafilter on the set of natural numbers provides us a way to define an equivalence relation on the set of all sequences of elements of T, in such manner that an adequate order on the equivalence classes would generate an ordered set. As it is common in the literature, we will use an asterisk (*) to denote the operation that assigns its ultrapower to the original ordered set.

In the subsequent sections let T be an ordered set and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

Let $\mathcal{T} = T^{\mathbb{N}}$ denote the set of all sequences of elements of T.

PROPOSITION 3.4. Let us define the relation $\sim$ on $\mathcal{T}$ in the following way: for any $\langle a_n \rangle \in \mathcal{T}$ and $\langle b_n \rangle \in \mathcal{T}$,

$$\langle a_n \rangle \sim \langle b_n \rangle \iff \{n \in \mathbb{N} \mid a_n = b_n\} \in \mathcal{U}.$$

Then $\sim$ is an equivalence relation. Furthermore, let us denote the set of the equivalence classes by T^*, while the class of an element $\langle a_n \rangle \in \mathcal{T}$ be denoted by $\overline{\langle a_n \rangle}$. The relation $\leq$ on T^*, given by

$$\overline{\langle a_n \rangle} \leq \overline{\langle b_n \rangle} \iff \{n \in \mathbb{N} \mid a_n \leq b_n\} \in \mathcal{U},$$

is well-defined, and $(T^*, \leq)$ is an ordered set.

The statement follows from a general extension theorem by Łoś [4], or it can be easily verified by the reader.

3.3. Cantor's property for the extension

In this section we will show that the operation * always produces an ordered set that satisfies Cantor's property. At first we will prove that an arbitrary chain of nonempty open intervals of the extension has a common point.

THEOREM 3.5. Let T be an ordered set, moreover let $a_k = \big\langle (a_k)_n \big\rangle \in \mathcal{T}$ and $b_k = \big\langle (b_k)_n \big\rangle \in \mathcal{T}$ $(k \in \mathbb{N})$ be sequences such that, for every $k \in \mathbb{N}$,

$$\overline{a_k} \leq \overline{a_{k+1}} < \overline{b_{k+1}} \leq \overline{b_k} \qquad \text{and} \qquad \big]\overline{a_k}, \overline{b_k}\big[\neq \emptyset.$$

Then

$$\bigcap_{k \in \mathbb{N}} \big]\overline{a_k}, \overline{b_k}\big[\neq \emptyset.$$

Proof. Since $\big]\overline{a_k}, \overline{b_k}\big[\neq \emptyset$ is assumed, for every $k \in \mathbb{N}$, there exist $c_k = \big\langle (c_k)_n \big\rangle \in \mathcal{T}$ such that $\overline{a_k} < \overline{c_k} < \overline{b_k}$. Let us define the following sets:

$$A_i = \{n \in \mathbb{N} \mid (a_i)_n \leq (a_{i+1})_n\},$$
$$B_i = \{n \in \mathbb{N} \mid (b_i)_n \geq (b_{i+1})_n\},$$
$$\text{and} \qquad C_i = \{n \in \mathbb{N} \mid (a_i)_n < (c_i)_n < (b_i)_n\} \qquad \text{for every } i \in \mathbb{N}.$$

Defining, in addition, $A_0 := \mathbb{N}$ and $B_0 := \mathbb{N}$, we construct the following sets:

$$\mathcal{A}_k = \bigcap_{i=0}^{k-1} A_i, \qquad \mathcal{B}_k = \bigcap_{i=0}^{k-1} B_i, \qquad \mathcal{C}_k = \bigcap_{i=1}^{k} C_i \qquad (k \in \mathbb{N}).$$

Obviously, for every $k \in \mathbb{N}$, the sets $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}_k$ belong to $\mathcal{U}$, as they are intersections of finitely many sets from $\mathcal{U}$. For the same reason $\mathcal{A}_k \cap \mathcal{B}_k \cap \mathcal{C}_k =: \mathcal{D}_k \in \mathcal{U}$.

Let us observe that, for every $k \in \mathbb{N}$, the set $\mathcal{D}_k$ consists of the natural numbers n for which the following inequalities hold:

$$(a_1)_n \le \cdots \le (a_k)_n < (b_k)_n \le \cdots \le (b_1)_n \qquad \text{and} \qquad (a_1)_n < (c_1)_n < (b_1)_n, \ldots, (a_k)_n < (c_k)_n < (b_k)_n.$$

In the next step, for every $n \in \mathbb{N}$, we define another set I_n of natural numbers as follows: $I_n = \{k \in \mathbb{N} \mid n \in \mathcal{D}_k\}$. As it is easy to see from the definition of the sets $\mathcal{D}_k$, if $k \in I_n$ then $l \in I_n$ is also true for every natural number $l \le k$. Using these sets we assign a non-negative integer α_n to every $n \in \mathbb{N}$ as follows: let

$$\alpha_n = \begin{cases} 0, & \text{if } I_n = \emptyset, \\ n, & \text{if } I_n \text{ has no upper bound}, \\ \min\{n, \max I_n\} & \text{if } I_n \text{ is non-empty and bounded from above.} \end{cases}$$

We should note that if n is an element of $\mathcal{D}_k$ and $k \le n$ then $\alpha_n \ge k$ ($k, n \in \mathbb{N}$). This also means that $n \notin \mathcal{D}_1$ holds if and only if $I_n = \emptyset$. Hence $n \in \mathcal{D}_k \setminus \{m \in \mathbb{N} \mid m < k\}$ implies

$$(a_1)_n \le \ldots \le (a_k)_n \le (a_{\alpha_n})_n < (c_{\alpha_n})_n < (b_{\alpha_n})_n \le (b_k)_n \le \ldots \le (b_1)_n.$$

It is trivial that $\mathcal{D}_k \setminus \{m \in \mathbb{N} \mid m < k\}$ is in the free ultrafilter $\mathcal{U}$.

After these remarks it is rather easy to construct a common point of the interval chain. We define the sequence $d = \langle d_n \rangle : \mathbb{N} \longrightarrow T$ as follows:

$$d_n = \begin{cases} (c_{\alpha_n})_n, & \text{if } n \in \mathcal{D}_1, \\ (a_1)_1, & \text{if } n \notin \mathcal{D}_1. \end{cases}$$

We will show that $\overline{a_k} < \overline{d} < \overline{b_k}$ for any $k \in \mathbb{N}$. This is a straightforward corollary of our previous remark, namely that

$$(a_1)_n \le \ldots \le (a_k)_n \le (a_{\alpha_n})_n < (c_{\alpha_n})_n = d_n < (b_k)_n \le \ldots \le (b_1)_n$$

holds if $n \in \mathcal{D}_k \setminus \{m \in \mathbb{N} \mid m < k\}$. Since $\mathcal{D}_k \setminus \{m \in \mathbb{N} \mid m < k\} \in \mathcal{U}$, the sets

$$\{n \in \mathbb{N} \mid (a_k)_n < d_n\} \quad \text{and} \quad \{n \in \mathbb{N} \mid d_n < (b_k)_n\}$$

are also elements of $\mathcal{U}$ (obviously they are supersets of $\mathcal{D}_k \setminus \{m \in \mathbb{N} \mid m < k\}$).

The final step of the proof is to use the definition of the ordering relation on T^*, so we obtain

$$\overline{d} \in \bigcap_{k \in \mathbb{N}} \left]\overline{a_k}, \overline{b_k}\right[. \qquad \square$$

THEOREM 3.6. If T is an ordered set then its extension T^* satisfies Cantor's property, i.e. if $a_k = \left\langle (a_k)_n \right\rangle \in \mathcal{T}$ and $b_k = \left\langle (b_k)_n \right\rangle \in \mathcal{T}$ ($k \in \mathbb{N}$) are such that for every $k \in \mathbb{N}$

$$\overline{a_k} \le \overline{a_{k+1}} \le \overline{b_{k+1}} \le \overline{b_k},$$

then

$$\bigcap_{k \in \mathbb{N}} \left[\overline{a_k}, \overline{b_k}\right] \ne \emptyset.$$

Proof. Let us first consider the case when, for every $k \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that $k < r$ and $\overline{a_k} < \overline{a_r}$. Then, for every $k \in \mathbb{N}$, there exist $p \in \mathbb{N}$ and $q \in \mathbb{N}$ such that $k < p < q$ and

$$\overline{a_k} < \overline{a_p} < \overline{a_q} \le \overline{b_q} \le \overline{b_p} \le \overline{b_k},$$

thus $\overline{a_k} < \overline{a_p} < \overline{b_k}$. Therefore, $\left]\overline{a_k}, \overline{b_k}\right[\ne \emptyset$ is also fulfilled for all $k \in \mathbb{N}$. Hence, due to Theorem 3.5, we have

$$\bigcap_{k \in \mathbb{N}} \left[\overline{a_k}, \overline{b_k}\right] \supseteq \bigcap_{k \in \mathbb{N}} \left]\overline{a_k}, \overline{b_k}\right[\ne \emptyset.$$

Otherwise, if there exists $m \in \mathbb{N}$ such that $\overline{a_m} = \overline{a_l}$ for all $l \in \mathbb{N}$, $l > m$ then clearly

$$\bigcap_{k \in \mathbb{N}} \left[\overline{a_k}, \overline{b_k}\right] \supseteq \{\overline{a_m}\} \ne \emptyset. \qquad \square$$

REMARK 3.7. It seems reasonable to make a similar proposition and replace the closed intervals by open intervals. However if we do so then we must require the ordered set T be dense everywhere. Otherwise a trivial counterexample can be made as an empty open interval exists.

On the other hand, the criterion concerning the density of T is sufficient to prove the alternate form of the previous theorem (i.e. open completeness).

We sum up these perceptions in the following theorem:

THEOREM 3.8. Let T be an ordered set. The following statements are equivalent:

(a) T is dense everywhere.

(b) if $a_k = \left\langle (a_k)_n \right\rangle \in \mathcal{T}$ and $b_k = \left\langle (b_k)_n \right\rangle \in \mathcal{T}$ ($k \in \mathbb{N}$) such that for every $k \in \mathbb{N}$

$$\overline{a_k} \leq \overline{a_{k+1}} < \overline{b_{k+1}} \leq \overline{b_k},$$

then

$$\bigcap_{k\in\mathbb{N}} \left]\overline{a_k}, \overline{b_k}\right[\neq \emptyset.$$

Proof. To show (b) $\Longrightarrow$ (a) we explain the counterexample which was mentioned in Remark 3.7. Let $p, q \in T$ such that $p < q$ and there is no element of T in the open interval $]p, q[$.

This means that the open interval $]\overline{p}, \overline{q}[$ is also empty, where $\overline{p}$ and $\overline{q}$ are the classes of the constant sequences $\langle p_n \rangle$ and $\langle q_n \rangle$ defined by $p_n = p$ and $q_n = q$ for every $n \in \mathbb{N}$.

Thus if $p_k := \langle p_n \rangle \in \mathcal{T}$ and $q_k := \langle q_n \rangle \in \mathcal{T}$ for every $k \in \mathbb{N}$, then

$$\bigcap_{k\in\mathbb{N}}]\overline{p_k}, \overline{q_k}[=]\overline{p}, \overline{q}[= \emptyset.$$

To show the reverse implication, we may observe the following. If T is dense everywhere and for some sequences $a = \langle a_n \rangle \in \mathcal{T}$ and $b = \langle b_n \rangle \in \mathcal{T}$ it holds that $\overline{a} < \overline{b}$, then $\left]\overline{a}, \overline{b}\right[\neq \emptyset$. Indeed, let

$$C := \{n \in \mathbb{N} \mid a_n < b_n\} \in \mathcal{U}.$$

As T is dense, there exists $c_n \in]a_n, b_n[$ for all $n \in C$. Therefore if $d = \langle d_n \rangle : \mathbb{N} \longrightarrow T$ is defined by

$$d_n = \begin{cases} c_n, & \text{if } n \in C \\ c_1, & \text{if } n \notin C, \end{cases}$$

then one can easily check that $\overline{d} \in \left]\overline{a}, \overline{b}\right[$.

This means that if T is dense then every open interval $\left]\overline{a_k}, \overline{b_k}\right[$ in part (b) is nonempty, so applying Theorem 3.5 we get that their intersection is also nonempty. □

3.4. Completeness of the extension

Finally we will show that the operation * does not preserve completeness in general. Moreover the completeness of the ultrapower depends only on the cardinality of the initial ordered set.

LEMMA 3.9. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $A_j \subseteq \mathbb{N}$ ($j = 1, \dots, n$). If

$$\bigcup_{j=1}^{n} A_j = \mathbb{N},$$

then there exists $k \in \{1, \dots, n\}$ such that $A_k \in \mathcal{U}$.

The statement can be easily proved by the reader.

THEOREM 3.10. Let T be an ordered set. T^* is complete if and only if T is finite.

Proof. In the first place we prove that if T is infinite then T^* is not complete. We will use the following basic fact: in an infinite ordered set there exists a strictly monotone sequence of elements. In order to prove this, we may consider an obviously existing injective sequence $\langle x_n \rangle : \mathbb{N} \longrightarrow T$ (i.e., $x_n \neq x_m$ if $n \neq m$). It is a well-known fact that every sequence in an ordered set contains a monotone subsequence (we can apply the proof for real sequences [6] in this more general context as well). Clearly, such a monotone subsequence of $\langle x_n \rangle$ is strictly monotone.

Let $t_1 < t_2 < t_3 < \dots$ be a strictly increasing sequence of elements in T (the case of a strictly decreasing sequence can be handled similarly). It is easy to see that the equivalence classes of the constant sequences $s_k \in \mathcal{T}$ defined as

$$(s_k)_n = t_k \qquad (n, k \in \mathbb{N})$$

generate a subset

$$S = \{\overline{s_k} \mid k \in \mathbb{N}\}$$

of T^* which is bounded from above. Indeed, one can easily check that $\overline{\langle t_n \rangle}$ is an upper bound of S. Now we demonstrate that S has no least upper bound. Let $\langle b_n \rangle \in \mathcal{T}$ such that $\overline{\langle b_n \rangle}$ is an upper bound of S. We define some sets in a similar manner as we did in the proof of Theorem 3.5: let

$$\mathcal{D}_k = \{n \in \mathbb{N} \mid b_n \geq t_k\} \in \mathcal{U}, \qquad I_n = \{k \in \mathbb{N} \mid b_n \geq t_k\} \qquad (k, n \in \mathbb{N}).$$

We should note that if $k \in I_n$ then $l \in I_n$ for every natural number $l \leq k$. Another easy observation is that $m \in \mathcal{D}_k$ if and only if $k \in I_m$ (we will use these two remarks later on).

Now we can define a sequence $\alpha : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$ as follows: let

$$\alpha_n = \begin{cases} 0, & \text{if } I_n = \emptyset, \\ n, & \text{if } I_n \text{ has no upper bound}, \\ \min\{n, \max I_n\}, & \text{otherwise.} \end{cases}$$

With the notations $t_0 := t_1$ and $\beta_n = \left\lfloor \frac{\alpha_n}{2} \right\rfloor$ it is possible to construct an upper bound for S which is smaller than $\overline{b}$. We define $\langle c_n \rangle \in \mathcal{T}$ as follows:

$$c_n = t_{\beta_n} \qquad (n \in \mathbb{N}).$$

For any natural number k the following argumentation can be made: if $\alpha_n \geq 2k$ then $\beta_n \geq k$ and therefore $c_n \geq t_k$. Since

$$\{n \in \mathbb{N} \mid c_n \geq t_k\} \supseteq \{n \in \mathbb{N} \mid \alpha_n \geq 2k\} = \mathcal{D}_{2k} \setminus \{m \in \mathbb{N} \mid m < 2k\} \in \mathcal{U}$$

follows from the two simple remarks that were stated earlier, $\overline{\langle c_n \rangle}$ is an upper bound of S. On the other hand, for every $m \in \mathcal{D}_2 \setminus \{1\}$, the value c_m is indeed smaller than b_m, because $2 \leq \alpha_m \in I_m$, and thus

$$c_m = t_{\beta_m} < t_{\alpha_m} \leq b_m,$$

so $\overline{\langle c_n \rangle} < \overline{\langle b_n \rangle}$. Therefore S has no least upper bound.

In the second part of the proof we will verify the reverse implication, namely that if T is finite then T^* is complete. Since a finite ordered set is always complete, it is sufficient to show that, for any finite ordered set T, T^* is finite as well.

Let $k \in \mathbb{N}$, $T = \{t_1, \dots, t_k\}$, and for each $j \in \{1, \dots, k\}$, let $s_j \in \mathcal{T}$ such that $(s_j)_n = t_j$ for all $n \in \mathbb{N}$ (a constant sequence). Now let us consider an arbitrary sequence $\langle a_n \rangle \in \mathcal{T}$. For every $j \in \{1, \dots, k\}$ we define the sets $A_j = \{n \in \mathbb{N} \mid a_n = t_j\}$. Obviously, $\bigcup_{j=1}^{k} A_j = \mathbb{N}$. According to Lemma 3.9, there exists an index $m \in \{1, \dots, k\}$ such that $A_m \in \mathcal{U}$ and therefore $\overline{\langle a_n \rangle} = \overline{s_m}$ (i.e., $\langle a_n \rangle$ is equivalent with the constant t_m sequence). So we may conclude that T^* contains only the equivalence classes of finitely many constant sequences, which implies that T^* is complete as well. □

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