Binomial Thue equations, ternary equations and their applications

Doktori (PhD) értekezés

Bazsó András

Témavezető: Dr. Bérczes Attila

Debreceni Egyetem
Természettudományi Doktori Tanács
Matematika- és Számítástudományok Doktori Iskola
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Ezen értekezést a Debreceni Egyetem Természettudományi Doktori Tanács *Matematika- és Számítástudományok* Doktori Iskola *Diofantikus és Konstruktív Számelmélet* programja keretében készítettem a Debreceni Egyetem természettudományi doktori (PhD) fokozatának elnyerése céljából.

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a jelölt aláírása

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Debrecen, 2010.

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a témavezető aláírása

Binomial Thue equations, ternary equations and their applications

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a matematika tudományágban

Írta: Bazsó András okleveles matematikus, matematika tanár

Készült a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskolája Diofantikus és Konstruktív Számelmélet programja keretében

Témavezető: Dr. Bérczes Attila

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A doktori szigorlat időpontja: 201 . . ..

Az értekezés bírálói:

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Köszönetnyilvánítás

Ezúton szeretnék köszönetet mondani mindazoknak, akik bármi módon hozzájárultak disszertációm elkészítéséhez.

Elsősorban szüleimnek, akik szeretetben felneveltek, szépre-jóra tanítottak és elindítottak az életben,

Témavezetőmnek, Dr. Bérczes Attilának az állandó támogatásáért, hasznos tanácsaiért és a dolgozatom elkészítésében nyújtott segítségéért,

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Kollégáimnak az Algebra és Számelmélet Tanszéken, akikhez bármikor fordulhattam szakmai segítségért,

végül de nem utolsó sorban Zsófinak, a bátyámnak, Gergőnek és barátaimnak a bíztatásért és hogy mellettem álltak.

SZÜLEIMNEK

Contents

Introduction

Our PhD dissertation consists of three chapters each containing new results. These results have been published in our papers [2], [3] and [4], respectively. In this introduction we shall give an overview of the contents of the chapters, but before doing so we make some introductory notes on the subject of our thesis.

Each of the three chapters is devoted to the study of the solutions of various important classes of diophantine equations, namely Fermat-type ternary equations, binomial Thue equations and norm form equations, respectively. The proofs of the new results in Chapters 2 and 3 use the results and techniques presented in the first chapter. This is the reason for dealing first with ternary equations of the form

$$Ax^n + By^n = Cz^m \text{ with } m \in \{2, 3, n\}, \tag{1}$$

where A, B, C are given nonzero integers, $n \geq 3$ and x, y, z are unknown integers. Equation (1) has a famous special case, namely the Fermat-equation

$$x^n + y^n = z^n$$

which has been widely studied for centuries due to Fermat's Last Theorem (FLT) which was finally proved in 1995 by Wiles [53]. The so-called modular approach that arose from the proof of Wiles for FLT, has many applications in the theory of diophantine equations, since it can be used for proving the unsolvability of (1) for several concrete values of A, B, C, n and m. For the details of the modular approach we refer to the papers of Bennett [6], Siksek [47] and the book of Stein [49]. Further central objects of our thesis are binomial Thue equations, i.e. diophantine equations of the form

$$Ax^n - By^n = C, \tag{2}$$

where A, B, C, n are nonzero integers and $n \geq 3$ is either fixed or also unknown. Thue equations and among them binomial Thue equations have a rich literature. Thue's classical ineffective result [51] on diophantine approximation of algebraic numbers implied that Thue equations have only finitely many solutions. Baker [1] was the first who gave effective upper bounds for the size of the solutions of Thue equations. Both of these results imply that equation (2) has only finitely many solutions if n is fixed. Tijdeman [52] considered the case when, in (2), the exponent n is also unknown, and gave effective upper bounds for $\max\{|x|, |y|, n\}$, where (x, y, n) are integer solutions of (2) with $|xy| > 1$. For other related results on binomial Thue equations and their applications we refer to [39], [46], [5], [34], [7], [9], [28], [10], [15], [2], [31] and the references given there.

We note that an integer solution (x, y, n) to (2) induces a solution to (1) of the type $(x, y, 1, n, m)$. So if, for some values of A, B, C, n, m, the corresponding ternary equation (1) proves to be unsolvable in integers (x, y, z) with Ax, By and Cz pairwise coprime, $|xy| > 1$, then it follows that with those values of A, B, n (2) also has no solutions (x, y).

In what follows, we briefly summarize the main results of our thesis.

In the first chapter ternary diophantine equations of the form (1) are investigated. The modular approach to these equations including Frey-curves [24] and modular forms which has many applications in the literature, is one of the main tools in the proofs of our results in all the three chapters. An outline of this method is therefore given in Section 1.1 based on the survey paper of Bennett [6], which summarizes some deep results of Bennett and Skinner [11], Kraus [35], and Bennett, Vatsal and Yazdani [12]. We note that the applicability of the above modular approach depends only on the prime factors of the coefficients A, B, C in (1). In Sections 1.2 - 1.5, we restrict our attention to the equation

$$(3) \qquad Ax^n - By^n = z^m$$

in the cases when $m = n$ and $m = 3$. In both cases we survey the results known on the solutions of (3).

First let $m = n$ and let α, β be nonnegative integers. For $AB = p^\alpha$ with a prime $p \le 29$ or $p = 53, 59$, the resolution of (3) follows from the results of Serre [45], Wiles [53], Darmon and Merel [23], and Ribet [43]. In [35], Kraus considered the case when $AB = 2^\alpha q^\beta$ with q being a prime and with n sufficiently large compared to q. Bennett, Győry, Mignotte and Pintér [10] solved (3) when $AB = 2^\alpha q^\beta$ with primes $3 \le q \le 13$. Győry and Pintér [31] recently generalized this result to the case $3 \le q \le 29$ in the sense that if n is a prime, then apart from 8 explicitly given possible exceptions (q, α), for every integer solution (x, y, z, A, B, n) of equation (3) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 11$. Our Theorem 1.1 extends these results in the following way.

Theorem 1.1. *Let $AB = 2^\alpha q^\beta$ with a prime $3 \le q \le 151$, $q \ne 31, 127$ and with nonnegative integers α, β. If n is a prime, then for every integer solution (x, y, z, A, B, n) of the equation (3) with $|xy| > 1$ and with Ax, By and z pairwise coprime we have $n \le 53$.*
Moreover, apart from 31 possible exceptions (q, n, α) given in Table 1.1, for every integer solution (x, y, z, A, B, n) of equation (3) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 13$.

The case $AB = p^\alpha q^\beta$ with primes $5 \le p < q \le 29$ was considered by Győry and Pintér [31], who proved in this case that if n is a prime, then apart from 10 explicitly given possible exceptions (p, q), for every solution (x, y, z, A, B, n) of (3) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 11$. We generalize this result in our Theorem 1.2 as follows.

Theorem 1.2. *Let $AB = p^\alpha q^\beta$ with primes $5 \le p, q \le 71$ and nonnegative integers α, β. If n is a prime, then apart from 28 possible exceptions (p, q, n) given in Table 1.2, for every integer solution*

(x, y, z, A, B, n) of (3) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \leq 13$.

Consider now equation (3) with $m = 3$ and $AB = p^{\alpha}q^{\beta}$ where p, q are primes and α, β are nonnegative integers. This case was studied for $3 \leq p, q \leq 13$ by Bennett, Győry, Mignotte and Pintér [10]; and for $3 \leq p < q \leq 29$ by Győry and Pintér [31]. Their results can be summarized in the following way: if n is a prime and $AB = p^{\alpha}q^{\beta}$ with primes $3 \leq p < q \leq 29$ such that either $p \leq 7$ or $(p, q) \in \{(11, 13), (11, 17), (11, 19), (13, 17), (13, 19), (17, 23)\}$, then apart from 14 possible exceptions (p, q, n), for every integer solution (x, y, z, A, B, n) of equation (3) with $|xy| > 1$, xy even and Ax, By and z pairwise coprime we have $n \leq 11$. Our third theorem in Section 1.3 extends the above results to much larger primes p, q.

Theorem 1.3. *Let $AB = p^{\alpha}q^{\beta}$ with nonnegative integers α, β and primes $3 \leq p < q \leq 71$ such that $pq \leq 583$. If n is a prime, then apart from 29 possible exceptions (p, q, n) given in Table 1.3, for every integer solution (x, y, z, A, B, n) of equation (3) with $|xy| > 1$, xy even and Ax, By and z pairwise coprime we have $n \leq 13$.*

The results of this chapter will be published in our paper [3].

In the second chapter we deal with (generalized) binomial Thue equations of the form (2) which have many important applications in number theory (see the above references). After surveying the previously mentioned ineffective and effective results on binomial Thue equations with fixed or unknown exponent, we turn to the resolution of such equations. It is important to note that even the best known effective upper bounds for the solutions are too large to determine the solutions of a concrete binomial Thue equation. For this purpose several other methods are needed. In this direction, the first general result is due to Bennett [5] who showed by means of the hypergeometric method that for $B = A + 1$, the equation

$$Ax^n - By^n = \pm 1 \tag{4}$$

has no solutions with $|xy| > 1$. The case when, in (2), the coefficients are bounded positive integers was first studied by Győry and Pintér in [29]. Using a local approach (described in Section 2.4), they derived a relatively sharp upper bound for n for concrete values of A, B, C provided that (2) has no solutions with $|xy| \leq 1$. Further, they determined all solutions of (2) with $|xy| > 1$ under some natural conditions in case of various upper bounds on the coefficients. The next new results are shortened versions of our theorems from Section 2.3, which are considerable extensions of the ones in [29] to much larger upper bounds on the coefficients.

Theorem 2.1'. *If* $1 < B \leq 400$, *then all integer solutions* (x, y, n) *of the equation*

$$x^n - By^n = \pm 1 \tag{5}$$

with $|xy| > 1, n \geq 3$ *and with* $(B, n) \notin \{(235, 23), (282, 23), (295, 29), (329, 23), (354, 29)\}$ *are with* $n \in \{3, 4, 5, 6, 7, 8\}$.

Theorem 2.2'. *(i) If* $400 < B < 800$ *is odd, then all integer solutions* (x, y, n) *of equation* (5) *with* $|xy| > 1, n \geq 3$ *and with the possible exceptions* (B, n) *listed in Table 2.1 are with* $n = 3, 9$.
(ii) Let $800 < B < 2000$ *be odd. If* $n < 13$, *then all integer solutions* (x, y, n) *of equation* (5) *with* $|xy| > 1, n \geq 3$ *are with* $n \in \{3, 5, 10\}$.
If $n > 100$ *is a prime, then equation* (5) *has no solutions in integers* (x, y, n) *with* $|xy| > 1, n \geq 3$ *and with the possible exceptions* (B, n) *listed in Table 2.2.*

Theorem 2.3'. *If* $1 \leq A < B \leq 50$ *and* $\gcd(A, B) = 1$, *then all integer solutions* (x, y, n) *to equation* (4) *with* $|xy| > 1,\ n \geq 3$ *and with* $(A, B, n) \notin \{(21, 38, 17), (26, 41, 17), (22, 43, 17), (17, 46, 17), (31, 46, 17), (21, 38, 19)\}$ *are with* $n = 3, 4$.

In the original versions all the integer solutions are given instead of the statements *"... are with* $n \in \{\ldots\}$*"*.

In Chapter 2, we present two further results on the solutions of the binomial Thue equations (2) and (4) with bounded coefficients. Both of them states that there are no solutions (with $|xy| > 1$) if $n > 19$, apart from some possible exceptions which are explicitly given in our Theorems 2.4 and 2.5, respectively. We note that in our Theorems 2.1 to 2.5 we arrived at the limit of the applicability of the currently available methods.

In our proofs, almost all techniques of the modern diophantine analysis are involved. We adopt some of the methods of [29], such as local methods, a Baker-type effective result of Pintér [42] on the solutions of (2), and the modular approach. Beside these, a main ingredient of our proofs is the following new result of ours concerning the solvability of equation (5).

Theorem 2.6. *Suppose that in equation* (5) *n is a prime and that each of the following conditions holds:*

(i) $n \geq 17$,

(ii) $B \leq \exp\{3000\}$,

(iii) $n \nmid B\phi(B)$,

(iv) $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$,

(v) $r^{n-1} \not\equiv 1 \pmod{n^2}$ *for some divisor r of B.*

Then equation (5) *has no solutions in integers* (x, y, n) *with* $|xy| > 1$.

The proof of this result requires the above-mentioned effective bound of Pintér [42], some results concerning cyclotomic fields, a recent theorem of Mihăilescu [38], and computational results of Buhler, Crandall, Ernvall, Metsänkylä and Shokrollahi [22].

In Section 2.7, another aspect of the resolution of binomial Thue equations of the form (4) is considered: the case when the coefficients A, B are allowed to be arbitrary large but they can have only fixed prime divisors. In other words, when the coefficients are unknown S-units for some set of primes S of small cardinality. For $S = \{p\}$ with $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59\}$, it follows from the work of Wiles [53], Darmon and Merel [23] and Ribet [43] on ternary equations that (4) has no solutions with $|xy| > 1$ and $n \geq 3$. Bennett [7] solved (4) for $S = \{2, 3\}$. Bennett, Győry, Mignotte and Pintér [10] solved (4) in the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 13$. Independently, Bugeaud, Mignotte and Siksek [21] solved (4) when $A = 2^\alpha, B = q^\beta$ with a prime $3 \leq q < 100$, or $A = p^\alpha, B = q^\beta$ with primes $3 \leq p < q \leq 31$, and in both cases α, β are nonnegative integers. Recently, Győry and Pintér [31] generalized the above results of [10] to the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 29$. Applying our Theorem 2.6 and the results of Chapter 1 combined with other methods, in our Theorem 2.7, we give reasonable upper bounds on the exponent n in (4) in the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 71$. This result may be a useful tool in solving concrete binomial Thue equations of such type.

The results presented in Section 2.3 are joint with A. Bérczes, K. Győry and Á. Pintér, and are published in our joint paper [4]. Theorem 2.7 will be published in our paper [3].

In the third chapter, we present an application of binomial Thue equations to norm form equations having solutions whose coordinates form an arithmetic progression. A *norm form* is a form of the type

$$F := a_0 N_{K/\mathbb{Q}}(\alpha_1 X_1 + \ldots + \alpha_m X_m),$$

where $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly independent elements of a number field K of degree n over $\mathbb{Q}$, and $a_0 \in \mathbb{Z} \setminus \{0\}$ is chosen so that F has integer coefficients. By a *norm form equation* we mean a diophantine equation of the form

$$a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \ldots + \alpha_m x_m) = b, \tag{6}$$

where $b \in \mathbb{Z} \setminus \{0\}$. Norm form equations may have infinitely many solutions. Schmidt [44] proved, in an ineffective way, a criterion for (6) to have only finitely many solutions. Later, Győry and Papp [26] derived effective finiteness results and explicit bounds for the solutions of a large class of norm form equations.

The idea of searching for arithmetic progressions among the solutions of norm form equations is due to Attila Pethő. Bérczes and

Pethő considered the equation

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = m. \tag{7}$$

where α is an algebraic number of degree n, $K = \mathbb{Q}(\alpha)$, $m \in \mathbb{Z}$ and $(x_0, \ldots, x_{n-1}) \in \mathbb{Z}^n$. In [14], among others they proved a nearly complete finiteness result on those solutions of (7) whose coordinates form an arithmetic progression. They also considered the corresponding solutions of the equation

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = 1, \tag{8}$$

and they determined all solutions in question when, in (8), α is a zero of either $x^n - 2$ or $x^n - 3$ $(n \geq 3)$. Further, in [15], they proved the lack of such solutions of (8) when α is a zero of the polynomial $x^n - a$, with $n \geq 3$ and $4 \leq a \leq 100$. For further results in this topic, see the papers of Bérczes, Pethő and Ziegler [16] and Bérczes, Hajdu and Pethő [13], respectively.

Our goal in Chapter 3 is to extend the result of [15] on the norm form equation (8). More precisely, we find all solutions of (8) forming an arithmetic progression when α is a zero of the polynomial $x^n - a$ for $-100 \leq a \leq -2$. Our Theorem 3.1 states that the only solutions $(x_0, \ldots, x_{n-1}) \in \mathbb{Z}^n$ where the coordinates x_i are consecutive terms in an arithmetic progression are $(2, 1, 0)$ when $(n, a) = (3, -7)$, and $(-2, -1, 0)$ when $(n, a) = (3, -9)$. In the case when $(n, a) = (11, -67)$ our result depends on the generalized Riemann Hypothesis (GRH). The proof of this theorem is based in one hand on the idea of reducing the solutions of (8) under consideration to the solutions of the (generalized) binomial thue equation

$$X^n - aY^n = (a-1)^2. \tag{9}$$

On the other hand the proof depends on our Theorem 3.2, in which all integer solutions of (9) are given, in the case $(n, a) = (11, -67)$, assuming GRH. This latter theorem is proved by first giving a Baker-type bound for the degree, then solving all equations under consideration up to this bound by means of either the local or the modular approach. For solving some equations with small exponents we used the computer packages PARI [40] and MAGMA [18]. In some cases, PARI is able to give only conditional result assuming GRH. This is the reason for the condition in our theorems.

The results of the third chapter are published in [2].

Chapter 1

Ternary equations

In this chapter we consider ternary Diophantine eqiations of the form

$$Ax^n + By^n = Cz^m$$

where A, B, C are nonzero integers, $m \in \{2, 3, n\}$ and x, y, z are unknown integers. The triple of integers (n, n, m) is usually referred to as the *signature* of the equation. The investigation of such ternary equations gained a high level of interest since Wiles [53] published his proof of Fermat's Last Theorem.

In Sections 1.1 and 1.2, starting from Wiles' result, we survey results on the resolution of ternary equations. Then in the rest of this chapter we present our new results on ternary equations, which will be applied, in Section 2.7, to binomial Thue equations with S-unit coefficients.

1.1 On the modular approach to ternary equations

Frey [24] observed the connection between a putative nonzero integer solution x, y, z of the equation

$$x^n + y^n = z^n \tag{1.1}$$

and an elliptic equation of the form

$$X(X - x^n)(X + y^n) = Y^2 \quad \text{in unknown integers } X, Y. \tag{1.2}$$

The Taniyama-Weil conjecture (TW) states that every rational elliptic curve is modular. In other words, every rational elliptic curve can be associated to a special Fourier series, a modular form. Wiles proved the TW conjecture for semistable elliptic curves, which implied that if there exists a semistable elliptic curve of the form (1.2), then there exists a modular form with level 2. However, since such modular forms do not exist, this means that for all integer solutions of (1.1) we have $xyz = 0$.

Equation (1.1) is a special case of the equation

$$Ax^n + By^n = Cz^m \text{ with } m \in \{2, 3, n\}, \tag{1.3}$$

where A, B, C are given nonzero integers, $n \geq 3$ and x, y, z are unknown integers. Approaches to solve such equations, analogous to that employed by Wiles [53], can be found in numerous papers. For a survey on this topic, see papers of Bennett [6], Siksek [47] or the book of Stein [49].

The so-called *modular approach* will be used in almost all of our proofs so here we give its outline in virtue of the paper of Bennett [6]. We also adopt his notation.

For a given prime q and non-zero integer u, set

$$Rad_q(u) := \prod_{\substack{p|u \\ p \neq q}} p,$$

where the product is taken over all positive primes p different from q and dividing u, and write $ord_q(u)$ for the largest integer k with $q^k|u$. Suppose that for given A, B, C and $n \geq 3$, we have a solution (x, y, z) to (1.3) in nonzero integers.

If $m = 2$, following the method of Bennett and Skinner [11] we first associate elliptic curves to solutions (x, y, z) of (1.3) as follows. We assume that Ax, By and Cz are pairwise coprime, and that C is squarefree. Without loss of generality, we may suppose we are in one of the following situations:

(i) $xyABC \equiv 1 \pmod 2$ and $y \equiv -BC \pmod 4$,

(ii) $xy \equiv 1 \pmod 2$ and either $ord_2(B) = 1$ or $ord_2(C) = 1$,

(iii) $xy \equiv 1 \pmod 2$, $ord_2(B) = 2$ and $z \equiv -By/4 \pmod 4$,

(iv) $xy \equiv 1 \pmod 2$, $ord_2(B) \in 3, 4, 5$ and $z \equiv C \pmod 4$,

(v) $ord_2(By^n) \geq 6$ and $z \equiv C \pmod 4$.

In cases (i) and (ii), we will consider the curve

$$E_1(x,y,z):\ Y^2 = X^3 + 2CzX^2 + BCy^nX\ .$$

In cases (iii) and (iv), we will consider

$$E_2(x,y,z):\ Y^2 = X^3 + CzX^2 + \frac{BCy^n}{4}X\ ,$$

and in (v),

$$E_3(x,y,z):\ Y^2 + XY = X^3 + \frac{Cz-1}{4}X^2 + \frac{BCy^n}{64}X\ .$$

After this via Galois representations, we can associate modular forms to these elliptic curves, so in this way in fact we associate modular forms to the solutions (x, y, z) of equation 1.3.

Put

$$N_2 = Rad_2\,(AB)\,Rad_2\,(C)^2\,\varepsilon_2,$$

where

$$\varepsilon_2 := \begin{cases} 1 & \text{if } ord_2\,(By^n) = 6 \\ 2 & \text{if } ord_2\,(By^n) \geq 7 \\ 4 & \text{if } ord_2\,(B) = 2 \text{ and } y \equiv -BC/4 \pmod 4 \\ 8 & \text{if } ord_2\,(B) = 2 \text{ and } y \equiv BC/4 \pmod 4 \\ & \text{or if } ord_2\,(B) \in \{4,5\} \\ 32 & \text{if } ord_2\,(B) = 3 \text{ or if } BCy \text{ is odd} \\ 128 & \text{if } ord_2\,(B) = 1 \\ 256 & \text{if } C \text{ is even.} \end{cases}$$

If $m = 3$ (see Bennett, Vatsal and Yazdani [12]) we assume, without loss of generality, that $3 \nmid Ax$ and $By^n \not\equiv 2 \pmod 3$. Further, suppose that C is cube-free, A and B are nth-power free and that equation (1.3) does not correspond to one of the following identities:

$$1 \cdot 2^5 + 27 \cdot (-1)^5 = 5 \cdot 1^3 \text{ or } 1 \cdot 2^7 + 3 \cdot (-1)^7 = 1 \cdot 5^3.$$

We consider the elliptic curve

$$E : Y^2 + 3CzXY + By^nY = X^3,$$

and set

$$N_3 = Rad_3(AB)Rad_3(C)^2\varepsilon_3,$$

where

$$\varepsilon_3 := \begin{cases} 3^2 & \text{if } 9 \mid (2 + C^2By^n - 3Cz), \\ 3^3 & \text{if } 3 \parallel (2 + C^2By^n - 3Cz), \\ 3^4 & \text{if } ord_3\,(By^n) = 1, \\ 3^3 & \text{if } ord_3\,(By^n) = 2, \\ 1 & \text{if } ord_3\,(B) = 3, \\ 3 & \text{if } ord_3\,(By^n) > 3 \text{ and } ord_3\,(B) \neq 3, \\ 3^5 & \text{if } 3 \mid C. \end{cases}$$

If $m = n$ (see Kraus [35]), then we may assume without loss of generality that $Ax^n \equiv -1 \pmod 4$ and $By^n \equiv 0 \pmod 2$. The corresponding Frey curve is

$$E : Y^2 = X(X - Ax^n)(X + By^n).$$

Put

$$N_n = Rad_2(ABC)\varepsilon_n,$$

where

$$\varepsilon_n := \begin{cases} 1 & \text{if } ord_2\,(ABC) = 4, \\ 2 & \text{if } ord_2\,(ABC) = 0 \text{ or } ord_2\,(ABC) \geq 5, \\ 2 & \text{if } 1 \leq ord_2\,(ABC) \leq 3 \text{ and } xyz \text{ even}, \\ 8 & \text{if } ord_2\,(ABC) = 2 \text{ or } 3 \text{ and } xyz \text{ odd}, \\ 32 & \text{if } ord_2\,(ABC) = 1 \text{ and } xyz \text{ odd}. \end{cases}$$

We note that for each of the three signatures, the numbers N_m are closely related to the conductors of the above curves.

The following Proposition 1.1 summarizes some results obtained by Kraus [35] ($m = n$), Bennett and Skinner [11] ($m = 2$), and Bennett, Vatsal and Yazdani [12] ($m = 3$).

Proposition 1.1. *Suppose that A, B, C, x, y and z are nonzero integers with Ax, By and Cz pairwise coprime, $xy \neq \pm 1$, satisfying equation* (1.3) *with a prime $n \geq 5$ (for $m \in \{3, n\}$) or $n \geq 7$ (if $m = 2$). Then there exists a cuspidal newform $f = \sum_{r=1}^{\infty} c_r q^r$ $(q := e^{2\pi i z})$ of weight 2, trivial Nebentypus character and level N_m for N_m $(m \in \{2, 3, n\})$ given as above. Moreover, if we write K_f for the field of definition of the Fourier coefficients c_r of the form f and suppose that p is a prime coprime to nN_m, then*

$$Norm_{K_f/\mathbb{Q}}(c_p - a_p) \equiv 0 \pmod{n}$$

with $a_p = \pm(p+1)$ (if $p \mid xy$) or $a_p \in S_{p,m}$ (if $p \nmid xy$), where

$$S_{p,2} = \{u \ : \ |u| < 2\sqrt{p},\ u \equiv 0 \pmod{2}\},$$

$$S_{p,3} = \{u \ : \ |u| < 2\sqrt{p},\ u \equiv p+1 \pmod{3}\}$$

and

$$S_{p,n} = \{u \ : \ |u| < 2\sqrt{p},\ u \equiv p+1 \pmod{4}\}.$$

Further, if $m = 2$ and the solution (x, y, z) arises from a rational cuspidal newform corresponding to an elliptic curve $E/\mathbb{Q}$ then if $p \nmid xy$ we have $a_p = a_p(E) = p + 1 - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p)$ is the number of points on E over the finite field $\mathbb{F}_p$.

Proof. This deep result was proved in [35] (for $m = n$), [11] (for $m = 2$) and [12] (for $m = 3$). □

We note that the applicability of the above modular approach depends only on the prime factors of the coefficients A, B, C.

1.2 On the resolution of ternary equations

For our purposes, we restrict our attention to the equation

$$Ax^n - By^n = z^m \tag{1.4}$$

in the cases when $m = n$ and $m = 3$.

1.2.1 The case $m = n$

The resolution of equation (1.4) for $AB = p^\alpha$ with a prime $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59\}$ and nonnegative integer α follows from the papers of Serre [45], Wiles [53], Darmon and Merel [23], and Ribet [43].

The first results on the resolution of (1.4) when AB has two prime factors are due to Kraus [35]. Bennett, Győry, Mignotte and Pintér [10] considered the case when $AB = 2^\alpha q^\beta$ with primes $3 \le q \le 13$. Their result was recently extended by Győry and Pintér [31] who proved that if $AB = 2^\alpha q^\beta$ with primes $3 \le q \le 29$, $\alpha \notin \{1, 2, 3, 4\}$ and n is a prime, then for every integer solutions (x, y, z, A, B, n) of equation (1.4) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 11$. Actually, they proved a more precise statement (Cf. Lemma 1.1).

In [31], Győry and Pintér also considered the case when AB has two odd prime factors. They proved that if $AB = p^\alpha q^\beta$ with primes $5 \le p < q \le 29$ and n is a prime, then apart from 10 explicitly given possible exceptions (p, q), for every solution (x, y, z, A, B, n) of (1.4) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \le 11$. For the exact statement see Lemma 1.2.

1.2.2 The case $m = 3$

In this direction we first refer to the result of Bennett, Győry, Mignotte and Pintér [10]. They proved that if $AB = p^\alpha q^\beta$ with primes $3 \le p, q \le 13$ and $n > 7$ is a prime coprime to pq, then equation (1.4) has no solutions in integers (x, y, z) with $|xy| > 1$, xy even, and Ax, By and z pairwise coprime.

The above result was generalized by Győry and Pintér [31] to the situation when n is a prime and $AB = p^\alpha q^\beta$ with primes $3 \leq p < q \leq 29$ such that either $p \leq 7$ or

$$(p, q) \in \{(11, 13), (11, 17), (11, 19), (13, 17), (13, 19), (17, 23)\} .$$

Their result implies that in this case for every solution (x, y, z, A, B, n) of (1.4) with $|xy| > 1, xy$ even and Ax, By and z pairwise coprime we have $n \leq 29$ (Cf. Lemma 1.3).

1.3 New results on ternary equations

By means of the modular method we establish new results on the solutions of equation (1.4) both for $m = n$ and for $m = 3$. These results will be crucial in the proof of Theorem 2.7.

Theorem 1.1. *Let* $AB = 2^\alpha q^\beta$ *with a prime* $3 \leq q \leq 151$*,* $q \neq 31, 127$ *and with nonnegative integers* α, β*. If* n *is a prime, then for every integer solution* (x, y, z, A, B, n) *of the equation*

$$Ax^n - By^n = z^n \tag{1.5}$$

with $|xy| > 1$ *and with* Ax, By *and* z *pairwise coprime we have* $n \leq 53$*.*

Moreover, apart from 31 possible exceptions (q, n, α) *given in Table 1.1 below, for every integer solution* (x, y, z, A, B, n) *of equation* (1.5) *with* $|xy| > 1$ *and* Ax, By *and* z *pairwise coprime we have* $n \leq 13$*.*

Table 1.1

(q,n,α)	(q,n,α)	(q,n,α)	(q,n,α)	(q,n,α)
$(3,n,1)$	$(17,n,4)$	$(73,17,1)$	$(109,29,1)$	$(149,37,4)$
$(3,n,2)$	$(37,19,\alpha)$	$(73,37,\alpha)$	$(113,19,\alpha)$	$(149,41,1)$
$(3,n,3)$	$(47,23,4)$	$(83,41,4)$	$(137,17,4)$	$(151,19,\alpha)$
$(5,n,2)$	$(53,17,1)$	$(97,29,1)$	$(137,23,\alpha)$	
$(5,n,3)$	$(59,29,4)$	$(101,17,\alpha)$	$(137,29,1)$	
$(7,n,2)$	$(61,31,\alpha)$	$(103,17,4)$	$(139,23,4)$	
$(7,n,3)$	$(67,17,\alpha)$	$(107,53,4)$	$(149,17,1)$	

For $q \leq 13$ and $n > 13$, this gives Theorem 2.2 of [10]; and for $q \leq 29$, $n > 13$, this implies Theorem 3 of [31] (cf. Lemma 1.1). Further, our Theorem 1.1 can be compared with the corresponding results of [45], [53], [43] and [7].

Theorem 1.2. *Let $AB = p^\alpha q^\beta$ with primes $5 \leq p, q \leq 71$ and non-negative integers α, β. If n is a prime, then apart from 28 possible exceptions (p, q, n) given in Table 1.2 below, for every integer solution (x, y, z, A, B, n) of (1.5) with $|xy| > 1$ and Ax, By and z pairwise coprime we have $n \leq 13$.*

Table 1.2

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(5,7,n)$	$(17,23,n)$	$(p,47,23)$	$(17,61,31)$	$(61,67,n)$
$(7,11,n)$	$(5,37,n)$	$(17,47,n)$	$(29,61,n)$	$(7,71,n)$
$(5,13,n)$	$(5,41,n)$	$(11,53,n)$	$(31,61,17)$	$(17,71,n)$
$(7,13,n)$	$(13,41,n)$	$(5,59,n)$	$(43,61,31)$	$(43,71,17)$
$(7,17,n)$	$(23,41,n)$	$(p,59,29)$	$(5,67,17)$	
$(13,19,n)$	$(11,43,n)$	$(5,61,n)$	$(53,67,17)$	

This is a generalization of Theorem 4 of [31] (cf. Lemma 1.2). For $\max\{p, q\} \leq 29$, $n > 13$ our result possesses two exceptions (p, q, n) fewer.

Theorem 1.3. *Let $AB = p^\alpha q^\beta$ with nonnegative integers α, β and primes $3 \leq p < q \leq 71$ such that $pq \leq 583$. If n is a prime, then apart from 29 possible exceptions (p, q, n) given in Table 1.3 below, for every integer solution (x, y, z, A, B, n) of the equation*

$$Ax^n - By^n = z^3 \tag{1.6}$$

with $|xy| > 1$, xy even and Ax, By and z pairwise coprime we have $n \leq 13$.

Table 1.3

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(11,23,17)$	$(11,31,19)$	$(7,43,19)$	$(11,47,23)$	$(5,61,31)$
$(13,23,17)$	$(3,37,19)$	$(13,43,17)$	$(3,59,29)$	$(7,61,31)$
$(11,29,17)$	$(5,37,19)$	$(3,47,23)$	$(5,59,29)$	$(3,67,17)$
$(11,29,23)$	$(7,37,19)$	$(5,47,23)$	$(7,59,19)$	$(5,67,17)$
$(13,29,19)$	$(11,37,19)$	$(7,47,23)$	$(7,59,29)$	$(7,67,17)$
$(19,29,23)$	$(13,37,19)$	$(11,47,17)$	$(3,61,31)$	

For $q \leq 13$ and $n > 13$, this gives Theorem 2.1 of [10]. Further, Theorem 1.3 is a considerable extension of Theorem 5 of [31] (cf. Lemma 1.3). Under the assumptions of Theorem 5 of [31] on p, q our result implies that if $n > 13$ is a prime, then (1.6) has no solutions with xy even and $|xy| > 1$, without any exception (p, q, n).

1.4 Auxiliary results I.

In this section we formulate three results of Győry and Pintér [31] in order to apply them in our proofs in the next section.

Lemma 1.1. *Suppose that $AB = 2^\alpha q^\beta$, where q is a prime with $3 \leq q \leq 29$ and α, β are nonnegative integers. If $n > 11$ is a prime, then equation (1.5) has no solutions in integers (x, y, z) with $|xy| > 1$ and Ax, By and z pairwise coprime, unless, possibly,*

$$(q, \alpha) \in \{(3,1), (3,2), (3,3), (5,2), (5,3), (7,2), (7,3), (17,4)\}$$

and xy is odd.

Proof. See Theorem 3 in [31]. □

Lemma 1.2. *Suppose that $AB = p^\alpha q^\beta$, where p, q are primes with $5 \leq p < q \leq 29$ and α, β are nonnegative integers. If $n > 11$ is a prime, then equation (1.5) has no solutions in integers (x, y, z) with $|xy| > 1$ and Ax, By and z pairwise coprime, unless, possibly $(p, q, n) = (19, 29, 13)$ or*

$(p,q) \in \{(5,7),(5,13),(7,11),(7,13),(7,17),(7,23),(13,17),(13,19),$ $(17,23)\}$.

Proof. See Theorem 4 in [31]. □

Lemma 1.3. *Suppose that $AB = p^\alpha q^\beta$, where α, β are nonnegative integers and p, q are primes with $3 \le p < q \le 29$ such that either $p \le 7$ or*

$$(p,q) \in \{(11,13),(11,17),(11,19),(13,17),(13,19),(17,23)\}.$$

If $n > 11$ is a prime, then equation (1.6) has no solutions in integers (x,y,z) with $|xy| > 1$, xy even, and Ax, By and z pairwise coprime, unless, possibly
$(p,q,n) \in \{(3,23,13),(5,19,13),(5,23,23),(5,29,13),(5,29,23),$ $(7,17,17),(7,17,19),(7,19,13),(11,13,13),(11,17,23),(11,19,13),$ $(11,19,31),(13,17,17),(13,19,13)\}$.

Proof. This is Theorem 5 in [31]. □

1.5 Proofs

In our proofs, we apply the results of the preceding section and some of Section 1.1.

Proof of Theorem 1.1. Suppose that for some prime $n > 13$ and for some A, B under consideration, equation (1.5) has a nontrivial solution $(x,y,z,A,$
$B,n)$ with Ax, By, and z coprime. By Lemma 1.1 we may assume that $31 \le q \le 151$. Further, we may assume that $\alpha > 0$ and $\beta > 0$, since otherwise the assertion of Theorem 1.1 follows from the results of [53], [43] and [23].

By Proposition 1.1, there exists a cuspidal newform f of level $N = 2^\gamma q$ with $\gamma \in \{0,1,3,5\}$. Using the notation of Proposition 1.1 with

$m = n$, set

$$A_{r,n} := Norm_{K_f/\mathbb{Q}}(c_r - (r+1)) \cdot Norm_{K_f/\mathbb{Q}}(c_r + (r+1)) \cdot \prod_{a_r \in S_{r,n}} Norm_{K_f/\mathbb{Q}}(c_r - a_r),$$

where r is a prime, coprime to $2nq$. In fact, in $A_{r,n}$, the index n is used only to indicate that we are dealing with the case $m = n$. In view of Proposition 1.1, n must be a divisor of $A_{r,n}$ for every prime r with $r \nmid 2nq$. In the following Table 1.4 we give the common prime divisors of the nonzero values of $A_{3,n}, A_{5,n}, \ldots, A_{47,n}$ for every level N under consideration. There is „$\emptyset$" in those cells for which all corresponding values of $A_{r,n}$ are equal to 0. One can see that in these cases $x = y = 1$ is a solution to (1.5) for every $n \geq 3$.

Table 1.4

$q\backslash N$	q	$2q$	$8q$	$32q$
31	5	$\emptyset$	2, 3	2, 3, 7
37	3	3, 19	2, 3, 5	2, 3, 5
41	2, 5	2, 3, 7	2, 3, 5	2, 3, 7, 13
43	3, 7	3, 5, 11	2, 3, 5	2, 3, 5, 11
47	3, 23	2, 3	2, 5	2, 3, 5
53	3, 13	3	2, 7	2, 3, 5, 17
59	29	3, 5	2, 3, 5, 7	2, 3, 5, 7
61	3, 5	3, 31	2, 5, 7	2, 3, 5, 13
67	3, 5, 11	3, 17	2, 3, 5	2, 3, 5, 17
71	3, 5, 7	2, 3, 5	2, 3, 5, 7	2, 3, 7
73	2, 3, 5	2, 3, 37	2	2, 3, 5, 13, 17
79	3, 5, 13	2, 3, 5	2, 3, 5	2, 3, 5
83	3, 41	3, 5, 7	2, 3, 5	2, 3, 5, 7
89	2, 3, 5, 11	2, 3, 5	2, 3, 5	2, 3, 5, 7
97	2	2, 3, 5, 7	2, 3, 5	2, 3, 5, 7, 29
101	3, 5	3, 7, 17	2, 3	2, 3, 5, 13
103	5, 17	2, 3, 5, 7, 13	2, 3, 5	2, 3, 5, 13
107	5, 53	3, 5	2, 3	2, 3, 5

Table 1.4 (continued)

$q\backslash N$	q	$2q$	$8q$	$32q$
109	3	3, 5, 11	2, 3	2, 3, 5, 13, 29
113	2, 3, 7	2, 3, 19	2, 3, 5	2, 3, 5
127	3, 7	$\oslash$	2, 3, 5	2, 3, 5
131	3, 5, 13	3, 5, 7, 11	2, 3, 5	2, 3, 5, 11
137	2, 7, 17	2, 3, 5, 23	2, 3, 5	2, 3, 5, 29
139	3, 7, 23	3, 5, 7	2, 3, 7	2, 3, 5, 7
149	3, 37	3, 5	2, 5	2, 3, 5, 17, 41
151	3, 5	2, 3, 5, 19	2, 3	2, 3, 5, 7, 19

Now Table 1.4 shows that $n \leq 53$ for all (q, α) under consideration, and that nontrivial solutions with $n > 13$ may occur only in the cases (q, n, α) which are listed in Table 1.1. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Suppose that for some prime $n > 13$ and for some A, B having the required properties, equation (1.5) has a nontrivial solution (x, y, z, A, B, n) with Ax, By, and z coprime. Again we may assume that, in $AB = p^\alpha q^\beta$, both α and β are positive. In view of Lemma 1.2 we may further assume that $31 \leq \max\{p, q\} \leq 71$ or that $(p, q) \in \{(5, 7), (5, 13), (7, 11), (7, 13), (7, 17), (7, 23), (13, 17), (13, 19), (17, 23)\}$. As in the proof of Theorem 1.1, we apply Proposition 1.1 with $m = n$. Under the assumptions of Theorem 1.2 the level N of the corresponding modular forms is $2pq$. In Table 1.5, for all the 134 pairs (p, q) under consideration, we list the common prime divisors (briefly *CPD*'s) of $A_{r,n}$ (defined in the proof of Theorem 1.1) for primes $r \in \{3, 5, 7, \ldots, 47\}$ which are coprime to pq. Again „$\oslash$" indicates the case that all corresponding values of $A_{r,n}$ are equal to 0.

Table 1.5

(p,q)	CPD's	(p,q)	CPD's	(p,q)	CPD's
$(5,7)$	$\oslash$	$(37,43)$	$2,3,5,7$	$(17,61)$	$2,3,5,7,31$
$(7,11)$	$\oslash$	$(41,43)$	$2,3,5,7$	$(19,61)$	$2,3,5,7,11$
$(5,13)$	$\oslash$	$(5,47)$	$2,3,5,23$	$(23,61)$	$2,3,5,7,11$
$(7,13)$	$\oslash$	$(7,47)$	$2,3,5,23$	$(29,61)$	$\oslash$
$(7,17)$	$\oslash$	$(11,47)$	$2,3,5,7,23$	$(31,61)$	$2,3,5,7,17$
$(13,17)$	$2,3,5$	$(13,47)$	$2,3,5,7,23$	$(37,61)$	$2,3,5,7$
$(13,19)$	$\oslash$	$(17,47)$	$\oslash$	$(41,61)$	$2,3,5,7,11$
$(7,23)$	$2,3,5,11$	$(19,47)$	$2,3,5,23$	$(43,61)$	$2,3,5,7,13,31$
$(17,23)$	$\oslash$	$(23,47)$	$2,3,5,11,23$	$(47,61)$	$2,3,5,7,23$
$(5,31)$	$2,3,5$	$(29,47)$	$2,3,5,7,23$	$(53,61)$	$2,3,5,7,13$
$(7,31)$	$2,3,5,7$	$(31,47)$	$2,3,5,7,23$	$(59,61)$	$2,3,5,7,11,29$
$(11,31)$	$2,3,5,7,11$	$(37,47)$	$2,3,5,7,11,23$	$(5,67)$	$2,3,5,7,11,17$
$(13,31)$	$2,3,5,7$	$(41,47)$	$2,3,5,7,23$	$(7,67)$	$2,3,5,11$
$(17,31)$	$2,3,5$	$(43,47)$	$2,3,5,7,23$	$(11,67)$	$2,3,5,7,11$
$(19,31)$	$2,3,5$	$(5,53)$	$2,3,5,7,11,13$	$(13,67)$	$2,3,5,11$
$(23,31)$	$2,3,5,7,11$	$(7,53)$	$2,3,5,7,13$	$(17,67)$	$2,3,5,7,11$
$(29,31)$	$2,3,5,7$	$(11,53)$	$\oslash$	$(19,67)$	$2,3,5,7,11$
$(5,37)$	$\oslash$	$(13,53)$	$2,3,5,13$	$(23,67)$	$2,3,5,7,11$
$(7,37)$	$2,3,5,7$	$(17,53)$	$2,3,5,13$	$(29,67)$	$2,3,5,7,11$
$(11,37)$	$2,3,5,7$	$(19,53)$	$2,3,5,7,13$	$(31,67)$	$2,3,5,7,11$
$(13,37)$	$2,3,5$	$(23,53)$	$2,3,5,11,13$	$(37,67)$	$2,3,5,7,11$
$(17,37)$	$2,3,5,7$	$(29,53)$	$2,3,5,7,13$	$(41,67)$	$2,3,5,7,11,13$
$(19,37)$	$2,3,5,7$	$(31,53)$	$2,3,5,7,13$	$(43,67)$	$2,3,5,7,11$
$(23,37)$	$2,3,5,11$	$(37,53)$	$2,3,5,11,13$	$(47,67)$	$2,3,5,11,23$
$(29,37)$	$2,3,5,7$	$(41,53)$	$2,3,5,7,13$	$(53,67)$	$2,3,5,11,13,17$
$(31,37)$	$2,3,5,7,13$	$(43,53)$	$2,3,5,7,13$	$(59,67)$	$2,3,5,11,29$
$(5,41)$	$\oslash$	$(47,53)$	$2,3,5,7,13,23$	$(61,67)$	$\oslash$
$(7,41)$	$2,3,5,7$	$(5,59)$	$\oslash$	$(5,71)$	$2,3,5,7$
$(11,41)$	$2,3,5,7$	$(7,59)$	$2,3,5,7,29$	$(7,71)$	$\oslash$
$(13,41)$	$\oslash$	$(11,59)$	$2,3,5,13,29$	$(11,71)$	$2,3,5,7$
$(17,41)$	$2,3,5,7$	$(13,59)$	$2,3,5,7,29$	$(13,71)$	$2,3,5,7$
$(19,41)$	$2,3,5,7$	$(17,59)$	$2,3,5,7,29$	$(17,71)$	$\oslash$
$(23,41)$	$\oslash$	$(19,59)$	$2,3,5,29$	$(19,71)$	$2,3,5,7$
$(29,41)$	$2,3,5,7$	$(23,59)$	$2,3,5,11,29$	$(23,71)$	$2,3,5,7,11$
$(31,41)$	$2,3,5$	$(29,59)$	$2,3,5,7,29$	$(29,71)$	$2,3,5,7$
$(37,41)$	$2,3,5,7$	$(31,59)$	$2,3,5,7,29$	$(31,71)$	$2,3,5,7,11$
$(5,43)$	$2,3,5,7,11$	$(37,59)$	$2,3,5,7,29$	$(37,71)$	$2,3,5,7$
$(7,43)$	$2,3,5,7$	$(41,59)$	$2,3,5,7,29$	$(41,71)$	$2,3,5,7$

Table 1.5 (continued)

(p,q)	CPD's	(p,q)	CPD's	(p,q)	CPD's
$(11,43)$	$\oslash$	$(43,59)$	$2,3,5,7,29$	$(43,71)$	$2,3,5,7,17$
$(13,43)$	$2,3,5,7,11$	$(47,59)$	$2,3,5,7,23,29$	$(47,71)$	$2,3,5,7,11,23$
$(17,43)$	$2,3,5,7$	$(53,59)$	$2,3,5,13,29$	$(53,71)$	$2,3,5,7,11,13$
$(19,43)$	$2,3,5,7,11$	$(5,61)$	$\oslash$	$(59,71)$	$2,3,5,7,29$
$(23,43)$	$2,3,5,7,11$	$(7,61)$	$2,3,5$	$(61,71)$	$2,3,5,7$
$(29,43)$	$2,3,5,7,11$	$(11,61)$	$2,3,5$	$(67,71)$	$2,3,5,7,11$
$(31,43)$	$2,3,5,7,13$	$(13,61)$	$2,3,5,11$		

By Proposition 1.1, n must divide $A_{r,n}$ for each r in question. However, as is seen from Table 1.5, apart from the exceptions listed in Table 1.2, we get a contradiction since $n > 13$. Thus Theorem 1.2 is proved. $\square$

Proof of Theorem 1.3. Suppose that for some A, B under consideration, equation (1.6) has a nontrivial solution (x, y, z, A, B, n) with xy even, Ax, By and z coprime, and with $n > 13$. Lemma 1.3 proves the assertion for those primes p, q for which either $p \leq 7$ and $q \leq 29$ or $(p,q) \in \{(11,13), (11, 17), (11,19), (13,17), (13,19), (17,23)\}$, unless

$$(p,q) \in \{(5,23), (5,29), (7,17), (11,17), (11,19), (13,17)\}.$$

We use again Proposition 1.1 but now with $m = 3$. First we study the case when, in $AB = p^\alpha q^\beta$, either $p = 3,\ \alpha > 0,\ q \in \{31, 37, 41, 43, 47, 53, 59, 61, 67, 71\}$ or $\alpha\beta = 0$. Then we have to consider modular forms f of level $N = 3^\gamma q$ with $\gamma \in \{0, 1, 2, 3, 4\}$. With the notation of Proposition 1.1, put

$$B_{r,3} := Norm_{K_f/\mathbb{Q}}(c_r - (r+1)) \cdot Norm_{K_f/\mathbb{Q}}(c_r + (r+1)).$$

Since xy is even, in the case $r = 2$, it is enough to consider $B_{2,3}$ instead of the product

$$A_{2,3} := Norm_{K_f/\mathbb{Q}}(c_2 - 3) \cdot Norm_{K_f/\mathbb{Q}}(c_2) \cdot Norm_{K_f/\mathbb{Q}}(c_2 + 3).$$

Moreover, the Hasse-Weil (HW) bound yields $n \leq 2\sqrt{2}+3$ for all rational newforms f, so we deal only with the non-rational ones. We note that all the newforms of level $N = 37$ are one dimensional. The following Table 1.6 contains the common prime divisors of $B_{2,3}$ and $A_{r,3}$ for primes $r \in \{5, 7, \ldots, 47\}$ different from q.

Table 1.6

$q\backslash N$	$1q$	$3q$	$9q$	$27q$	$81q$
31	5	$2,7$	$2,3,5,7$	$2,3,5$	$2,3,5,7$
37	HW	$2,19$	$2,3,5,19$	$2,3,7$	$2,3,5$
41	$2,5$	$2,7$	$2,5,7$	$2,3,7$	$2,3,7,11$
43	7	$2,7,11$	$2,3,7,11$	$2,3,5$	$2,3,5,7$
47	23	2	$2,23$	$2,3,13$	$2,3,7$
53	$2,5,13$	$2,3$	$2,3,5,13$	$2,3,5$	$2,3,5,13$
59	$2,29$	$2,5,7$	$2,5,7,29$	$2,3,5,11$	$2,3,5,7$
61	$2,5$	$2,5,31$	$2,3,5,31$	$2,3,5$	$3,7$
67	$5,11$	$2,17$	$2,3,5,11,17$	$3,5,7,11$	$2,3,7,13$
71	$5,7$	$2,3,5$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7$

In view of Proposition 1.1 Table 1.6 shows that we get a contradiction with $n > 13$ unless

$$(p, q, n) \in \{(3, 37, 19), (3, 47, 23), (3, 59, 29), (3, 61, 31), (3, 67, 17)\}.$$

In the remaining cases we have in $AB = p^\alpha q^\beta$ that $p \geq 5$ and $\alpha, \beta > 0$. By virtue of Lemma 1.3, it suffices to deal with the pairs (p, q) which are not considered there and with

$$(p, q) \in \{(5, 23), (5, 29), (7, 17), (11, 17), (11, 19), (13, 17)\}.$$

For each of the remaining pairs (p, q) we use again Proposition 1.1 with $m = 3$, and collect the common prime divisors of $B_{2,3}$ and $A_{r,n}$ with primes $r \in \{5, 7, 11, \ldots, 47\}$ for each occuring newform of level $N = 3pq,\ 9pq,\ 27pq$. To these computations we used MAGMA and its results are listed in the following Table 1.7.

Table 1.7

(p,q)	$3pq$	$9pq$	$27pq$
$(5,23)$	$2,3,7,11$	$2,3,7,11$	$2,3,5,7$
$(5,29)$	$2,5,7$	$2,3,5,7$	$2,3,5,7$
$(5,31)$	$2,5,7$	$2,3,5,7$	$2,3,5$
$(5,37)$	$2,5,11,19$	$2,3,5,7,11,19$	$2,3,5,7,11$
$(5,41)$	$2,3,5$	$2,3,5,7,11$	$2,3,5,7$
$(5,43)$	$2,5,7$	$2,3,5,7,11$	$2,3,5,7$
$(5,47)$	$2,5,23$	$2,3,5,7,23$	$2,3,5,13$
$(5,53)$	$2,5,7,13$	$2,3,5,7,13$	$2,3,5$
$(5,59)$	$2,3,5,29$	$2,3,5,29$	$2,3,5,7,11$
$(5,61)$	$2,3,5,7$	$2,3,5,7,31$	$2,3,5,7$
$(5,67)$	$2,5,7,11$	$2,3,5,7,11,13,17$	$2,3,5,7,11$
$(5,71)$	$2,3,5,7,13$	$2,3,5,7,13$	$2,3,5,7$
$(7,17)$	$2,7$	$2,3,5,7$	$2,3,5$
$(7,31)$	$2,5,7$	$2,3,5,7$	$2,3,5,7$
$(7,37)$	$2,3,7$	$2,3,5,7,19$	$2,3,5$
$(7,41)$	$2,5,7$	$2,3,5,7,11$	$2,3,5,7$
$(7,43)$	$2,5,7,11$	$2,3,5,7,11$	$2,3,5,11,19$
$(7,47)$	$2,5,7,13,23$	$2,3,5,7,13,23$	$2,3,5$
$(7,53)$	$2,5,7,13$	$2,3,5,7,13$	$2,3,5,7$
$(7,59)$	$2,5,7,11,19,29$	$2,3,5,7,11,19,29$	$2,3,5,7,11,13$
$(7,61)$	$2,3,5,7$	$2,3,5,7,13,31$	$2,3,5,13$
$(7,67)$	$2,3,11$	$2,3,5,11,17$	$2,3,5,7$
$(7,71)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7,11$
$(11,17)$	$2,3,5$	$2,3,5$	$2,3,5$
$(11,23)$	$2,3,5,7,11$	$2,3,5,7,11,17$	$2,3,5,7$
$(11,29)$	$2,3,5,7,13,17$	$2,3,5,7,13,17$	$2,3,23$
$(11,31)$	$2,5,7$	$2,3,5,7,19$	$2,3,5,13$
$(11,37)$	$2,3,5,7,13$	$2,3,5,7,11,13,19$	$2,3,5,7$
$(11,41)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7$
$(11,43)$	$2,5,7,11$	$2,3,5,7,11$	$2,3,5$
$(11,47)$	$2,3,5,7,17,23$	$2,3,5,7,17,23$	$2,3,5,7,13$
$(11,53)$	$2,3,5,7,13$	$2,3,5,7,13$	$2,3,5,7$
$(13,17)$	$2,5$	$2,3,5,7$	$3,5,7$

Table 1.7 (continued)

(p,q)	$3pq$	$9pq$	$27pq$
$(13,23)$	$2,5,11,13$	$2,3,5,7,11,13$	$2,3,5,11,17$
$(13,29)$	$2,5,7$	$2,3,5,7$	$2,3,5,7,19$
$(13,31)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7$
$(13,37)$	$2,3,5,7,19$	$2,3,5,7,19$	$2,3,5,7$
$(13,41)$	$2,5,7$	$2,3,5,7$	$2,3,5,7,11,13$
$(13,43)$	$2,3,5,7$	$2,3,5,7,11,17$	$2,3,5,7,13$
$(17,19)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7,13$
$(17,29)$	$2,3,5,7,11$	$2,3,5,7,11$	$2,3,5,7$
$(17,31)$	$2,5,11$	$2,3,5,11$	$2,3,5,7,11,13$
$(19,23)$	$2,3,5,11$	$2,3,5,7,11$	$2,3,5$
$(19,29)$	$2,3,5,7$	$2,3,5,7$	$2,3,5,7,11,23$

Proposition 1.1 now implies that equation (1.6) has no solutions for those triples (p,q,n) for which n does not occur in Table 1.7 as a common prime divisor. It is seen from Tables 1.6 and 1.7 that there are 29 triples (p,q,n) with $n > 13$ which are those listed in our Theorem 1.3 as possible exceptions. This proves Theorem 1.3. $\square$

Chapter 2

Binomial Thue equations

In this chapter we deal with binomial Thue equations. We first summarize the corresponding ineffective and effective finiteness results then we turn our attention to results on the resolution of such equations. Finally, we present our new theorems with their proofs.

2.1 Introduction and finiteness results

Let us consider the Diophantine equation

$$Ax^n - By^n = C, \tag{2.1}$$

where A, B, C, n are nonzero integers and $n \geq 3$ is either fixed or also unknown. Equation (2.1) with fixed n is called a *binomial Thue equation.* We use the same terminology also for the case of unknown n. We may assume that

$$1 \leq A < B \quad \text{and} \quad \gcd(A, B) = 1. \tag{2.2}$$

Thue equations and generalized Thue equations have many applications in number theory, see e.g. [39], [46], [5], [34], [7], [9], [28], [10], [15], [2], [31] and the references given there. By a classical theorem of Thue [51], for fixed n, equation (2.1) has at most finitely many solutions in integers x, y. This result is ineffective in the sense that it does not provide any algorithm for finding the solutions. The first effective upper bounds for the size of the solutions of (2.1) are due to Baker [1] for n fixed. For n also unknown, Tijdeman [52] proved that $\max\{|x|, |y|, n\}$ can be still effectively bounded for every integer solution (x, y, n) of (2.1) with $|xy| > 1$. This effective finiteness results is extended in [27] by Győry, Pink and Pintér to the case when the numbers A, B, C are taken to be unknown S-units (i.e. all their prime factors lie in S, where S is a finite set of primes).

Using Baker's theory of linear forms in logarithms, the results of [1] and [52] have been improved several times, but even the best known upper bounds are too large for finding the solutions of (2.1) in concrete cases.

2.2 The resolution of binomial Thue equations

The first results on the complete resolution of equation (2.1) with unknown $n \geq 3$ were obtained with $C = \pm 1$. In [5], Bennett showed by means of the hypergeometric method that for $B = A + 1$, the equation

$$Ax^n - By^n = \pm 1 \tag{2.3}$$

has no solutions with $|xy| > 1$. In [7], [9] and [42], (2.1) has been resolved for some choices of the coefficients (A, B). For certain sets S of primes all solutions of (2.3) with S-unit coefficients A, B have been determined by Bennett [7], Bennett, Győry, Mignotte and Pintér [10], Bugeaud, Mignotte and Siksek [21], and Győry and Pintér [31]. These results, together with our related result, will be discussed in detail in Section 2.7. Now we turn to the case when, in (2.1), the coefficients A, B and C are bounded positive integers. This situation was first considered by Győry and Pintér in [29]. They first derived for concrete values of A, B, C a relatively sharp upper bound for n, provided that (2.1) has no solutions with $|xy| \leq 1$. Then they explicitly solved equation (2.1) in integers x, y and n with $|xy| > 1$, $n \geq 3$ for a collection of coefficients A, B, C. Under the assumptions (2.2) and $\max\{A, B, |C|\} \leq 10$ they gave all integer solutions (x, y, n) to (2.1) with $|xy| > 1, n \geq 3$ and with

$$B \pm A \neq C \quad \text{if} \quad C \geq 2. \tag{2.4}$$

For $C = \pm 1$, assuming (2.2) and $\max\{A, B\} \leq 20$, they determined all solutions (x, y, n) to equation (2.3) with $|xy| > 1$ and $n \geq 3$. Finally, in the case $A = |C| = 1$, $B \leq 70$, they gave all solutions to the equation

$$x^n - By^n = \pm 1 \tag{2.5}$$

in integers x, y, n with $|xy| > 1$ and $n \geq 3$. Their proofs require a wide variety of powerful techniques including local arguments, some classical results in cyclotomic fields, lower bounds for linear forms in logarithms of algebraic numbers, computational methods for finding the solutions of Thue equations of small degree, the hypergeometric method and results on ternary equations based on Galois representations and modular forms. We note that these statements of [29] cannot be deduced from the results of [5], [7], [10], [21] and [31].

2.3 New results

In this section, our purpose is to extend the above-mentioned results of [29] to much larger values of A, B, C. The theorems of this section are joint results of Bérczes, Győry, Pintér and the author, and were published in [4]. The main novelty in our proofs is a new result of ours (Theorem 2.6) concerning the solvability of binomial Thue equations of the form (2.5). The use of our Theorem 2.6 is crucial in proving Theorems 2.1 and 2.2. It is important to note that in our Theorems 2.1 to 2.5 we arrived at the limit of the applicability of the currently available methods.

For equation (2.5) we prove the following results.

Theorem 2.1. *If* $1 < B \leq 400$, *then all integer solutions* (x, y, n) *of equation* (2.5) *with* $|xy| > 1, n \geq 3$ *and with* $(B, n) \notin \{(235, 23), (282, 23), (295, 29), (329, 23), (354, 29)\}$ *are given by*

$$\begin{aligned}
n = 3,\ (B,x,y) &= (7,\pm(2,1)), (9,\pm(2,1)), (17,\pm(18,7)), (19,\pm(8,3)),\\
&(20,\pm(19,7)), (26,\pm(3,1)), (63,\pm(4,1)), (91,\pm(9,2)), (124,\pm(5,1)),\\
&(126,\pm(5,1)), (182,\pm(17,3)), (215,\pm(6,1)), (217,\pm(6,1)),\\
&(254,\pm(19,3)), (342,\pm(7,1)), (344,\pm(7,1)),\\
n = 4,\ (B,x,y) &= (5,\pm 3,\pm 2), (15,\pm 2,\pm 1), (17,\pm 2,\pm 1), (39,\pm 5,\pm 2),\\
&(80,\pm 3,\pm 1), (150,\pm 7,\pm 2), (255,\pm 4,\pm 1),\\
n = 5,\ (B,x,y) &= (31,\pm(2,1)), (242,\pm(3,1)), (244,\pm(3,1)),\\
n = 6,\ (B,x,y) &= (63,\pm 2,\pm 1),\\
n = 7,\ (B,x,y) &= (127,\pm(2,1)), (129,\pm(2,1)),\\
n = 8,\ (B,x,y) &= (255,\pm 2,\pm 1).
\end{aligned}$$

This is a considerable extension of Theorem 4 of [29]. In the proofs of our Theorems 2.1 and 2.2 the method of modular forms and Theorem 2.6 play very important roles. In Theorem 2.1, and in Theorems 2.2 to 2.5 below, there are some exceptions (B, n) resp. (A, B, n) for which our methods do not work. This is partly due to the fact that the necessary data concerning the arising modular forms of too high levels are not at our disposal.

In the next theorem we restrict ourselves to the case when B is odd. Then xy is even, which fact considerably extends the applicability of our method of proof.

Theorem 2.2. *(i) If* $400 < B < 800$ *is odd, then all integer solutions* (x, y, n) *of equation* (2.5) *with* $|xy| > 1, n \geq 3$ *and apart from the possible exceptions* (B, n) *listed in Table 2.1 below are given by*

$$\begin{aligned}
n = 3,\ (B,x,y) &= (511,\pm(8,1)), (513,\pm(8,1)), (635,\pm(361,42)),\\
&(651,\pm(26,3)),\\
n = 9,\ (B,x,y) &= (511,\pm(2,1)), (513,\pm(2,1)).
\end{aligned}$$

Table 2.1

(B,n)	(B,n)	(B,n)	(B,n)	(B,n)
$(413,29)$	$(519,43)$	$(649,29)$	$(695,23)$	$(757,379)$
$(415,41)$	$(535,53)$	$(669,37)$	$(699,29)$	$(767,29)$
$(417,23)$	$(537,89)$	$(681,113)$	$(717,17)$	$(789,131)$
$(447,37)$	$(573,19)$	$(683,31)$	$(721,17)$	$(799,23)$
$(501,83)$	$(581,41)$	$(685,17)$	$(745,37)$	
$(517,23)$	$(611,23)$	$(687,19)$	$(749,53)$	

(ii) Let $800 < B < 2000$ be odd. If $n < 13$, then all integer solutions (x,y,n) of equation (2.5) with $|xy| > 1, n \geq 3$ are given by

$$n=3,\ (B,x,y) = (813,\pm(28,3)),\ (999,\pm(10,1)),\ (1001,\pm(10,1)),$$
$$(1521,\pm(23,2)),(1657,\pm(71,6)),(1727,\pm(12,1)),(1729,\pm(12,1)),$$
$$(1801,\pm(73,6)),(1953,\pm(25,2))$$
$$n=5,\ (B,x,y) = (1023,\pm(4,1)),(1025,\pm(4,1)),$$
$$n=10,\ (B,x,y) = (1023,\pm2,\pm1),(1025,\pm2,\pm1).$$

If $n > 100$ is a prime, then equation (2.5) has no solutions in integers (x,y,n) with $|xy| > 1, n \geq 3$ and apart from the possible exceptions (B,n) listed in Table 2.2 below.

Table 2.2

(B,n)	(B,n)	(B,n)
$(1041,173)$	$(1509,251)$	$(1795,179)$
$(1077,179)$	$(1527,127)$	$(1821,101)$
$(1135,113)$	$(1589,113)$	$(1841,131)$
$(1149,191)$	$(1671,139)$	$(1857,103)$
$(1315,131)$	$(1689,281)$	$(1915,191)$
$(1401,233)$	$(1735,173)$	$(1929,107)$
$(1437,239)$	$(1761,293)$	$(1959,163)$

In case (*ii*), solving equation (2.5) with our present methods, for $13 \leq n \leq 100$ we obtained so many exceptions that we disregard that case.

For equation (2.3), we have the following.

Theorem 2.3. *Under the assumptions* (2.2) *and* $\max\{A,B\} \leq 50$, *all integer solutions* (x,y,n) *to equation* (2.3) *with* $|xy| > 1$, $n \geq 3$ *and with* $(A,B,n) \notin \{(21,38,17),(26,41,17),(22,43,17),(17,46,17),(31,46,17),(21,38,19)\}$ *are given by*

$$
\begin{aligned}
& n = 3,\ (A,B,x,y) = (1,7,\pm(2,1)),(1,9,\pm(2,1)),(1,17,\pm(18,7)),\\
&\quad (1,19,\pm(8,3)),(1,20,\pm(19,7)),(1,26,\pm(3,1)),(2,15,\pm(2,1)),\\
&\quad (2,17,\pm(2,1)),(3,10,\pm(3,2)),(5,13,\pm(11,8)),(5,17,\pm(3,2)),\\
&\quad (8,17,\pm(9,7)),(8,19,\pm(4,3)),(11,19,\pm(6,5))\\
& n = 4,\ (A,B,x,y) = (1,5,\pm 3,\pm 2),(1,15,\pm 2,\pm 1),(1,17,\pm 2,\pm 1),\\
&\quad (1,39,\pm 5,\pm 2).
\end{aligned}
$$

The next theorem can be regarded as an extension of Theorem 2.3 to the case $\max\{A,B\} \leq 100$. For $n = 17$ and 19, there are, however, many exceptions (A,B,n) when none of our methods works. Hence we consider only the situation when n is a prime greater than 19.

Theorem 2.4. *Let* A, B *be integers with* $\max\{A,B\} \leq 100$ *and* (2.2), *and let* n *be a prime.*

- *(i) If* $n > 41$, *then equation* (2.3) *has no integer solutions* (x,y,n) *with* $|xy| > 1$.
- *(ii) If* $19 < n \leq 41$, *then equation* (2.3) *has no integer solutions* (x,y,n) *with* $|xy| > 1$, *apart from the possible exceptions* $(A,B,n) = (35,58,29), (8,75,31), (11,76,31), (23,78,31), (31,58,31), (39,71,31)$ *and* $(17,82,41)$.

We conjecture that for $\max\{A,B\} \leq 100$, equation (2.3) possesses only the solutions listed in Theorem 2.3.

Finally, we consider the case when C is not necessarily ± 1.

Theorem 2.5. *Let A, B, C be integers with* $\max\{A, B, |C|\} \leq 30$ *and with* (2.2), (2.4), *and let n be a prime.*

(i) If $n > 31$, then equation (2.1) *has no integer solutions* (x, y, n) *with* $|xy| > 1$.

(ii) If $19 < n \leq 31$, then equation (2.1) *has no integer solutions* (x, y, n) *with* $|xy| > 1$, *apart from the possible exceptions* $(A, B, C, n) = (1, 19, 26, 31), (1, 26, 19, 31), (2, 15, 14, 31), (2, 23, 6, 31), (6, 23, 2, 31)$ *and* $(13, 21, 30, 31)$.

In [9] and [29], some special cases of our Theorems 2.1 and 2.3 were used to solve, for certain values of k and D, the equations $1^k + 2^k + \ldots + x^k = y^n$ and $x(x+1) = Dy^n$. Here x, y and n are unknown positive integers with $n \geq 2$.

The following result, which may have independent interest, will be crucial in solving equation (2.5) in many cases. Let $\phi(\ )$ denote Euler's function.

Theorem 2.6. *Suppose that in equation* (2.5) *n is a prime and that each of the following conditions holds:*

(i) $n \geq 17$,

(ii) $B \leq \exp\{3000\}$,

(iii) $n \nmid B\phi(B)$,

(iv) $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$,

(v) $r^{n-1} \not\equiv 1 \pmod{n^2}$ *for some divisor r of B.*

Then equation (2.5) *has no solutions in integers* (x, y, n) *with* $|xy| > 1$.

We remark that our results and their proofs provide the theoretical background of a possible implementation of a binomial Thue equation solver subroutine in certain computer algebraic systems like MAGMA [18] or SAGE [50]. For a computational approach of modular forms we refer to [49].

2.4 Auxiliary results II.

To prove our theorems we need several lemmas.

Lemma 2.1. *Set* $M = \max\{A, B, 3\}$ *and* $\lambda = \log(1 + \frac{\log M}{|\log(A/B)|})$. *Suppose that* (x, y, n) *is an integer solution to* (2.1) *with*

$$x > |y| > 0,\ \ 3\log(1.5\,|C/B|) \le 7400\,\frac{\log M}{\lambda} \ \textit{and}\ \frac{\log 2C}{\log 2} \le 8\,\log M.$$

Then we have

$$n \le \min\left(7400\,\frac{\log M}{\lambda},\ 3106\,\log M\right).$$

Proof. A similar result was proved by Mignotte [37] with a weaker upper bound for n. Mignotte's estimate has been improved in [42] by Pintér by iterated application of Baker's theory of logarithmic forms. □

Combining Lemma 2.1 with local arguments Győry and Pintér ([29], Theorem 1) obtained considerably sharper upper bounds for n whenever $|xy| > 1$. We now formulate this result of [29]. Lemmas 2.1 and 2.2 will be used to bound the exponent n in our equations.

Lemma 2.2. *Suppose that* (2.2) *holds and*

$$C \notin \{A, B, B \pm A\}\,. \tag{2.6}$$

For the pairs $(M_1, n_1), (M_2, n_2)$ *given in Table 2.3, and for every integer solution* (x, y, n) *of* (2.1) *with* $n \ge 3$ *prime, we have*
(i) $n \le n_1$ *if* $\max\{A, B, C\} \le M_1$, *and*
(ii) $n \le n_2$ *if* $C = 1$ *and* $\max\{A, B\} \le M_2$.

Table 2.3

M_1	n_1	M_2	n_2
100	71	200	79
35	43	100	53
20	37	50	31
10	19	20	19

Proof. See Győry and Pintér [29]. The proof depends on Lemma 2.1 and on a short MAGMA program which is based on the following version of the local method. For each quadruple (A, B, C, n) one can search for a local obstruction by considering (2.1) modulo a prime of the form $p = 2kn + 1$, coprime to A, B and C, with $k \in \mathbb{N}$. For such a prime, there are at most $(2k+1)^2$ possible residue classes for $Ax^n - By^n$. If none of these contains C, then equation (2.1) is impossible modulo p. If one cannot find such a prime with $k \leq 150$, then one can test the solvability of the equation modulo n^2. We note that using the method of the proof, Table 2.3 can be extended to larger values of M_1 and M_2 as well. □

Combining several powerful techniques, Bennett [5] obtained the following results.

Lemma 2.3. *If A, B and n are nonzero integers and $n \geq 3$, then equation* (2.3) *has at most one solution in positive integers x, y.*

Proof. See Theorem 1.1 in [5]. We shall use this lemma in the special case $B = A+1$. Then $x = y = 1$ is a solution to (2.3), hence no further solution exists. □

Lemma 2.4. *Let $b > a$ be positive, coprime integers and suppose that*

$$17 \leq n \leq 43 \textit{ is prime, } m = \left[\frac{n+1}{3}\right],$$

and define $c_1(n), d(n)$ via

n	$c_1(n)$	$d(n)$	n	$c_1(n)$	$d(n)$
17	8.93	13.06	31	17.92	30.55
19	9.40	15.46	37	21.92	32.51
23	13.03	17.66	41	25.83	36.08
29	17.39	29.95	43	26.62	33.95

If we have

$$\left(\sqrt[m]{b} - \sqrt[m]{a}\right)^m e^{c_1(n)} < 1$$

then, if x and $y > 0$ are integers, we may conclude that

$$\left|\left(\frac{b}{a}\right)^{\frac{1}{n}} - \frac{x}{y}\right| > \left(3.15 \cdot 10^{24}(m-1)^2 n^{m-1} e^{c_1(n)+d(n)} \left(\sqrt[m]{b} + \sqrt[m]{a}\right)^m\right)^{-1} y^{-\lambda},$$

where

$$\lambda = (m-1)\left(1 - \frac{\log(\left(\sqrt[m]{b} + \sqrt[m]{a}\right)^m e^{c_1(n)+1/20})}{\log(\left(\sqrt[m]{b} - \sqrt[m]{a}\right)^m e^{c_1(n)})}\right).$$

Proof. This is a special case of Theorem 7.1 in [5] which is stated and proved for primes $17 \leq n \leq 347$. □

Recently, Bennett [8] improved this result by giving a sharper lower bound for primes $37 \leq n \leq 73$. However, in our applications we cannot benefit from this improvement.

We recall that for a finite set of primes S, an integer u is an S-unit if all its prime factors lie in S. The following result is due to Bennett, Győry, Mignotte and Pintér [10] for $2 \leq p, q \leq 13$, and to Győry and Pintér [31] for $2 \leq p, q \leq 29$.

Lemma 2.5. *Let $S = \{p, q\}$ for primes p and q with $2 \leq p, q \leq 29$. If A, B, x, y and n are positive integers with A, B S-units, $A < B$ and $n \geq 3$, then the only solutions to equation* (2.3) *are those with*

$$n \geq 3,\ A \in \{1, 2, 3, 4, 7, 8, 16\},\ x = y = 1$$

and

$n=3,\ (A,x)=(1,2),(1,3),(1,4),(1,9),(1,19),(1,23),(3,2),(5,11),$
$n=4,\ (A,x)=(1,2),(1,3),(1,5),(3,2),$
$n=5,\ (A,x)=(1,2),(1,3),$
$n=6,\ (A,x)=(1,2).$

Proof. This is Theorem 1 in [31]; see also Theorem 1.1 in [10]. □

The following two lemmas are special cases of two theorems of Bugeaud, Mignotte and Siksek [21] and will be used in the proofs of our Theorems 2.3 and 2.4.

Lemma 2.6. *Suppose* $3 \le q < 100$ *is a prime. The equation*

$$q^u x^n - 2^v y^n = \pm 1$$

has no solutions in integers x, y, u, v, n *with* $x, y > 0$, $|xy| > 1$, $u, v \ge 0$ *and* $n > 5$.

Proof. Cf. Theorem 1.1 in [21]. □

Lemma 2.7. *Suppose* $3 \le p < q \le 31$ *are primes. The equation*

$$p^u x^n - q^v y^n = \pm 1$$

has no solutions in integers x, y, u, v, n *with* $x, y > 0$, $u, v \ge 0$ *and* $n > 5$.

Proof. Cf. Theorem 1.2 in [21]. □

We note that in contrast with Lemma 2.5, the Lemmas 2.6 and 2.7 cannot be applied to equations of the form (2.3) when $A = 1$ and B has two distinct prime factors. Further, in case $A = 1$ equation (2.3) cannot be solved by the methods used in [10], [31] and [21] when B is divisible by more than two distinct primes.

We now consider the equation

$$x^n + y^n = Bz^n, \tag{2.7}$$

where $n > 3$ is a prime, B is a nonzero integer and x, y, z are coprime nonzero rational integers.

Lemma 2.8. *Suppose that n is coprime to $B\phi(B)$, $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$ and (2.7) has a solution in pairwise coprime nonzero integers x, y and z. Then either (i) $n \mid z$ or (ii) $n \mid xy$, Bz is odd and $r^{n-1} \equiv 1 \pmod{n^2}$ for each divisor r of B.*

Proof. This lemma was proved in [9] (see also [25]). □

Assume that in (2.7) $n \mid B$ but $n \nmid z$. Let $n, p_1, \ldots, p_r$ denote the distinct prime factors of B. For $r \geq 1$, denote by $f_1, \ldots, f_r$ the smallest positive integers for which

$$p_i^{f_i} \equiv 1 \pmod{n}, \quad i = 1, \ldots, r,$$

and set $ord_n(B) = N$.

Remark. If $N = 1$, then (2.7) has no solution x, y, z with $n \nmid z$. Indeed, in the opposite case (2.7) implies $n \mid x + y$ whence $n \mid \frac{x^n+y^n}{x+y}$, a contradiction.

Let $\zeta = e^{2\pi/n}$. We recall that a prime n is called regular if n does not divide the class number of the cyclotomic field $\mathbb{Q}(\zeta)$. The next assertion is due to Maillet [36].

Lemma 2.9. *Suppose that the prime n is regular. If $N \geq 1$, $N \equiv 0$ or $1 \pmod{n}$ and, for $r \geq 1$,*

$$\sum_{i=1}^{r} \frac{1}{f_i} \leq \frac{n-3}{n-1}, \tag{2.8}$$

then (2.7) has no solutions in coprime nonzero rational integers x, y, z not divisible by n.

Proof. See [36]. □

Denote by h_n^+ the class number of the maximal real subfield $\mathbb{Q}(\zeta + \zeta^{-1})$ of $\mathbb{Q}(\zeta)$. The following result has been recently proved by Mihăilescu [38].

Lemma 2.10. *Let $n \geq 17$ be a prime. If the equation*

$$\frac{x^n - 1}{x - 1} = n^e \cdot w^n,\ e \in \{0, 1\}$$

has an integer solution (x, w) with $x \equiv 0, 1$ or $-1 \pmod{n}$, then

$$n \mid h_n^+. \tag{2.9}$$

Remark. It follows from the results of Buhler, Crandall, Ernvall, Metsänkylä and Shokrollahi [22] that condition (2.9) implies that $n > 12 \cdot 10^6$.

2.5 Proofs

In this section we prove our Theorems 2.1 - 2.6. The Tables A1-A10 needed in the proofs are collected in Section 2.6.

First we prove Theorem 2.6 because this theorem will be one of the main tools of the proof of Theorems 2.1, 2.2 and 2.5.

Proof of Theorem 2.6. Suppose that in equation (2.5) n and B satisfy the conditions of Theorem 2.6 and that we have an integer solution (x, y, n) to (2.5) with $|xy| > 1$. It is clear that then the equation

$$x^n - 1 = By^n \tag{2.10}$$

also has an integer solution (x, y, n) with $|xy| > 1$. We can apply Lemma 2.1 with the choice $A = C = 1$ to obtain that $n \leq 3106 \log B$. Together with condition (ii) this yields $n \leq 9.318 \cdot 10^6$. Furthermore, in view of $(iii), (iv)$ and (v), Lemma 2.8 implies that $n \mid y$. Thus we have $n \mid x - 1$ and hence $n \mid \frac{x^n-1}{x-1}$. It is known that

$$\gcd\left(\frac{x^n - 1}{x - 1}, x - 1\right) \mid n.$$

Further, each prime factor of $\frac{x^n-1}{x-1}$ is either n or $\equiv 1 \pmod{n}$ and $n^2 \nmid \frac{x^n-1}{x-1}$. Since by assumption $n \nmid \phi(B)$, we infer from (2.10) that

$$\frac{x^n - 1}{x - 1} = nw^n \tag{2.11}$$

with some nonzero integer w. Now, since $n \geq 17$ by (i), one can apply Lemma 2.10 to equation (2.11) which implies that $n \mid h_n^+$. But as is remarked after Lemma 2.10, it then follows that $n > 12 \cdot 10^6$ which is a contradiction. Thus Theorem 2.6 is proved. □

To prove our Theorems 2.1, 2.3 and Theorem 2.2 (i), it will be enough to solve the corresponding equations for $n = 4$ and for odd primes n. From the values of the solutions x, y so obtained one can easily determine all solutions (x, y, n) with composite $n \geq 3$.

Proof of Theorem 2.1. In view of Theorem 4 of [29] it suffices to deal with the case when $71 \leq B \leq 400$. For $n \leq 13$ we resolved the corresponding Thue equations using PARI [40] or MAGMA.

In case of $n \geq 17$, we obtained an upper bound n_0 on n for each B by means of Lemma 2.1. Then combining Theorem 2.6 with the modular approach (Lemma 1.1) with signature (n, n, n) we could exclude the solvability of most of the equations under consideration with $17 \leq n \leq n_0$.

To illustrate our method we give an example. Set $B = 119$. Then Lemma 2.1 implies that $n \leq n_0 = 14843$. With an easy MAGMA program we checked that each of the conditions of Theorem 2.6 is fulfilled for each such n, except $n = 17$. Thus Theorem 2.6 implies that the equation $x^n - 119y^n = \pm 1$ has no solutions with $|xy| > 1$, unless possibly when $n = 17$. Then we considered the equation $x^{17} - 119y^{17} = \pm 1$ as a ternary equation and applied Lemma 1.1 with signature (n, n, n). The level of the corresponding newforms is 238. There are 6 newforms of level 238. If (x, y) is a solution of the equation then one can show by local arguments that $103 \mid xy$. We recall that K_f denotes the field generated by the Fourier coefficients c_r of a modular form f. In the case $|xy| > 1$, Proposition 1.1 implies that

$$103 \mid Norm_{K_f/\mathbb{Q}}(c_{103} - 104) \cdot Norm_{K_f/\mathbb{Q}}(c_{103} + 104)$$

for some cuspidal newform f of level 238. However, an easy calculation shows that the above relation is impossible for each newform under consideration, hence no nontrivial x, y solutions exist.

After these computations, it remained to consider equation (2.5) in the following cases
$(B, n) \in \{(141, 23), (177, 29), (235, 23), (249, 41), (268, 17), (274, 29), (282, 23), (295, 29), (309, 17), (321, 53), (329, 23), (354, 29)\}$.

For $(B, n) = (268, 17)$ and $(309, 17)$, we resolved the corresponding Thue equations using PARI.

For $(B, n) \in \{(141, 23), (177, 29), (249, 41), (274, 29), (321, 53)\}$ we applied Proposition 1.1 with signature $(n, n, 3)$ and used MAGMA to get a contradiction in each case. For instance, when $(B, n) = (249, 41)$, one can see that $2 \mid xy$ for each solution x, y of the equation $x^{41} - 249y^{41} = \pm 1$. In view of Lemma 1.1 it is enough to check the relation

$$2 \mid Norm_{K_f/\mathbb{Q}}(c_2 - 3) \cdot Norm_{K_f/\mathbb{Q}}(c_2 + 3), \tag{2.12}$$

for each newform f of level $N \in \{249, 6723\}$, where c_2 denotes the second Fourier coefficient of f. There are 5 and 22 newforms at levels 249 and 6723 respectively. It is easy to check that condition (2.12) does not hold for any of those newforms, hence there are no nontrivial solutions x, y.

Finally, in the exceptional cases listed in the theorem we were unable to solve the corresponding Thue equations. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. (*i*) For $n \leq 13$, $(B, n) \neq (649, 13)$, we resolved the corresponding Thue equations of the form (2.5) one by one using PARI or MAGMA. When $(B, n) = (649, 13)$, PARI cannot handle the corresponding Thue equation. We then applied Lemmas 2.8 and 2.9 in the following way. It is easy to check that $(13, 649 \cdot \phi(649)) = 1$, $649^{12} \not\equiv 2^{12} \pmod{13^2}$ and $649^{12} \not\equiv 1 \pmod{13^2}$. Thus Lemma 2.8 gives that in (2.5) 13 must divide y. Then we can rewrite our equation $x^{13} - 649y^{13} = 1$ as $x^{13} - 649 \cdot 13^N y_1^{13} = 1$ where $13 \mid N$ and $13 \nmid y_1$. With the notation of Lemma 2.9 we have $r = 2$, $p_1 = 11$, $p_2 = 59$, $f_1 = f_2 = 12$ and since $2/12 < 10/12$, Lemma 2.9 yields a contradiction.

Consider now the case $n \geq 17$. As in the proof of Theorem 2.1, we obtained an upper bound n_0 on n for each B using Lemma 2.1. Then we combined again Theorem 2.6 with the modular approach (Lemma 1.1) with signature (n, n, n). We used this sieve for each of the equations of the form (2.5) with odd $400 < B < 800$ and with primes $17 \leq n \leq n_0$. We considered the pairs

$$(B, n) \in \{(411, 17), (423, 23), (509, 17), (531, 29), (747, 41)\} .$$

For $(B, n) = (411, 17)$ and $(509, 17)$, the corresponding Thue equations of the form (2.5) can be solved using PARI. When $(B, n) = (423, 23), (531, 29)$ or $(747, 41)$, we applied Lemma 1.1 with signature $(n, n, 3)$ to prove that the corresponding Thue equations of the form (2.5) have no integer solutions x, y with $|xy| > 1$.

Unfortunately, for the remaining pairs (B, n) which are listed in Table 2.1 we failed to resolve the corresponding Thue equations. This completes the proof of part (i) of Theorem 2.2.

(ii) For $n < 13$, $(B, n) \neq (1799, 11)$, we resolved the corresponding Thue equations of the form (2.5) one by one using PARI. In the case when $(B, n) = (1799, 11)$, PARI cannot handle the occuring Thue equation, hence we applied Proposition 1.1 with signature (n, n, n) to prove that no nontrivial solutions exist.

In the sequel we assume that $n > 100$. For each odd B under consideration we deduced first an upper bound n_0 on n using Lemma 2.1. Then we applied again the sieve consisting of Theorem 2.6 and the modular technique (Lemma 1.1) with signature (n, n, n) for each of the equations of the form (2.5) with odd $800 < B < 2000$ and with primes $101 \leq n \leq n_0$. We obtained that all equations of the form (2.5) under consideration can have integer solutions (x, y, n) with $|xy| \leq 1$ only, except possibly for the pairs (B, n) listed in Table 2.2. This completes the proof of part (ii) of Theorem 2.2. $\square$

Proof of Theorem 2.3. Our Theorem 2.1 provides all solutions of equation (2.3) with $A = 1$ and $B \leq 50$. Further Győry and Pintér [29] gave, under the assumption (2.2), all solutions to equation (2.3) for $\max\{A, B\} \le 20$. In view of these results we may assume that $A > 1$ and $\max\{A, B\} \ge 21$. If $B - A = 1$ then $x = y = 1$ is a solution of (2.3), and Lemma 2.3 gives that equation (2.3) has no solution with $|xy| > 1$. Hence we may also assume that $B - A > 1$. Then Lemma 2.2, (ii) yields that $n \le 31$.

We used the local method described in the proof of Lemma 2.2 to prove that under the assumptions of Theorem 2.3 equation (2.3) has no solutions (x, y, n) with $|xy| > 1, n \ge 3$, except for the triples (A, B, n) contained in Table A1 of Section 2.6.

Using PARI, we resolved the corresponding Thue equations (2.3) for $n \le 19$ wherever it was possible. We note that this subroutine of PARI that we used is based on theoretical work of Hanrot [33], and it works without assuming the GRH if the right-hand side of the Thue equation is 1 or if the conditional class group is trivial.

If $(A, B, n) \in \{(2, 37, 19), (4, 23, 13), (8, 43, 31), (11, 32, 19),$ $(17, 32, 17)\}$ or $(A, B, n) \in \{(7, 23, 13), (13, 23, 13), (17, 29, 17),$ $(23, 25, 13), (23, 29, 13), (23, 29, 19), (23, 49, 19), (31, 49, 19)\}$, then the corresponding Thue equations are impossible by Lemmas 2.6 or 2.7, respectively.

In the case when (A, B, n) is one of the triples listed in Table A2 of Section 2.6, we applied Lemma 2.4 to show that equation (2.3) has no solutions. For example, when $(A, B, n) = (19, 26, 31)$, we applied Lemma 2.4 with $b = 26,\ a = 19$. Then one can check that the condition $\left(\sqrt[m]{26} - \sqrt[m]{19}\right)^m e^{c_1(31)} < 1$ is fulfilled for $m = \left[\frac{31+1}{3}\right] = 10$ and $c_1(31) = 17.92$. Thus Lemma 2.4 yields that

$$\left|\left(\frac{26}{19}\right)^{\frac{1}{31}} - \frac{x}{y}\right| > \frac{1}{1.7259 \cdot 10^{65}}\, y^{-27.5338}.$$

On the other hand, the equation $|19x^{31} - 26y^{31}| = 1$ implies that

$$\left|\left(\frac{26}{19}\right)^{\frac{1}{31}} - \frac{x}{y}\right| < \frac{1}{y^{31}},$$

i.e. we have $y < 5.5677 \cdot 10^{21}$. Then in each case we used an algorithm developed by Pethő [41] for finding the small solutions of Thue equations to resolve our corresponding equation.

When $(A, B, n) = (27, 37, 19)$ or $(27, 47, 19)$, we considered the corresponding equations as ternary equations with signature $(n, n, 3)$ and we applied Proposition 1.1 as in the proof of Theorem 2.1 to solve our equations.

For (A, B, n) contained in Table A3, we considered the corresponding equations as ternary equations with signature (n, n, n). Applying again Proposition 1.1, we proved using MAGMA that the equations under consideration have no nontrivial solutions.

In the exceptional cases excluded in the theorem, we were unable to prove with the above-mentioned methods that the corresponding Thue equations have no nontrivial integer solutions. □

Proof of Theorem 2.4. As in the proof of Theorem 2.3, it suffices to consider the case when $A > 1$ and $B - A > 1$. Then, in view of Lemma 2.2 we may assume that $19 < n \leq 53$. By Theorem 2.3 we may further assume that $\max\{A, B\} \geq 51$. Using the local method, we obtained that for most of the triples (A, B, n) under consideration, the corresponding equation (2.3) has no solutions. Those triples (A, B, n) for which the local method does not work are listed in Table A4.

In the cases corresponding to the triples of Table A5 we applied Lemma 2.4 and the above-mentioned algorithm of [41] to show the impossibility of equation (2.3).

When $(A, B, n) = (27, 91, 31)$ one can see that $2 \mid xy$ for all solutions. Then we applied Proposition 1.1 with signature $(n, n, 3)$ to infer that if x, y is a solution to equation (2.3) with $|xy| > 1$ then (2.12) holds for the Fourier coefficient c_2 of some newform f of level 91. There are 4 newforms of level 91 and using MAGMA we arrived at a contradiction with (2.12) in each case.

For $(A, B, n) \in \{(6, 67, 31), (31, 73, 31), (31, 77, 53), (31, 89, 31), (37, 88, 31), (40, 79, 31), (44, 83, 31), (52, 83, 31), (64, 99, 31)\}$ we applied Proposition 1.1 with signature (n, n, n). Here, for the computation of the corresponding Fourier coefficients of the arising newforms, we used again MAGMA.

Unfortunately, in the remaining 7 cases mentioned in the theorem, we could not find any way to solve the corresponding equations. □

Proof of Theorem 2.5. Let A, B, C be positive integers with $\max\{A, B, C\} \leq 30$ which satisfy conditions (2.2) and (2.4) and let (x, y, n) be a fixed solution of the corresponding equation (2.1). By Theorem 2.3 it suffices to consider the case when $C > 1$.

1) First assume that $(A - C)(B - C) \neq 0$. In this case Lemma 2.2 yields $19 < n \leq 43$. Using the local method, we showed that for most of the quadruples (A, B, C, n) under consideration, the corresponding equation (2.1) has no solutions. Those quadruples for which the local method does not work can be found in Table A6 of Section 2.6.

For (A, B, C, n) listed in Table A7 we applied Proposition 1.1 with signature (n, n, n), and using MAGMA we arrived at a contradiction in each case.

When $(A, B, C, n) = (1, 15, 21, 31)$ or $(1, 21, 15, 31)$, we applied Proposition 1.1 with signature $(n, n, 3)$. We note that here $2 \mid xy$ for every solution x, y of both Thue equations. Computing again in MAGMA we checked the impossibility of relation (2.12) for all arising newforms f.

To exclude the cases $(A, B, C, n) \in \{(11, 14, 17, 37), (13, 21, 30, 31), (14, 15, 2, 31), (14, 17, 11, 37), (15, 19, 21, 31), (17, 24, 21, 31), (18, 19, 22, 31), (19, 21, 15, 31), (21, 29, 26, 31)\}$, we combined Lemma 2.4 as above with Pethő's algorithm [41] to get a contradiction.

In the remaining 6 exceptional cases that are listed in the theorem, we were unable to solve the corresponding Thue equations. This completes the first part of the proof.

2) Next consider the case when $(A - C)(B - C) = 0$. In this case equation (2.1) leads to an equation of the form

$$x_1^n - B_1 y_1^n = \pm 1 \text{ in integers } x_1, y_1, \tag{2.13}$$

where B_1 is a positive integer having no prime factors greater than 29. If in (2.1) AB has at most 2 prime factors, then Lemma 2.5 applies to the new equation (2.13) and gives the possible solutions. The remaining cases for (A, B) in (2.1) are listed in Table A8 of Section 2.6. For the pairs (A, B) occurring in Table A8 we infer that $B_1 = A \cdot B^{n-1}$ or $B \cdot A^{n-1}$.

Since equation (2.13) always has the trivial solution $(x, y) = (1, 0)$, the local method cannot be used for showing the unsolvability of such an equation. However, since $n > 19$, we can apply Theorem 2.6 to equation (2.13). In this way we could exclude on one hand each case when $A = C$ except the ones corresponding to the triples (A, B, n) of Table A9. On the other hand, when $B = C$ we could exclude each case but the ones corresponding to the triples (A, B, n) occurring in Table A10.

When $(A, B, C, n) = (29, 30, 29, 29)$ or $(29, 30, 29, 67)$, we applied as above Lemma 2.4 combined with Pethő's algorithm.

For the rest of the equations corresponding to the triples in Tables A9 and A10 we applied Proposition 1.1 using MAGMA for the computations. For $(A, B, C, n) = (8, 21, 21, 31)$, we considered equation (2.1) as a ternary equation with signature $(n, n, 3)$ and arrived at the desired contradiction. In the remaining cases we applied Proposition 1.1 with signature (n, n, n) to prove that there is no nontrivial integer solutions of the corresponding equations. This completes the proof of Theorem 2.5. □

2.6 Tables

In this section we present the tables cited in the proofs in the preceeding section. They contain all the data which is necessary to reconstruct the computations done for proving the theorems of this chapter. For those readers who are not interested in this reconstruction, this section can be skipped.

Table A1

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(2,23,13)$	$(7,47,19)$	$(15,23,13)$	$(20,49,19)$	$(23,50,13)$	$(35,47,11)$
$(2,37,19)$	$(8,43,17)$	$(15,26,13)$	$(21,29,13)$	$(24,41,17)$	$(36,49,17)$
$(3,26,13)$	$(8,43,31)$	$(15,26,17)$	$(21,38,17)$	$(25,36,31)$	$(37,46,17)$
$(3,35,19)$	$(8,45,11)$	$(15,32,11)$	$(21,38,19)$	$(26,41,17)$	$(37,46,19)$
$(3,37,19)$	$(9,31,11)$	$(15,38,13)$	$(21,44,17)$	$(27,34,19)$	$(38,41,19)$
$(3,43,19)$	$(9,38,11)$	$(17,29,17)$	$(22,39,11)$	$(27,37,19)$	$(38,47,11)$
$(3,50,11)$	$(9,40,19)$	$(17,32,17)$	$(22,43,17)$	$(27,47,19)$	$(38,49,17)$
$(4,23,13)$	$(10,33,13)$	$(17,37,13)$	$(23,25,13)$	$(28,43,19)$	$(39,44,13)$
$(5,22,31)$	$(10,37,11)$	$(17,46,17)$	$(23,29,13)$	$(29,33,17)$	$(39,44,17)$
$(5,27,11)$	$(11,32,19)$	$(18,29,17)$	$(23,29,19)$	$(29,37,19)$	$(39,46,17)$
$(5,39,13)$	$(11,34,11)$	$(18,41,13)$	$(23,34,13)$	$(29,41,11)$	$(39,50,13)$
$(5,42,11)$	$(12,23,13)$	$(18,47,17)$	$(23,35,13)$	$(29,47,11)$	$(40,47,11)$
$(5,46,17)$	$(13,23,13)$	$(19,22,11)$	$(23,37,13)$	$(31,37,11)$	$(41,43,19)$
$(7,23,13)$	$(13,36,13)$	$(19,24,19)$	$(23,37,29)$	$(31,46,17)$	$(43,46,23)$
$(7,29,11)$	$(13,37,11)$	$(19,26,31)$	$(23,38,13)$	$(31,49,19)$	$(44,49,11)$
$(7,33,17)$	$(13,41,13)$	$(19,37,19)$	$(23,39,13)$	$(33,47,11)$	$(44,49,19)$
$(7,37,19)$	$(13,42,13)$	$(19,41,13)$	$(23,47,13)$	$(33,47,13)$	
$(7,41,31)$	$(14,23,31)$	$(19,49,11)$	$(23,48,13)$	$(34,49,13)$	
$(7,47,11)$	$(15,22,11)$	$(20,27,11)$	$(23,49,19)$	$(35,44,19)$	

Table A2

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(19,24,19)$	$(29,33,17)$	$(37,46,17)$	$(39,44,17)$	$(43,46,23)$
$(19,26,31)$	$(29,37,19)$	$(37,46,19)$	$(39,46,17)$	$(44,49,19)$
$(27,34,19)$	$(35,44,19)$	$(38,41,19)$	$(41,43,19)$	

Table A3

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(5,22,31)$	$(7,41,31)$	$(15,26,17)$	$(21,44,17)$	$(28,43,19)$	$(38,49,17)$
$(5,46,17)$	$(7,47,19)$	$(18,47,17)$	$(23,37,29)$	$(33,47,11)$	
$(7,33,17)$	$(8,43,17)$	$(19,37,19)$	$(24,41,17)$	$(33,47,13)$	
$(7,37,19)$	$(14,23,31)$	$(20,49,19)$	$(25,36,31)$	$(36,49,17)$	

Table A4

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(6,67,31)$	$(27,91,31)$	$(35,58,29)$	$(44,83,31)$	$(64,99,31)$	$(79,84,41)$
$(8,75,31)$	$(31,58,31)$	$(37,88,31)$	$(45,59,31)$	$(67,82,41)$	$(82,91,31)$
$(11,76,31)$	$(31,73,31)$	$(39,71,31)$	$(52,83,31)$	$(68,95,43)$	$(93,95,31)$
$(17,82,41)$	$(31,77,53)$	$(40,79,31)$	$(55,82,41)$	$(69,91,31)$	$(95,98,37)$
$(23,78,31)$	$(31,89,31)$	$(44,53,31)$	$(61,79,23)$	$(79,82,41)$	

Table A5

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(44,53,31)$	$(55,82,41)$	$(67,82,41)$	$(69,91,31)$	$(79,84,41)$	$(93,95,31)$
$(45,59,31)$	$(61,79,23)$	$(68,95,43)$	$(79,82,41)$	$(82,91,31)$	$(95,98,37)$

Table A6

(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)
$(1,4,28,31)$	$(2,23,4,31)$	$(7,24,4,31)$	$(13,15,9,31)$	$(17,24,18,29)$
$(1,5,22,31)$	$(2,23,6,31)$	$(7,25,13,31)$	$(13,20,11,31)$	$(17,24,21,31)$
$(1,8,6,31)$	$(3,4,24,31)$	$(8,19,23,31)$	$(13,21,30,31)$	$(17,27,21,31)$
$(1,14,23,31)$	$(3,7,16,31)$	$(8,27,16,31)$	$(13,30,4,31)$	$(17,29,9,31)$
$(1,15,21,31)$	$(4,7,24,31)$	$(9,13,26,31)$	$(14,15,2,31)$	$(18,19,22,31)$
$(1,19,26,31)$	$(4,13,10,31)$	$(9,17,29,31)$	$(14,17,11,37)$	$(19,21,15,31)$
$(1,21,15,31)$	$(4,13,30,31)$	$(9,29,17,31)$	$(15,19,21,31)$	$(19,22,26,31)$
$(1,23,14,31)$	$(4,23,2,31)$	$(10,13,4,31)$	$(16,19,27,31)$	$(19,23,8,31)$
$(1,26,19,31)$	$(4,27,16,31)$	$(11,14,17,37)$	$(16,27,4,31)$	$(19,26,22,31)$
$(1,28,4,31)$	$(5,27,2,31)$	$(11,15,30,31)$	$(16,27,8,31)$	$(19,27,16,31)$
$(2,5,27,31)$	$(6,23,2,31)$	$(11,20,13,31)$	$(16,27,19,31)$	$(21,29,26,31)$
$(2,9,13,31)$	$(7,10,18,31)$	$(11,28,30,23)$	$(16,27,20,43)$	$(25,28,5,31)$
$(2,13,9,31)$	$(7,13,25,31)$	$(11,30,15,31)$	$(17,20,11,37)$	
$(2,15,14,31)$	$(7,16,3,31)$	$(12,13,26,31)$	$(17,24,6,31)$	

Table A7

(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)
$(1,4,28,31)$	$(4,13,10,31)$	$(9,17,29,31)$	$(16,27,8,31)$
$(1,5,22,31)$	$(4,13,30,31)$	$(9,29,17,31)$	$(16,27,19,31)$
$(1,8,6,31)$	$(4,23,2,31)$	$(10,13,4,31)$	$(16,27,20,43)$
$(1,14,23,31)$	$(4,27,16,31)$	$(11,15,30,31)$	$(17,20,11,37)$
$(1,23,14,31)$	$(5,27,2,31)$	$(11,20,13,31)$	$(17,24,6,31)$
$(1,28,4,31)$	$(7,10,18,31)$	$(11,28,30,23)$	$(17,24,18,29)$
$(2,5,27,31)$	$(7,13,25,31)$	$(11,30,15,31)$	$(17,27,21,31)$
$(2,9,13,31)$	$(7,16,3,31)$	$(12,13,26,31)$	$(17,29,9,31)$
$(2,13,9,31)$	$(7,24,4,31)$	$(13,15,9,31)$	$(19,22,26,31)$
$(2,23,4,31)$	$(7,25,13,31)$	$(13,20,11,31)$	$(19,23,8,31)$
$(3,4,24,31)$	$(8,19,23,31)$	$(13,30,4,31)$	$(19,26,22,31)$
$(3,7,16,31)$	$(8,27,16,31)$	$(16,19,27,31)$	$(19,27,16,31)$
$(4,7,24,31)$	$(9,13,26,31)$	$(16,27,4,31)$	$(25,28,5,31)$

Table A8

(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)
$(2,15)$	$(5,28)$	$(8,15)$	$(11,15)$	$(13,21)$	$(15,28)$	$(19,24)$	$(23,26)$
$(2,21)$	$(6,7)$	$(8,21)$	$(11,18)$	$(13,22)$	$(15,29)$	$(19,26)$	$(23,28)$
$(3,10)$	$(6,11)$	$(9,10)$	$(11,20)$	$(13,24)$	$(16,21)$	$(19,28)$	$(23,30)$
$(3,14)$	$(6,13)$	$(9,14)$	$(11,21)$	$(13,28)$	$(17,18)$	$(19,30)$	$(24,25)$
$(3,20)$	$(6,17)$	$(9,20)$	$(11,24)$	$(13,30)$	$(17,20)$	$(20,21)$	$(24,29)$
$(3,22)$	$(6,19)$	$(9,22)$	$(11,26)$	$(14,15)$	$(17,21)$	$(20,23)$	$(25,26)$
$(3,26)$	$(6,23)$	$(9,26)$	$(11,28)$	$(14,17)$	$(17,22)$	$(20,27)$	$(25,28)$
$(3,28)$	$(6,25)$	$(9,28)$	$(11,30)$	$(14,19)$	$(17,24)$	$(20,29)$	$(26,27)$
$(4,15)$	$(6,29)$	$(10,11)$	$(12,13)$	$(14,23)$	$(17,26)$	$(21,22)$	$(26,29)$
$(4,21)$	$(7,10)$	$(10,13)$	$(12,17)$	$(14,25)$	$(17,28)$	$(21,23)$	$(27,28)$
$(5,6)$	$(7,12)$	$(10,17)$	$(12,19)$	$(14,27)$	$(17,30)$	$(21,25)$	$(28,29)$
$(5,12)$	$(7,15)$	$(10,19)$	$(12,23)$	$(14,29)$	$(18,19)$	$(21,26)$	$(29,30)$
$(5,14)$	$(7,18)$	$(10,21)$	$(12,25)$	$(15,16)$	$(18,23)$	$(21,29)$	
$(5,18)$	$(7,20)$	$(10,23)$	$(12,29)$	$(15,17)$	$(18,25)$	$(22,23)$	
$(5,21)$	$(7,22)$	$(10,27)$	$(13,14)$	$(15,19)$	$(18,29)$	$(22,25)$	
$(5,22)$	$(7,24)$	$(10,29)$	$(13,15)$	$(15,22)$	$(19,20)$	$(22,27)$	
$(5,24)$	$(7,26)$	$(11,12)$	$(13,18)$	$(15,23)$	$(19,21)$	$(22,29)$	
$(5,26)$	$(7,30)$	$(11,14)$	$(13,20)$	$(15,26)$	$(19,22)$	$(23,24)$	

Table A9

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(2,15,2081)$	$(6,25,811)$	$(10,23,269)$	$(14,23,23)$	$(18,23,23)$	$(22,27,5393)$
$(2,21,31)$	$(6,29,29)$	$(10,27,43)$	$(14,23,47)$	$(18,25,3457)$	$(22,29,29)$
$(3,10,383)$	$(6,29,67)$	$(10,27,229)$	$(14,23,4513)$	$(18,29,29)$	$(23,24,23)$
$(3,20,61)$	$(7,10,593)$	$(10,27,263)$	$(14,27,2437)$	$(18,29,151)$	$(23,24,2731)$
$(3,22,359)$	$(7,15,5749)$	$(10,29,29)$	$(14,29,29)$	$(18,29,157)$	$(23,26,23)$
$(3,22,4813)$	$(7,18,113)$	$(10,29,283)$	$(14,29,617)$	$(18,29,173)$	$(23,28,23)$
$(3,26,269)$	$(7,20,41)$	$(10,29,8387)$	$(14,29,677)$	$(18,29,191)$	$(23,28,509)$
$(4,21,131)$	$(7,20,97)$	$(11,21,2711)$	$(14,29,2273)$	$(18,29,5261)$	$(23,28,599)$
$(5,6,383)$	$(7,20,653)$	$(11,24,107)$	$(15,17,2617)$	$(19,24,829)$	$(23,28,5197)$
$(5,12,3457)$	$(7,30,31)$	$(11,24,4637)$	$(15,19,281)$	$(19,24,1663)$	$(23,30,23)$
$(5,14,593)$	$(7,30,47)$	$(11,26,47)$	$(15,19,2999)$	$(19,26,83)$	$(24,29,29)$
$(5,21,89)$	$(7,30,73)$	$(11,26,79)$	$(15,23,23)$	$(19,26,8329)$	$(24,29,601)$
$(5,21,719)$	$(7,30,491)$	$(11,26,2053)$	$(15,23,293)$	$(19,28,3499)$	$(25,26,29)$
$(5,21,2857)$	$(7,30,1987)$	$(12,17,2273)$	$(15,28,5749)$	$(19,30,2399)$	$(25,26,233)$
$(5,22,1531)$	$(9,14,113)$	$(12,23,23)$	$(15,29,29)$	$(20,21,71)$	$(25,28,61)$
$(5,26,89)$	$(9,20,67)$	$(12,23,43)$	$(15,29,73)$	$(20,21,137)$	$(26,27,103)$
$(5,26,3607)$	$(9,20,887)$	$(12,23,179)$	$(15,29,101)$	$(20,21,2339)$	$(26,29,29)$
$(5,26,6619)$	$(9,20,9257)$	$(12,23,1637)$	$(15,29,6217)$	$(20,23,23)$	$(26,29,2287)$
$(5,28,43)$	$(9,26,727)$	$(12,25,353)$	$(16,21,173)$	$(20,29,29)$	$(27,28,149)$
$(6,11,107)$	$(9,28,439)$	$(12,29,29)$	$(17,20,401)$	$(21,23,23)$	$(27,28,1291)$
$(6,11,4637)$	$(10,11,1279)$	$(12,29,6833)$	$(17,21,71)$	$(21,23,41)$	$(28,29,29)$
$(6,17,1231)$	$(10,13,61)$	$(13,15,2297)$	$(17,21,251)$	$(21,23,73)$	$(29,30,29)$
$(6,17,1493)$	$(10,13,157)$	$(13,18,727)$	$(17,21,2851)$	$(21,25,31)$	$(29,30,67)$
$(6,19,829)$	$(10,17,31)$	$(13,21,89)$	$(17,24,1231)$	$(21,26,331)$	
$(6,19,1663)$	$(10,17,71)$	$(13,22,47)$	$(17,24,1493)$	$(21,29,29)$	
$(6,23,23)$	$(10,19,269)$	$(13,22,79)$	$(17,26,23)$	$(21,29,1601)$	
$(6,23,2731)$	$(10,21,29)$	$(13,22,2053)$	$(17,26,1117)$	$(22,23,23)$	
$(6,25,23)$	$(10,23,23)$	$(14,15,61)$	$(17,28,257)$	$(22,27,4793)$	

Table A10

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(3,14,223)$	$(6,25,4253)$	$(10,23,6857)$	$(13,28,6211)$	$(17,28,9209)$	$(21,29,467)$
$(3,20,29)$	$(6,29,29)$	$(10,29,29)$	$(13,30,701)$	$(17,28,9623)$	$(22,23,23)$
$(3,20,2311)$	$(7,10,4889)$	$(10,29,367)$	$(14,15,31)$	$(17,30,179)$	$(22,25,137)$
$(3,22,2203)$	$(7,12,4253)$	$(11,12,31)$	$(14,15,47)$	$(18,19,947)$	$(22,27,47)$
$(3,22,8111)$	$(7,15,29)$	$(11,12,1321)$	$(14,15,73)$	$(18,23,23)$	$(22,27,89)$
$(3,26,137)$	$(7,15,109)$	$(11,14,281)$	$(14,15,491)$	$(18,25,383)$	$(22,29,23)$
$(3,28,23)$	$(7,18,1039)$	$(11,21,37)$	$(14,15,1987)$	$(18,29,29)$	$(22,29,29)$
$(4,15,163)$	$(7,18,2131)$	$(11,24,1289)$	$(14,17,733)$	$(18,29,109)$	$(23,24,23)$
$(5,14,193)$	$(7,20,37)$	$(11,28,271)$	$(14,23,23)$	$(19,20,101)$	$(23,26,23)$
$(5,18,1291)$	$(7,20,487)$	$(11,30,61)$	$(14,25,431)$	$(19,21,53)$	$(23,28,23)$
$(5,22,43)$	$(7,20,7699)$	$(12,13,61)$	$(14,27,2111)$	$(19,21,2861)$	$(23,28,53)$
$(5,22,97)$	$(7,22,2897)$	$(12,13,1889)$	$(14,29,29)$	$(19,22,5839)$	$(23,30,23)$
$(5,22,157)$	$(7,24,103)$	$(12,19,29)$	$(15,17,1879)$	$(19,24,47)$	$(23,30,41)$
$(5,24,47)$	$(7,24,797)$	$(12,19,163)$	$(15,19,41)$	$(19,26,163)$	$(23,30,47)$
$(5,24,113)$	$(7,30,571)$	$(12,19,193)$	$(15,19,233)$	$(19,28,23)$	$(23,30,139)$
$(5,24,1481)$	$(8,15,2081)$	$(12,23,23)$	$(15,19,5297)$	$(19,28,659)$	$(24,25,23)$
$(5,26,31)$	$(8,21,31)$	$(12,23,199)$	$(15,22,53)$	$(19,28,7079)$	$(24,25,811)$
$(5,26,101)$	$(9,10,1733)$	$(12,23,5867)$	$(15,22,487)$	$(19,30,89)$	$(24,29,29)$
$(5,28,83)$	$(9,14,29)$	$(12,25,3967)$	$(15,22,5431)$	$(19,30,1163)$	$(24,29,67)$
$(5,28,89)$	$(9,14,41)$	$(12,29,29)$	$(15,23,23)$	$(19,30,8599)$	$(25,26,41)$
$(5,28,163)$	$(9,20,41)$	$(12,29,179)$	$(15,23,421)$	$(20,21,89)$	$(25,26,347)$
$(6,11,359)$	$(9,22,227)$	$(12,29,2837)$	$(15,28,151)$	$(20,21,719)$	$(25,28,53)$
$(6,11,4813)$	$(9,28,61)$	$(13,14,199)$	$(15,28,1229)$	$(20,21,2857)$	$(25,28,59)$
$(6,13,269)$	$(10,11,1531)$	$(13,15,53)$	$(15,28,1291)$	$(20,23,23)$	$(26,27,47)$
$(6,17,157)$	$(10,13,89)$	$(13,15,743)$	$(15,29,29)$	$(20,23,659)$	$(26,27,2243)$
$(6,19,23)$	$(10,13,3607)$	$(13,18,37)$	$(15,29,71)$	$(20,29,29)$	$(26,29,29)$
$(6,19,263)$	$(10,13,6619)$	$(13,18,8563)$	$(16,21,131)$	$(21,22,151)$	$(27,28,43)$
$(6,23,23)$	$(10,17,1657)$	$(13,20,7603)$	$(17,20,107)$	$(21,22,1013)$	$(28,29,29)$
$(6,23,79)$	$(10,17,2237)$	$(13,21,563)$	$(17,20,151)$	$(21,23,23)$	$(28,29,73)$
$(6,23,151)$	$(10,19,47)$	$(13,22,263)$	$(17,20,241)$	$(21,23,5419)$	$(29,30,29)$
$(6,23,673)$	$(10,19,601)$	$(13,24,43)$	$(17,21,47)$	$(21,25,103)$	
$(6,25,197)$	$(10,19,821)$	$(13,28,89)$	$(17,26,173)$	$(21,26,2347)$	
$(6,25,313)$	$(10,23,23)$	$(13,28,569)$	$(17,28,3931)$	$(21,29,29)$	

2.7 The case of S-unit coefficients (A new result)

We end this chapter by considering binomial Thue equations of the form (2.3) in which the coefficients A, B are allowed to be arbitrary large but we specify the set of their prime divisors.

Let S be a finite set of primes. We recall that an integer with no prime factors outside S is an *S-unit.* As we mentioned in Section 2.1, binomial Thue equations have finitely many solutions which can be effectively bounded even in the case when the coefficients are unknown S-units. In the sequel we restrict our attention to the equation

$$Ax^n - By^n = \pm 1 \tag{2.14}$$

in unknown S-units $A, B \in \mathbb{Z}$, and unknown integers x, y, n with $|xy| \geq 1$ and $n \geq 3$. For $S = \{p\}$ with a prime $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59\}$, it follows from the work of Wiles [53], Darmon and Merel [23] and Ribet [43] on Fermat-type equations that (2.14) has no solutions with $|xy| > 1$ and $n \geq 3$. For $S = \{2, 3\}$, (2.14) was solved by Bennett [7]. His result was extended by Bennett, Győry, Mignotte and Pintér [10] to the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 13$. Independently, Bugeaud, Mignotte and Siksek [21] solved (2.14) in the case when, in (2.14), $A = 2^\alpha, B = q^\beta$ with a prime $3 \leq q < 100$, or $A = p^\alpha, B = q^\beta$ with primes $3 \leq p < q \leq 31$, and in both cases α, β are nonnegative integers. Recently, Győry and Pintér [31] generalized the results of [10] to the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 29$ (Cf. Lemma 2.5).

As an application of our Theorems 1.1, 1.2, 1.3 and 2.6, combining them with some other techniques, we extend the above results by studying the solutions of equation (2.14) in the case when $S = \{p, q\}$ with primes $2 \leq p, q \leq 71$. Although our Theorem 2.7 does not give the resolution of equation (2.14), we give reasonable upper bounds for n which may be useful if someone needs to solve concrete binomial Thue equations of such type. Our result is the following.

Theorem 2.7. *Let $n \geq 3$ be a prime, $S = \{p, q\}$ with primes $2 \leq p, q \leq 71$ and let A, B be coprime integer S-units with $A < B$.*

If $(A, p, q) \neq (1, 2, 31)$ and

$$(p, q) \notin \{(23, 41), (17, 47), (29, 61), (61, 67), (17, 71)\},$$

then for every integer solution (x, y, A, B, n) of equation (2.14) with $|xy| > 1$ we have $n \leq 31$.

Moreover,

(i) if $A = 1$ and
$(p, q, n) \notin \{(47, q, 23), (59, q, 29), (2, 61, 31), (17, 61, 31), (43, 61, 31), (53, 67, 17)\}$, then for every integer solution (x, y, A, B, n) of equation (2.14) with $|xy| > 1$ we have $n \leq 13$;

(ii) if $A > 1$ and
$(p, q, n) \notin \{(3, 37, 19), (5, 37, 19), (3, 61, 31), (17, 61, 31), (43, 61, 31)\}$, then for every integer solution (x, y, A, B, n) of equation (2.14) with $|xy| > 1$ we have $n \leq 17$.

Proof. In view of Lemmas 2.5 and 2.7 it is enough to solve equation (2.14) for primes $31 \leq \max\{p, q\} \leq 71$. Let x, y, A, B, n be a solution of equation (2.14) with $|xy| > 1, n \geq 3$ and A, B coprime S-units. Then clearly, $(x, y, \pm 1)$ is a solution of the ternary equations (1.5) and (1.6) respectively. Then Theorems 1.1, 1.2 and 1.3 imply that $n \leq 31$ unless

$$(p, q) \in \{(2, 31), (23, 41), (17, 47), (29, 61), (61, 67), (17, 71)\}.$$

For $A > 1$ and $(p, q) = (2, 31)$ one can apply Lemma 2.6 to obtain that $n < 6$ is true for all solutions of $2^\alpha x^n - 31^\beta y^n = \pm 1$, thus the first statement of the theorem is proved.

For the proof of the stronger statements (i) and (ii) of Theorem 2.7, by Theorems 1.1, 1.2 and 1.3 we have to consider the equation

$$Ax^n - By^n = \pm 1$$

for 50 cases of (p, q, n) which are listed in Table 2.15.

Table 2.15

(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)	(p,q,n)
$(2,37,19)$	$(23,47,23)$	$(7,59,29)$	$(43,59,29)$	$(59,61,29)$
$(3,37,19)$	$(29,47,23)$	$(11,59,29)$	$(47,59,23)$	$(2,67,17)$
$(5,37,19)$	$(31,47,23)$	$(13,59,29)$	$(47,59,29)$	$(3,67,17)$
$(2,47,23)$	$(37,47,23)$	$(17,59,29)$	$(53,59,29)$	$(5,67,17)$
$(3,47,23)$	$(41,47,23)$	$(19,59,29)$	$(2,61,31)$	$(47,67,23)$
$(5,47,23)$	$(43,47,23)$	$(23,59,29)$	$(3,61,31)$	$(53,67,17)$
$(7,47,23)$	$(2,53,17)$	$(29,59,29)$	$(17,61,31)$	$(59,67,29)$
$(11,47,23)$	$(47,53,23)$	$(31,59,29)$	$(31,61,17)$	$(43,71,17)$
$(13,47,23)$	$(2,59,29)$	$(37,59,29)$	$(43,61,31)$	$(47,71,23)$
$(19,47,23)$	$(3,59,29)$	$(41,59,29)$	$(47,61,23)$	$(59,71,29)$

For each such triple, we have to consider the equation for

$$A = 1, B = p^\alpha q^\beta; \text{ and for } A = p^\alpha, B = q^\beta$$

with every $(\alpha,\beta) \in \{1,\ldots,n-1\}^2$. For example, that means $2\cdot 28^2 = 1568$ equations to solve when $n = 29$.

First, let $A = 1$. For $(p,q,n) \in \{(3,37,19),(5,37,19),(2,53,17),(3,61,31),(31,61,17),(3,67,17),(43,71,17)\}$ we applied Theorem 2.6 combined with the modular method with signature (n,n,n) to exclude the solvability of all equations under consideration. To illustrate how this approach works we give the details for the case $(p,q,n) = (5,37,19)$. We checked that apart from the pairs (α,β) in Table 2.16 below, for each $(\alpha,\beta) \in \{1,\ldots,18\}^2$ the equations

$$x^{19} - 5^\alpha 37^\beta y^{19} = \pm 1 \tag{2.15}$$

fullfill the conditions $(i)-(v)$ of Theorem 2.6, so they do not have nontrivial integer solutions. For each pair in Table 2.16, by local arguments we found two distinct primes p_1, p_2 which divide xy, where x,y is a putative nontrivial solution of the corresponding equation (2.15). These primes are also listed in Table 2.16.

Table 2.16

(α,β)	p_1, p_2	(α,β)	p_1, p_2	(α,β)	p_1, p_2
$(1,5)$	$419,457$	$(8,1)$	$191,761$	$(14,3)$	$191,229$
$(2,18)$	$191,229$	$(9,14)$	$191,229$	$(15,16)$	$191,229$
$(3,12)$	$229,419$	$(10,8)$	$191,229$	$(16,10)$	$191,229$
$(4,6)$	$191,419$	$(11,2)$	$191,229$	$(17,4)$	$191,419$
$(6,13)$	$191,419$	$(12,15)$	$229,457$	$(18,17)$	$191,229$
$(7,7)$	$457,571$	$(13,9)$	$229,1483$		

There are 16 cuspidal newforms f at level $2\cdot 5\cdot 37$. We recall that K_f denotes the number field generated by the Fourier coefficients c_r of the modular form f. Using the program package MAGMA for each pairs (α,β) of Table 2.16, we obtained that

$$19 \nmid Norm_{K_f/\mathbb{Q}}(c_{p_i}-(p_i+1))\cdot Norm_{K_f/\mathbb{Q}}(c_{p_i}+(p_i+1))$$

with either $i=1$ or $i=2$ for all 16 newforms. Thus, Proposition 1.1 implies that the equations (2.15) corresponding to the pairs (α,β) in Table 2.16 have no solutions with $|xy|>1$.

In the case $(p,q,n)\in\{(2,37,19),(2,67,17),(5,67,17)\}$, we combined Theorem 2.6 with the routine of PARI for solving Thue equations of low degree. For example, Theorem 2.6 implies that the equation

$$x^{19}-2^{\alpha}37^{\beta}y^{19}=\pm 1$$

has no nontrivial solutions, unless
$(\alpha,\beta)\in\{(3,16),(5,13),(6,2),(7,10),(8,18),(9,7),(10,15),(11,4),(12,12),(13,1),(14,9),(15,17),(16,6),(17,14),(18,13)\}$. We solved each equation corresponding to these pairs using PARI.

In the sequel let $A>1$. For $(p,q,n)\in\{(2,37,19),(2,47,23),(2,59,29),(2,61,31)\}$ we can apply again Lemma 2.6 to exclude the solvability of the corresponding equations.

For the remaining 46 triples of Table 2.15, and for each corresponding binomial Thue equation we used a similar local approach as was introduced in the proof of Lemma 2.2. We chose a small integer k such that $p = 2kn+1$ is a prime. Since in this case both x^n and y^n are either $2k$-th roots of unity (mod p) or zero, we had to check the congruence

$$Ax^n - By^n \equiv \pm 1 \pmod{p}$$

only in $(2k+1)^2$ cases, which was quickly done with a short MAGMA program. (We note that this method cannot be used when $A = 1$, because $x^n - By^n = 1$ always has the solution $(x, y) = (1, 0)$.) These computations proved the unsolvability of each binomial Thue equation under consideration, except the ones with

$$(p, q, n) \in \{(3, 37, 19), (5, 37, 19), (3, 61, 31), (17, 61, 31), (43, 61, 31)\}.$$

This completes the proof of Theorem 2.7. □

For the exceptional (p, q, n), the methods used in the proof of Theorem 2.7 proved to be inefficient to solve equation (2.14) for arbitrary nonnegative integer exponents of the primes p, q. However, they work for several particular exponents. We further note that binomial Thue equations with degree at most 17 can be solved in most cases by using a powerful computer and the program packages MAGMA [18], PARI [40] or SAGE [50].

Chapter 3

An application of binomial Thue equations

As we have pointed out in the beginning of Chapter 2, many number theoretical problems lead to (generalized) binomial Thue equations. In this chapter we present an application to norm form equations, another important type of Diophantine equations. Under some conditions, such equations have infinitely many solutions. In this case we study those solutions whose coordinates form an arithmetic progression. For a certain family of norm form equations we determine all such solutions.

3.1 On norm form equations with solutions forming arithmetic progressions

Let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ be linearly independent algebraic numbers over $\mathbb{Q}$ and put $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$. Denote by n the degree $[K : \mathbb{Q}]$ of the field K over the rationals. Let $\sigma_1, \ldots, \sigma_n$ be the $\mathbb{Q}$-isomorphisms of K into $\mathbb{C}$. For any $\alpha \in K$ put $\alpha^{(i)} = \sigma_i(\alpha)$. Consider the linear forms

$$l^{(i)}(\underline{X}) = X_1 + \alpha_2^{(i)} X_2 + \ldots + \alpha_n^{(i)} X_n, \quad i = 1, \ldots, n,\ \underline{X} = (X_1, \ldots, X_n).$$

Then there exists a nonnegative integer a_0 such that the form

$$F(\underline{X}) := a_0 N_{K/\mathbb{Q}}(\alpha_1 X_1 + \ldots + \alpha_m X_m) = a_0 \prod_{i=1}^{n} l^{(i)}(\underline{X})$$

has integer coefficients. Such a form is called a *norm form*, and the equation

$$a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \ldots + \alpha_m x_m) = b \tag{3.1}$$

is called a *norm form equation.*

We call (3.1) *degenerate* if the $\mathbb{Q}$-vector space generated by $\alpha_1, \ldots, \alpha_m$ has a subspace, which is proportional to a full $\mathbb{Z}$-module of an algebraic number field, different from $\mathbb{Q}$ and the imaginary quadratic fields. In this case it is easy to see, that there exists $b \in \mathbb{Z}$, such that (3.1) has infinitely many solutions. For non-degenerate norm form equations Schmidt [44] proved in an ineffective way that the number of their solutions is finite. For a large class of norm form equations Győry and Papp [26] gained effective finiteness results and explicit bounds for the solutions. In the sequel, from the broad literature of norm form equations we mention only those results which motivated our investigations.

The study of searching for arithmetic progressions among the solutions of norm form equations has been initiated by Attila Pethő. The problem itself first raised when Buchmann and Pethő [19] found, as a byproduct of a search for independent units, that in the field $K := \mathbb{Q}(\alpha)$ with $\alpha^7 = 3$ the integer

$$10 + 9\alpha + 8\alpha^2 + 7\alpha^3 + 6\alpha^4 + 5\alpha^5 + 4\alpha^6$$

is a unit. This means that the equation

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_6\alpha^6\right) = 1$$

has a solution $(x_0, \ldots, x_6) \in \mathbb{Z}^7$ such that the coordinates form an arithmetic progression.

This led Bérczes and Pethő in [14] to investigate in more general context norm form equations with solutions whose coordinates form an arithmetic progression. They considered the equation

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = m. \tag{3.2}$$

where α is an algebraic number of degree n, $K = \mathbb{Q}(\alpha)$, $m \in \mathbb{Z}$ and $\{x_0, \ldots, x_{n-1}\} \in \mathbb{Z}^n$. Put $X = \max\{|x_0|, \ldots, |x_{n-1}|\}$. The sequence $\{x_0, \ldots, x_{n-1}\}$ is said to be *nearly an arithmetic progression* if there exists $d \in \mathbb{Z}$ and $0 < \delta \in \mathbb{R}$ such that

$$|(x_i - x_{i-1}) - d| < X^{1-\delta}, \quad i = 1, \ldots, n-1. \tag{3.3}$$

They proved an effective finiteness result on the solutions of (3.2) with property (3.3) provided that

$$\beta := \frac{n\alpha^n}{\alpha^n - 1} - \frac{\alpha}{\alpha - 1}$$

is an algebraic number of degree at least 3, over $\mathbb{Q}$. In the special case when $\delta = 1$ they proved a nearly complete finiteness result.

Bérczes and Pethő also considered arithmetic progressions arising from the norm form equation

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = 1.$$

In [14], they determined all such solutions when α is a zero of either $x^n - 2$ or $x^n - 3$, both with $n \geq 3$. Further they proved in [15] that there are no such solutions at all when α is a zero of the polynomial $x^n - a$, with $n \geq 3$ and $4 \leq a \leq 100$.

In the case when α is a zero of the polynomial

$$f_a(x) = x^3 - (a-1)x^2 - (a+2)x - 1, \quad a \in \mathbb{Z},$$

Bérczes, Pethő and Ziegler [16] determined all primitive solutions of the norm form inequality

$$|N_{K/\mathbb{Q}}(x_0 + x_1\alpha + x_2\alpha^2)| \leq |2a + 1|$$

such that $x_0 < x_1 < x_2$ is an arithmetic progression.

We note that arithmetic progressions may occur not only in a solution but in the solution set of a norm form equation if we consider some fixed coordinate of the solutions. For results in this direction we refer to Bérczes, Hajdu and Pethő [13].

3.2 New results

In this section our aim is to extend the result of Bérczes and Pethő [15] on equation (3.4).

Let α be an algebraic integer of degree $n \geq 3$, $K = \mathbb{Q}(\alpha)$, and consider the equation

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = 1 \tag{3.4}$$

in $x_0, \ldots, x_{n-1} \in \mathbb{Z}$. Let α be a zero of the polynomial $x^n - a$, where a is an integer such that $x^n - a$ is irreducible. As was mentioned above, Bérczes and Pethő [15] proved that equation (3.4) has no solution in integers forming an arithmetic progression when $4 \leq a \leq 100$. In the following theorem, we extend their result to negative values of the parameter a. More precisely, for $-100 \leq a \leq -2$ we determine all such solutions of equation (3.4) for which $x_0, \ldots, x_{n-1} \in \mathbb{Z}$ are consecutive terms of an arithmetic progression.

Theorem 3.1. *Let $n \geq 3$ be an integer, let α be a zero of the irreducible polynomial $x^n - a \in \mathbb{Z}[x]$. Put $K = \mathbb{Q}(\alpha)$ and suppose that $-100 \leq a \leq -2$. Then the only solutions of equation (3.4) which form an arithmetic progression are $(2, 1, 0)$ when $(n, a) = (3, -7)$, and $(-2, -1, 0)$ when $(n, a) = (3, -9)$. In the case when $(n, a) = (11, -67)$ our result is conditional and depends on the truth of the generalized Riemann Hypothesis.*

We will deduce Theorem 3.1 from the following result concerning a special family of generalized binomial Thue-equations.

Theorem 3.2. *The only solutions of the generalized binomial Thue-equation*

$$X^n - aY^n = (a-1)^2 \tag{3.5}$$

in (n, a, X, Y) for $-100 \leq a \leq -2$, are those listed in Table 3.1 below. In the case when $(n, a) = (11, -67)$ our result depends on the truth of the generalized Riemann Hypothesis.

Table 3.1

n	a	(X, Y)
3	-97	(35, -7)
3	-63	(4, 4), (16,0), (64, -16)
3	-62	(1, 4)
3	-61	(13, 3)
3	-39	(16, -4)
3	-35	(11, -1), (46, -14)
3	-27	(10, -2)
3	-26	(3, 3), (9, 0), (27, -9)
3	-25	(1, 3)
3	-18	(-5, 3), (7, 1)
3	-12	(-11, 5)
3	-9	(7, -3)
3	-7	(-5, 3), (2, 2), (4, 0), (8, -4)
3	-6	(1, 2)
3	-3	(-2, 2)
4	-99	(-10, 0), (10, 0)
4	-80	(-9, 0), (-3, -3), (-3, 3), (3, -3), (3, 3), (9, 0)
4	-79	(-1, -3), (-1, 3), (1, -3), (1, 3)
4	-63	(-8, 0), (8, 0)
4	-48	(-7, 0), (7, 0)
4	-35	(-6, 0), (6, 0)
4	-24	(-5, 0), (5, 0)
4	-15	(-4, 0), (-2, -2), (-2, 2), (2, -2), (2, 2), (4, 0)

Table 3.1 (Continued)

n	a	(X, Y)
4	-14	(-1, -2), (-1, 2), (1, -2), (1, 2)
4	-8	(-3, 0), (3, 0)
4	-3	(-2, 0), (2, 0)
5	-31	(2, 2), (4, 0), (8, -4)
5	-30	(1, 2)
6	-63	(2, 2), (-2, -2), (2, -2), (-2, 2), (4, 0), (-4, 0)
6	-26	(3, 0), (-3, 0)
6	-7	(2, 0), (-2, 0)
8	-80	(3, 0), (-3, 0)
8	-15	(2, 0), (-2, 0)
10	-31	(2, 0), (-2, 0)
12	-63	(2, 0), (-2, 0)

3.3 Proofs

To prove Theorem 3.2 we need some results from Sections 1.1 and 2.4.

Proof of Theorem 3.2. Clearly, it is enough to solve equation (3.5) for $n = 4$ and in the cases when n is an odd prime. The other cases are simple consequences of these.

As a first step we use Lemma 2.1. Clearly, the conditions of Lemma 2.1 are fulfilled, so it provides an upper bound $B(a)$ for the degree n of the Thue-equation (3.5) in terms of a. Since $|a| \leq 100$, this shows that in order to prove Lemma 3.2 we have to consider only finitely many cases for n. The following table contains the approximate value of the bound $B(a)$ for some values of $|a|$.

Table 3.2

$\|a\|$	10	20	30	40	50	60	70	80	90	100
$B(a)$	7151	9304	10564	11457	12150	12717	13195	13610	13976	14303

The second step is to use a local argument (that we already used in former chapters) to prove that apart from a few exceptions equation 3.5 has no solutions in X, Y for $-100 \le a \le -2$ and $11 \le n \le B(a)$. For sake of completeness, we sketch the main idea of this local method. Choose a small integer k such that $p = 2kn + 1$ is a prime. Then both X^n and Y^n are either $2k$-th roots of unity (mod p) or zero. The congruence

$$X^n - aY^n \equiv (a-1)^2 \pmod{p}$$

then have to be checked only in $(2k+1)^2$ cases. Those values of $11 \le n \le B(a)$ and $-100 \le a \le -2$ for which this method did not prove the unsolvability of equation (3.5) are listed in Table 3.3.

Table 3.3

n	11	11	11	11	11	11	11	11	13	13	13	13	13	13	13	13
a	-2	-36	-45	-46	-55	-67	-78	-89	-8	-12	-21	-28	-52	-71	-76	-81

n	13	13	17	17	17	17	17	17	19	19	19	19	19	19	23
a	-82	-91	-9	-42	-45	-46	-52	-100	-14	-51	-60	-68	-77	-99	-94

The third step is to solve one by one the remaining equations. Wherever it was possible we used the Thue-solver implemented in the computer algebra packages MAGMA [18] and PARI [40].

To solve the equations corresponding to pairs (n, a) with $n \in \{3, 4, 5, 7\}$ and $-100 \le a \le -2$ we used the package MAGMA. In order to solve the "exceptional" equations corresponding to pairs (n, a) listed in Table 3.3 we used the Thue-solver included in PARI. (For the main ideas behind the latest improvements on this Thue-solver implemented by G. Hanrot see [33] and [17].)

In the case when $(n, a) = (23, -94)$ the Thue-solvers of the mentioned computer algebra packages were unable to solve equation (3.5), and if

$$(n, a) \in \{(11, -89), (11, -67), (11, -46), (13, -82), (19, -77)\},$$

using PARI we were able to get only conditional result assuming the generalized Riemann Hypothesis. To get an unconditional result, we considered the above cases as ternary equations with signature $(n, n, 2)$.

If $n = 23$ and $a = -94$, we used Proposition 1.1 with $p = 599$. First using the local approach we proved that equation (3.5) might only have solutions with $xy \equiv 0 \pmod{599}$ and then we used Proposition 1.1 with this value of p. Here both for the local computations and for the computation of the needed Fourier coefficients of all occurring newforms we used again MAGMA.

If

$$(n, a) \in \{(11, -89), (11, -46), (13, -82), (19, -77)\} ,$$

we also used Proposition 1.1 but in these cases we found no primes which divide xy. For instance, we consider the case when $n = 11$, $a = -89$. Equation (3.5) then takes the form

$$x^{11} + 89y^{11} = (-90)^2 . \tag{3.6}$$

Let us suppose that we have a solution (x, y, z) of (3.6) with $z = -90$ and with the conditions of Proposition 1.1. Then $\varepsilon_2 = 32$, since $89y$ must be odd, and we have to consider the space of modular forms of level

$$N = Rad_2(1 \cdot 89) \cdot Rad_2(1)^2 \cdot 32 = 89 \cdot 32 = 2848 .$$

17 cuspidal newforms occur at this level. Let us denote them by $f_1, \ldots, f_{17}$ and put $p = 23$. Then using MAGMA we get a contradiction with

$$Norm_{K_f/\mathbb{Q}}(c_{23} - a_{23}) \equiv 0 \pmod{11}, \tag{3.7}$$

on the case of all of these newforms if $a_{23} = \pm 24$ or $a_{23} \in \{x : |x| < 2\sqrt{23},\ x \equiv 0 \pmod 2\}$ except f_1 and f_4 that are both rational newforms. Analyzing the conditions on (x, y, z, A, B, C) we get that we can only be in case (i) among the above mentioned (i)-(v) cases. So we associate to the solution (x, y, z) of equation (3.6) the elliptic curve E_1 that now takes the form

$$E_1(x, y, z) : \ Y^2 = X^3 - 180X^2 + 89y^{11}X .$$

The local method shows that $y^{11} \equiv 22 \pmod{23}$ always holds. Thus the curve E_1 has the following form over $\mathbb{F}_{23}$:

$$E_1 : \; Y^2 = X^3 + 4X^2 + 3X \; ,$$

which is independent of (x, y, z). For the number of points on this curve over $\mathbb{F}_{23}$ we get that $\#E_1(\mathbb{F}_{23}) = 24$ so we have

$$a_{23} = 23 + 1 - 24 = 0 \; .$$

The Fourier series and the 23rd Fourier coefficient of f_1 and f_4 are

$$f_1 = q+2q^5-2q^7-3q^9+4q^{11}+4q^{13}-2q^{17}+8q^{19}+6q^{23}+O\left(q^{24}\right), c_{23} = 6$$

and

$$f_4 = q+2q^5+2q^7-3q^9-4q^{11}+4q^{13}-2q^{17}-8q^{19}-6q^{23}+O\left(q^{24}\right), c_{23} = -6,$$

respectively. Thus we get a contradiction with (3.7) in both cases since the corresponding norms are not divisible by 11. In the other cases we did similar computations.

Unfortunately in the case $n = 11$, $a = -67$ we found no way to prove the result unconditionally. Theorem 3.2 is proved. □

Now, using an idea of Bérczes and Pethő [14] and Theorem 3.2, we prove our main theorem.

Proof of Theorem 3.1. Let $x_0, \ldots, x_{n-1} \in \mathbb{Z}$ be a solution of equation (3.4) which forms an arithmetic progression and put $d = x_{i+1} - x_i$ for $i = 1, \ldots, n-1$. Then equation (3.4) has the form

$$N_{K/\mathbb{Q}}\Big((1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1})x_0 \\ + (\alpha + 2\alpha^2 + \ldots + (n-1)\alpha^{n-1})d\Big) = 1 \; . \tag{3.8}$$

In [14], Bérczes and Pethő showed that any solution x_0, d of equation (3.8) leads to a solution X, Y of equation (3.5) and these solutions are related to each other by the formulas

$$X := -x_0\,(a-1) - dan$$

and

$$Y := -x_0\,(a-1) - dan + d\,(a-1)\,.$$

Theorem 3.2 gives us all solutions of equation (3.5), and these solutions are listed in Table 3.1.

Now to prove our Theorem 3.1, we have to check whether a solution of equation (3.5) leads to an integral solution of equation (3.4), which has coordinates forming an arithmetic progression, or not. Using Table 3.1 one can verify that this condition is fulfilled only if $(n, a, X, Y) \in \{(3, -7, -5, 3)\,, (3, -9, 7, -3)\}$. Any other solution (X, Y) of equation (3.5) leads to a pair (x_0, d), where x_0 and d are not both integers. This concludes the proof of Theorem 3.1. $\square$

Summary

Our dissertation consists of three chapters.

In the first chapter results concerning (n, n, m) signature ternary diophantine equations of the form (1.4) are presented. The unsolvability of such an equation can often be proved via an analogue of the method used by Wiles [53] to prove Fermat's Last Theorem, the so-called modular approach including Frey-curves and modular forms. The applicability of this approach depends only on the prime factors of the coefficients in (1.4).

Let $m = n$. The works of Serre [45], Wiles [53], Darmon and Merel [23], and Ribet [43] give all solutions of (1.4) when the coefficients A, B are of the form $AB = p^\alpha$ with a prime p for which either $p \leq 29$ or $p = 53, 59$. Bennett, Győry, Mignotte and Pintér [10] solved (1.4) in the case $n > 7$ is a prime and $AB = 2^\alpha q^\beta$ with a prime $3 \leq q \leq 13$. Győry and Pintér [31] recently extended this result to primes $3 \leq q \leq 29$, and showed that in this case, apart from 8 possible exceptions (q, n, α) every nontrivial solution (x, y, z, A, B, n) of (1.4) has $n \leq 11$. We were able to considerably generalize this result to primes $3 \leq q \leq 151$, $q \neq 31, 127$ by showing then that apart from 31 explicitly given possible exceptions (q, n, α) every nontrivial solution (x, y, z, A, B, n) of (1.4) has $n \leq 13$. The case when, in (1.4), $AB = p^\alpha q^\beta$ with two odd primes p, q was first considered by Kraus [35]. For $5 \leq p < q \leq 29$, Győry and Pintér [31] proved that apart from 10 possible exceptional pairs (p, q) every nontrivial solution (x, y, z, A, B, n) of (1.4) has $n \leq 11$. We give the bound $n \leq 13$ on the solutions of (1.4) in the more general case when $5 \leq p, q \leq 71$ with 28 explicitly given possible exceptions (p, q, n).

Suppose that $m = 3$. Those types of equations (1.4) were investigated by Bennett, Győry, Mignotte és Pintér [10] for primes $3 \leq p, q \leq 13$, and later by Győry és Pintér [31] for primes $3 \leq p < q \leq 29$. Their results can be summarized as follows: if $AB = p^\alpha q^\beta$ with primes $3 \leq p < q \leq 29$ such that either $p \leq 7$ or $(p, q) \in \{(11, 13), (11, 17), (11, 19), (13, 17), (13, 19), (17, 23)\}$, then apart from 14 possible exceptions (p, q, n), every nontrivial solution (x, y, z, A, B, n) of (1.4) has $n \leq 11$. In Chapter 1, an extension of this result is also presented. We show that apart from 29 explicitly given possible exceptions, every nontrivial solution (x, y, z, A, B, n) of (1.4) has $n \leq 11$, when we allow the coefficients to have prime factors $3 \leq p < q \leq 71$ with $pq \leq 583$. The results of this chapter will be published in our paper [3].

The second chapter is devoted to the resolution of generalized binomial Thue equations of the form (2.1), where generalized means that in (2.1), the exponent is also unknown. The first general result on the resolution of binomial Thue equations is due to Bennett [5] who showed by means of the hypergeometric method that for $B = A + 1$, the equation (2.3) has no solutions with $|xy| > 1$. The case when, in $Ax^n - By^n = C$, the coefficients A, B, C are bounded positive integers was first studied by Győry and Pintér in [29], in which, using a local considerations, they derived a relatively sharp upper bound for n for concrete values of $A, B, C \leq 100$ provided that (2.1) has no trivial solutions with $|xy| \leq 1$. Further, they determined all nontrivial solutions of (2) under some natural conditions in case of various upper bounds on the coefficients.

We were able to significantly improve on the results of [29]. Among other things we completely solved equation (2.1) when $A = |C| = 1,\ B < 235$ and $C = \pm 1,\ 1 < A < B \leq 37$, respectively. More precisely, in the second chapter we present results on the complete and incomplete resolution of the following cases of coefficients: $A = |C| = 1,\ B \leq 400$; $A = |C| = 1,\ 400 < B < 2000$ with B being odd; $C = \pm 1,\ \max\{A, B\} \leq 50, 100$; and $\max\{A, B, |C|\} \leq 30$. In our proofs, we combine almost all techniques of the modern diophantine analysis, including local methods, linear forms in logarithms, the modular approach and computational methods to solve Thue equations of low degree. Beside these, a main ingredient of our proofs is a new result of ours, namely Theorem 2.6, concerning the solvability of equation (5).

In the second chapter, we also present results concerning the resolution of generalized binomial Thue equations of the form (2.14) $Ax^n - By^n = \pm 1$, when the coefficients $A, B \in \mathbb{Z}$ are unknown S-units for certain sets S of prime numbers. Using our results from the first chapter combined with our Theorem 2.6, we establish reasonable bounds on n in the solutions of (2.14), when $S = \{p, q\}$ with primes $2 \leq p, q \leq 71$. The results presented in Section 2.3 have been obtained jointly with A. Bérczes, K. Győry and Á. Pintér, and have been published in [4]. The result concerning binomial Thue equations with S-unit coefficients will be published in [3].

In the third chapter we are concerned with searching for arithmetical progressions among the solutions of norm form equations. This problem was first considered by Bérczes and Pethő [14], [15]. For an algebraic number α of degree n, $K = \mathbb{Q}(\alpha)$ and $m \in \mathbb{Z}$, they proved a nearly complete finiteness result on those solutions of (3.2) $N_{K/\mathbb{Q}}(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}) = m$ whose coordinates form an arithmetic progression. They also considered (3.2) with $m = 1$ and determined all solutions in question when α is a root of the polynomial $x^n - a$, with $n \geq 3$ and $2 \leq a \leq 100$. Further related results can be found in [16] and [13].

In Chapter 3, we extend the result of [15] on (3.2) with $m = 1$, by showing that there are only two solutions of that norm form equation whose coordinates are consecutive terms of an arithmetic progression in the case when α is a root of the polynomial $x^n - a$ with $n \geq 3$ and $-100 \leq a \leq -2$. These solutions are $(2, 1, 0)$ when $(n, a) = (3, -7)$, and $(-2, -1, 0)$ when $(n, a) = (3, -9)$. The proof of this result of ours is based on the idea of reducing the corresponding solutions of the norm form equation to solutions of the generalized binomial Thue equation (3.5) $X^n - aY^n = (a - 1)^2$. On the other hand another own result is applied, in which we give all integer solutions of (3.5) for $-100 \leq a \leq -2$. The results of the third chapter are published in [2].

Összefoglaló

Értekezésem három fejezetből áll. Ebben az Összefoglalóban az egyenletek a bevezetés (Introduction) szerinti számozással szerepelnek.

Az **első fejezet**ben

$$Ax^n + By^n = Cz^m, \quad m \in \{2, 3, n\} \tag{1}$$

alakú ternér diofantikus egyenletek megoldásaival foglalkozunk, ahol A, B és C nemnulla egészek, $n \geq 3$ és x, y, z pedig ismeretlen egészek. A Fermat-sejtés Wiles [53] által adott bizonyításának Frey-görbéken [24] és moduláris formákon alapuló módszere, az ún. moduláris módszer számos konkrét esetben alkalmazható annak bizonyítására, hogy az (1) egyenletnek nincs olyan x, y, z egész megoldása, melyre $|xy| > 1$ teljesül. Mivel ezt a módszert minden későbbi fejezetben alkalmazzuk, így az 1.1 szakaszban Bennett [6] nyomán vázoljuk az általunk használt moduláris technikákat, amelyek Bennett és Skinner [11], Kraus [35], és Bennett, Vatsal és Yazdani [12] mély eredményein alapulnak. Megjegyezzük, hogy az említett moduláris módszer alkalmazhatósága csak az (1)-beli A, B, C együtthatók prímosztóitól függ, a nagyságuktól nem. A fejezet további részében a

$$Ax^n - By^n = z^m \tag{3}$$

egyenlet megoldásaira koncentrálunk. Legyen $m = n$. Serre [45], Wiles [53], Darmon és Merel [23], és Ribet [43] munkái alapján ismert a (3) összes megoldása abban az esetben, ha $AB = p^\alpha$, ahol p egy prím, amelyre $p \leq 29$ vagy $p = 53, 59$. Bennett, Győry, Mignotte és Pintér [10] megoldották (3)-at $n > 7$ prím és $AB = 2^\alpha q^\beta$ esetén, ha

$3 \le q \le 13$ egy prím. Győry és Pintér [31] nemrég általánosították ezt az eredményt a $3 \le q \le 29$ esetre abban az értelemben, hogy 8 kivételes (q, α)-tól eltekintve a (3) egyenlet minden olyan (x, y, z, A, B, n) megoldására, ahol $|xy| > 1$, és Ax, By, z páronként relatív prímek, teljesül, hogy $n \le 11$. A következő tételünk tovább általánosítja a fenti eredményeket.

Tétel (Theorem 1.1). *Legyen $AB = 2^\alpha q^\beta$, ahol q egy prím, melyre $3 \le q \le 151$, $q \ne 31, 127$ továbbá legyenek α, β nemnegatív egészek.*
Ha n prím, akkor a (3) egyenlet minden olyan (x, y, z, A, B, n) megoldására, amelyre $|xy| > 1$, és Ax, By, z páronként relatív prímek, $n \le 53$ teljesül.
Továbbá, az 1.1 táblázatban szereplő 31 lehetséges (q, n, α) kivételtől eltekintve a (3) egyenlet minden olyan (x, y, z, A, B, n) megoldására, amelyre $|xy| > 1$, és Ax, By, z páronként relatív prímek, $n \le 13$ teljesül.

Az $AB = p^\alpha q^\beta$ esetet először Győry és Pintér [31] vizsgálták abban az esetben, amikor $5 \le p < q \le 29$ prímek. Belátták, hogy ha ezenfelül n is prím, akkor 10 megadott (p, q) lehetséges kivételtől eltekintve az (3) egyenlet minden olyan (x, y, z, A, B, n) megoldására, ahol $|xy| > 1$, és Ax, By, z páronként relatív prímek, $n \le 11$ teljesül. A következőképpen sikerült kiterjesztenünk ezt az eredményt nagyobb prímekre.

Tétel (Theorem 1.2). *Legyen $AB = p^\alpha q^\beta$, ahol $5 \le p, q \le 71$ prímek és α, β nemnegatív egészek. Ha n prím, akkor az 1.2 táblázatban szereplő 28 lehetséges (p, q, n) kivételtől eltekintve a (3) egyenlet minden olyan (x, y, z, A, B, n) megoldására, melyre $|xy| > 1$, és Ax, By, z páronként relatív prímek, $n \le 13$ teljesül.*

Tekintsük a (3) egyenletet $m = 3$ és $AB = p^\alpha q^\beta$ mellett, ahol p, q prímek és α, β nemnegatív egészek. Ezt az esetet $3 \le p, q \le 13$ prímekre Bennett, Győry, Mignotte és Pintér [10] vizsgálta; majd később $3 \le p < q \le 29$ prímekre Győry és Pintér [31]. Belátták, hogy ha n prím és $AB = p^\alpha q^\beta$, ahol $3 \le p < q \le 29$ olyan prímek, melyekre vagy $p \leq 7$ vagy $(p,q) \in \{(11,13), (11,17), (11,19), (13,17), (13,19), (17,23)\}$, akkor 14 megadott (p,q,n) lehetséges kivételtől eltekintve a (3) egyenlet minden olyan (x,y,z,A,B,n) megoldására, melyre $|xy| > 1$, és Ax, By, z páronként relatív prímek, teljesül, hogy $n \leq 11$. A következő általánosítást sikerült bizonyítanunk.

Tétel (Theorem 1.3). *Legyen $AB = p^\alpha q^\beta$ ahol α, β nemnegatív egészek és $3 \leq p < q \leq 71$ olyan prímek, melyekre $pq \leq 583$. Ha n prím, akkor az 1.3 táblázatban szereplő 29 lehetséges (p,q,n) kivételtől eltekintve a (3) egyenlet minden olyan (x,y,z,A,B,n) megoldására, melyre $|xy| > 1$, xy páros, és Ax, By, z páronként relatív prímek, teljesül, hogy $n \leq 13$.*

Az első fejezetben szereplő eredményeink a [3] dolgozatban fognak megjelenni.

A **második fejezet** bizonyos (általánosított) binom Thue-egyenletek teljes megoldásával kapcsolatos eredményeket tartalmaz. Tekintsük a

$$(2) \qquad Ax^n - By^n = C$$

diofantikus egyenletet, ahol $A, B, C, n \in \mathbb{Z} \setminus \{0\}$ és $n \geq 3$ rögzített vagy ismeretlen. Rögzített n esetén (2) egy binom Thue-egyenlet. A Thue-egyenletek és általánosított Thue-egyenletek irodalma meglehetősen gazdag. Thue [51] 1909-es ineffektív eredményéből tudjuk, hogy a Thue-egyenleteknek csak véges sok megoldása lehet. Baker [1] 1968-ban elsőként adott effektív felső korlátot a Thue-egyenletek megoldásainak a méretére. Mindkét eredményből következik, hogy (2)-nek rögzített n mellett csak véges sok megoldása lehet. Tijdeman [52] ismeretlen n kitevő esetén effektív korlátot adott $\max\{|x|, |y|, n\}$ értékére, ahol (x,y,n) a (2) egy olyan megoldása, melyre $|xy| > 1$. További binom Thue-egyenletekkel és alkalmazásaikkal kapcsolatos eredmények találhatók a [39], [46], [5], [34], [7], [9], [28], [10], [15], [2], [31] dolgozatokban és az ezekben található hivatkozásokban.

A második fejezetben (általánosított) binom Thue-egyenletek teljes megoldását tárgyaljuk, amihez általában önmagukban nem elegendőek akár a legélesebb effektív korlátok sem, mivel túlságosan nagyok. Így a megoldáshoz további módszerek szükségesek. Az első általános eredmény ebben az irányban Bennett [5] nevéhez fűződik, aki a hipergeometrikus módszerrel megmutatta, hogy $B = A + 1$ esetén az

$$Ax^n - By^n = \pm 1 \tag{4}$$

egyenletnek nincs olyan x, y megoldása, melyre $|xy| > 1$. Győry és Pintér [29] vizsgálta először azt az esetet, amikor (2)-ben az együtthatók korlátos pozitív egészek. Egy - a 2.4 szakaszban tárgyalt - lokális módszerrel sikerült konkrét A, B, C értékek mellett viszonylag éles felső korlátot adniuk n-re, feltéve, hogy az (2) egyenletnek nincs triviális, $|xy| \le 1$ tulajdonságú megoldása. Továbbá, természetes feltételek mellett meghatározták a (2) összes $|xy| > 1$ tulajdonságú megoldását az együtthatókra vonatkozó különböző felső korátok esetén. Az utóbbi eredményeket sikerült jelentős mértékben kiterjeszteni. Legfontosabb új eredményeink egyszerűsített formában a következők:

Tétel (Theorem 2.1'). *Ha* $1 < B \le 400$, *akkor az*

$$x^n - By^n = \pm 1 \tag{5}$$

egyenlet összes (x, y, n) *megoldásaira, melyre* $|xy| > 1, n \ge 3$ *és* $(B, n) \notin \{(235, 23), (282, 23), (295, 29), (329, 23), (354, 29)\}$, *teljesül, hogy* $n \in \{3, 4, 5, 6, 7, 8\}$.

Tétel (Theorem 2.2'). *(i) Ha* $400 < B < 800$ *páratlan, és* (B, n) *nem szerepel a 2.1 táblázatban, akkor az* (5) *egyenlet összes* (x, y, n), $|xy| > 1, n \ge 3$ *tulajdonságú megoldásaira* $n = 3, 9$.
(ii) Legyen $800 < B < 2000$ *páratlan. Ha* $n < 13$, *akkor az* (5) *egyenlet összes* $(x, y, n), |xy| > 1, n \ge 3$ *tulajdonságú megoldásaira* $n \in \{3, 5, 10\}$.
Ha $n > 100$ *egy prím, akkor eltekintve a 2.2 táblázatbeli lehetséges* (B, n) *kivételektől, az* (5) *egyenletnek nincs* $(x, y, n), |xy| > 1, n \ge 3$ *tulajdonságú megoldása.*

Tétel (Theorem 2.3'). *Az* $1 \leq A < B \leq 50$, $\gcd(A, B) = 1$ *feltételek mellett, a* (4) *egyenlet összes olyan* $(x, y, n), |xy| > 1, n \geq 3$ *megoldásaira, amelyre* $(A, B, n) \notin \{(21, 38, 17), (26, 41, 17), (22, 43, 17), (17, 46, 17), (31, 46, 17), (21, 38, 19)\}$, *teljesül, hogy* $n = 3, 4$.

Az eredeti megfogalmazásokban az n-re vonatkozó következtetések helyett a megoldások listája szerepel.

Ebben a fejezetben bizonyítunk még két további eredményt korlátos együtthatójú, (2) illetve (4) alakú binom Thue-egyenletekre vonatkozóan. Mindkét tétel azt állítja, hogy a megfelelő egyenleteknek nincs $|xy| > 1$ tulajdonságú megoldása, ha $n > 19$, eltekintve bizonyos lehetséges kivételektől, amelyeket a Theorem 2.4 illetve Theorem 2.5 eredményeinkben felsorolunk. Megjegyezzük, hogy tételeinkben elértük a jelenlegi módszereink alkalmazhatóságának a határát.

Bizonyításainkban alkalmazzuk a modern diofantikus számelmélet majdnem minden technikáját. A [29]-ben használt módszerek közül használjuk a lokális módszert, Pintér [42] egy effektív eredményét és a moduláris technikát. Ezek és mások mellett, az (5) egyenletre vonatkozó tételek bizonyításai elsősorban a következő saját eredményünkön alapulnak:

Tétel (Theorem 2.6). *Tegyük fel, hogy az* (5) *egyenletben* n *prím és hogy az alábbi feltételek mindegyike teljesül:*

(i) $n \geq 17$,

(ii) $B \leq \exp\{3000\}$,

(iii) $n \nmid B\phi(B)$,

(iv) $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$,

(v) $r^{n-1} \not\equiv 1 \pmod{n^2}$ *valamely* $r|B$ *esetén.*

Ekkor (5)*-nek nincs olyan* (x, y, n) *megoldása, melyre* $|xy| > 1$.

A 2.7 szakaszban (4) alakú binom Thue-egyenletek teljes megoldásának egy másik megközelítését tárgyaljuk, amelyben az A, B együtthatók tetszőlegesen nagyok lehetnek, de csak adott prímekkel lehetnek oszthatók. Más szóval, az együtthatók ismeretlen S-egységek, prímszámok valamely kis elemszámú S halmaza esetén. Ha $S = \{p\}$ és $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59\}$, a (4) egyenletnek nincs $|xy| > 1$, $n \geq 3$ tulajdonságú megoldása Wiles [53], Darmon és Merel [23] és Ribet [43] ternér egyenletekre vonatkozó eredményei alapján. Bennett [7] megoldotta (4)-et az $S = \{2, 3\}$ esetben, amit Bennett, Győry, Mignotte és Pintér [10] kiterjesztett az $S = \{p, q\}, 2 \leq p, q \leq 13$ esetre. Bugeaud, Mignotte és Siksek [21] más módszerrel megoldotta (4)-et, ha $A = 2^\alpha, B = q^\beta$ ahol $3 \leq q < 100$ egy prím, vagy $A = p^\alpha, B = q^\beta$ ahol $3 \leq p < q \leq 31$ prímek. Győry és Pintér [31]-ben általánosította a [10]-beli eredményeket az $S = \{p, q\}, 2 \leq p, q \leq 29$ esetre. A második fejezet lezárásaként éles felső korlátot adunk a (4) egyenletbeli n kitevőre, ha $S = \{p, q\}$ és $2 \leq p, q \leq 71$. Ez a tételünk hasznos eszköz lehet konkrét S-egység együtthatós binom Thue-egyenletek megoldása során.

A 2.3 szakaszban szereplő tételek Bérczes Attilával, Győry Kálmánnal és Pintér Ákossal közösen elért eredmények, amelyek a [4] közös cikkben jelentek meg. A Theorem 2.7 [3]-ban fog megjelenni.

A **harmadik fejezet** binom Thue-egyenletek egy alkalmazását tárgyalja norma forma egyenletek számtani sorozatot alkotó megoldásainak meghatározására. Legyenek $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ $\mathbb{Q}$ fölött lineárisan független elemei egy n-edfokú K algebrai számtestnek és tekintsük az

$$a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \ldots + \alpha_m x_m) = b, \quad b \in \mathbb{Z} \setminus \{0\} \tag{6}$$

norma forma egyenletet, ahol a_0-t úgy választjuk meg, hogy a (6) baloldalán szereplő polinom egész együtthatós legyen. 1971-ben Schmidt [44] egy ineffektív kritériumot adott arra, hogy a (6) egyenletnek véges sok megoldása legyen. Később, Győry és Papp [26] effektív végességi eredményeket és explicit felső korlátokat nyertek norma forma egyenletek széles osztályainak megoldásaira.

A norma forma egyenletek megoldásai közötti számtani sorozatok vizsgálatának ötletét Pethő Attila vetette fel, és először Bérczes és

Pethő [14], [15] végeztek ilyen vizsgálatokat a

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = m \tag{7}$$

egyenlettel kapcsolatban, ahol α egy n-edfokú algebrai szám, $K = \mathbb{Q}(\alpha)$, $m \in \mathbb{Z}$ és $(x_0, \ldots, x_{n-1}) \in \mathbb{Z}^n$. [14]-ben egyebek mellett egy majdnem teljes effektív végességi eredményt bizonyítottak (7) azon megoldásaira, melynek koordinátái számtani sorozatot alkotnak. Ezenkívül meghatározták a

$$N_{K/\mathbb{Q}}\left(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}\right) = 1 \tag{8}$$

egyenlet összes számtani sorozatot alkotó megoldását, abban az esetben, ha (8)-ban az α gyöke az $x^n - 2$ illetve $x^n - 3$ $(n \geq 3)$ polinomok valamelyikének. Továbbá, [15]-ben megmutatták, hogy (8)-nak nincs ilyen megoldása, ha α gyöke az $x^n - a$ polinomnak, ahol $n \geq 3$ és $4 \leq a \leq 100$. A témában további eredmények találhatók a [16] és [13] dolgozatokban.

A harmadik fejezetben kiterjesztjük a [15]-beli eredményeket arra az esetre, ha (8)-ban az α gyöke az $x^n - a$, $(n \geq 3)$ polinomnak, ahol $-100 \leq a \leq -2$. Ekkor megmutatjuk, hogy a (8) egyenlet összes olyan $(x_0, \ldots, x_{n-1}) \in \mathbb{Z}^n$ megoldása, amelyre az x_i koordináták egy számtani sorozat egymást követő elemei a $(2, 1, 0)$ amikor $(n, a) = (3, -7)$, illetve $(-2, -1, 0)$ amikor $(n, a) = (3, -9)$. Az $(n, a) = (11, -67)$ esetben állításunk az általánosított Riemann hipotézis (GRH) feltételezése mellett igazolható. E tételünk bizonyításában először a (8) megfelelő megoldásait visszavezetjük az

$$X^n - aY^n = (a-1)^2, \quad -100 \leq a \leq -2 \tag{9}$$

általánosított binom Thue-egyenlet megoldásaira, majd használjuk a fejezetbeli másik saját eredményünket, amely megadja a (9) egyenlet összes egész megoldását ($(n, a) = (11, -67)$ esetén a GRH mellett).

A harmadik fejezetben tárgyalt eredményeink a [2] dolgozatban jelentek meg.

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1. A. Bazsó, *Further computational experiences on norm form equations with solutions forming arithmetic progressions*, Publ. Math. Debrecen 71(3-4) (2007), 489-497.

2. A. Bazsó, A. Bérczes, K. Győry and Á. Pintér, *On the resolution of equations $Ax^n - By^n = C$ in integers x, y and $n \geq 3$, II*, Publ. Math. Debrecen 76(1-2) (2010), 227-250.

3. A. Bazsó, *On binomial Thue equations and ternary equations with S-unit coefficients*, Publ. Math. Debrecen, accepted fo publication.

List of conference talks of the author

1. *Norm form equations with solutions forming arithmetic progressions*, The 18th Czech and Slovak International Conference on Number Theory, 27-31 August 2007, Smolenice (Slovakia).

2. *Norma forma egyenletek számtani sorozatot alkotó megoldásairól*, Egri Számelméleti és Kriptográfiai Napok, 6 October 2007, Eger.

3. *On the resolution of equations of the form $Ax^n - By^n = C$ in integers x, y and $n \geq 3$*, The 7th Polish, Slovak and Czech Conference on Number Theory, 10-12 June 2008, Ostravice (Czech Republic).

4. *Binom Thue-egyenletek megoldásáról*, II. Soproni Diofantikus és Kriptográfiai Napok, 10-12 October 2008, Sopron.

5. *Solving binomial Thue equations*, Winter School on Explicit Methods in Number Theory, 26-30 January 2009, Debrecen.

6. *On the resolution of binomial Thue equations*, 26th Journées Arithmétiques, 6-10 July 2009, Saint-Étienne (France).

7. *On ternary and binomial Thue equations with S-unit coefficients*, The 19th Czech and Slovak International Conference on Number Theory, 31 August-4 September 2009, Hradec nad Moravicí (Czech Republic).

8. *On ternary and binomial Thue equations with S-unit coefficients*, Debreceni Diofantikus és Kriptográfiai Napok, 7 November 2009, Debrecen.

9. *On ternary equations and binomial Thue equations*, The 8th Czech, Polish and Slovak Conference on Number Theory, 21-24 June 2010, Bukowina Tatrzańska (Poland).