The r-Lah numbers

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Abstract

In this paper we present a detailed study of r-Lah numbers, which give the number of partitions of a finite set into a fixed number of nonempty ordered subsets such that r distinguished elements belong to distinct ordered blocks.

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1. Introduction

It is well-known that Stirling number of the first kind $\left[{n \atop k}\right]$ counts the number of those permutations of n elements which are the product of k disjoint cycles, and Stirling number of the second kind $\left\{{n \atop k}\right\}$ gives the number of partitions of a set with n elements into k nonempty subsets. Both kinds of Stirling numbers are of basic importance in enumerative combinatorics.

If we modify the latter problem such that the blocks in the partition are ordered, then we arrive at the somewhat less known Lah number $\left\lfloor{n \atop k}\right\rfloor$. These numbers are named after the Slovenian mathematician Ivo Lah [8], [9]. Lah numbers are often denoted by $L(n,k)$, but we use the Karamata-Knuth type notation introduced in [17]. Lah numbers are rarely called Stirling numbers of the third kind.

Stirling numbers have various generalizations. Among them, r-Stirling numbers were defined by A. Z. Broder [5], later rediscovered by R. Merris [10], when r distinguished elements have to be in distinct cycles or blocks. The similar idea led to closely related combinatorial numbers, r-Bell numbers by I. Mező [13] and r-Fubini numbers by I. Mező and G. Nyul [14]. For further generalization of r-Stirling numbers, the variants of the so-called r-Whitney numbers, see

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[6], [12]. An alternative way to introduce r-Stirling numbers of the second kind through graphs is described in [7].

Naturally, r-Lah numbers also can be defined. Although they are once mentioned in [6] (together with r-Whitney-Lah numbers), in a very recent paper H. Belbachir and A. Belkhir [3] studied a few properties of them, and they appear in [16] under the name of restricted Lah numbers, to the best of our knowledge, there is no systematic treatment of these numbers. We should remark that they also can be reached by a special substitution into the so-called partial r-Bell polynomials in [15], or as a special case of the solution to a general bivariate recurrence in [1], [2].

In this paper we do it in details, in a self-contained way, we prove properties of r-Lah numbers, derive several identities, study their log-concavity, etc. Some of these properties could be shown also by tedious calculation, or in certain cases they could be derived from particular results of the above mentioned papers, but we give always purely combinatorial proofs when it is possible. It is important to underline that these are not only simple translation to the r-generalized case, for example, the standard proof of the explicit formula for classical Lah numbers does not work here, we need a new idea to prove Theorem 3.7. However, along this note we find some new identities for r-Stirling numbers, too.

2. r-Stirling numbers

For completeness, in the following table we briefly summarize the properties of r-Stirling numbers, some of which seem to be new identities. For nonnegative integers $k \leq n$ and r, n and r not both 0, $\left[{n \atop k}\right]_r$ is the number of those permutations of $n+r$ elements which are the product of $k+r$ disjoint cycles and r distinguished elements belong to distinct cycles, while $\left\{{n \atop k}\right\}_r$ is the number of partitions of a set with $n+r$ elements into $k+r$ nonempty subsets such that r distinguished elements belong to distinct blocks, with the convention $\left[{0 \atop 0}\right]_0 = \left\{{0 \atop 0}\right\}_0 = 1$. We notice that A. Z. Broder [5] and R. Merris [10] used different parametrizations, but we prefer the latter notation. These properties, including the new identities, can be usually proved similarly as it will be presented in the next section for r-Lah numbers.

r-Stirling numbers of the first kind	r-Stirling numbers of the second kind
$\left[{n \atop k}\right]_0 = \left[{n \atop k}\right]$ [5],[10], $\left[{n \atop k}\right]_1 = \left[{n+1 \atop k+1}\right]$ [5]	$\left\{{n \atop k}\right\}_0 = \left\{{n \atop k}\right\}$ [5],[10], $\left\{{n \atop k}\right\}_1 = \left\{{n+1 \atop k+1}\right\}$ [5]
$\left[{n \atop 0}\right]_r = r^{\overline{n}}$ [5],[10], $\left[{n \atop 1}\right]_r = n!H_n^{(r)}$ [4]	$\left\{{n \atop 0}\right\}_r = r^n$ [5],[10], $\left\{{n \atop 1}\right\}_r = (r+1)^n - r^n$
$\left[{n \atop n-1}\right]_r = \binom{n}{2} + nr$, $\left[{n \atop n}\right]_r = 1$ [5],[10]	$\left\{{n \atop n-1}\right\}_r = \binom{n}{2} + nr$, $\left\{{n \atop n}\right\}_r = 1$ [5],[10]
$\sum_{k=0}^{n} \left[{n \atop k}\right]_r = (r+1)^{\overline{n}}$	$\sum_{k=0}^{n} \left\{{n \atop k}\right\}_r = B_{n,r}$ [13]
$\left[{n+1 \atop k}\right]_r = \left[{n \atop k-1}\right]_r + (n+r)\left[{n \atop k}\right]_r$ [5],[10]	$\left\{{n+1 \atop k}\right\}_r = \left\{{n \atop k-1}\right\}_r + (k+r)\left\{{n \atop k}\right\}_r$ [5],[10]
$(x+r)^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right]_r x^k$ [5],[10] $(x-r)^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k}\right]_r x^k$ [5]	$(x+r)^n = \sum_{k=0}^{n} \left\{{n \atop k}\right\}_r x^{\underline{k}}$ [5] $(x-r)^n = \sum_{k=0}^{n} (-1)^{n-k} \left\{{n \atop k}\right\}_r x^{\overline{k}}$ [5]
$\left[{n+1 \atop k+1}\right]_r = \sum_{j=k}^{n} (n+r)^{\underline{n-j}} \left[{j \atop k}\right]_r$	$\left\{{n+1 \atop k+1}\right\}_r = \sum_{j=k}^{n} (k+r+1)^{n-j} \left\{{j \atop k}\right\}_r$
$\left[{n \atop k}\right]_r = \sum_{j=k}^{n} \binom{n}{j} \left[{j \atop k}\right]_{r-s} s^{\overline{n-j}}$ [5] $\left[{n \atop k}\right]_r = \sum_{j=k}^{n} \binom{n}{j} \left[{j \atop k}\right]_{r-1} (n-j)!$ [5] $\left[{n \atop k}\right]_r = \sum_{j=k}^{n} \binom{n}{j} \left[{j \atop k}\right] r^{\overline{n-j}}$ [5],[10]	$\left\{{n \atop k}\right\}_r = \sum_{j=k}^{n} \binom{n}{j} \left\{{j \atop k}\right\}_{r-s} s^{n-j}$ [5] $\left\{{n \atop k}\right\}_r = \sum_{j=k}^{n} \binom{n}{j} \left\{{j \atop k}\right\}_{r-1}$ [5] $\left\{{n \atop k}\right\}_r = \sum_{j=k}^{n} \binom{n}{j} \left\{{j \atop k}\right\} r^{n-j}$ [5],[10]
$\left[{n \atop k}\right]_r = \sum_{j=k}^{n} \left[{n \atop j}\right]_{r-s} \binom{j}{k} s^{j-k}$ $\left[{n \atop k}\right]_r = \sum_{j=k}^{n} \left[{n \atop j}\right]_{r-1} \binom{j}{k}$ $\left[{n \atop k}\right]_r = \sum_{j=k}^{n} \left[{n \atop j}\right] \binom{j}{k} r^{j-k}$ [10]	$\left\{{n \atop k}\right\}_r = \sum_{j=k}^{n} \left\{{n \atop j}\right\}_{r-s} \binom{j}{k} s^{\underline{j-k}}$ $\left\{{n \atop k}\right\}_r = \left\{{n \atop k}\right\}_{r-1} + (k+1)\left\{{n \atop k+1}\right\}_{r-1}$ $(k \le n-1)$ [5] $\left\{{n \atop k}\right\}_r = \sum_{j=k}^{n} \left\{{n \atop j}\right\} \binom{j}{k} r^{\underline{j-k}}$
$\binom{k+l}{k} \left[{n \atop k+l}\right]_{r+s} = \sum_{j=k}^{n-l} \binom{n}{j} \left[{j \atop k}\right]_r \left[{n-j \atop l}\right]_s$ [5]	$\binom{k+l}{k} \left\{{n \atop k+l}\right\}_{r+s} = \sum_{j=k}^{n-l} \binom{n}{j} \left\{{j \atop k}\right\}_r \left\{{n-j \atop l}\right\}_s$ [5]
	$\left\{{n \atop k}\right\}_r = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+r-j)^n$
$\left(\left[{n \atop k}\right]_r\right)_{k=0}^{n}$ strictly log-concave, unimodal [11]	$\left(\left\{{n \atop k}\right\}_r\right)_{k=0}^{n}$ strictly log-concave, unimodal [11]
$\sum_{n=k}^{\infty} \left[{n \atop k}\right]_r \frac{x^n}{n!} = \frac{1}{k!} \left(\ln\left(\frac{1}{1-x}\right)\right)^k \frac{1}{(1-x)^r}$ [5],[10]	$\sum_{n=k}^{\infty} \left\{{n \atop k}\right\}_r \frac{x^n}{n!} = \frac{1}{k!} (\exp(x) - 1)^k \exp(rx)$ [5],[10]
$\sum_{j=k}^{n} (-1)^{j-k} \left[{n \atop j}\right]_r \left\{{j \atop k}\right\}_s = \binom{n}{k} (r-s)^{\overline{n-k}}$ [5], $\sum_{j=k}^{n} (-1)^{j-k} \left\{{n \atop j}\right\}_r \left[{j \atop k}\right]_s = \binom{n}{k} (r-s)^{n-k}$ [5] $\sum_{j=k}^{n} (-1)^{j-k} \left[{n \atop j}\right]_r \left\{{j \atop k}\right\}_r = \delta_{nk}$ [5],[10], $\sum_{j=k}^{n} (-1)^{j-k} \left\{{n \atop j}\right\}_r \left[{j \atop k}\right]_r = \delta_{nk}$ [5],[10]	
$b_n = \sum_{k=0}^{n} \left[{n \atop k}\right]_r a_k$ $(n \ge 0)$ if and only if $a_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{{n \atop k}\right\}_r b_k$ $(n \ge 0)$ $b_n = \sum_{k=0}^{n} \left\{{n \atop k}\right\}_r a_k$ $(n \ge 0)$ if and only if $a_n = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k}\right]_r b_k$ $(n \ge 0)$	

3. *r*-Lah numbers

The r-Lah numbers can be defined similarly to r-Stirling numbers: For integers $0 \le k \le n$ and $r \ge 0$, if n, r are not both 0, then $\left\lfloor{n \atop k}\right\rfloor_r$ counts the number of partitions of a set with $n+r$ elements into $k+r$ nonempty ordered subsets such that r distinguished elements have to be in distinct ordered blocks. Moreover, let $\left\lfloor{0 \atop 0}\right\rfloor_0 = 1$.

If $r = 0$ or $r = 1$, then distinguished elements give no restriction, hence $\left\lfloor{n \atop k}\right\rfloor_0 = \left\lfloor{n \atop k}\right\rfloor$, $\left\lfloor{n \atop k}\right\rfloor_1 = \left\lfloor{n+1 \atop k+1}\right\rfloor$. By simple combinatorial observation, we obtain r-Lah numbers for special small and large values of k: $\left\lfloor{n \atop 0}\right\rfloor_r = (2r)^{\overline{n}}$, $\left\lfloor{n \atop 1}\right\rfloor_r = (2r+1)^{\overline{n}} - (2r)^{\overline{n}}$ $(n \ge 1)$, $\left\lfloor{n \atop n-1}\right\rfloor_r = n(n-1) + 2nr$ $(n \ge 1)$, $\left\lfloor{n \atop n}\right\rfloor_r = 1$.

The r-Lah numbers satisfy the following recurrence relation.

Theorem 3.1. *If $1 \le k \le n$ and $r \ge 0$, then*

$$\left\lfloor \begin{matrix} n+1 \\ k \end{matrix} \right\rfloor_r = \left\lfloor \begin{matrix} n \\ k-1 \end{matrix} \right\rfloor_r + (n+k+2r) \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r .$$

Proof. Consider an $(n+r+1)$-element set with r distinguished elements, which will be partitioned into $k+r$ nonempty ordered subsets such that the distinguished elements are in distinct ordered blocks.

Choose a non-distinguished element. If this element is in a singleton, then there are $\left\lfloor \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\rfloor_r$ possibilities to partition the other elements. If it is contained in an ordered block with at least two elements, then the other elements can be partitioned in $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r$ ways and the chosen element can be put to $n+k+2r$ places (before any element or to the end of any ordered block). □

Connection between shifted falling and rising factorials can be described by r-Lah numbers.

Theorem 3.2. *If $n, r \ge 0$, then*

$$(x+2r)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r x^{\underline{k}}, \quad (x-2r)^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r x^{\overline{k}}.$$

Proof. We can suppose that n, r are not both 0. Consider an $(n+r)$-element set with r distinguished elements. We will partition it into nonempty ordered subsets such that the distinguished elements are in distinct ordered blocks, and colour it with $m+r$ colours such that two elements share their colours if and only if they are in the same ordered block.

Partitioning our set can be done in $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r$ ways if we have $k+r$ ordered blocks ($k = 0, \ldots, n$). Then the number of colourings block by block is $(m+r)^{\underline{k+r}}$. Therefore there are $\sum_{k=0}^{n} \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r (m+r)^{\underline{k+r}}$ coloured partitions.

On the other hand, there are $(m+r)^{\underline{r}}$ possibilities to colour the distinguished elements. Taking the jth non-distinguished element, suppose that the first $j-1$ non-distinguished elements together with the distinguished ones are coloured and partitioned into $l+r$ ordered blocks. We can put this element into one of these ordered blocks to $l+2r+j-1$ places (before any element or to the end of any ordered block) and in this case its colour is given. While if this element opens a new block, then we can colour it with $m-l$ colours. Hence the jth non-distinguished element can be placed and coloured in $m+2r+j-1$ ways, therefore the non-distinguished elements altogether in $(m+2r)^{\overline{n}}$ ways. Summarizing, the number of coloured partitions is $(m+r)^{\underline{r}}(m+2r)^{\overline{n}}$.

From the equality of the differently derived values, $(m+r)^{\underline{r}}(m+2r)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r (m+r)^{\underline{k+r}}$, a division by $(m+r)^{\underline{r}}$ gives $(m+2r)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r m^{\underline{k}}$.

Finally, the second equation follows by substituting $(-x)$. □

The next theorem can be viewed as a vertical recurrence for r-Lah numbers.

Theorem 3.3. *If $0 \le k \le n$ and $r \ge 0$, then*

$$\left\lfloor \begin{matrix} n+1 \\ k+1 \end{matrix} \right\rfloor_r = \sum_{j=k}^{n} (n+k+2r+1)^{\underline{n-j}} \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor_r .$$

Proof. Consider an $(n+r+1)$-element set with the first r elements distinguished. We will partition it into $k+r+1$ nonempty ordered subsets such that the first r elements are in distinct ordered blocks.

Let $j+r+1$ be the largest among the smallest indices of elements of the ordered blocks ($j = k, \ldots, n$). Then the first $j+r$ elements have to be partitioned into $k+r$ nonempty ordered subsets such that the distinguished elements are in distinct ordered blocks, which can be done in $\left\lfloor \begin{smallmatrix} j \\ k \end{smallmatrix} \right\rfloor_r$ ways. The other elements can be put before any element or to the end of any ordered block, hence there are $(n+k+2r+1)^{\underline{n-j}}$ possibilities to place the last $n-j$ elements. □

The following two theorems express r-Lah numbers by $(r-s)$-Lah numbers.

Theorem 3.4. *If $0 \le k \le n$ and $0 \le s \le r$, then*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \sum_{j=k}^{n} \binom{n}{j} \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor_{r-s} (2s)^{\overline{n-j}}.$$

Remark 3.1. In the special cases $s = 1$ and $s = r$ this gives

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \sum_{j=k}^{n} \binom{n}{j} \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor_{r-1} (n-j+1)!, \quad \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \sum_{j=k}^{n} \binom{n}{j} \left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor (2r)^{\overline{n-j}}.$$

Proof. We can suppose that n, r are not both 0. Consider an $(n+r)$-element set with r distinguished elements, which will be partitioned into $k+r$ nonempty ordered subsets such that the distinguished elements are in distinct ordered blocks.

Begin with placing the first s distinguished elements into distinct ordered blocks, and denote by j the number of those non-distinguished elements which do not belong to these ordered blocks ($j = k, \ldots, n$). There are $\binom{n}{j}$ possibilities to choose these elements, and together with the last $r-s$ distinguished elements they can be partitioned in $\left\lfloor \begin{smallmatrix} j \\ k \end{smallmatrix} \right\rfloor_{r-s}$ ways. The remaining $n-j$ non-distinguished elements have to be placed into the ordered blocks of the first s distinguished elements, which can be done in $(2s)^{\overline{n-j}}$ ways since we can put them before any element or to the end of any ordered block. □

Theorem 3.5. *If $0 \le k \le n$ and $0 \le s \le r$, then*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \sum_{j=k}^{n} \left\lfloor \begin{matrix} n \\ j \end{matrix} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}}.$$

Remark 3.2. Especially, for $s=1$ under the assumption $k \le n-2$ and for $s=r$ we get

$$\left\lfloor {n \atop k} \right\rfloor_r = \left\lfloor {n \atop k} \right\rfloor_{r-1} + 2(k+1)\left\lfloor {n \atop k+1} \right\rfloor_{r-1} + (k+1)(k+2)\left\lfloor {n \atop k+2} \right\rfloor_{r-1},$$

$$\left\lfloor {n \atop k} \right\rfloor_r = \sum_{j=k}^{n} \left\lfloor {n \atop j} \right\rfloor \binom{j}{k} (2r)^{\underline{j-k}}.$$

Proof. We prove the theorem by induction on n. It can be easily checked for $n=0$ and assume that it holds for some n.

In case of $n+1$, the assertion follows from Theorem 3.2 for $k=0$, and it is straightforward for $k=n+1$. For $1 \le k \le n$, we use Theorem 3.1, the induction hypothesis and the recurrence relation of binomial coefficients to have

$$\left\lfloor {n+1 \atop k} \right\rfloor_r = \left\lfloor {n \atop k-1} \right\rfloor_r + (n+k+2r)\left\lfloor {n \atop k} \right\rfloor_r =$$

$$\sum_{j=k-1}^{n} \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j}{k-1} (2s)^{\underline{j-k+1}} + (n+k+2r)\sum_{j=k}^{n} \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}} =$$

$$\left\lfloor {n \atop k-1} \right\rfloor_{r-s} + \sum_{j=k}^{n} \left\lfloor {n \atop j} \right\rfloor_{r-s} \left(\binom{j}{k-1} + \binom{j}{k} \right) (2s)^{\underline{j-k+1}} + \sum_{j=k}^{n} (n+j+2r-2s) \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}} =$$

$$\sum_{j=k-1}^{n} \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j+1}{k} (2s)^{\underline{j-k+1}} + \sum_{j=k}^{n} (n+j+2r-2s) \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}} =$$

$$\binom{n+1}{k} (2s)^{\underline{n-k+1}} + \sum_{j=k}^{n} \left(\left\lfloor {n \atop j-1} \right\rfloor_{r-s} + (n+j+2r-2s)\left\lfloor {n \atop j} \right\rfloor_{r-s} \right) \binom{j}{k} (2s)^{\underline{j-k}} =$$

$$\sum_{j=k}^{n+1} \left\lfloor {n+1 \atop j} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}}.$$

□

A binomial convolutional type identity holds for r-Lah numbers.

Theorem 3.6. *If $n, k, l, r, s \ge 0$ and $k+l \le n$, then*

$$\binom{k+l}{k} \left\lfloor {n \atop k+l} \right\rfloor_{r+s} = \sum_{j=k}^{n-l} \binom{n}{j} \left\lfloor {j \atop k} \right\rfloor_r \left\lfloor {n-j \atop l} \right\rfloor_s.$$

Proof. We can suppose that n, r, s are not all 0. Consider an $(n+r+s)$-element set with $r+s$ distinguished elements. We will partition it into $k+l+r+s$ nonempty ordered subsets such that the distinguished elements belong to distinct ordered blocks, and colour it using only colours green and red such that the ordered blocks are monochromatic, the first r distinguished elements are green, the other s distinguished elements are red, finally the number of green and red ordered blocks are $k+r$ and $l+s$, respectively.

First, there are $\left\lfloor{n \atop k+l}\right\rfloor_{r+s}$ possibilities to partition the elements. The $r+s$ ordered blocks containing distinguished elements get their colours automatically, while we can choose further k green ordered blocks among the other $k+l$ ordered blocks in $\binom{k+l}{k}$ ways.

Now, we begin with colouring the elements. The colours of distinguished elements are given. Let j be the number of green non-distinguished elements ($j = k, \dots, n-l$), there are $\binom{n}{j}$ possibilities to choose them. Then $j+r$ green elements have to be partitioned into $k+r$ nonempty ordered subsets and $n-j+s$ red elements have to be partitioned into $l+s$ nonempty ordered subsets such that the distinguished elements belong to distinct ordered blocks, which can be done in $\left\lfloor{j \atop k}\right\rfloor_r$ and $\left\lfloor{n-j \atop l}\right\rfloor_s$ ways, respectively. □

The r-Lah numbers satisfy the following explicit formula.

Theorem 3.7. *If $0 \le k \le n$, $r \ge 0$ and k, r are not both 0, then*

$$\left\lfloor{n \atop k}\right\rfloor_r = \frac{n!}{k!}\binom{n+2r-1}{k+2r-1}.$$

Proof. Consider an $(n+r)$-element set with r distinguished elements, which will be partitioned into $k+r$ nonempty ordered subsets such that the distinguished elements are in distinct ordered blocks.

First, place the distinguished elements into r distinct ordered blocks, and permute the non-distinguished elements, which can be done in $n!$ ways. Thereafter we decide how many non-distinguished elements we place before and after distinguished elements in their ordered blocks and how many non-distinguished elements go to the k new ordered blocks. This means that we have to split this permutation into $k+2r$ pieces such that the last k pieces are nonempty, for which we have $\left(\!\binom{k+2r}{n-k}\!\right) = \binom{n+2r-1}{n-k} = \binom{n+2r-1}{k+2r-1}$ possibilities. But we have to divide by the number of lists of the k new ordered blocks, since their order is unimportant. □

This explicit formula enables us to study relations among r-Lah numbers when the upper parameter is fixed.

Theorem 3.8. *Let $n \ge 1$ and $r \ge 0$. Then the sequence $\left(\left\lfloor{n \atop k}\right\rfloor_r\right)_{k=0}^{n}$ is strictly log-concave, therefore it is unimodal.*

Proof. We have to prove that $\left\lfloor{n \atop k-1}\right\rfloor_r \left\lfloor{n \atop k+1}\right\rfloor_r < \left\lfloor{n \atop k}\right\rfloor_r^2$ for $1 \le k \le n-1$, where we can assume that $n \ge 2$.

It is easy to prove when $k=1$ and $r=0$. In the remaining cases, after some calculation, Theorem 3.7 shows that this inequality is equivalent to $k(k+2r-1)(n-k) < (k+1)(k+2r)(n-k+1)$, which is obviously true. □

From strictly log-concavity, it follows that $\left(\left\lfloor{n \atop k}\right\rfloor_r\right)_{k=0}^{n}$ has one or two consecutive maximum points, which can be completely described.

Theorem 3.9. *Let $n \geq 1$ and $r \geq 0$. If $n + r^2 + 1$ is not a square number, then the sequence $\left(\left\lfloor {n \atop k} \right\rfloor_r\right)_{k=0}^{n}$ has maximum only at $k = \left\lfloor \sqrt{n + r^2 + 1} \right\rfloor - r$, while if $n + r^2 + 1$ is a square number, then it has maxima at $k = \sqrt{n + r^2 + 1} - r - 1$ and $\sqrt{n + r^2 + 1} - r$.*

Proof. Apart from $\left\lfloor {n \atop 0} \right\rfloor_0 = 0$, $\left\lfloor {n \atop k} \right\rfloor_r$ is always positive, hence it is enough to handle these values. Then, again after some calculation, Theorem 3.7 implies that $\left\lfloor {n \atop k-1} \right\rfloor_r \lesseqqgtr \left\lfloor {n \atop k} \right\rfloor_r$ is equivalent to $k^2 + 2kr - n - 1 \lesseqqgtr 0$, which gives $(k+r)^2 \lesseqqgtr n + r^2 + 1$, that is $k \lesseqqgtr \sqrt{n + r^2 + 1} - r$. □

Theorem 3.10. *For $k, r \geq 0$, the exponential generating function of $\left(\left\lfloor {n \atop k} \right\rfloor_r\right)_{n=k}^{\infty}$ is*

$$\sum_{n=k}^{\infty} \left\lfloor {n \atop k} \right\rfloor_r \frac{x^n}{n!} = \frac{1}{k!} \left(\frac{x}{1-x} \right)^k \left(\frac{1}{1-x} \right)^{2r}.$$

Proof. We can assume that k, r are not both 0. Then applying Theorem 3.7 and the binomial series, we have

$$\sum_{n=k}^{\infty} \left\lfloor {n \atop k} \right\rfloor_r \frac{x^n}{n!} = \sum_{n=k}^{\infty} \frac{n!}{k!} \binom{n+2r-1}{k+2r-1} \frac{x^n}{n!} = \frac{x^k}{k!} \sum_{n=k}^{\infty} \binom{n+2r-1}{n-k} x^{n-k} =$$

$$\frac{x^k}{k!} \sum_{n=0}^{\infty} \binom{n+k+2r-1}{n} x^n = \frac{x^k}{k!} \sum_{n=0}^{\infty} \binom{-k-2r}{n} (-x)^n = \frac{x^k}{k!} (1-x)^{-k-2r}.$$

□

Finally, we describe the self-orthogonality of r-Lah numbers and their connections with r-Stirling numbers.

Theorem 3.11. *Let $0 \leq k \leq n$ and $r, s \geq 0$. Then*

- $\displaystyle\sum_{j=k}^{n} (-1)^{j-k} \left\lfloor {n \atop j} \right\rfloor_r \left\lfloor {j \atop k} \right\rfloor_s = \binom{n}{k} (2r-2s)^{\overline{n-k}}$,
- $\displaystyle\begin{bmatrix} n \\ k \end{bmatrix}_{2r-s} = \sum_{j=k}^{n} (-1)^{j-k} \left\lfloor {n \atop j} \right\rfloor_r \begin{bmatrix} j \\ k \end{bmatrix}_s$ *if $2r \geq s$,*
- $\displaystyle\begin{Bmatrix} n \\ k \end{Bmatrix}_{2s-r} = \sum_{j=k}^{n} (-1)^{n-j} \begin{Bmatrix} n \\ j \end{Bmatrix}_r \left\lfloor {j \atop k} \right\rfloor_s$ *if $2s \geq r$,*
- $\displaystyle\left\lfloor {n \atop k} \right\rfloor_{\frac{r+s}{2}} = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r \begin{Bmatrix} j \\ k \end{Bmatrix}_s$ *if r and s have the same parity.*

Proof. Using Theorem 3.2 twice, we have

$$(x+2r-2s)^{\overline{n}} = \sum_{j=0}^{n} \left[\begin{matrix} n \\ j \end{matrix}\right]_r (x-2s)^{\underline{j}} = \sum_{j=0}^{n} \left[\begin{matrix} n \\ j \end{matrix}\right]_r \sum_{k=0}^{j} (-1)^{j-k} \left[\begin{matrix} j \\ k \end{matrix}\right]_s x^{\overline{k}} =$$

$$\sum_{k=0}^{n} \sum_{j=k}^{n} (-1)^{j-k} \left[\begin{matrix} n \\ j \end{matrix}\right]_r \left[\begin{matrix} j \\ k \end{matrix}\right]_s x^{\overline{k}}.$$

Comparing it with the binomial theorem for rising factorials, $(x+2r-2s)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} (2r-2s)^{\overline{n-k}}$, we proved the first assertion.

The other statements follow similarly from Theorem 3.2 and the polynomial identities for r-Stirling numbers. □

Corollary 3.1. *If $0 \le k \le n$ and $r \ge 0$, then*

$$\sum_{j=k}^{n} (-1)^{n-j} \left[\begin{matrix} n \\ j \end{matrix}\right]_r \left[\begin{matrix} j \\ k \end{matrix}\right]_r = \delta_{nk}, \qquad \left[\begin{matrix} n \\ k \end{matrix}\right]_r = \sum_{j=k}^{n} (-1)^{j-k} \left[\begin{matrix} n \\ j \end{matrix}\right]_r \left[\begin{matrix} j \\ k \end{matrix}\right]_r,$$

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}_r = \sum_{j=k}^{n} (-1)^{n-j} \left\{\begin{matrix} n \\ j \end{matrix}\right\}_r \left[\begin{matrix} j \\ k \end{matrix}\right]_r, \qquad \left[\begin{matrix} n \\ k \end{matrix}\right]_r = \sum_{j=k}^{n} \left[\begin{matrix} n \\ j \end{matrix}\right]_r \left\{\begin{matrix} j \\ k \end{matrix}\right\}_r.$$

Proof. It is an immediate consequence of Theorem 3.11 with $s = r$. However, the last statement can be proved combinatorially as well.

We can suppose that n, r are not both 0. Partition an $(n+r)$-element set into $k+r$ ordered sets such that r distinguished elements are in distinct ordered blocks in the following way: First split the elements into $j+r$ disjoint cycles such that the distinguished elements belong to distinct cycles ($j = k, \dots, n$), then partition these cycles into $k+r$ blocks such that the r cycles containing distinguished elements are in distinct blocks. The product of disjoint cycles in a certain block gives a list of the occuring elements, that is an ordered block. Therefore, the number of partitions into ordered sets is $\sum_{j=k}^{n} \left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_r \left\{\begin{smallmatrix} j \\ k \end{smallmatrix}\right\}_r$. □

Theorem 3.12. *Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be sequences of complex numbers and let $r \ge 0$. Then $b_n = \sum_{k=0}^{n} \left\lfloor\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rfloor_r a_k$ $(n \ge 0)$ if and only if $a_n = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rfloor_r b_k$ $(n \ge 0)$.*

Proof. If $b_n = \sum_{k=0}^{n} \left\lfloor\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rfloor_r a_k$ $(n \ge 0)$, then the first equation of Corollary 3.1 gives

$$\sum_{k=0}^{n} (-1)^{n-k} \left\lfloor\begin{matrix} n \\ k \end{matrix}\right\rfloor_r b_k = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor\begin{matrix} n \\ k \end{matrix}\right\rfloor_r \sum_{j=0}^{k} \left\lfloor\begin{matrix} k \\ j \end{matrix}\right\rfloor_r a_j =$$

$$\sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{n-k} \left\lfloor\begin{matrix} n \\ k \end{matrix}\right\rfloor_r \left\lfloor\begin{matrix} k \\ j \end{matrix}\right\rfloor_r a_j = \sum_{j=0}^{n} \delta_{nj} a_j = a_n.$$

For the opposite direction, apply this for the sequences $((-1)^n b_n)_{n=0}^{\infty}$ and $((-1)^n a_n)_{n=0}^{\infty}$. □

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