

On a problem of Recaman and its generalization

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Abstract

We solve some cases of a conjecture of Pomerance concerning reduced residue systems modulo k consisting of the first $\varphi(k)$ primes not dividing k . We cover the case when k is a prime, thus giving a complete solution to a problem of Recaman.

Key words: the problem of Recaman, the problem of Pomerance, Jacobsthal function, primes in residue classes

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Dedicated to Professor K. Gyóry on the occasion of his 70th birthday

1 Introduction

Let $k > 1$ be an integer and denote by $\varphi(k)$ Euler's totient function. We say that k is a P -integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k . Note that a prime p is a P -integer if and only if the first p primes form a complete residue system modulo p . In 1980, Pomerance [3] showed that there are only finitely many P -integers. Thereby he qualitatively solved the problem of finitely many prime P -integers which was raised earlier in 1978 by Recaman [4]. In his paper Pomerance conjectured that the largest

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P -integer is $k = 30$. It is easy to check that the only P -integers less than or equal to 30 are $k = 2, 4, 6, 12, 18, 30$.

In this paper we prove the conjecture of Pomerance in two “opposite” extremal cases: when k is composed of “large” prime factors (i.e. when all the prime divisors of k are above $\log(k)$), and when k is composed of “small” prime factors (i.e. k is the product of all primes $\leq x$ for some x). As a trivial consequence of the first result we get a complete quantitative solution for the problem of Recaman. Further, we verify the conjecture of Pomerance for all $k < 5.5 \cdot 10^5$. We note that Pomerance’s finiteness result for P -integers [3] in principle can be made effective: one can possibly get an explicit upper bound for P -integers k . However, according to our calculations, this bound is rather huge, and it seems that to cover the remaining gap some additional (theoretical and/or computational) arguments are needed. So the complete resolution of the problem of Pomerance still remains an open quest; we plan to attack it in a future paper.

The proofs of our results depend on some properties of the Jacobsthal function $g(m)$ as in [3]. Among others we use the exact values of $g(m)$ when m is the product of first $h \leq 46$ primes, which were recently obtained by Hagedorn [1]. Further, we apply several formulas of Rosser and Schoenfeld [5], concerning various functions involving primes.

2 Main results

Our first result solves Recaman’s problem completely.

Theorem 1 *The only prime P -integer is 2.*

In fact Theorem 1 is a trivial consequence of the following much more general result. For $k > 1$ let $\ell(k)$ be the least prime divisor of k .

Theorem 2 *Let $k > 1$ be an integer with $\ell(k) > \log(k)$. Then k is a P -integer if and only if $k \in \{2, 4, 6\}$.*

For fixed positive integer r and positive real X write

$$N_r := \{n \mid \omega(n) = r\} \quad \text{and} \quad N_r(X) := \{n \in N_r \mid n \leq X\},$$

where $\omega(n)$ denotes the number of distinct prime divisors of n . Further, for any positive real x , we let $\log_1(x) = \log(x)$ and for $t \geq 2$, $\log_t(x) = \log(\log_{t-1}(x))$.

By a result of Landau it is known that

$$|N_r(X)| \sim \frac{X(\log_2(X))^{r-1}}{\log(X)(r-1)!}$$

(see Theorem 437, p. 368 of [2]). Let $N'_r(X)$ denote the set of integers n in $N_r(X)$ with $\ell(n) \leq \log(n)$. Then for any $n \in N'_r(X)$ we have $\ell(n) \leq \log(X)$ and $n/\ell(n) \in N_{r-1}(X/\ell(n))$. Applying Landau's result to $N_{r-1}(X/p)$ for every $p \leq \log(X)$, and noting that $\frac{(\log_2(x))^{r-2}}{\log(x)}$ is a decreasing function of x for sufficiently large x , we find that

$$\begin{aligned} |N'_r(X)| &\leq c_1 \sum_{p \leq \log(X)} \frac{\frac{X}{p} \left(\log_2\left(\frac{X}{p}\right)\right)^{r-2}}{\log\left(\frac{X}{p}\right)(r-2)!} \leq c_2 \frac{X(\log_2(X))^{r-2}}{\log(X)(r-2)!} \sum_{p \leq \log(X)} \frac{1}{p} \leq \\ &\leq c_3 \frac{X(\log_2(X))^{r-2}}{\log(X)(r-2)!} \log_3(X) \end{aligned}$$

where c_1 , c_2 and c_3 are absolute constants. Thus we see that almost all integers in N_r has $\ell(n) > \log(n)$. In particular, k is not a P -integer whenever k is the product of twin primes.

Our third theorem verifies the conjecture of Pomerance for integers k being the products of the first few primes.

Theorem 3 *Let k be the product of the primes $\leq x$ for some $x \geq 2$. Then k is a P -integer if and only if $k \in \{2, 6, 30\}$.*

Finally, we formulate a statement concerning the solution of the problem of Pomerance for “small” values of k . Our main motivation of doing so is that this result will be very useful in the proof of Theorem 2.

Proposition 4 *Suppose that $1 < k < 5.5 \cdot 10^5$. Then k is a P -integer if and only if $k \in \{2, 4, 6, 12, 18, 30\}$.*

3 Lemmas

We need many lemmas of different types to prove our theorems. We shall make use of several estimates of Rosser and Schoenfeld [5] concerning various functions related to prime numbers. Further, we need certain results due to Stevens [6] and Hagedorn [1] about the Jacobsthal function. Finally, we need a theorem of Pomerance about primes in residue classes modulo m .

3.1 Lemmas concerning functions involving primes

The following four lemmas are estimates from Rosser and Schoenfeld [5] which we need later on.

Lemma 5 *Let p_n denote the n -th prime. Then*

- (i) $p_n > n(\log(n) + \log_2(n) - \frac{3}{2})$ for $n > 1$;
- (ii) $p_n < n(\log(n) + \log_2(n))$ for $n \geq 6$.

Lemma 6 *For any $x \geq 59$ we have*

$$\frac{x}{\log(x)} \left(1 + \frac{1}{2\log(x)}\right) < \pi(x) < \frac{x}{\log(x)} \left(1 + \frac{3}{2\log(x)}\right).$$

Lemma 7 *For $x \geq 2$ write $\vartheta(x) = \sum_{p \leq x} \log(p)$. For any $x \geq 563$ we have*

$$x \left(1 - \frac{1}{2\log(x)}\right) < \vartheta(x) < x \left(1 + \frac{1}{2\log(x)}\right).$$

Lemma 8 *For any $x > 1$ we have*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{0.56146}{\log(x)} \left(1 + \frac{1}{2\log^2(x)}\right).$$

Note that here 0.56146 could be replaced by any number exceeding $e^{-\gamma}$, where γ is Euler's constant.

3.2 Lemmas about the Jacobsthal function

For $n \geq 1$ the Jacobsthal function $g(n)$ is defined as the smallest integer such that any sequence of $g(n)$ consecutive integers contains an element which is coprime to n . This function has been studied by many authors, and good lower as well as upper bounds are known (see e.g. [6], [3] and [1] for history). Further, the exact values of $g(n)$ when n is the product of the first $h < 50$ primes is given in Table 1 of [1].

It was observed by Jacobsthal that for integers k with $\ell(k) > \log(k)$ we have $g(k) = \omega(k) + 1$. Further, $g(k) \geq \omega(k) + 1$ is obviously valid for any k . We shall use these assertions throughout the paper without any further reference.

Our first lemma concerning the Jacobsthal function is a reformulation of the Theorem of Stevens [6].

Lemma 9 *We have $g(k) \leq 2\omega(k)^{2+2e \log(\omega(k))}$ for all $k > 1$.*

The next lemma is Proposition 1.1 of Hagedorn [1].

Lemma 10 *We have*

$$g\left(\prod_{i=1}^h p_i\right) \geq 2p_{h-1} \quad \text{for } h > 2.$$

3.3 A result of Pomerance

Let k and l be positive integers with $\gcd(k, l) = 1$. Denote by $p(k, l)$ the least prime $p \equiv l \pmod{k}$. We write $P(k)$ for the maximal value of $p(k, l)$ for all l . Observe that k is a P -integer if and only if $P(k)$ equals the $\varphi(k)$ -th prime not dividing k . Since the number of primes dividing k is $\omega(k)$, we get that if k is a P -integer then

$$p_{\varphi(k)} \leq P(k) \leq p_{\varphi(k)+\omega(k)}$$

holds. Note also that since $\varphi(k) + \omega(k) \leq k$, we have $P(k) \leq p_k$ whenever k is a P -integer.

To prove the finiteness of k 's which are P -integers, Pomerance [3] derived a lower bound for $P(k)$ which (for large k) turns to be larger than standard upper bounds for $p_{\varphi(k)+\omega(k)}$, obtained by using estimates from [5]. This lower bound of Pomerance is based upon the following result from [3].

Lemma 11 *Let k and m be integers with $0 < m \leq \frac{k}{1+g(k)}$ and $\gcd(m, k) = 1$. Then $P(k) > (g(m) - 1)k$.*

4 Proofs

Since in the proof of Theorem 2 we use Proposition 4, we start with the proof of the latter result.

Proof of Proposition 4. Let k be arbitrary with $1 < k < 5.5 \cdot 10^5$. Let $q_1 < q_2 < q_3 < \dots$ be the primes $> tk$ with $t = 1$ if k is even and $t = 2$ if k is odd, respectively. We find the first index i such that $q_i - tk$ is a prime. For all k in the considered interval we found $i \leq 34$. If $k + 2$ is a prime then let $q = k + 2$, otherwise set $q = q_i$ with the above defined index i . A calculation with Maple based upon Lemma 5 ensures that for $k > 210$ we have $q \leq p_{\varphi(k)}$. Thus there exist two primes $\leq p_{\varphi(k)}$ being coprime to k in the same residue class modulo k , which proves that k is not a P -integer in this case. Finally, for $k \leq 210$ we

check by Maple the first $\varphi(k)$ primes not dividing k to get the assertion of the proposition. \square

Proof of Theorem 2. Let k be a P -integer with $\ell(k) > \log k$. Assume first that $k \geq 10^{90}$. We split the proof of this case into two parts. Suppose first that $k < (\omega(k) + 2)^{20}$. Then, since we know that $\omega(k) \log(\ell(k)) \leq \log(k)$, we obtain

$$\omega(k) \leq \frac{\log(k)}{\log_2(k)}.$$

Hence using our assumption for k we get

$$k < \left(\frac{\log(k)}{\log_2(k)} + 2 \right)^{20}.$$

This implies that $k < 10^{90}$, which is a contradiction, and the statement follows in this case. Suppose next that we have $k \geq (\omega(k) + 2)^{20}$. Let

$$h = \left\lfloor \frac{0.92 \log(k)}{\log_2(k)} \right\rfloor + 1.$$

Then

$$h < \frac{0.946 \log(k)}{\log_2(k)} < \log(k).$$

Hence by Lemma 5 (ii)

$$p_h < 0.946 \log(k) < \log(k).$$

Let m be the product of the first h primes coprime to k . Since $p_h < \log(k) < \ell(k)$, by assumption, we see that m is indeed the product of all the first h primes. Hence

$$m < p_h^h < e^{0.946 \log(k)} < \frac{k}{\omega(k) + 2}$$

since we assumed $\omega(k) + 2 \leq k^{\frac{1}{20}}$. Thus by Lemmas 10 and 11, we have

$$P(k) > (g(m) - 1)k \geq (2p_{h-1} - 1)k.$$

Now

$$h - 1 \geq 0.92 \frac{\log(k)}{\log_2(k)} - 1 > 0.894 \frac{\log(k)}{\log_2(k)}.$$

Hence by Lemma 5 (i)

$$p_{h-1} \geq X \left(\log(X) + \log_2(X) - \frac{3}{2} \right)$$

where $X = 0.894 \frac{\log(k)}{\log_2(k)}$. Let

$$F(k) = 2X \left(\log(X) + \log_2(X) - \frac{3}{2} \right) k - k \log(k) - k \log_2(k) - k.$$

Then $F(k) = k \log(k) f(k)$ with

$$f(k) := \frac{1.788}{\log_2(k)} \left(\log(X) + \log_2(X) - \frac{3}{2} \right) - 1 - \frac{\log_2(k)}{\log(k)} - \frac{1}{\log(k)}.$$

Observe that $f(k)$ is an increasing function of k and hence $f(k) \geq f(10^{90})$, since $k \geq 10^{90}$. As $f(10^{90}) \geq 0.0803$, we find that $F(k) > 0$ which implies that $P(k) > k \log(k) + k \log_2(k) > p_k \geq p_{\varphi(k)+\omega(k)}$. Hence k is not a P -integer. This contradiction proves the theorem for $k \geq 10^{90}$ with $\ell(k) > \log(k)$.

Assume now that $k < 10^{90}$. By Proposition 4 we may suppose that $5.5 \cdot 10^5 \leq k < 10^{90}$. We divide the interval $[5.5 \cdot 10^5, 10^{90})$ into sub-intervals and assign a value h to each interval as follows. Let $v_0 = 10^{90}$. The largest integer h such that $p_h < \log(10^{90})$ is 46. We set our initial sub-interval as $[u_0, v_0) = [10^{87}, 10^{90})$, $\alpha_0 = 87$ and $h_0 = h = 46$. For any k with $\ell(k) > \log(k)$ in this interval we have $g(k) = \omega(k) + 1 < \log(k) + 1 < 209$. We check that

$$m_0 := \prod_{j=1}^{46} p_j < \frac{10^{87}}{210} \leq \frac{k}{g(k) + 1}.$$

Now we proceed inductively. Let $i \geq 1$ and take $h_i = h_0 - i$. We define the sub-interval $[u_i, v_i)$ as $[10^{\alpha_i}, 10^{\alpha_{i-1}})$ satisfying the following properties:

$$p_{h_i} < \alpha_i \log(10) \tag{1}$$

and

$$m_i := \prod_{j=1}^{h_0-i} p_j < \frac{10^{\alpha_i}}{(\alpha_{i-1} \log(10) + 2)}. \tag{2}$$

Let $k \in [u_i, v_i)$ with $\ell(k) > \log(k)$. Then $p_{h_i} < \log(k)$ and hence by the assumption on k , m_i is the product of the first h_i primes, and $\gcd(m_i, k) = 1$. Suppose that

$$g(m_i) - 1 - \alpha_{i-1} \log(10) - \log(\alpha_{i-1} \log(10)) > 0. \tag{3}$$

Then, since $k \leq 10^{\alpha_{i-1}}$, we find by Lemma 11 and Lemma 5 (ii) that

$$P(k) > k \log(k) + k \log_2(k) > p_k \geq p_{\varphi(k)+\omega(k)}$$

and hence $k \in [u_i, v_i)$ is not a P -integer.

In Table 1 we give the values $h_i = h$, $\alpha_i = \alpha$, and the exact value of $g(m)$ with $m = m_i$ from Table 1 of [4]. For these values, we check that (1), (2)

and (3) are satisfied and hence we conclude that $k < 10^8$. Now consider k in

Table 1

h	7	8	9	10	11	12	13	14	15	16
$g(m)$	26	34	40	46	58	66	74	90	100	106
α	8	9	10	13	14	17	18	19	21	24
h	17	18	19	20	21	22	23	24	25	26
$g(m)$	118	132	152	174	190	200	216	234	258	264
α	26	27	30	31	32	35	37	39	43	44
h	27	28	29	30	31	32	33	34	35	36
$g(m)$	282	300	312	330	354	378	388	414	432	450
α	45	47	48	55	56	57	60	61	65	66
h	37	38	39	40	41	42	43	44	45	46
$g(m)$	476	492	510	538	550	574	600	616	642	660
α	69	71	73	76	78	79	83	84	86	87

the intervals $[3 \cdot 10^7, 10^8)$ with $h = 7$ and $[5.5 \cdot 10^5, 3 \cdot 10^7)$ with $h = 6$ and $g(m) = 22$, respectively. Then conditions (1), (2) and (3) are satisfied again, showing that k is not a P -integer. Hence the statement follows. \square

Proof of Theorem 3. Assume first that $x \geq 1000$ and put $k = \prod_{p \leq x} p$. Set $m := \prod_{x < p \leq y} p$ with $y = 1.777x$. First we show that by these choices we have $m \leq k/(1 + g(k))$. This inequality can be rewritten as

$$1 + g(k) \leq \frac{\exp(2\vartheta(x))}{\exp(\vartheta(y))}.$$

Using Lemma 9, it is sufficient to show that

$$1 + 2\pi(x)^{2+2e \log(\pi(x))} \leq \exp(2\vartheta(x) - \vartheta(1.777x)).$$

With the help of Maple, by Lemmas 6 and 7 this can be seen to be true whenever $x \geq 12000$. For $1000 \leq x < 12000$ the assertion can be checked by calculating the exact values of the functions $\pi(x)$ and $\vartheta(x)$.

Now we show that (still with $x \geq 1000$) we have $(g(m) - 1)k \geq p_{\varphi(k)+\omega(k)}$. By Lemma 11 this implies the statement. To prove this, observe that $g(m) >$

$\omega(m) = \pi(y) - \pi(x)$. Hence using Lemma 5 (ii) it is sufficient to check that

$$\pi(1.777x) - \pi(x) \geq \left(\prod_{p \leq x} \left(1 - \frac{1}{p} \right) + \frac{\pi(x)}{\prod_{p \leq x} p} \right) (\vartheta(x) + \log(\vartheta(x)))$$

for $x \geq 1000$. Again, by the help of Lemmas 6, 7 and 8 this inequality can be verified for $x \geq 12000$ with Maple. Further, for $1000 \leq x < 12000$ the assertion can be proved by calculating the exact values of the expressions involved. Hence the statement is valid when $x \geq 1000$.

Assume now that $x < 1000$. Then we check the values of k one by one. For k given, let $q_1 = p_{\pi(k)+1}$ and $q_2 = p_{\pi(k)+2}$. A calculation by Maple shows that for $k > 30$ we have $q_2 \leq p_{\varphi(k)+\omega(k)}$, and also that one of $q_1 - k$, $q_2 - k$ is a prime. Finally, as $k = 2, 6, 30$ are P -integers indeed, the statement follows. \square

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