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Sandbox method for factorial moments and anomalous fractal dimensions

F. Kun [a,1], H. Sorge [b], K. Sailer [a], G. Bardos [a], W. Greiner [b]

[a] *Department of Theoretical Physics, Kossuth Lajos University, H-4010 Debrecen, P.O. Box 5, Hungary*

[b] *Institut für Theoretische Physik, Johann Wolfgang Goethe Universität, D-60054 Frankfurt am Main, Postfach 111932, Germany*

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Abstract

For the purpose of the intermittency analysis of high-energy heavy-ion data, the definition of factorial moments is reformulated in terms of horizontal averages determined by the sandbox method. It is shown that the sandbox method gives the correct anomalous fractal dimensions for exactly solvable models of geometrical multifractals when the box-counting method usually used in heavy-ion physics fails. For the RQMD generated event sample with two-particle correlations, the anomalous fractal dimensions obtained by the both methods differ significantly.

1. Motivations

Bialas and Peschanski proposed to study the scaled factorial moments of the (pseudo) rapidity distribution of the final state particles to investigate properties of dynamical fluctuations in high-energy heavy-ion reactions and introduced the concept of intermittency into high-energy nuclear physics [1]. The scaled factorial moments F_i of the (pseudo) rapidity distribution of the produced particles are defined by

$$F_i(M) = M^{i-1}\left\langle \sum_{m=1}^{M} \frac{n_m(n_m-1)\dots(n_m-i+1)}{N(N-1)\dots(N-i+1)} \right\rangle \tag{1}$$

where the pseudorapidity interval $\Delta\eta$ considered was split into M bins of equal size $\delta\eta = \Delta\eta/M$, n_m is the multiplicity in the m-th bin, N is the total multiplicity of an event and the brackets $\langle \dots \rangle$ denote the average over events. The most important property of F_i discovered in Ref. [1] is that it filters out the statistical fluctuations, thus any nontrivial behaviour of F_i is a direct consequence of some features of the dynamics of particle production. Another significant aspect of F_i that only such bins can contribute to its value for which $n_m \geq i$. That is why higher order moments at high resolution give information about the spikes. The disadvantage of the factorial moments is the inability to extract any information about dips, although it is recognized that the rapidity gaps are also important to study. The power law behaviour of the scaled factorial moments with increasing resolution is called intermittency, namely,

$$F_i \sim (\delta\eta)^{-f_i} \tag{2}$$

(or $F_i \sim M^{f_i}$). The intermittency indices f_i are related to the so-called anomalous fractal dimensions $d_i = f_i/(i-1)$.

[1] E-mail: KUN@TIGRIS.KLTE.HU.

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As it can be seen in Eq. (1) the calculation of the factorial moments F_i contains two different kinds of averages: a horizontal one (within one event) and a vertical one (over the event ensemble). As a rule, the horizontal average is calculated using the so-called box-counting method, putting a fixed grid onto the pseudorapidity interval of interest and increasing the resolution (decreasing the bin size) up to the experimental resolution.

In the present letter we show that this procedure may lead to incorrect values of the anomalous fractal dimensions in the analysis of the rapidity distributions of particles created in high-energy heavy-ion collisions. We propose to use the sandbox method for horizontal averaging, which was successfully applied in the study of strange attractors [13–15] and geometrical multifractals [7–10]. We show using a data set calculated for nucleus-nucleus collisions at 200 A GeV with the transport model RQMD [4,5] that the sandbox method results in values of the anomalous fractal dimensions rather different from that obtained by the box-counting method.

The organization of the remaining part of the paper is as follows. The concept of horizontal averaging with the sandbox method will be introduced applying it to geometrical multifractals as an example. We review the connection of the sandbox average for the moments G_q to correlation integrals. As to the next, it will be shown for a deterministic multifractal model that the sandbox method can give a much better estimate of the anomalous fractal dimensions in case of a single event with very high multiplicity. Furtheron, it will be demonstrated for an exactly solvable statistical multifractal model that the sandbox method converges faster to the exact values of the fractal dimensions with increasing number of (low multiplicity) events. We close this letter by comparing the factorial moments and the anomalous fractal dimensions obtained by both methods using them to the analysis of RQMD events. The significant differences found and the tests performed on exactly solvable multifractal models show that the sandbox method must be favoured for the intermittency analysis of rapidity distributions of particles produced in high-energy heavy-ion collisions independently whether they were obtained either in an experiment or in the framework of a theoretical model.

2. Sandbox method

Non-equilibrium physical processes usually create complex structures which can be described in terms of fractal geometry in many cases. The fractal is characterized by a non-integer exponent called fractal dimension. In addition to this single fractal dimension of the object itself, any singular distribution of a physical quantity (any measure) defined on the fractal determines an infinite set of fractal dimensions each one corresponding to the distribution of a given kind of singularity of the measure. Different distributions determine different multifractal spectra. That is why multifractality in general manifests itself via non-geometrical properties.

The concepts of geometrical multifractality was introduced by Tél and Vicsek [7]. In the case of fractals on which the measure (e.g. the mass, in systems formed by particles unit mass is associated to each particle) is uniformly distributed, multifractality manifests itself via purely geometrical properties, via the density fluctuations in the embedding space. Then the multifractal spectra can be considered to characterize the fractal support itself. These fractals are called geometrical or mass multifractals. It is evident that particle spectra created by heavy-ion collisions are directly related to geometrical multifractals. To determine multifractal properties of an object there are two basic methods: the box-counting (mentioned above) and the sandbox method (reviewed below). Tél and Vicsek showed [8] that the box-counting method usually fails for geometrical multifractals. It is not suitable to calculate the multifractal spectra correctly. They demonstrated on an exactly solvable geometrical multifractal model [8] that in the determination of the D_q spectra the sandbox method can reproduce the exact values at any q but the box-counting method can give good results only in the case of high q values ($q \geq 8$). Because of the analogy of geometrical multifractals with particle spectra, the sandbox method can be a more powerful tool to study the dynamical correlations in heavy-ion physics as well.

We review briefly the concept of the sandbox method [8,15] and then we apply it to the calculation of the factorial moments.

Total description of the scaling properties of a distribution can be given in terms of the G_q moments [2] which are defined by

$$G_q(M) = \sum_{m=1}^{M} p_m^q \tag{3}$$

where $p_m = n_m/N$ at a given resolution and q can be any arbitrary real number. Rewriting Eq. (3) we find:

$$G_q^{\mathrm{bc}} = \sum_{m=1}^{M} p_m^q = \sum_{m=1}^{M} \left(\frac{n_m}{N}\right)^q = \sum_{m=1}^{M} \frac{n_m}{N} \left(\frac{n_m}{N}\right)^{q-1} \sim \left(\frac{\delta\eta}{\Delta\eta}\right)^{(q-1)D_q} . \tag{4}$$

Since n_m/N can be considered as a probability distribution we get:

$$G_q^{\mathrm{bc}}(\delta\eta) = \left\langle \left(\frac{n_m}{N}\right)^{q-1} \right\rangle \sim \left(\frac{\delta\eta}{\Delta\eta}\right)^{(q-1)D_q^{\mathrm{bc}}} , \tag{5}$$

where the average is taken according to the distribution $p_m = n_m/N$. This expectation value can be calculated in another way using 'sandboxes' of radius $\delta\eta < \Delta\eta$ centered on the particles of the object, instead of a grid of lattice unit $\delta\eta$:

$$G_q^{\mathrm{sb}}(\delta\eta) = \left\langle \left(\frac{n(\delta\eta)}{N}\right)^{q-1} \right\rangle_c \sim \left(\frac{\delta\eta}{\Delta\eta}\right)^{(q-1)D_q^{\mathrm{sb}}} \tag{6}$$

The subscript c denotes the average over the centers. (It should be noted that in principle, applying the sandbox method it is not necessary to put sandboxes onto all the elements of the set. One can choose among them according to a unique distribution.)

Both methods of horizontal averaging give the same result in general, but they are not equivalent for geometrical multifractals.

The n-th order correlation integral is defined as

$$C_n(r) = N^{-n} \; [\text{number of } n\text{-tuplets } (x_{i_1}, \dots, x_{i_n}) \text{ of particles with distances } |x_{i_\alpha} - x_{i_\beta}| \le r \text{ for all } (i_\alpha, i_\beta)] \tag{7}$$

In the special case of $n = 2$:

$$C_2(r) = \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(r - |x_i - x_j|) \tag{8}$$

where Θ is the Heaviside-function. The number of particles in the sphere of radius $\delta\eta$ and centered at x_i:

$$n_i(\delta\eta) = \frac{1}{N} \sum_{j=1}^{N} \Theta(\delta\eta - |x_i - x_j|). \tag{9}$$

Evidently $C_2(\delta\eta) \equiv \langle n(\delta\eta)\rangle_c \equiv G_2^{\mathrm{sb}}(\delta\eta)$, i.e. the sandbox average G_2^{sb} of the moment G_2 accounts for the two particle correlations exactly. For higher order correlation integrals:

$$C_n(\delta\eta) = \frac{1}{N} \sum_{i=1}^{N} C_n^i(\delta\eta) \tag{10}$$

where

$$C_n^i(r) = \frac{1}{N^n} \; [\text{number of } n\text{-tuplets } (x_{i_1}, \dots, x_{i_n}) \text{ of particles in the } i\text{-th sphere with distances } |x_{i_\alpha} - x_{i_\beta}| \le r \text{ for all } (i_\alpha, i_\beta)] \tag{11}$$

If the distribution of the particles in the i-th box can be considered uniform then $C_n^i(\delta\eta) \sim n_i^n(\delta\eta)$ and $G_n^{\mathrm{sb}}(\delta\eta) \sim C_n(\delta\eta)$. In the future heavy-ion experiments where the event multiplicities can be very high the multifractal analysis is reliable within one event. In the evaluation of those experiments the sandbox averaged G_q moments can be a useful tool besides the factorial moments.

The definition of the factorial moments (Eq. (1)) can also be reformulated in terms of the sandbox average similarly to the moments G_q. Without the event average:

$$F_i^{\mathrm{bc}} = M^{i-1} \sum_{m=1}^{M} \frac{n_m(n_m - 1)\dots(n_m - i + 1)}{N(N-1)\dots(N-i+1)} = M^{i-1} \sum_{m=1}^{M} \frac{n_m}{N} \left(\frac{(n_m - 1)\dots(n_m - i + 1)}{(N-1)\dots(N-i+1)} \right) . \tag{12}$$

The last expression can also be considered as an expectation value and we can again use the sandbox method for its calculation. Thus we can define F_i^{sb} as

$$F_i^{\rm sb} = \left(\frac{\Delta\eta}{\delta\eta}\right)^{i-1} \times \left\langle \frac{(n(\delta\eta)-1)\dots(n(\delta\eta)-i+1)}{(N-1)\dots(N-i+1)} \right\rangle_c \sim \left(\frac{\Delta\eta}{\delta\eta}\right)^{f_i^{\rm sb}}. \tag{13}$$

We have to note that in this expression $\Delta\eta/\delta\eta$ is not equal to the number of boxes.

3. Results and discussion

In this section we would like to present the advantages of the usage of the sandbox method for the calculation of horizontal averages in heavy-ion physics. We restrict ourselves to the study of factorial moments because there are several examples in the literature for the study of multifractality [15,7–9]. At first the role of the sandbox averaging in the determination of the anomalous fractal dimensions will be shown on exactly solvable model systems.

To study the case of a single but very high multiplicity event we used the model of the so-called asymmetric growing Cantor set [7,8] to generate density fluctuations in one dimension. The number of generations n in the model calculations was chosen $n = 7$. We calculated the anomalous fractal dimension as a function of the resolution with the sandbox ($d_i^{\rm sb}$) and the box-counting methods ($d_i^{\rm bc}$) and compared them to the exact solutions $d_i^{\rm ex}$:

$$d_i^{\rm sb} = \frac{1}{i-1}\frac{\ln F_i^{\rm sb}(\delta\eta)}{\ln\left(\frac{\Delta\eta}{\delta\eta}\right)},$$

$$d_i^{\rm bc} = \frac{1}{i-1}\frac{\ln F_i^{\rm bc}(\delta\eta)}{\ln\left(\frac{\Delta\eta}{\delta\eta}\right)}. \tag{14}$$

This comparison is shown in Fig. 1 for $i = 2,3,4$. We see that the sandbox method gives much better estimates of the d_i's, the $d_i^{\rm bc}$'s are far away from the exact values. Increasing the rank i of the moments the differences between $d_i^{\rm bc}$ and $d_i^{\rm ex}$ decrease, but the $d_i^{\rm sb}$'s are close to the exact solutions for all ranks i. Due to the finiteness of the sample the curves for $d_i^{\rm sb}$'s and $d_i^{\rm bc}$'s break down for small and large resolution as well. Nevertheless, for $d_i^{\rm sb}$'s we find a wide window of

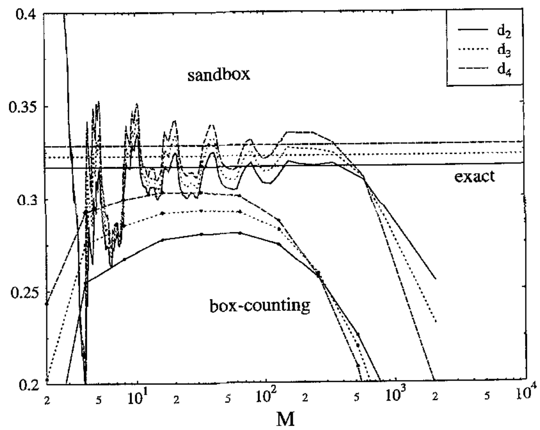

Fig. 1. The comparison of the anomalous fractal dimensions obtained by different methods of horizontal averages in the case of a single but very high multiplicity 'event'. The results for d_2, d_3, and d_4 are shown.

resolution within which the average of $d_i^{\rm sb}$ is in good agreement with $d_i^{\rm ex}$.

To study the ensemble of low multiplicity events we used a simple one-parameter multifractal model, the so-called p-model introduced in Ref. [11]. (The p-model was also studied previously by Lipa and Buschbeck in the context of factorial moments [12].) By means of this model we generated a one-particle distribution of width $\Delta\eta = 3$ in seven iteration steps and filled $\overline{N} = 35$ particles according to this distribution in the resulting 128 intervals. The multiplicity distribution of the event sample was Poissonian with the mean value $\overline{N} = 35$. Varying the number of 'events' considered in the analysis we studied the convergence of $d_i^{\rm sb}$ and $d_i^{\rm bc}$ to $d_i^{\rm ex}$ as a function of the number of events. These results are shown in Fig. 2 for 20, 50, 80, 100 events. The sandbox method gives a reliable estimate of the anomalous fractal dimensions even for small event-numbers and increasing the number of events its convergence to the exact values is faster than that of the box-counting method.

Finally, we analysed RQMD events for the quasi-central collision of ^{32}S beam at 200 GeV/nucleon energy with Ag/Br emulsion. RQMD is a microscopic phase space approach based on resonance and string excitations, fusion of neighbouring strings into so-called ropes and subsequent reinteractions of all secondaries with each other and with the original ingoing baryons (see [4] and [5] for a description of the recently updated version RQMD 1.08 which was used

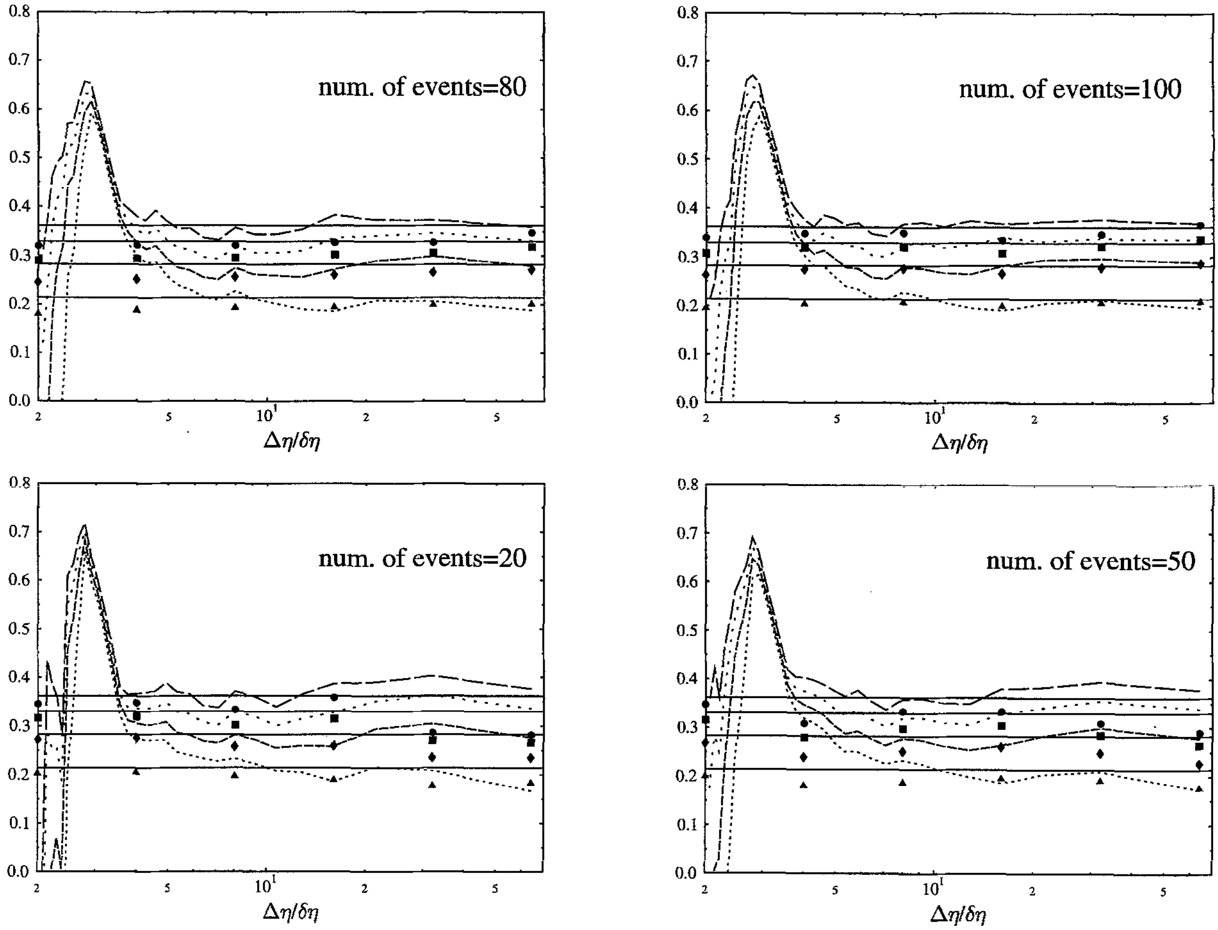

Fig. 2. The test of the convergence of the anomalous fractal dimensions obtained by the sandbox and the box - counting methods to the exact solutions in the case of the low multiplicity event sample. The horizontal continuous lines represent the exact solutions for d_2, d_3, d_4, d_5, the different types of dashed lines are for the sandbox results and the filled symbols are for the box-counting results.

for the calculations presented here). In our previous paper [6] we studied the role of quantum correlations in intermittency in the framework of the RQMD model. In Ref. [6] we presented a procedure to determine Bose-Einstein correlations by making use of the spacetime and momentum coordinates of the particles at freeze-out obtained in RQMD and to generate events improved by such correlations. Both the original and the improved sets of events were analysed with the method of factorial moments using both algorithms of horizontal averaging. Vertically we averaged over 500 events. The detailed description of our method to put two-particle correlations into the RQMD generated data sample can be found in Ref. [6]. Fig 3 shows the comparison of the factorial moments for the original (Fig. 3a)) and the 'correlated' data sets (Fig. 3b)) calculated with the box-counting and sandbox methods. For the correlated data set (Fig. 3b) we calculated the anomalous fractal dimensions. In the region 1.5–3.5 straight lines were fitted to the $\ln F_i$'s as a function of $\ln \Delta\eta/\delta\eta$. The results are presented in Fig. 4. With the box-counting method we reproduced our previous results (see Ref. [6]). It can be seen that there are significant differences between the anomalous fractal dimensions calculated with both methods of horizontal averaging. $d_2^{\rm sb}$ is approximately three times larger than $d_2^{\rm bc}$ but on increasing the rank i of the moments the differences between the $d_i^{\rm sb}$ and

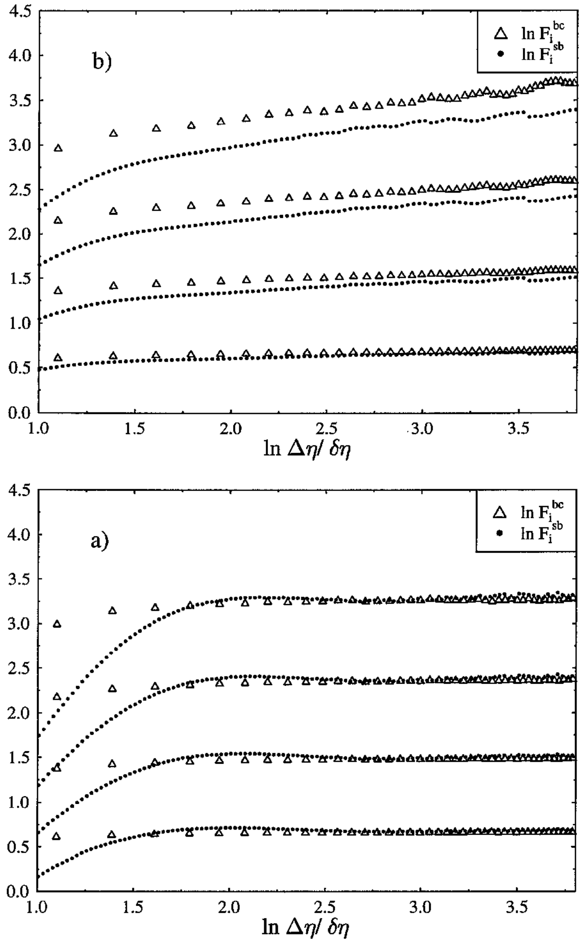

Fig. 3. The comparison of the factorial moments obtained by different types of horizontal averages in the case of the RQMD simulated data sample. The rank of the moments are $i = 2, 3, 4, 5$.

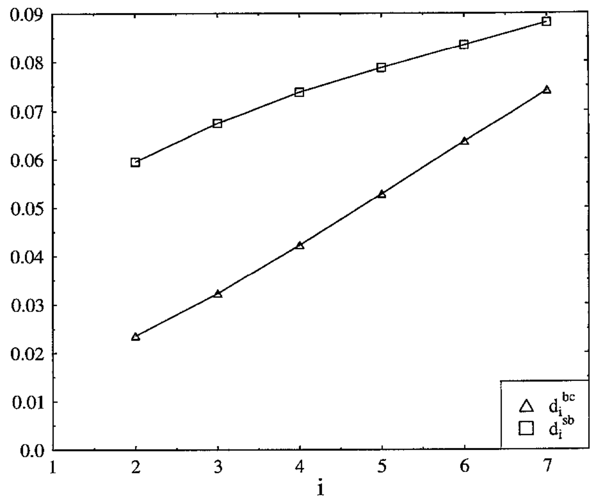

Fig. 4. The anomalous fractal dimensions extracted from F_i^{bc} and F_i^{sb} in the case of the 'correlated' RQMD sample (Fig. 3b). The ranks are $i = 2, \ldots, 7$. Significant differences can be seen between the d_i^{bc}'s and d_i^{sb}'s.

d_i^{bc} decrease as expected, because only second-order correlations are included in the data sets.

The sandbox method for horizontal averaging is even more relevant for the intermittency analysis of experimental data in which generally the higher-order correlations can be also present.

Summarizing, in the present letter we reformulated the definition of factorial moments in terms of the sandbox method for horizontal averaging. Tests of the method have been performed in the cases of a single but very high multiplicity event, of the ensemble of low multiplicity events and of RQMD data samples with two-particle correlations. They revealed the reliability and advantages of the sandbox method in intermittency analysis of data on high-energy heavy-ion collisions:

- For a single high multiplicity event the sandbox averaging leads to reliable anomalous fractal dimensions in a wide range of resolution when box-counting is completely unreliable.
- For an ensemble of low multiplicity events the sandbox averaging leads to a much better convergence of the anomalous fractal dimensions to their exact value with increasing number of events.
- Sandbox averaging results in values of the anomalous fractal dimensions for RQMD data samples with Bose-Einstein correlation which differ significantly from those obtained by the box-counting method.

We also argued that sandbox type horizontal averages take into account two-particle correlations exactly. Thus we conclude that rather the sandbox method than the box-counting one should be used for a reliable analysis of correlations and intermittency based on either experimental data or some theoretical framework.

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