

## On a functional equation with a symmetric component

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*Dedicated to Professor Zoltán Daróczy on the occasion of his seventieth birthday*

**Abstract.** Let  $I \subset \mathbb{R}$  be a nonvoid open interval and  $r \neq 0, 1$ ,  $q \in (0, 1)$ , such that  $r \neq q$ ,  $r \neq \frac{1}{2}$  and  $q \neq \frac{1}{2}$ . In this paper we give all the functions  $f, g : I \rightarrow \mathbb{R}_+$  such that

$$f\left(\frac{x+y}{2}\right) [r(1-q)g(y) - (1-r)qg(x)] = \frac{r-q}{1-2q} [(1-q)f(x)g(y) - qf(y)g(x)]$$

for all  $x, y \in I$ .

### 1. Introduction

Let  $J \subset \mathbb{R}$  be a nonvoid open interval and denote the class of continuous and strictly monotone real valued functions defined on the interval  $J$  by  $\mathcal{CM}(J)$ . A function  $M : J^2 \rightarrow J$  is called a *weighted quasi-arithmetic mean* on  $J$  if there exist  $0 < p < 1$  and  $\varphi \in \mathcal{CM}(J)$  such that

$$M(x, y) = \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)) =: A_\varphi(x, y; p).$$

for all  $x, y \in J$ . The number  $p$  is said to be the *weight* and the function  $\varphi$  is called the *generating function* of the weighted quasi-arithmetic mean  $M$ .

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Now we can formulate the general problem as follows: determine all  $M, N : J^2 \rightarrow J$  weighted quasi-arithmetic means and the constants  $\mu \neq 0, 1$  and  $r \neq 0, 1$ , such that

$$\mu M(u, v) + (1 - \mu)N(u, v) = ru + (1 - r)v$$

holds for all  $u, v \in J$ . In detail this equation means the following: determine all the functions  $\varphi, \psi \in \mathcal{CM}(J)$  and the constants  $r \neq 0, 1$ ,  $(p, q) \in (0, 1)^2$ ,  $\mu \neq 0, 1$  such that

$$\mu\varphi^{-1}(p\varphi(u) + (1 - p)\varphi(v)) + (1 - \mu)\psi^{-1}(q\psi(u) + (1 - q)\psi(v)) = ru + (1 - r)v$$

holds for all  $u, v \in J$ .

If we suppose that  $\varphi, \psi \in \mathcal{CM}(J)$  are differentiable on  $J$  and  $\varphi'(u) > 0$ ,  $\psi'(u) > 0$  for all  $u \in J$ , then with the notations  $f := \varphi' \circ \varphi^{-1}$ ,  $g := \psi' \circ \psi^{-1}$ ,  $I := \varphi(J)$  for the unknown functions  $f, g : I \rightarrow \mathbb{R}_+$  and  $\varphi(u) = x$  and  $\varphi(v) = y$  ( $x, y \in I$ ), from the above equation we have

$$\begin{aligned} f(px + (1 - p)y)[r(1 - q)g(y) - (1 - r)qg(x)] \\ = \mu[p(1 - q)f(x)g(y) - (1 - p)qf(y)g(x)] \end{aligned} \quad (1)$$

for all  $x, y \in I$ . The functional equation (1) depends on the parameters  $r \neq 0, 1$ ,  $(p, q) \in (0, 1)^2$  and  $\mu \neq 0, 1$  for which, if  $x = y$  in (1), by  $f(x) > 0$ ,  $g(x) > 0$  we have

$$\mu(p - q) = r - q. \quad (2)$$

The functional equation (1) was studied in the following special cases:

- (i)  $p = q = r = \mu = 1/2$ , by J. MATKOWSKI [11], then by Z. DARÓCZY and Zs. PÁLES [5] under much weaker conditions.
- (ii)  $p = q$ ,  $(p, q, r) \in (0, 1)^3$  (then by (2)  $r = q$ ) by Z. DARÓCZY and Zs. PÁLES in [6], [5].
- (iii)  $\mu = r$ ,  $(p, q, r) \in (0, 1)^3$  J. JARCZYK and J. MATKOWSKI in [8], and J. JARCZYK [7], P. BURAI [1].
- (iv)  $\mu = r$  and  $p = 1/2$ ,  $q \neq 1/2$ ,  $(q, r) \in (0, 1)^2$  by Z. DARÓCZY in [3] without any conditions.
- (v)  $p = 1/2$ ,  $q \neq 1/2$  and  $q, r \in (0, 1)^2$ ,  $r \neq q$ ,  $r \neq 1/2$  and  $\mu = \frac{2(r-q)}{1-2q}$  by Z. DARÓCZY and J. DASCĂL in [4].

In this paper we study the functional equation (1) in the case  $p = 1/2$  and  $p \neq q$ . Hence, by (2) we have  $r \neq q$  and  $r \neq \frac{1}{2}$  and

$$\mu = \frac{r - q}{\frac{1}{2} - q} = 2 \cdot \frac{r - q}{1 - 2q}.$$

This means we have to determine all the functions  $f, g : I \rightarrow \mathbb{R}_+$  ( $I \subset \mathbb{R}$  nonvoid open interval) and the constants  $r \neq 0, 1$ ,  $q \in (0, 1)$ , such that

$$\begin{aligned} f\left(\frac{x+y}{2}\right) [r(1-q)g(y) - (1-r)qg(x)] \\ = \frac{r-q}{1-2q} [(1-q)f(x)g(y) - qf(y)g(x)] \end{aligned} \quad (3)$$

holds for all  $x, y \in I$ .

## 2. Main result

**Theorem 1.** *Let  $I \subset \mathbb{R}$  be a nonvoid open interval and  $r \neq 0, 1$ ,  $q \in (0, 1)$ , such that  $r \neq q$ ,  $r \neq \frac{1}{2}$  and  $q \neq \frac{1}{2}$ . If the functions  $f, g : I \rightarrow \mathbb{R}_+$  are solutions of the functional equation (3) then the following cases are possible:*

- (1) *If  $r \neq \frac{q^2}{q^2+(1-q)^2}$  and  $r \neq \frac{q}{2q-1}$  then there exist constants  $a, b \in \mathbb{R}_+$  such that*

$$f(x) = a \quad \text{and} \quad g(x) = b \quad \text{for all } x \in I;$$

- (2) *If  $r = \frac{q^2}{q^2+(1-q)^2}$  then there exists an additive function  $A : \mathbb{R} \rightarrow \mathbb{R}$  and positive real numbers  $c_1, c_2$  such that*

$$g(x) = c_1 e^{A(x)} \quad \text{and} \quad f(x) = c_2 e^{2A(x)} \quad \text{for all } x \in I;$$

- (3) *If  $r = \frac{q}{2q-1}$  then there exist real numbers  $d_1, d_2, d_3$  such that*

$$g(x) = \frac{1}{d_1 x + d_2} > 0 \quad \text{and} \quad f(x) = d_3 \frac{1}{d_1 x + d_2} > 0 \quad \text{for all } x, y \in I.$$

Conversely, the functions given in the above cases are solutions of equation (3).

To prove Theorem 1 we need the following lemmas.

**Lemma 1.** *Let  $I \subset \mathbb{R}$  be a nonvoid open interval and  $r \neq 0, 1$ ,  $0 < q < 1$ ,  $r \neq q$ ,  $r, q \neq 1/2$ . If the functions  $f, g : I \rightarrow \mathbb{R}_+$  satisfy the functional equation (3) then*

$$f\left(\frac{x+y}{2}\right) [g(x) + g(y)] = [f(x)g(y) + f(y)g(x)] \quad (4)$$

holds for all  $x, y \in I$ .

**Lemma 2.** *Let  $I \subset \mathbb{R}$  be a nonvoid open interval and  $r \neq 0, 1$ ,  $0 < q < 1$ ,  $r \neq q$ ,  $r, q \neq 1/2$ . If the functions  $f, g : I \rightarrow \mathbb{R}_+$  satisfy the functional equation (3) then*

$$\begin{aligned} f(x)g(y)\{q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x)\} \\ = f(y)g(x)\{q(1-q)(1-2r)g(x) - [r(1-2q) - q^2(1-2r)]g(y)\} \end{aligned} \quad (5)$$

holds for all  $x, y \in I$ .

These lemmas are proved in [4].

Proof of Theorem 1:

The proof of cases (1) and (2) is the same as the proof of Theorem 1 from [4].

In case (3), when  $r = \frac{q}{2q-1}$ , by Lemma 2 the equation (5) becomes

$$f(x)g(y)\frac{q(1-q)}{1-2q}[g(x) + g(y)] = f(y)g(x)\frac{q(1-q)}{1-2q}[g(x) + g(y)].$$

for all  $x, y \in I$ . Hence  $f(x)g(y) = f(y)g(x)$ , thus

$$f(x) = d_3g(x) \quad \text{for some } d_3 > 0 \quad \text{and for all } x \in I. \quad (6)$$

Replacing this form of  $f$  in (4) we have

$$g\left(\frac{x+y}{2}\right) = \frac{2}{\frac{1}{g(x)} + \frac{1}{g(y)}},$$

consequently, by [9], [10] there exist an additive function  $B : \mathbb{R} \rightarrow \mathbb{R}$  and a real number  $d_2$  such that  $\frac{1}{g(x)} = B(x) + d_2 > 0$ , thus  $g(x) = \frac{1}{B(x)+d_2} > 0$  for all  $x \in I$ , that is, there exists  $d_1 \in \mathbb{R}$  such that  $B(x) = d_1x$  for all  $x \in I$ , thus  $g(x) = \frac{1}{d_1x+d_2}$  for all  $x \in I$ . Finally, (6) completes the proof of case (3).

### 3. Application

Returning to the generalized problem we need the following definitions.

*Definition 1.* Let  $\varphi, \psi \in \mathcal{CM}(J)$ . If there exist  $a \neq 0$  and  $b$  such that

$$\psi(x) = a\varphi(x) + b \quad \text{if } x \in J$$

then we say that  $\varphi$  is equivalent to  $\psi$  on  $J$  and denote it by  $\varphi(x) \sim \psi(x)$  if  $x \in J$  or in short  $\varphi \sim \psi$  on  $J$ .

It is well-known that if  $0 < p < 1$  and  $\varphi, \psi \in \mathcal{CM}(J)$ , then  $A_\varphi(x, y; p) = A_\psi(x, y; p)$  for all  $x, y \in J$  if and only if  $\varphi \sim \psi$  on  $J$ .

We define the following sets:

$$T_+(J) := \{t \in \mathbb{R} \mid J + t \subset \mathbb{R}_+\}$$

$$T_-(J) := \{t \in \mathbb{R} \mid -J + t \subset \mathbb{R}_+\}.$$

With the help of these notations, set

$$\gamma_t^+(x) := \sqrt{x+t} \quad \text{if } t \in T_+(J) \quad (x \in J)$$

$$\gamma_t^-(x) := \sqrt{-x+t} \quad \text{if } t \in T_-(J) \quad (x \in J).$$

The general problem is as follows: determine all the functions  $\varphi, \psi \in \mathcal{CM}(J)$  and the constants  $r \neq 0, 1$ ,  $(p, q) \in (0, 1)^2$ ,  $\mu \neq 0, 1$  such that

$$\mu\varphi^{-1}(p\varphi(u) + (1-p)\varphi(v)) + (1-\mu)\psi^{-1}(q\psi(u) + (1-q)\psi(v)) = ru + (1-r)v$$

holds for all  $u, v \in J$ . If either  $p$  or  $q$  equals  $1/2$ , the following theorem gives the solutions of this equation. If  $(\varphi, \psi)$  is the solution of the above functional equation with  $p = 1/2$ ,  $q \neq 1/2$ , then  $(\psi, \varphi)$  is the solution of the equation with  $p \neq 1/2$ ,  $q = 1/2$ . So it is enough to state our theorem for the case  $p = 1/2$ ,  $q \neq 1/2$ . In [4] the above equation (with  $p = 1/2$ ) is solved for  $0 < r < 1$ , but here we take  $r \neq 0, 1$  and we get further solutions, which solutions are also found by Z. DARÓCZY in [2] without the assumption of differentiability of the functions  $\varphi$  and  $\psi$ .

**Theorem 2.** *Let  $J \subset \mathbb{R}$  be a nonvoid open interval and  $r \neq 0, 1$ ,  $0 < q < 1$ ,  $r, q \neq \frac{1}{2}$ ,  $r \neq q$ . If  $\varphi, \psi \in \mathcal{CM}(J)$  are solutions of the functional equation*

$$\frac{2(r-q)}{1-2q} \varphi^{-1} \left( \frac{\varphi(u) + \varphi(v)}{2} \right) + \left( 1 - \frac{2(r-q)}{1-2q} \right) \psi^{-1}(q\psi(u) + (1-q)\psi(v)) = ru + (1-r)v \quad (7)$$

for all  $u, v \in J$  and  $\varphi, \psi$  are differentiable on  $J$  and  $\varphi'(u) > 0$ ,  $\psi'(u) > 0$  for all  $u \in J$  then  $\varphi \sim \text{id}$  and  $\psi \sim \text{id}$  on  $J$ , furthermore in the case  $r = \frac{q^2}{q^2 + (1-q)^2}$  the following cases are also possible:

$$\varphi \sim \log \gamma_t^+, \psi \sim \gamma_t^+ \quad \text{if } t \in T_+(J) \quad \text{or} \quad \varphi \sim \log \gamma_t^-, \psi \sim \gamma_t^- \quad \text{if } t \in T_-(J)$$

and in the case  $r = \frac{q}{2q-1}$  the following cases are also possible:

$$\varphi \sim \gamma_t^+, \psi \sim \gamma_t^+ \quad \text{if } t \in T_+(J) \quad \text{or} \quad \varphi \sim \gamma_t^-, \psi \sim \gamma_t^- \quad \text{if } t \in T_-(J).$$

PROOF. It is enough to solve the functional equation (7) up to the equivalence of the functions  $\varphi$  and  $\psi$ . With the notations  $f := \varphi' \circ \varphi^{-1}$ ,  $g := \psi' \circ \varphi^{-1}$ ,  $I := \varphi(J)$  we get that equation (3) holds. Due to the definition of  $f$ , we obtain the differential equation for the function  $\varphi$ :

$$\varphi'(x) = f(\varphi(x)) \quad x \in J. \quad (8)$$

By Theorem 1, the case  $r \neq \frac{q^2}{q^2+(1-q)^2}$ ,  $r \neq \frac{q}{2q-1}$  gives the constant solutions, which implies that  $\varphi \sim \text{id}$ ,  $\psi \sim \text{id}$ .

If  $r = \frac{q^2}{q^2+(1-q)^2}$  the proof is found in [4].

If  $r = \frac{q}{2q-1}$  then

$$f(x) = d_3 \frac{1}{d_1 x + d_2} \text{ and } g(x) = \frac{1}{d_1 x + d_2} \quad \text{for all } x \in I, \quad (9)$$

where  $d_1, d_2, d_3 \in \mathbb{R}$ ,  $d_3 > 0$ .

In the case  $d_1 = 0$ ,  $\varphi \sim \text{id}$  and  $\psi \sim \text{id}$ .

In the case  $d_1 \neq 0$  from (8) we have

$$\varphi'(u) = d_3 \frac{1}{d_1 \varphi(u) + d_2} > 0 \quad \text{for all } u \in J,$$

which implies that  $\varphi(u) \sim \sqrt{C_2 u + C_3}$ , from which we deduce that either there exists  $t \in T_+(J)$  such that  $\varphi \sim \gamma_t^+$  on  $J$  or there exists  $t \in T_-(J)$  such that  $\varphi \sim \gamma_t^-$  on  $J$ .

Due to the definition of  $g$ , by (9) we obtain that

$$\psi'(u) = \frac{1}{d_1 \varphi(u) + d_2} > 0 \quad \text{for all } u \in J,$$

which implies that either there exists  $t \in T_+(J)$  such that  $\psi \sim \gamma_t^+$  on  $J$  or there exists  $t \in T_-(J)$  such that  $\psi \sim \gamma_t^-$  on  $J$ .  $\square$

## References

- [1] P. BURAI, A Matkowski–Sutô type equation, *Publ. Math. Debrecen* **70** (2007), 233–247.
- [2] Z. DARÓCZY, Mean values and functional equations, *Differential Equations & Dynamical Systems – An International Journal for Theory, Applications and Computer Simulations*, accepted.
- [3] Z. DARÓCZY, On a class of means of two variables, *Publ. Math. Debrecen* **55** (1999), 177–197.

- [4] Z. DARÓCZY and J. DASCĂL, On the general solution of a family of functional equations with two parameters and its application, *Math. Pannonica*, accepted.
- [5] Z. DARÓCZY and J. DASCĂL, Gauss-composition of means and the solution of the Matkowski–Sutô problem, *Publ. Math. Debrecen* **61** (2002), 157–218.
- [6] Z. DARÓCZY and J. DASCĂL, On functional equations involving means, *Publ. Math. Debrecen* **62** (2003), 363–377.
- [7] J. JARCZYK, Invariance of weighted quasi-arithmetic means with continuous generators, *Publ. Math. Debrecen* **71** (2007), 279–294.
- [8] J. JARCZYK and J. MATKOWSKI, Invariance in the class of weighted quasi-arithmetic means, *Ann. Polon. Math.* **88**(1) (2006), 39–51. .
- [9] M. KUCZMA, An Introduction to the Theory of Functional Equations and Inequalities, Vol. 489, Prace Naukowe Uniwersytetu Śląskiego w Katowicach, *Państwowe Wydawnictwo Naukowe – Uniwersytet Śląski, Warszawa – Kraków – Katowice*, 1985.
- [10] K. LAJKÓ, Applications of extensions of additive functions, *Aequationes Math.* **11** (1974), 68–76.
- [11] J. MATKOWSKI, Invariant and complementary quasi-arithmetic means, *Aequationes Math.* **57** (1999), 87–107.

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