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Jensen type results concerning generalized convex functions

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JENSEN TYPE RESULTS CONCERNING GENERALIZED CONVEX FUNCTIONS

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„You see, one thing is, I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers which might be wrong... but I do not have to know an answer. I do not feel frightened by not knowing things, by being lost in a mysterious universe without having any purpose which is the way it really is as far as I can tell – possibly. It does not frighten me."

Richard Feynman

Conventions and notation

We are going to use the usual notation for the special subsets of the set of real numbers $\mathbb{R}$, namely, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will stand for the *set of natural*, *integer*, and *rational numbers*, respectively. Sometimes we shall need only the positive elements of $\mathbb{Q}$ and $\mathbb{R}$. These subsets will be denoted shortly by $\mathbb{Q}_+$ and $\mathbb{R}_+$. We note that, in the light of this convention, $\mathbb{N}$ can be also interpreted as $\mathbb{Z}_+$. Some of the considered functions will map their domain into the *set of extended real numbers* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

The function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ will stand for the identity function, that is, we have $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}$. The restrictions of id for any given subset of $\mathbb{R}$ will be denoted also by id.

For given numbers $n, m \in \mathbb{Z}$, we define the finite set $\{n, \dots, m\}$ by the intersection $\{k \in \mathbb{Z} \mid n \le k\} \cap \{k \in \mathbb{Z} \mid k \le m\}$. According to this, $\{n, \dots, m\}$ is the empty set if $m < n$, and it equals to the singleton $\{n\}$ if $n = m$. The set $\{1, \dots, n\}$ will be denoted simply by $\mathbb{N}_n$.

For a given subset $H \subseteq \mathbb{N}$, let $\mathbb{1}_H : \mathbb{N} \to \{0, 1\}$ stand for *the characteristic function* of H, that is,

$$\mathbb{1}_H(n) := 1 \quad \text{if } n \in H \qquad \text{and} \qquad \mathbb{1}_H(n) := 0 \quad \text{if } n \in \mathbb{N} \setminus H.$$

Jensen's Theorem as motivation

To motivate our investigations, first I would like to recall a celebrated theorem belonging to the theory of convex functions, which is due to the Danish mathematician Johan L. W. V. Jensen from 1906. To formulate the theorem precisely, first we need some notions, which shall play a crucial role also in the whole dissertation.

Let X be a *linear space*, that is, a vector space over the field $\mathbb{R}$. We say that a subset $D \subseteq X$ is *convex* if, for all $t \in [0,1]$, the inclusion $tD+(1-t)D \subseteq D$ is satisfied. This definition implies that the whole space X and the empty set are convex.

Having a nonempty convex subset $D \subseteq X$ and $t \in [0,1]$, we say that a function $f : D \to \mathbb{R}$ is *t-convex on D* if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{1}$$

holds for all $x, y \in D$. The function f will be called *midpoint convex* or, in honor of Jensen, *Jensen convex* on its domain if the inequality (1) is valid for all $x, y \in D$ under $t = \frac{1}{2}$. Finally, we will say that f is *convex on D* if (1) holds for all $t \in [0,1]$. We note that any function defined on D trivially satisfies the inequality (1) under the parameters $t = 0$ and $t = 1$.

After introducing these concepts, Jensen's Theorem [**10**] sounds as follows.

THEOREM 0.1. *Let X be a linear space and $D \subseteq X$ be a nonempty convex subset. Then the following statements are pairwise equivalent.*

(1) The function $f : D \to \mathbb{R}$ is Jensen convex.

(2) For any given positive integer $n \in \mathbb{N}$, the function $f : D \to \mathbb{R}$ fulfills the n-variable Jensen Inequality, that is, for all $x_1, \dots, x_n \in D$, we have

$$f\Big(\frac{x_1+\cdots+x_n}{n}\Big) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}.$$

(3) The function $f : D \to \mathbb{R}$ is rationally convex on D, that is, for all $r \in [0,1] \cap \mathbb{Q}$ and for all $x, y \in D$, we have

$$f(rx + (1-r)y) \le rf(x) + (1-r)f(y).$$

The above theorem draws attention, among others, two interesting phenomenon concerning real functions. Now I only would like to highlight them, the details will be clarified later.

I. Firstly, in view of the assertions (1) and (3), if f satisfies the inequality (1) with $t = \frac{1}{2}$, then there are infinitely many other parameters, for which (1) is fulfilled by f too. In addition, it also turns out that this set of parameters is at least countable and it forms a dense subset of the closed unit interval.

   Introducing the notation

$$\mathcal{C}_f := \{t \in [0,1] \mid f \text{ is } t\text{-convex on } D\} \subseteq [0,1],$$

   the equivalence of (1) and (3) of Theorem 0.1 can be reformulated as follows: the inclusion $\frac{1}{2} \in \mathcal{C}_f$ holds if and only if $[0,1] \cap \mathbb{Q}$ is contained in $\mathcal{C}_f$. Now the following question arises naturally, namely, for a given function $f : D \to \mathbb{R}$, what kind of implications are valid among the members of $\mathcal{C}_f$ or, more general, what can we state about the algebraic and topological structure of the set $\mathcal{C}_f$? As we will see, these questions were totally answered in the previous years, but only in the case of standard convexity.

II. The second is the connection of the statements (1) and (2). More precisely, in view of the above theorem, the n-variable Jensen Inequality, as a convexity property, is *reducible* in the sense that, for any fixed $n \in \mathbb{N}$, it implies the two-variable Jensen Inequality, namely the midpoint convexity of the function. We will see that this property strongly depends on the behavior of the arithmetic mean.

Based on I. and II., with regard its subject, my dissertation can be divided in two main parts. The first part consists of Chapter 1. and Chapter 2., where we are going to introduce a *„convexity parameter set"*, which is very similar to $\mathcal{C}_f$ but concern a more general concept of convexity. Then we will deduce algebraic and topological properties of it pointing to the similarities and differences with the standard case.

In Chapter 3., which gives the second part of the dissertation, we are going to deal with reducibility of general mean values and generalized convexity properties. Here we also introduce a possible generalization of *standard deviation means*, more precisely, we extend them from the subintervals of $\mathbb{R}$ to convex subsets of any topological vector spaces of Hausdorff type. It turns out that our results about reducibility can be naturally applied in this general class. Finally, as a consequence, we also formulate and prove an abstract Hölder–Minkowski type inequality.

Introduction to the First Part

In this short introduction I would like to expound the historical background of the statement about the structure of $\mathcal{C}_f$ leaded up in I. I also recall some analogs of this result related to different generalizations of standard convexity. This chapter of my dissertation is based on the paper [**11**].

By its definition, the set $\mathcal{C}_f$ is never empty, namely $\{0, 1\}$ is trivially contained in $\mathcal{C}_f$. Furthermore, our function f is convex if and only if $\mathcal{C}_f = [0, 1]$. As we mentioned, Jensen proved in [**10**] that the inclusion $\mathbb{Q} \cap [0, 1] \subseteq \mathcal{C}_f$ holds provided that f is midpoint convex. This result immediately has a crucial consequence concerning $\mathcal{C}_f$, namely, if $\frac{1}{2}$ is contained in $\mathcal{C}_f$ then it forms automatically a dense subset of the closed unit interval. After this the exact algebraic structure of $\mathcal{C}_f$ remained still hidden.

The next step in a very similar direction was due to Jürg Rätz in the year 1976. In [**34**] the author proved that the homogeneity set of an additive function automatically possesses some nice algebraic structure, more precisely, if A and B are modules over some non-trivial ring R then the set of parameters

$$\mathcal{H}_f := \{t \in R \mid f(tx) = tf(x) \text{ for all } x \in A\}$$

forms a subring of R, provided that $f : A \to B$ is additive, that is, it satisfies the Cauchy Functional Equation. Motivated by this, $\mathcal{H}_f$ is called the *homogeneity ring* of the function in question. In the same paper it also turned out that, for any given ring R and for any R-modules A and B, one can construct a function $f : A \to B$ such that $\mathcal{H}_f$ and R coincide with each other, assuming that A has a basis over R. Particular cases of this phenomenon are also discussed in [**14**].

In view of Jensen's previous result and the definition of $\mathcal{C}_f$, we have the chain of inclusions

$$\mathbb{Q} \cap [0, 1] \subseteq \mathcal{C}_f \subseteq [0, 1] = \mathbb{R} \cap [0, 1],$$

which suggest the investigation of $\mathcal{C}_f$ from the view point of Jürg Rätz, namely if $\mathcal{C}_f$ can be written as an intersection of some proper subfield of $\mathbb{R}$ and the closed unit interval or not.

In 1980, the question, related to Jensen convex functions, was answered affirmatively by Roman Ger. More precisely, Ger proved that, for any Jensen convex function $f : D \to \mathbb{R}$ defined on a nonempty convex subset D of a real linear space X, the set $\mathcal{C}_f$ can be written as $F \cap [0,1]$, where F is a suitable subfield of $\mathbb{R}$. Similarly to the result of Rätz, the reverse statement turned to be also true, that is, having any subfield $F \subseteq \mathbb{R}$ and a nonempty convex subset $D \subseteq X$, one can construct a function $f : D \to \mathbb{R}$, such that $\mathcal{C}_f = F \cap [0,1]$. Obviously, such a function is necessarily Jensen convex.

However, this result of Ger was not so complete as in the case of additive functions, because an additional technical assumption related to $\mathcal{C}_f$ was needed. The final, satisfactory answer was given by Norbert Kuhn in the year 1984 (cf. [**15**]).

THEOREM 0.2. (Kuhn, 1984) *For any function $f : I \to \mathbb{R}$, the convexity parameter set $\mathcal{C}_f$ is either $\{0,1\}$ or it can be written as $F \cap [0,1]$, where F is the subfield of $\mathbb{R}$ generated by $\mathcal{C}_f$.*

This, of course, implies Jensen's result about the relationship of $\mathbb{Q} \cap [0,1]$ and $\mathcal{C}_f$ in the case, when $\mathcal{C}_f$ contains $\frac{1}{2}$. The proof of Kuhn is transparent but it is quite long. In 1987, Zoltán Daróczy and Zsolt Páles, using an elegant one-row-calculation, showed that the same conclusion can be obtained supposing that $\mathcal{C}_f$ has at least three elements.

Obviously, the above questions can be formulated in terms of more general concept of convexity. Keeping the previous notations, we say that a function $f : D \to \mathbb{R}$ is *t-Wright convex on D* for some given $t \in [0,1]$ if, for all $x, y \in D$, we have the inequality

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y).$$

It is easy to see that any t-convex function, and hence any convex function, is t-Wright convex. Indeed, taking the inequality (1), interchanging the points x and y and finally adding up the two inequalities so obtained, we get the above inequality of t-Wright convexity. The function f is called *Wright-convex* if it is t-Wright convex for all $t \in [0,1]$. Now, for a given $f : D \to \mathbb{R}$, let us define the set

$$\mathcal{W}_f := \{t \in [0,1] \mid f \text{ is } t\text{-Wright convex on } D\}.$$

The above set is never empty, because $\{0,1\} \subseteq \mathcal{W}_f$. In addition, Gyula Maksa, Kazimierz Nikodem and Zsolt Páles proved in [**21**] that $\mathcal{W}_f$ is always symmetric with respect to $\frac{1}{2}$, it is dense in $[0,1]$ if $\mathcal{W}_f \setminus \{0,1\}$ is nonempty, it is closed under the binary operation $(s,t) \mapsto st+(1-s)(1-t)$, and $\frac{1}{2}$ is contained in $\mathcal{W}_f$ provided that it has at least one rational element different from 0 and 1. On the other hand, if $t \in]0,1[$ is transcendental or it is algebraic such that one of its algebraic conjugates does not belong to the disc $\{z \in \mathbb{C} \mid |z - \frac{1}{2}| < \frac{1}{2}\}$, then there exists a t-Wright convex function $f : D \to \mathbb{R}$ such that $\frac{1}{2} \notin \mathcal{W}_f$.

As a generalization of t-Wright-convexity, we can consider the class of t-Schur convex functions. We say that a function $F : D \times D \to \mathbb{R}$ is *t-Schur convex* for some given $t \in [0,1]$ if, for all $x, y \in D$, the inequality

$$F(tx + (1-t)y, (1-t)x + ty) \leq F(x,y)$$

is valid. Setting $F(u,v) := f(u) + f(v)$, we get back the notion of t-Wright convexity. In the paper [**3**], Pál Burai and Judit Makó, showed that the set

$$\mathcal{S}_F := \{t \in [0,1] \mid F \text{ is } t\text{-Schur convex on } D\}$$

is symmetric with respect to $\frac{1}{2}$, it is closed under the binary operation $(s,t) \mapsto st + (1-s)(1-t)$, and if X is a normed linear space, F is lower semi-continuous with nonempty $\mathcal{S}_F \setminus \{0,1\}$, then $\frac{1}{2}$ is contained in $\mathcal{S}_F$. The density of $\mathcal{S}_F$ in $[0,1]$ forms still an open problem.

In the first chapter of the dissertation, we are going to consider and investigate the previous questions related to the concept of *lower and upper convexity of extended real valued functions*. After defining the certain convexity families, we will earn some algebraic and topological properties of them. Then we will apply our theorems for *asymmetrically t-convex functions* and give also an example for a function, whose parameter set fails to satisfy Kuhn's theorem in this extended sense. More precisely, we are going to construct an asymmetrically upper convex function, where the parameter set is not closed under the addition of its elements, but it forms a dense subgroup of $[0,1]$ with respect to the usual multiplication of real numbers.

CHAPTER 1

Constructing new means from given ones

Throughout the dissertation, let I stand for a nonempty subinterval of $\mathbb{R}$ having at least two distinct elements. In this chapter we are going to present a general method which, having a given finite sequence of two-variable means $M_1, \dots, M_n$ defined on a certain subset of $I \times I$, is suitable to derive further means of two variables defined on the same domain. To avoid the trivial cases, we will always assume that $n \geq 2$. Before we perform the main idea, we clarify the notion of means and partial means under the more general setting.

Let X be a linear space and $H \subseteq X$ be any subset. The smallest convex subset of X, which contains H is called *the convex hull of* H and is denoted by conv(H). The intersection of any family of convex subsets of X is convex again, furthermore X is convex itself, hence the notion of the convex hull of a set is well-defined. It can be easily checked that the convex hull of H, provided it is nonempty, is nothing else but the set of all convex combinations made of its elements. More precisely, we have $u \in \text{conv}(H)$ if and only if there exist $m \in \mathbb{N}$, $t_1, \dots, t_m \geq 0$ with $t_1 + \cdots + t_m = 1$ and $x_1, \dots, x_m \in H$, such that $u = t_1x_1 + \cdots + t_mx_m$. If each element of $\{t_1, \dots, t_m\}$ is positive, then we say that u belongs to *the relative interior of* conv(H).

Let $n \geq 2$ be a fixed integer number and $S \subseteq X$ be a nonempty subset. A function $M : S^n \to X$ will be called an *n-variable mean on S* if, for all $(x_1, \dots, x_n) \in S^n$, we have the inclusion

$$M(x_1, \dots, x_n) \in \text{conv}(x_1, \dots, x_n), \tag{1}$$

The mean M is said to be *strict* if $M(x_1, \dots, x_n)$ belongs to the relative interior of conv$(x_1, \dots, x_n)$, whenever the set $\{x_1, \dots, x_n\} \subseteq S$ has at least two elements. We will say that M is *symmetric* if, for any bijection $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$, we have $M(x_{\pi(1)}, \dots, x_{\pi(n)}) = M(x_1, \dots, x_n)$.

We note that, by its definition, an n-variable mean M is a *reflexive function*, that is, $M(x_1, \dots, x_n) = x$ if $\{x_1, \dots, x_n\}$ is the singleton $\{x\}$.

Now, as an extension of the notion of means, we define partial means. Let $P \subseteq S^n$ be any nonempty subset and $M : S^n \to X$ be a mean. We say that *$M : S^n \to X$ is a partial mean on S with respect to P* or, shortly, *$M : P \to X$ is a partial mean on S* if the restriction $M|_P$ is a mean on P.

Now we present some classes of n-variable means, which will be crucial to formulate our further results.

1.1. The class of Daróczy means

The means appearing in the title were introduced by Zoltán Daróczy in 1971 in the paper [**5**]. This class is rather wide and it contains the well-known, usual means. Furthermore it has many interesting properties which were investigated by several authors, cf. Aczél and Daróczy [**1**], Daróczy [**5, 4**], Daróczy–Losonczi [**6**], Daróczy–Páles [**7, 8**], Losonczi [**18, 17, 19, 20**], and Páles [**25, 24, 26, 27, 28, 29, 30, 32, 31**]. To interpret them, we need the notion of deviation functions.

A two-place function $E : I \times I \to \mathbb{R}$ is called *a deviation function on* I or, simply, *a deviation on* I, if

(D1) E vanishes on the diagonal of $I \times I$, that is, $E(u, u) = 0$ for all $u \in I$ and,

(D2) for any fixed element $u \in I$, the function $v \mapsto E(u, v)$ is continuous and strictly decreasing on the interval I.

The class of all deviation functions defined on I will be denoted by $\mathbf{D}(I)$.

Note that the properties (D1) and (D2) together imply that a deviation function $E \in \mathbf{E}(I)$ always possesses the so-called *sign-property*

$$\operatorname{sgn} E(u, v) = \operatorname{sgn}(u - v), \qquad (u, v \in I). \tag{2}$$

Indeed, if $u = v$, then due to (D1), the statement in (2) is trivial. Hence we may assume that say $u < v$. Then $\operatorname{sgn}(u - v) = -1$ and, by the strict decreasingness of E in the second variable, we have $0 = E(u, u) > E(u, v)$, that is, $\operatorname{sgn} E(u, v)$ equals to -1 too. The case $v < u$ can be treated similarly.

For $(E_1, \dots, E_n) \in \mathbf{D}(I)^n$ and $(x_1, \dots, x_n) \in I^n$, the unique value $y \in I$, satisfying the equation

$$E_1(x_1, y) + \cdots + E_n(x_n, y) = 0, \tag{3}$$

is called *the* $(E_1, \dots, E_n)$*-deviation mean* or *the* $(E_1, \dots, E_n)$*-Daróczy mean* of the elements $x_1, \dots, x_n$, and is denoted by $\mathcal{D}^{(E_1,\dots,E_n)}(x_1, \dots, x_n)$.

Now we shortly show that the notion of the $(E_1, \dots, E_n)$-Daróczy mean is well-defined. Let $x := (x_1, \dots, x_n) \in I^n$ be arbitrarily fixed, and denote the minimum and the maximum of the set $\{x_1, \dots, x_n\}$ by α and β, respectively. Let us further define the function

$$\Sigma_{E,x} : I \to \mathbb{R}, \qquad \Sigma_{E,x}(u) := E_1(x_1, u) + \cdots + E_n(x_n, u). \tag{4}$$

The sign-property (2) of the deviation functions $E_1, \dots, E_n$ implies that $\Sigma_{E,x}(\beta) \le 0 \le \Sigma_{E,x}(\alpha)$. Thus, due to the continuity in the second variable of the deviation functions, we get that the equation (3) has at least one solution in the convex hull of the set $\{x_1, \dots, x_n\}$. Finally, the strict monotonicity-property of the deviations provides that this solution has to be unique.

Now we recall the most classical subclasses of the class of deviation means.

1.1.1. The class of Matkowski means. This class of means was introduced by Janusz Matkowski in the paper [**22**] in 2010. A function $M : I^n \to \mathbb{R}$ is said to be *an n-variable generalized quasi-arithmetic mean* or, shortly, *an n-variable Matkowski mean* if there exist continuous functions $f_1, \dots, f_n : I \to \mathbb{R}$, which are strictly monotone in the same sense and for which

$$M(x_1, \dots, x_n) = (f_1 + \dots + f_n)^{-1}(f_1(x_1) + \dots + f_n(x_n))$$

holds for all $x_1, \dots, x_n \in I$. In this case, the n-tuple $(f_1, \dots, f_n)$ is called *the generator of the Matkowski mean*, furthermore the Matkowski mean of the given points $x_1, \dots, x_n \in I$ is denoted by $\mathcal{M}^{(f_1,\dots,f_n)}(x_1, \dots, x_n)$.

If, for $i \in \{1, \dots, n\}$, we define $E_i : I \times I \to \mathbb{R}$ by the formula

$$E_i(u, v) := f_i(u) - f_i(v),$$

it can be proved that E_i is a deviation function for all $i \in \{1, \dots, n\}$, and that the corresponding deviation mean $\mathcal{D}^{(E_1,\dots,E_n)}$ is nothing else but the Matkowski mean $\mathcal{M}^{(f_1,\dots,f_n)}$.

Specializing the generator functions, this notion gives back the usual, well-known classes of means. If $f_1 = \dots = f_n =: f$ on I then we get the concept of *quasi-arithmetic mean*s, more precisely, for all $x_1, \dots, x_n \in I$, we have

$$\mathcal{M}^{(f,\dots,f)}(x_1, \dots, x_n) = f^{-1}\left(\frac{f(x_1) + \dots + f(x_n)}{n}\right) =: \mathcal{Q}^f(x_1, \dots, x_n).$$

If $I \subseteq [0, +\infty[$ and, for some fixed $p \in \mathbb{R}$, we have $f := \mathrm{id}^p$ on I, then the quasi-arithmetic mean generated by f is called *a Hölder mean* or *a power mean* and is denoted by $\mathcal{H}_p$. In detail, the definition is

$$\mathcal{H}_p(x_1, \dots, x_n) := \begin{cases} \left(\dfrac{x_1^p + \dots + x_n^p}{n}\right)^{\frac{1}{p}} & \text{if } p \neq 0, \\ \sqrt[n]{x_1 \dots x_n} & \text{if } p = 0, \end{cases}$$

where, setting $p = -1$ or $p = 1$, we obtain the notion of *harmonic mean* or *arithmetic mean*, respectively.

The Matkowski means, by the definition, are strict and continuous. Roughly speaking, a Matkowski mean is symmetric or homogeneous if and only if it is a quasi-arithmetic mean or a Hölder mean, respectively. The equality, comparison and invariance problem were also investigated in the class of

Matkowski means. The first two are completely, the last only partially resolved.

1.1.2. The class of Bajraktarević means. The other wide subclass of Daróczy means is the class of Bajraktarević means, which was originally introduced by the Bosnian mathematician Mahmut Bajraktarević in the paper [**2**] from 1958.

A function $M : I^n \to \mathbb{R}$ is called *an n-variable Bajraktarević mean* if there exist a continuous, strictly increasing function $f : I \to \mathbb{R}$ and an n-tuple of weight functions $\omega : (\omega_1, \dots, \omega_n) : I \to \mathbb{R}_+^n$ such that

$$M(x_1, \dots, x_n) = f^{-1}\left(\frac{\omega_1(x_1)f(x_1) + \dots + \omega_n(x_n)f(x_n)}{\omega_1(x_1) + \dots + \omega_n(x_n)}\right)$$

holds for all $x_1, \dots, x_n \in I$. The pair (f, ω) is called the generator of the Bajraktarević mean and, in the above case, the function M is denoted by $\mathcal{B}^{(f,\omega)}$.

To see that such a mean is indeed a deviation mean, for $i \in \{1, \dots, n\}$, let us define the function $E_i : I \times I \to \mathbb{R}$ by the formula

$$E_i(u, v) := \omega_i(u)(f(u) - f(v)).$$

Obviously, for all $i \in \{1, \dots, n\}$, the function E_i enjoys the properties listed as (D1) and (D2). Using the above definition of $E_1, \dots, E_n$, it is also easy to see that $\mathcal{D}^{(E_1,\dots,E_n)} = \mathcal{B}^{(f,\omega)}$ on I^n.

Similarly to the previous part, well-known classes of means can be obtained if the generator functions are specialized. If $\omega_i = 1$ on I for all $i \in \{1, \dots, n\}$, then we get back the class of quasi-arithmetic means. Formally, if $I \subseteq [0, +\infty[$, $p, q \in \mathbb{R}$ are distinct numbers, furthermore $f = \mathrm{id}^{p-q}$ and $\omega_i := \mathrm{id}^q$ for all $i \in \{1, \dots, n\}$, we get the notion of Gini means, which are exactly the homogeneous Bajraktarević means. The precise definition is given by

$$\mathcal{G}^{(p,q)}(x_1, \dots, x_n) := \begin{cases} \left(\dfrac{x_1^p + \dots + x_n^p}{x_1^q + \dots + x_n^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp\left(\dfrac{x_1^p \ln(x_1) + \dots + x_n^p \ln(x_n)}{x_1^p + \dots + x_n^p}\right) & \text{if } p = q, \end{cases}$$

where $x_1, \dots, x_n \in I \subseteq [0, +\infty[$. Putting $q = 0$, it can also easily seen that Hölder means are also Bajraktarević means. If the generator f is the identity function, then we get the notion of *functionally weighted arithmetic mean*, which will be denoted by $\mathcal{A}^\omega := \mathcal{B}^{(\mathrm{id},\omega)}$.

The equality, comparison, and invariance problems were also considered by several authors.

1.2. Tools from linear algebra

For $n \in \mathbb{N}$ and for given vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in $\mathbb{R}^n_+$, define the two-diagonal matrix $A(u, v)$ by

$$A(u,v) := \begin{pmatrix} 0 & u_1 & \dots & 0 & 0 \\ v_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & u_n \\ 0 & 0 & \dots & v_n & 0 \end{pmatrix}. \tag{5}$$

The next theorem is about the properties of the eigenvalues of $A(u, v)$.

THEOREM 1.1. *For all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{R}^n_+$, any eigenvalue of $A(u, v)$ is a real number. Furthermore, the eigenvalues of $A(u, v)$ are smaller than 1 if and only if $w_1, \dots, w_n > 0$, where*

$$w_k := w_{k-1} - u_k v_k w_{k-2}, \quad (k \in \{1, \dots, n\}), \tag{6}$$

provided that $w_{-1} := w_0 := 1$.

PROOF. In the sequel, for $k \in \mathbb{N}$, denote the unit matrix of the matrix algebra $\mathbb{R}^{k\times k}$ by I_k and, for a square matrix $S \in \mathbb{R}^{k\times k}$, let $P_S : \mathbb{R} \to \mathbb{R}$ stand for the characteristic polynomial of S defined by $P_S := \det(\mathrm{id} \cdot I_k - S)$.

Let $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ be any elements of $\mathbb{R}^n_+$, let $A_0(u, v) := 0$, and, for $k \in \{1, \dots, n\}$, define

$$A_k(u,v) := \begin{pmatrix} 0 & u_1 & \dots & 0 & 0 \\ v_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & u_k \\ 0 & 0 & \dots & v_k & 0 \end{pmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}_+. \tag{7}$$

In other words, $A_k(u, v)$ is the leading principal minor of order k of the matrix $A(u, v)$, where $k \in \{1, \dots, n\}$. Then, obviously, $A_n(u, v) = A(u, v)$. Observe that $P_{A_0(u,v)} = \mathrm{id}$ and $P_{A_1(u,v)} = \mathrm{id}^2 - u_1 v_1$. Expanding the determinant in the definition of the characteristic polynomial by its last row, we can easily deduce a recursive formula. More precisely, for $k \in \{1, \dots, n-1\}$, we have

$$P_{A_{k+1}(u,v)} = \mathrm{id} \cdot P_{A_k(u,v)} - u_{k+1} v_{k+1} P_{A_{k-1}(u,v)}. \tag{8}$$

Now, by induction on k, we are going to prove that, for all $k \in \{1, \dots, n\}$, the characteristic polynomials of the matrices $A_k(u, v)$ and $A_k(\sqrt{uv}, \sqrt{uv})$ are identical, where the notation $\sqrt{uv}$ stands for the vector $(\sqrt{u_1 v_1}, \dots, \sqrt{u_n v_n})$.

The statement is trivial for $k = 0$. If $k = 1$, then we have

$$P_{A_1(u,v)} = \mathrm{id}^2 - u_1 v_1 = \mathrm{id}^2 - \sqrt{u_1 v_1}\sqrt{u_1 v_1} = P_{A_1(\sqrt{uv},\sqrt{uv})}.$$

Assume that we have established the identity $P_{A_(u,v)} = P_{A_j(\sqrt{uv},\sqrt{uv})}$ for $j \leq k$. Using the recursive formula in (8) two times and then our inductive hypothesis, for $k \in \{1, \dots, n-1\}$, we obtain that

$$\begin{aligned} P_{A_{k+1}(u,v)} &= \mathrm{id}\, P_{A_k(\sqrt{uv},\sqrt{uv})} - \sqrt{u_{k+1}v_{k+1}}\sqrt{u_{k+1}v_{k+1}} P_{A_{k-1}(\sqrt{uv},\sqrt{uv})} \\ &= P_{A_{k+1}(\sqrt{uv},\sqrt{uv})}. \end{aligned}$$

This completes the proof of $P_{A_k(u,v)} = P_{A_k(\sqrt{uv},\sqrt{uv})}$ for all $k \in \{1, \dots, n\}$.

The matrix $A_n(\sqrt{uv}, \sqrt{uv})$ is symmetric with real entries, therefore its characteristic polynomial has only real roots, whence it follows that the eigenvalues of $A_n(u,v) = A(u,v)$ are also real. The eigenvalues of $A_n(\sqrt{uv}, \sqrt{uv})$ are smaller than 1 if and only if the eigenvalues of the symmetric matrix $I_{n+1} - A_n(\sqrt{uv}, \sqrt{uv})$ are positive, which is equivalent to the positive definiteness of $I_{n+1} - A_n(\sqrt{uv}, \sqrt{uv})$. In view of the *Sylvester's Criterion*, this holds if and only if all the leading principal minor determinants of $I_{n+1} - A_n(\sqrt{uv}, \sqrt{uv})$ are positive, that is, if

$$P_{A_k(u,v)}(1) = P_{A_k(\sqrt{uv},\sqrt{uv})}(1) > 0 \qquad (k \in \{0, \dots, n\}). \tag{9}$$

By the recursive formula (8) applied for $\lambda = 1$, it results that $P_{A_k(\sqrt{uv},\sqrt{uv})}(1) = w_k$ for all $k \in \{0, \dots, n\}$, therefore, (9) is equivalent to the inequalities $w_1, \dots, w_n > 0$. □

In the next result we give a sufficient condition in order that the inequalities $w_1, \dots, w_n > 0$ hold.

LEMMA 1.2. *Let $n \in \mathbb{N}$, $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ be vectors in $\mathbb{R}^n_+$ and assume that*

$$v_1 \leq 1, \quad \max\{u_1 + v_2, \dots, u_{n-1} + v_n\} \leq 1, \quad \textit{and} \quad u_n < 1. \tag{10}$$

Then the system of inequalities $w_1, \dots, w_n > 0$ holds, where $w_1, \dots, w_n$ are defined as in (6) *of Theorem 1.1.*

PROOF. Observe that the positivity of $v_2, \dots, v_n$ and (10) yield that $u_1, \dots, u_n < 1$. To show that w_k is positive for all $k \in \{1, \dots, n\}$, we shall prove that

$$w_k > 0 \qquad \text{and} \qquad (1 - u_k) w_{k-1} \leq w_k < w_{k-1} \tag{11}$$

hold for all $k \in \{1, \dots, n-1\}$. For $k = 1$, the second chain of inequalities is equivalent to $1 - u_1 \leq 1 - u_1 v_1 \leq 1$, which easily follows from $0 < v_1 \leq 1$ and $0 < u_1$. Hence $w_1 > 0$ also holds.

Assume that we have proved (11) for some $k \in \{1, \dots, n-1\}$. Then, using the recursion (6) and using the right hand side inequality in (11), we get that

$$\begin{aligned} w_{k+1} = w_k - u_{k+1}v_{k+1}w_{k-1} &< w_k - u_{k+1}v_{k+1}w_k \\ &= w_k(1 - u_{k+1}v_{k+1}) < w_k. \end{aligned}$$

On the other hand, using the upper estimate for w_{k-1} obtained from (11), it follows that

$$\begin{aligned} w_{k+1} &= w_k - u_{k+1}v_{k+1}w_{k-1} \\ &\geq w_k - u_{k+1}v_{k+1}\frac{w_k}{1-u_k} = w_k\frac{1-u_k-u_{k+1}v_{k+1}}{1-u_k} \\ &\geq w_k\frac{1-u_k-u_{k+1}(1-u_k)}{1-u_k} = w_k(1-u_{k+1}) > 0, \end{aligned}$$

which completes the proof of (11). □

LEMMA 1.3. *For all $n \in \mathbb{N}$ and for all vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in $\mathbb{R}^n_+$, there exists an eigenvector of $A(u, v)$ with positive components whose eigenvalue is also positive.*

PROOF. We follow the argument of the standard proof of the Perron–Frobenius Theorem. Consider the set

$$S_{n+1} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0, \dots, x_n \geq 0 \text{ and } x_0 + \cdots + x_n = 1\}.$$

Then S_{n+1} is a compact convex subset of $\mathbb{R}^{n+1}$. Let $u, v \in \mathbb{R}^n$ be fixed vectors with positive components and let $A_0, \dots, A_n$ be the row vectors of the matrix $A(u, v)$. Observe that

$$A(u,v)x = \big(\langle A_0, x\rangle, \dots, \langle A_n, x\rangle\big), \qquad (x \in \mathbb{R}^{n+1}), \tag{12}$$

furthermore the sum $\langle A_0, x\rangle + \cdots + \langle A_n, x\rangle$ does not vanish on S_{n+1}. Indeed, if for some $x \in S_{n+1}$ we have $\langle A_0, x\rangle + \cdots + \langle A_n, x\rangle = 0$, then, by the non-negativity of the terms on the left hand side of this equation, it follows that $\langle A_i, x\rangle = 0$ for all $i \in \{0, \dots, n\}$. Using the positivity of the parameters u_i and v_i, these equalities imply $x = 0$, which contradicts $x \in S_{n+1}$.

Consider now the mapping $F : S_{n+1} \to \mathbb{R}^{n+1}$ defined by

$$F(x) := \frac{A(u,v)x}{\langle A_0, x\rangle + \cdots + \langle A_n, x\rangle}, \qquad (x \in S_{n+1}).$$

Then F is continuous on S_{n+1} and, by (12), we have $F(S_{n+1}) \subseteq S_{n+1}$. Hence, in view of the Brouwer Fixed Point Theorem, there exists a fixed point $p \in S_{n+1}$ of the function F. Then we have

$$A(u,v)p = (\langle A_0, p\rangle + \cdots + \langle A_n, p\rangle)F(p) = (\langle A_0, p\rangle + \cdots + \langle A_n, p\rangle)p,$$

which shows that p is an eigenvector of $A(u,v)$ with the eigenvalue $\lambda := \langle A_0, p\rangle + \cdots + \langle A_n, p\rangle > 0$. Therefore, by $A(u,v)p = \lambda p$, the system of equations

$$\begin{array}{rcrcl} u_1 p_1 & & & = & \lambda p_0, \\ u_{i+1}p_{i+1} & + & v_i p_{i-1} & = & \lambda p_i, \qquad (i \in \{1,\dots,n-1\}), \\ & & v_n p_{n-1} & = & \lambda p_n \end{array} \tag{13}$$

hold, where $p = (p_0,\dots,p_n)$. If $p_i = 0$ for some $i \in \{0,\dots,n\}$, then the non-negativity of the terms on the left hand side of the ith equation yields that $p_j = 0$ for $j \in \{i-1, i+1\} \cap \{1,\dots,n-1\}$. This results that p has to be zero, which contradicts $p \in S_{n+1}$. □

1.3. Tools from fixed point theory

For our purposes, we recall some notions and results related to fixed point theorems. Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ will be called a *semimetric* if

(1) it is *positive definite*, that is, for all $x, y \in X$, we have $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$, and
(2) it is *symmetric*, that is, for all $x, y \in X$, the identity $d(x,y) = d(y,x)$ holds.

If d is a semimetric, then the pair (X,d) is called *semimetric space*. If (X, d_X) and (Y, d_Y) are semimetric spaces then a function $f : X \to Y$ is said to have the *Lipschitz property* if there exists $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x,y), \qquad (x, y \in X). \tag{14}$$

The *Lipschitz modulus* of f is defined by

$$\mathrm{Lip}(f) := \sup\left\{ \tfrac{d_Y(f(x),f(y))}{d_X(x,y)} \,\middle|\, x, y \in X \text{ and } x \neq y \right\}.$$

Obviously, f possesses the Lipschitz property if and only if $\mathrm{Lip}(f)$ is finite. The function f will be called a *contraction* if $\mathrm{Lip}(f) < 1$.

It is an immediate consequence of these definitions, that, for a subset $D \subseteq X$ and for a contraction $f : D \to X$ with respect to the semimetric d_X, the map f can have at most one fixed point in D. Indeed, if x and y are both fixed points of f in D, then

$$d_X(x,y) = d_X(f(x), f(y)) \leq \mathrm{Lip}(f) \cdot d_X(x,y),$$

which implies $d_X(x,y) \leq 0$. By property (1) of semimetrics, we get $x = y$.

The following lemma is useful to compute the Lipschitz modulus of differentiable real valued functions.

LEMMA 1.4. *Let $f, g : I \to \mathbb{R}$ be differentiable functions such that $0 \notin g'(I)$. Then, for the Lipschitz modulus of the function $f \circ g^{-1} : g(I) \to \mathbb{R}$, we have*

$$\operatorname{Lip}\left(f \circ g^{-1}\right) = \sup_{t \in I} \left|\frac{f'(t)}{g'(t)}\right|.$$

PROOF. Due to the assumptions, $g : I \to \mathbb{R}$ is continuous and strictly monotone. Therefore $g^{-1} : g(I) \to \mathbb{R}$ is well-defined. Thus, applying the Cauchy Mean Value Theorem, we have that

$$\begin{aligned}\operatorname{Lip}\left(f \circ g^{-1}\right) &= \sup_{\substack{x,y \in g(I)\\ x \neq y}} \frac{|f \circ g^{-1}(x) - f \circ g^{-1}(y)|}{|x-y|} \\ &= \sup_{\substack{u,v \in I\\ u \neq v}} \frac{|f(u) - f(v)|}{|g(u) - g(v)|} = \sup_{t \in I} \left|\frac{f'(t)}{g'(t)}\right|.\end{aligned}$$

□

In what follows, we recall first the following generalization of the Tychonov Fixed Point Theorem established by Halpern and Bergman [**9**]. For the formulation of this result, we define the notion of the *inward set of a convex subset* K of a locally convex space X by

$$\operatorname{Inw}_K(x) := x + \mathbb{R}_+(K - x), \qquad (x \in K).$$

Observe that the inclusion $K \subseteq \operatorname{Inw}_K(x)$ is valid for all $x \in K$. On the other hand, for an interior point $x \in K$, we have $\operatorname{Inw}_K(x) = X$, therefore $y \in \operatorname{Inw}_K(x)$ is always trivial, provided that $x \in K \setminus \partial K$, where ∂K stands for the set of boundary points of K.

We say that a function $f : K \to X$ is *weakly inward* if $f(x) \in \operatorname{cl} \circ \operatorname{Inw}_K(x)$ holds for all $x \in \partial K$.

THEOREM 1.5. *Let X be a locally convex Hausdorff space, $K \subseteq X$ be a compact convex subset and $f : K \to X$ be a continuous weakly inward map. Then the set of the fixed points of f forms a nonempty compact subset of K.*

If $f(K) \subseteq K$, then $f(x) \in \operatorname{cl} \circ \operatorname{Inw}_K(x)$ trivially holds for all $x \in \partial K$, therefore, in this case, the above result reduces to the Tychonov Fixed Point Theorem.

The fixed point theorem stated below, that we are going to use for the existence proofs in our main results, is consequence of the Halpern–Bergman Fixed Point Theorem. It establishes the existence of the fixed point for continuous maps defined over a convex polyhedron.

THEOREM 1.6. *Let* $c_1, \dots, c_m \in \mathbb{R}^n$, $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ *and assume that the polyhedron*

$$K := \big\{x \in \mathbb{R}^n \mid \langle c_k, x\rangle \le \gamma_k,\ k \in \{1, \dots, m\}\big\} \tag{15}$$

is bounded. Let further $f : K \to \mathbb{R}^n$ *be a continuous function such that*

$$\langle c_k, f(x)\rangle \le \gamma_k \tag{16}$$

for all $x \in K$ *and for all* $k \in \{1, \dots, m\}$ *with the property* $\langle c_k, x\rangle = \gamma_k$. *Then the set of the fixed points of* f *is a nonempty compact subset of* K.

PROOF. By our assumption, K is a compact convex set. Therefore, it is sufficient to show that the set $\mathrm{Inw}_K(x)$ and

$$\big\{u \in \mathbb{R}^n \mid \langle c_k, u\rangle \le \gamma_k \text{ for all } k \in \{1, \dots, m\} \text{ such that } \langle c_k, x\rangle = \gamma_k\big\} \tag{17}$$

coincide with each other for all $x \in K$. Having this, by condition (16), it follows that $f(x) \in \mathrm{Inw}_K(x)$ for all $x \in K$, whence the Halpern–Bergman Fixed Point Theorem yields the existence the fixed point of f. For the brevity, denote the set in (17) by H.

Let $x \in K$ be any point. If $u \in \mathrm{Inw}_K(x)$, then there exists $y \in K$ and $t \ge 0$ such that $u = (1-t)x+ty$. Then, for $k \in \{1, \dots, m\}$ with $\langle c_k, x\rangle = \gamma_k$, we have

$$\begin{aligned}\langle c_k, u\rangle = \langle c_k, (1-t)x + ty\rangle &= (1-t)\langle c_k, x\rangle + t\langle c_k, y\rangle \\ &= (1-t)\gamma_k + t\langle c_k, y\rangle \le (1-t)\gamma_k + t\gamma_k = \gamma_k,\end{aligned}$$

which proves the inclusion $\mathrm{Inw}_K(x) \subseteq H$.

For the inclusion $H \subseteq \mathrm{Inw}_K(x)$, pick up $u \in \mathbb{R}^n$ such that $\langle c_k, u\rangle \le \gamma_k$ for all $k \in \{1, \dots, m\}$ with $\langle c_k, x\rangle = \gamma_k$. Choose further $t > 0$ such that $t \ge \frac{\langle c_k, u-x\rangle}{\gamma_k - \langle c_k, x\rangle}$ for all $k \in \{1, \dots, m\}$ with $\langle c_k, x\rangle < \gamma_k$ and define the point $y \in \mathbb{R}^n$ by the formula $y := \frac{1}{t}(u-x) + x$. Then, distinguishing the cases whether $\langle c_k, x\rangle = \gamma_k$ or not, for every $k \in \{1, \dots, m\}$, we obtain that

$$\langle c_k, u - x\rangle \le t(\gamma_k - \langle c_k, x\rangle).$$

Therefore, for all $k \in \{1, \dots, m\}$, we have

$$\langle c_k, y\rangle = \langle c_k, \tfrac{1}{t}(u-x) + x\rangle \le (\gamma_k - \langle c_k, x\rangle) + \langle c_k, x\rangle = \gamma_k,$$

which proves that $y \in K$. On the other hand, by the definition of y, we have that $u = (1-t)x + ty$. Consequently, $u \in \mathrm{Inw}_K(x)$, which finishes the proof the theorem. $\square$

1.4. Descendants of means

In this section we are going to investigate two-variable means but only on a restricted domain. To formulate our results, let us introduce some notations.

For a given nonempty subset $S \subseteq \mathbb{R}$ and for $n \in \mathbb{N}$, denote the set of increasingly and strictly increasingly ordered n-tuples of S by $S^n_\leq$ and $S^n_<$, respectively, that is, we have

$$S^n_\leq := \{(t_1, \dots, t_n) \in S^n \mid t_1 \leq \dots \leq t_n\}$$

and

$$S^n_< := \{(t_1, \dots, t_n) \in S^n \mid t_1 < \dots < t_n\}.$$

We note that, if we have a two-variable mean $M : I \times I \to \mathbb{R}$ on I and we want to state something about the restriction $M|_{I^2_\leq}$, then we will simply say that $M : I^2_\leq \to \mathbb{R}$ is a mean on I. This means that, in our case, the values of the original mean on the complementary set $S^2_> := S^2 \setminus S^2_\leq$ are irrelevant.

Let $n \geq 2$ and $(M_1, \dots, M_n) : I^2_\leq \to \mathbb{R}^n$ be a given n-tuple of two-variable means on I. Then, we are going to deal with *the existence* and *the uniqueness* of two-variable means $N_1, \dots, N_n : I^2_\leq \to \mathbb{R}$ satisfying the system of functional equations

$$\begin{aligned} &N_1(x, y) = M_1\big(x, N_2(x, y)\big), \\ (18) \qquad &N_i(x, y) = M_i\big(N_{i-1}(x, y), N_{i+1}(x, y)\big), \quad (i \in \{2, \dots, n-1\}), \\ &N_n(x, y) = M_n\big(N_{n-1}(x, y), y\big) \end{aligned}$$

on I. In order to make the problem more manageable, we reformulate it as follows. Observe, that the validity of (18) states that, for any $(x, y) \in I^2_\leq$, the vector $\big(N_1(x, y), \dots, N_n(x, y)\big) \in [x, y]^n_\leq$ is a fixed point of $\varphi_{(x,y)} : [x, y]^n_\leq \to \mathbb{R}^n$, where, for $t = (t_1, \dots, t_n) \in [x, y]^n_\leq$, we have the definition

$$(19) \qquad \varphi_{(x,y)}(t) := \big(M_1(x, t_2), \dots, M_i(t_{i-1}, t_{i+1}), \dots, M_n(t_{n-1}, y)\big).$$

This means that, firstly, we have to investigate the fixed point set

$$(20) \qquad \Phi_{(x,y)} := \big\{\xi \in [x, y]^n_\leq \;\big|\; \varphi_{(x,y)}(\xi) = \xi\big\}.$$

There are many cases, when the fixed points can be elementary calculated. For example, let $M_1 := \dots = M_n := \mathcal{A}_2$ and $x, y \in \mathbb{R}^2_\leq$ be arbitrarily fixed. Then the fixed point equation in question is of the form

$$(t_1, \dots, t_n) := \Big(\frac{x + t_2}{2}, \frac{t_1 + t_3}{2}, \dots, \frac{t_{n-1} + y}{2}\Big),$$

where we assume that $(t_1,\dots,t_n) \in [x,y]^n_{\leq}$. Then one can easily calculate that $\Psi_{(x,y)}$ is the singleton $\{(\xi_1,\dots,\xi_n)\}$, where

$$\xi_i = \frac{(n+1-i)x+iy}{n+1}, \qquad (i \in \{1,\dots,n\}).$$

Due to our auxiliary results of the previous sections, we have the following theorem concerning the behavior of the set $\Phi_{(x,y)}$.

THEOREM 1.7. *Let $n \geq 2$, $(M_1,\dots,M_n) : I^2_{\leq} \to \mathbb{R}^n$ be an n-tuple of two-variable means on I, and $(x,y) \in I^2_{<}$ be any point. Then the following statements hold.*

(1) The fixed point set $\Phi_{(x,y)}$ is a nonempty compact subset of $[x,y]^n_{\leq}$ provided that all the means $M_1,\dots,M_n$ are continuous. In addition, if all the means $M_1,\dots,M_n$ are strict, then $\Phi_{(x,y)}$ is contained in $]x,y[^n_{<}$.

(2) The set $\Phi_{(x,y)}$ is a singleton provided that there exist semimetrics $d_1,\dots,d_n : [x,y]^2 \to \mathbb{R}_+$ such that the system of inequalities

$$\begin{aligned}
d_1(M_1(x,s),M_1(x,v)) &\leq b_1 d_2(s,v),\\
d_i(M_i(t,s),M_i(u,v)) &\leq a_i d_{i-1}(t,u) + b_i d_{i+1}(s,v), \quad (i \in \{2,\dots,n-1\}),\\
d_n(M_n(t,y),M_n(u,y)) &\leq a_n d_{n-1}(t,u)
\end{aligned}$$

hold for all $t,s,u,v \in [x,y]$ with some positive real numbers $a_2,\dots,a_n$ and $b_1,\dots,b_{n-1}$ such that $w_1,\dots,w_{n-1} > 0$, where, provided that $w_{-1} := w_0 := 1$, we have the recursion

$$w_i := w_{i-1} - a_{i+1} b_i w_{i-2}, \qquad (i \in \{1,\dots,n-1\}). \tag{21}$$

PROOF. Let $(x,y) \in I^2_{<}$ be arbitrarily fixed. Then the set $K := [x,y]^n_{\leq}$ is a compact convex set, which can be characterized using $n+1$ inequalities as follows. The vector $(t_1,\dots,t_n)$ belongs to K if and only if

$$-t_1 \leq -x, \quad t_1 - t_2 \leq 0, \quad \dots, \quad t_{n-1} - t_n \leq 0, \quad \text{and} \quad t_n \leq y. \tag{22}$$

Therefore, K is a polyhedron of the form (15) with $m = n+1$, suitably chosen vectors $c_1,\dots,c_{n+1} \in \mathbb{R}^n$ and scalars $\gamma_1,\dots,\gamma_{n+1} \in \mathbb{R}$. Thus, in order to show that the fixed point set of the continuous function $f := \varphi_{x,y}$ is a nonempty compact subset of $K = [x,y]^n_{\leq}$, we need to verify that condition (16) is satisfied.

For the sake of brevity, denote $t_0 := x$ and $t_{n+1} := y$. If, for some $k \in \{2,\dots,n\}$, the kth inequality holds with equality in (22), then $t_{k-1} = t_k$. Therefore, by the mean value property of the means M_{k-1} and M_k, we get

$$s_{k-1} = M_{k-1}(t_{k-2},t_k) \leq t_k = t_{k-1} \leq M_k(t_{k-1},t_{k+1}) = s_k,$$

which proves that the vector s satisfies the kth inequality in (22).

On the other hand, by the mean value properties of M_1 and M_n, we have $x \le M_1(x, t_2) = s_1$ and $s_n = M_n(t_{n-1}, y) \le y$, therefore, s also satisfies the first and last inequality in (22), and thus the verification of condition (16) is complete.

To prove the second part of the statement (1), assume that all the means $M_1, \dots, M_n$ are strict and let $(\xi_1, \dots, \xi_n) \in \Phi_{(x,y)}$. Then

$$(23) \qquad M_1(x, \xi_2) = \xi_1, \quad M_2(\xi_1, \xi_3) = \xi_2, \quad \dots, \quad M_n(\xi_{n-1}, y) = \xi_n.$$

If $x = \xi_1$, then the strict mean property of M_1 and the identity $M_1(x, \xi_2) = \xi_1$ imply that $\xi_1 = \xi_2$. Now, by the strict mean property of M_2 and the identity $M_2(\xi_1, \xi_3) = \xi_2$, it follows that $\xi_2 = \xi_3$. Continuing this argument, we get that $\xi_{n-1} = \xi_n$. Finally, the strict mean property of M_n and $M_n(\xi_{n-1}, y) = \xi_n$ imply that $\xi_n = y$. This leads to the contradiction $x = y$. Hence, we may assume that $x < \xi_1$. Applying the strict mean property of $M_1, \dots, M_n$ and the equalities in (23), we get $\xi_i < \xi_{i+1}$ recursively for $i \in \{1, \dots, n-1\}$ and finally $\xi_n < y$, which proves that $(\xi_1, \dots, \xi_n) \in]x, y[^n_<$.

To prove (2), assume that there exist semimetrics $d_1, \dots, d_n : [x, y]^2 \to \mathbb{R}_+$ such that the estimates listed in (2) of Theorem 1.7 hold and let $a := (a_2, \dots, a_n)$ and $b := (b_1, \dots, b_{n-1})$ such that each member of the sequence $w_1, \dots, w_{n-1}$, defined by (21), is positive. According to the previous lemmas, the matrix $A(a, b)$ has an eigenvector $p := (p_1, \dots, p_n)$ with positive components and with eigenvalue $0 < \lambda < 1$. This means that p and λ satisfy the following system of linear equations:

$$(24) \qquad \begin{array}{ccccccl} a_2 p_2 & & & = & \lambda p_1, & & \\ a_{i+1} p_{i+1} & + & b_{i-1} p_{i-1} & = & \lambda p_i & & (i \in \{2, \dots, n-1\}), \\ & & b_{n-1} p_{n-1} & = & \lambda p_n. & & \end{array}$$

We show that $\varphi_{(x,y)}$ is a contraction with modulus λ with respect to the semimetric $D_p : [x, y]^n \times [x, y]^n \to \mathbb{R}_+$ defined by

$$D_p\big((u_1, \dots, u_n), (v_1, \dots, v_n)\big) := p_1 d_1(u_1, v_1) + \dots + p_n d_n(u_n, v_n)$$

for all $(u_1, \dots, u_n), (v_1, \dots, v_n) \in [x, y]^n$. To prove this, let $(t_1, \dots, t_n)$ and $(s_1, \dots, s_n)$ be arbitrary elements of $[x, y]^n_\le$. For the sake of brevity, set $t_0 = s_0 = x$ and $t_n = s_n = y$. Using our estimates concerning the semimetrics and then the identities in (24), we obtain that

$$
\begin{aligned}
&D_p\big(\varphi_{(x,y)}(t_1,\dots,t_n),\varphi_{(x,y)}(s_1,\dots,s_n)\big)\\
&\quad=\sum_{i=1}^{n} c_i d_i\big(M_i(t_{i-1},t_{i+1}),M_i(s_{i-1},s_{i+1})\big)\\
&\quad\le c_1b_1d_2(t_2,s_2)+\left(\sum_{i=2}^{n-1} c_ia_id_{i-1}(t_{i-1},s_{i-1})+c_ib_{i+1}d_i(t_{i+1},s_{i+1})\right)\\
&\quad+c_na_nd_n(t_{n-1},s_{n-1})\\
&\quad=\lambda\big(c_1d_1(t_1,s_1)+\dots+c_nd_n(t_n,s_n)\big)=\lambda D_c((t_1,\dots,t_n),(s_1,\dots,s_n)).
\end{aligned}
$$

This results the uniqueness of the fixed point of $\varphi_{(x,y)}$. □

Now we turn to the definition of the descendants. Let $n \ge 2$ and let $(M_1,\dots,M_n) : I^2_{\le} \to \mathbb{R}^n$ be an n-tuple of continuous two-variable means. For $i \in \{1,\dots,n\}$, the mean $N : I^2_{\le} \to \mathbb{R}$ is said to be an ith *descendant of the n-tuple of means* $(M_1,\dots,M_n)$ if, for all $(x,y) \in I^2_{\le}$, we have

$$
\begin{gathered}
N(x,y) \in \bigcup\big\{\xi_i \mid (\xi_1,\dots,\xi_n)\in\Phi_{(x,y)}\big\} \text{ whenever } x<y\\
\text{and}\\
N(x,y)=x \text{ if } x=y,
\end{gathered}
\tag{25}
$$

where $\Phi_{(x,y)}$ stands for the fixed point set of $\varphi_{(x,y)} : [x,y]^n_{\le} \to \mathbb{R}^n$ defined by (19). The class of all such functions will be denoted by $\mathbf{D}_i(M_1,\dots,M_n)$.

Note that, in view of Theorem 1.7, the continuity of the means $M_1,\dots,M_n$ implies that the descendant functions are well-defined. As a direct consequence of the compactness of the fixed point set $\Phi_{(x,y)}$, we obtain that the family $\mathbf{D}_i(M_1,\dots,M_n)$ has a minimal and a maximal member in the following sense: there exist $N_i^-, N_i^+ \in \mathbf{D}_i(M_1,\dots,M_n)$ such that

$$N_i^-(x,y)\le N(x,y)\le N_i^+(x,y)$$

for all $x,y \in I$ and for all $N \in \mathbf{D}_i(M_1,\dots,M_n)$. It is also obvious that each element of $\mathbf{D}_i(M_1,\dots,M_n)$ is a strict mean provided that all the means $M_1,\dots,M_n$ are strict.

We also note that the uniqueness of the fixed point of the map $\varphi_{(x,y)}$ cannot be stated in general. For instance, let $n \ge 2$, $M_1 := \max$, $M_n := \min$, and let M_i be the two-variable arithmetic mean for each $i \in \{2,\dots,n-1\}$ over the interval $\mathbb{R}$. Then, for $(x,y) \in \mathbb{R}^2_{<}$, the fixed point equation $(t_1,\dots,t_n) =$

$\varphi_{(x,y)}(t_1,\dots,t_n)$ holds if and only if

$$(t_1,\dots,t_n)=\Big(t_2,\frac{t_1+t_3}{2},\dots,\frac{t_{n-2}+t_n}{2},t_{n-1}\Big).$$

One can easily compute that this equality is equivalent to $t_1=\dots=t_n$. Therefore we have infinitely many fixed points for all $x<y$, namely

$$\Phi_{(x,y)}=\{(t_1,\dots,t_n)\mid t_1=\dots=t_n\in[x,y]\}.$$

1.5. Descendants of Matkowski means

In this section we are going to apply our technique to two-variable Matkowski means, introduced in Section 1.1. More precisely, we present some useful corollaries of Theorem 1.7, stating that Matkowski means always have descendants.

THEOREM 1.8. *Let $n\geq 2$ and let $f_1,\dots,f_n,g_1,\dots,g_n:I\to\mathbb{R}$ be continuous, strictly increasing functions. For $(x,y)\in I^2_<$, define the function $\varphi_{(x,y)}:[x,y]^n_\leq\to\mathbb{R}^n$ as in* (19) *using the means $M_i:=\mathcal{M}^{(f_i,g_i)}$, where $i\in\{1,\dots,n\}$. Then, for $(x,y)\in I^2_<$, the set of fixed points $\Phi_{(x,y)}$, defined by* (20)*, is nonempty and compact. Furthermore, $\Phi_{(x,y)}$ is a singleton if*

$$\begin{aligned}a_i&:=\mathrm{Lip}\left[f_i\circ(f_{i-1}+g_{i-1})^{-1}\right]<+\infty \qquad (i\in\{2,\dots,n\}),\\ b_i&:=\mathrm{Lip}\left[g_i\circ(f_{i+1}+g_{i+1})^{-1}\right]<+\infty \qquad (i\in\{1,\dots,n-1\}),\end{aligned}\tag{26}$$

hold and if the constants $w_1,\dots,w_{n-1}$ defined by (21) *are positive.*

PROOF. The means $\mathcal{M}^{(f_1,g_1)},\dots,\mathcal{M}^{(f_n,g_n)}$ are continuous, thus, for all pair $(x,y)\in I^2_<$, the mapping $\varphi_{(x,y)}$ is also continuous. Based on the Theorem 1.7, the corresponding set $\Phi_{(x,y)}$ is a nonempty compact subset of $[x,y]^n_\leq$. Due to the strictness of Matkowski means it also follows that $\Phi_{(x,y)}\subseteq\,]x,y[^n_<$.

Now assume that (26) and $w_1,\dots,w_{n-1}>0$ hold and pick up a point $(x,y)\in I^2_<$ arbitrarily. To show that $\Phi_{(x,y)}$ is a singleton, for $i\in\{1,\dots,n\}$, define the semimetrics $d_i:I\times I\to\mathbb{R}_+$ by

$$d_i(s,t):=|(f_i+g_i)(s)-(f_i+g_i)(t)|,\qquad (s,t\in I).$$

Note that in our case, for all $i\in\{1,\dots,n\}$, the function d_i is a *metric*. This mean that in addition of the properties (1) and (2) of semimetrics, d_i also satisfies the *triangle inequality*, namely, for all $i\in\{1,\dots,n\}$, we have

$$d_i(s,t)\leq d_i(s,r)+d_i(r,t),\qquad (r,s,t\in I).$$

Let $t, s, u, v \in [x, y]$ be arbitrary. Then, for all $i \in \{2, \dots, n-1\}$, we have the following estimation:

$$\begin{aligned}
&d_i(M_i(t,s), M_i(u,v)) = |(f_i + g_i)(\mathrm{M}_{f_i,\,g_i}(t,s)) - (f_i + g_i)(\mathrm{M}_{f_i,\,g_i}(u,v))| \\
&= |f_i(t) + g_i(s) - f_i(u) - g_i(v)| \le |f_i(t) - f_i(u)| + |g_i(s) - g_i(v)| \\
&\le \mathrm{Lip}\left(f_i \circ (f_{i-1} + g_{i-1})^{-1}\right) d_{i-1}(t,u) + \mathrm{Lip}\left(g_i \circ (f_{i+1} + g_{i+1})^{-1}\right) d_{i+1}(s,v) \\
&= a_i d_{i-1}(t,u) + b_i d_{i+1}(s,v).
\end{aligned}$$

On the other hand, for $i = 1$ and $i = n$, we get that

$$d_1(M_1(x,s), M_1(x,v)) \le b_1 d_2(s,v)$$

and

$$d_n(M_n(t,y), M_n(u,y)) \le a_n d_{n-1}(t,u)$$

are valid. Therefore, all the estimates listed in (2) of Theorem 1.7 are satisfied. Thus, in view of the Theorem 1.7, for all $(x,y) \in I^2_<$, the fixed point set $\Phi_{(x,y)}$ is indeed a singleton. $\square$

Due to the fact that Lipschitz modulus can be easily calculated in case of differentiable functions, we have the following consequence of Theorem 1.8.

COROLLARY 1.9. *Let $n \ge 2$ and $f_1, \dots, f_n, g_1, \dots, g_n : I \to \mathbb{R}$ be differentiable, strictly increasing functions such that $0 \notin (f_i + g_i)'(I)$ for all $i \in \{1, \dots, n\}$. For $(x,y) \in I^2_<$, define the function $\varphi_{(x,y)} : [x,y]^n_\le \to \mathbb{R}^n$ as in* (19) *using the means $M_i := \mathcal{M}^{(f_i,g_i)}$, where $i \in \{1, \dots, n\}$, and, finally, assume that*

$$\begin{aligned}
a_i &:= \sup_{t\in I} \left[f_i' \cdot (f_{i-1}' + g_{i-1}')^{-1}\right](t) < +\infty \qquad (i \in \{2, \dots, n\}), \\
b_i &:= \sup_{t\in I} \left[g_i' \cdot (f_{i+1}' + g_{i+1}')^{-1}\right](t) < +\infty \qquad (i \in \{1, \dots, n-1\}).
\end{aligned} \tag{27}$$

Then, for all $(x,y) \in I^2_<$, the set of fixed points $\Phi_{(x,y)}$ defined by (20) *is a nonempty compact subset of $[x,y]^n_\le$, and, it is a singleton if the constants $w_1, \dots, w_{n-1}$ defined by* (21) *are positive.*

PROOF. In view of Theorem 1.8, we only need to verify that $\Phi_{(x,y)}$ is a singleton, which in turn is obvious. Using Lemma 1.4 and the conditions in (27), one can easily see that the estimations in (26) of Theorem 1.8 hold, that is, the constants $a_2, \dots, a_n$ and $b_1, \dots, b_{n-1}$ are real numbers. $\square$

In view of the next theorem, the descendants of a chain of Matkowski means are uniquely determined provided that they are weighted quasi-arithmetic means with a common generator function h. In this case, the descendants will be again weighted quasi-arithmetic means, where the new generators can be directly calculated using the original weights and h.

THEOREM 1.10. *Let $n \geq 2$, $s_1, \dots, s_n \in]0,1[$, and $h : I \to \mathbb{R}$ be a continuous, strictly increasing function. For $(x,y) \in I^2_<$, define the function $\varphi_{(x,y)} : [x,y]^n_\leq \to \mathbb{R}^n$ as in* (19) *using the means $M_i := \mathcal{M}^{(s_i h,(1-s_i)h)}$, where $i \in \{1, \dots, n\}$. Then, for all $(x,y) \in I^2_<$, the fixed point set $\Phi_{(x,y)}$ is the singleton $\big\{\big(\mathcal{M}^{(\sigma_1 h,\,(1-\sigma_1)h)}(x,y), \dots, \mathcal{M}^{(\sigma_n h,\,(1-\sigma_n)h)}(x,y)\big)\big\}$, where*

$$\sigma_i := \bigg(\sum_{j=i}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k}\bigg)\bigg(\sum_{j=0}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k}\bigg)^{-1}, \quad (i \in \{1, \dots, n\}). \tag{28}$$

PROOF. In order to apply Theorem 1.8, let $f_i := s_i h$ and $g_i := (1-s_i)h$ for $i \in \{1, \dots, n\}$. Then it immediately follows that the fixed point set $\Phi_{(x,y)}$ is nonempty and compact for all $(x,y) \in I^2_<$.

To show that $\Phi_{(x,y)}$ is a singleton, define the constants $a_2, \dots, a_n$, $b_1, \dots, b_{n-1}$, and $w_1, \dots, w_{n-1}$ as in Theorem 1.8. We need to show that conditions (26) and $w_1, \dots, w_{n-1} > 0$ hold. Observe that, for $i \in \{1, \dots, n\}$, we have $f_i + g_i = h$, furthermore

$$a_i = \text{Lip}\left(f_i \circ (f_{i-1} + g_{i-1})^{-1}\right) = \text{Lip}[s_i \cdot h \circ h^{-1}] = s_i$$

for all $i \in \{2, \dots, n\}$ and

$$b_i = \text{Lip}\left(g_i \circ (f_{i+1} + g_{i+1})^{-1}\right) = \text{Lip}((1-s_i) \cdot h \circ h^{-1}) = 1 - s_i$$

for all $i \in \{1, \dots, n-1\}$. Thus each of the constants $a_2, \dots, a_n$ and $b_1, \dots, b_{n-1}$ are finite, on the other hand, under the notation $(u_1, \dots, u_{n-1}) := (a_2, \dots, a_n)$ and $(v_1, \dots, v_{n-1}) := (b_1, \dots, b_{n-1})$, they also satisfy the condition (10) of Lemma 1.2. Therefore, the inequalities $w_1, \dots, w_{n-1} > 0$ and hence $\Phi_{(x,y)}$ has to be a singleton.

Finally, we verify that, for all $(x,y) \in I^2_<$, the vector

$$\big(\mathcal{M}^{(\sigma_1 h,\,(1-\sigma_1)h)}(x,y), \dots, \mathcal{M}^{(\sigma_n h,\,(1-\sigma_n)h)}(x,y)\big) \tag{29}$$

is a fixed point of $\varphi_{(x,y)}$. For this purpose, we show first that $\sigma_1, \dots, \sigma_n$ fulfill the following system of linear equations:

$$\begin{aligned} \sigma_1 &= s_1 + (1-s_1)\sigma_2, \\ \sigma_i &= s_i\sigma_{i-1} + (1-s_i)\sigma_{i+1} \qquad (i \in \{2, \dots, n-1\}), \\ \sigma_n &= s_n\sigma_{n-1}. \end{aligned} \tag{30}$$

We prove the above equality for $i \in \{2, \dots, n-1\}$. First observe that

$$\begin{aligned}\prod_{k=1}^{i} \frac{s_k}{1-s_k} &= \frac{s_i}{1-s_i} \prod_{k=1}^{i-1} \frac{s_k}{1-s_k} \\ &= s_i\Big(1 + \frac{s_i}{1-s_i}\Big) \prod_{k=1}^{i-1} \frac{s_k}{1-s_k} = s_i\bigg(\prod_{k=1}^{i-1} \frac{s_k}{1-s_k} + \prod_{k=1}^{i} \frac{s_k}{1-s_k}\bigg).\end{aligned}$$

Adding this identity to the equality

$$\sum_{j=i+1}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k} = s_i \sum_{j=i+1}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k} + (1-s_i) \sum_{j=i+1}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k}$$

side by side, we get the desired identity $\sigma_i = s_i\sigma_{i-1} + (1-s_i)\sigma_{i+1}$. In the cases $i = 1$ and $i = n$ the proof of (30) is completely analogous.

For the brevity, denote $\xi_i := \mathcal{M}^{(\sigma_i h, (1-\sigma_i)h)}(x, y)$ whenever $i \in \{1, \dots, n\}$. Using (30), after some calculation we easily get that

$$\begin{aligned}\xi_1 &= \mathcal{M}^{(s_1 h,\, (1-s_1)h)}(x, \xi_2), \\ \xi_i &= \mathcal{M}^{(s_i h,\, (1-s_i)h)}(\xi_{i-1}, \xi_{i+1}), \qquad (i \in \{2, \dots, n-1\}), \\ \xi_n &= \mathcal{M}^{(s_n h,\, (1-s_n)h)}(\xi_{n-1}, y),\end{aligned}$$

which proves that (29) is indeed a fixed point of $\varphi_{(x,y)}$. □

In the next theorem the means are not necessarily weighted quasi-arithmetic, but they are strongly related to each other by shifts. In this case the descendants are turned to be uniquely determined and easily calculated, but only recursively. Furthermore, the descendants are not necessarily Matkowski mean, only compositions of them.

THEOREM 1.11. *Let $n \geq 2$, $j \in \{1, \dots, n\}$ and $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ be continuous, strictly increasing functions, furthermore set $h_0 := h_n := 0$. For $(x, y) \in I^2_<$, define the mapping $\varphi_{(x,y)} : [x, y]^n_\leq \to \mathbb{R}^n$ by* (19)*, using the means*

$$M_i := \begin{cases} \mathcal{M}^{(p+h_{i-1},\, h_i)} & \text{if } i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(p+h_{i-1},\, h_i+q)} & \text{if } i = j, \\ \mathcal{M}^{(h_{i-1},\, h_i+q)} & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Then, for $(x, y) \in I^2_<$, the fixed point set $\Phi_{(x,y)}$ defined by (20) *is the singleton $\{(\xi_1, \dots, \xi_n)\}$, where $\xi_j := \mathcal{M}^{(p,q)}(x, y)$ and the rest of the coordinates are* *defined by the two-way recurrence*

$$\xi_i := \begin{cases} \mathcal{M}^{(p,h_i)}(x,\xi_{i+1}) & \textit{if } i \in \{1,\dots,j-1\}, \\ \mathcal{M}^{(h_{i-1},q)}(\xi_{i-1},y) & \textit{if } i \in \{j+1,\dots,n\}. \end{cases} \tag{31}$$

PROOF. Let $(x,y) \in I^2_<$ be fixed. By Theorem 1.7, the set $\Phi_{(x,y)}$ is nonempty. Let $(\xi_1,\dots,\xi_n) \in \Phi_{(x,y)}$ be arbitrary, furthermore denote $\xi_0 := x$ and $\xi_{n+1} := y$. Then, by the definition of Matkowski means, we have

$$(p + h_{i-1} + h_i)(\xi_i) = (p + h_{i-1})(\xi_{i-1}) + h_i(\xi_{i+1}),\ (i \in \{1,\dots,j-1\}),$$

$$(p + h_{i-1} + h_i + q)(\xi_i) = (p + h_{i-1})(\xi_{i-1}) + (h_i + q)(\xi_{i+1}), \quad (i = j),$$

$$(h_{i-1} + h_i + q)(\xi_i) = h_{i-1}(\xi_{i-1}) + (h_i + q)(\xi_{i+1}),\ (i \in \{j+1,\dots,n\}).$$

Adding up these equalities for $i \in \{1,\dots,n\}$ side by side, it follows that

$$p(\xi_j) + h_0(\xi_1) + h_n(\xi_n) + q(\xi_j) = p(\xi_0) + h_0(\xi_0) + h_n(\xi_{n+1}) + q(\xi_{n+1}),$$

which finally simplifies to

$$(p + q)(\xi_j) = p(x) + q(y).$$

This is equivalent to the equality on the left hand side of (31). By this computation it also follows that ξ_j is uniquely determined.

To prove the first equality on the right hand side of (31), let assume that $1 \le j - 1$ and let $k \in \{1,\dots,j-1\}$ be fixed. Adding up the previous system of equalities but only for $i \in \{1,\dots,k\}$, we arrive at

$$p(\xi_k) + h_0(\xi_1) + h_k(\xi_k) = p(\xi_0) + h_0(\xi_0) + h_k(\xi_{k+1}),$$

which reduces to $(p + h_k)(\xi_k) = p(x) + h_k(\xi_{k+1})$ proving the first equality on the right hand side of (31) for $i = k$.

Analogously, to verify the second equality on the right hand side of (31), assume that $j + 1 \le n$ and let $k \in \{j+1,\dots,n\}$ be fixed. Similarly, adding up the equalities in our system of equations for $i \in \{k,\dots,n\}$, we obtain that

$$h_{k-1}(\xi_k) + h_n(\xi_n) + q(\xi_k) = h_{k-1}(\xi_{k-1}) + h_n(\xi_{n+1}) + q(\xi_{n+1}).$$

This yields $(h_{k-1} + q)(\xi_k) = h_{k-1}(\xi_{k-1}) + q(y)$, which verifies the second equality on the right hand side of (31) for $i = k$.

In view of the uniqueness of ξ_j and the recursive system of equalities on the right hand side of (31), we can see that, for $i \neq j$, the value of ξ_i is also uniquely determined. $\square$

CHAPTER 2

Deriving new convexity properties

2.1. The class of upper and lower M-convex functions

To motivate the definition of our main notion, we recall a well-known characterization of standard convexity of real functions. It is easy to show that the concept of standard convexity can be characterized in terms of second order divided differences. More precisely, the function f is convex on I if and only if, for all elements $x < u < y$ from I, the corresponding second order divided difference $[x, u, y; f]$ is non-negative.

The upper and lower M-convexity will concern not necessarily real valued but extended real valued functions. Hence, to extend the above characterization, firstly, we have to adopt the definition of second order divided differences for the case of extended real valued functions. To do this, consider the following binary operations defined on the extended real line $\overline{\mathbb{R}}$. For given $x, y \in \overline{\mathbb{R}}$, let their *upper sum* and *lower sum* are defined by

$$x \dot{+} y := \begin{cases} x + y, & \text{if } \max\{x, y\} < +\infty, \\ +\infty, & \text{if } \max\{x, y\} = +\infty, \end{cases}$$

and

$$x \underset{\cdot}{+} y := \begin{cases} x + y, & \text{if } \min\{x, y\} > -\infty, \\ -\infty, & \text{if } \min\{x, y\} = -\infty, \end{cases}$$

respectively. We note that both of the operations $\dot{+}$ and $\underset{\cdot}{+}$ restricted to pairs of real numbers are the same as the standard addition of the reals. In fact, apart from the standard cases, the only difference between $\dot{+}$ and $\underset{\cdot}{+}$ is that

$$(-\infty) \dot{+} (+\infty) = (+\infty) \dot{+} (-\infty) = +\infty$$

and

$$(-\infty) \underset{\cdot}{+} (+\infty) = (+\infty) \underset{\cdot}{+} (-\infty) = -\infty.$$

It is also easy to see, that the pairs $(\overline{\mathbb{R}}, \dot{+})$ and $(\overline{\mathbb{R}}, \underset{\cdot}{+})$ are commutative semigroups. As direct consequences of the definitions we have the following easy-to-prove statement.

PROPOSITION 1. *For all $x, y \in \overline{\mathbb{R}}$, we have*

$$x \underset{\cdot}{+} y \leq x \dot{+} y \qquad \textit{and} \qquad -(x \underset{\cdot}{+} y) = (-x) \dot{+} (-y), \tag{32}$$

furthermore, we have the following equivalences.

(i) *The upper sum $x \dot{+} y$ is non-negative or it is non-positive if and only if*

$$-x \leq y \qquad \textit{or} \qquad \max(x, y) < +\infty \quad \textit{and} \quad x \leq -y,$$

respectively.

(ii) *The lower sum $x \underset{\cdot}{+} y$ is non-negative or it is non-positive if and only if*

$$-\infty < \min\{x, y\} \quad \textit{and} \quad -x \leq y \qquad \textit{or} \qquad x \leq -y,$$

respectively.

PROOF. The statements easily follow by the definition of lower and upper sum. □

Let $S \subseteq \mathbb{R}$ be a nonempty subset and $f : S \to \overline{\mathbb{R}}$. The *upper second-order divided difference* of f at the distinct points x, y and z of S is an extended real number defined by

$$\lceil x, y, z; f \rceil := \frac{f(x)}{(y-x)(z-x)} \dot{+} \frac{f(y)}{(x-y)(z-y)} \dot{+} \frac{f(z)}{(x-z)(y-z)}.$$

Similarly, the *lower second-order divided difference* of f at the points x, y and z of S is

$$\lfloor x, y, z; f \rfloor := \frac{f(x)}{(y-x)(z-x)} \underset{\cdot}{+} \frac{f(y)}{(x-y)(z-y)} \underset{\cdot}{+} \frac{f(z)}{(x-z)(y-z)}.$$

Obviously, the above second-order divided differences are symmetric functions of (x, y, z). Observe that if the inequalities $x < y < z$ hold, then the coefficients of $f(x)$ and $f(z)$ are positive and the coefficient to $f(y)$ is negative.

Using the definitions and Proposition 1, one can easily prove the following useful statement about connections between upper- and lower second-order divided differences.

PROPOSITION 2. *Let $S \subseteq \mathbb{R}$ and $f : S \to \overline{\mathbb{R}}$. Then we have*

$$\lfloor x, y, z; f \rfloor \leq \lceil x, y, z; f \rceil \qquad \textit{and} \qquad -\lfloor x, y, z; f \rfloor = \lceil x, y, z; -f \rceil$$

for all points $x < y < z$ of S.

The following proposition determine further relations among these extended versions of second-order divided differences. The corresponding result concerning the standard real valued case, among others, can be found in [**13**, Lemma XV.2.2, pp. 376-377].

PROPOSITION 3. (Extended Chain Inequality) *Let $S \subseteq \mathbb{R}$ and $f : S \to \overline{\mathbb{R}}$. Then, for all $n \in \mathbb{N}$ and $x_0 < x_1 < \cdots < x_{n+1}$ in S, and for all $i \in \{1, \dots, n\}$, the inequalities*

$$\min_{1 \le j \le n} \lfloor x_{j-1}, x_j, x_{j+1}; f \rfloor \le \lfloor x_0, x_i, x_{n+1}; f \rfloor \\ \le \lceil x_0, x_i, x_{n+1}; f \rceil \le \max_{1 \le j \le n} \lceil x_{j-1}, x_j, x_{j+1}; f \rceil$$

hold.

PROOF. We only need to prove the first inequality, because the second one is trivial, furthermore the last one is the consequence of the first and Proposition 2.

The statement is trivial for $n = 1$, therefore we may assume that $n \ge 2$. Let $x_0 < x_1 < \cdots < x_{n+1}$ be arbitrary elements of S and $i \in \{1, \dots, n\}$. If either the left hand side of the first inequality equals $-\infty$ or the right hand side equals $+\infty$, then there is nothing to prove. In the remaining case, for all $j \in \{1, \dots, n\}$, we have that $\lfloor x_{j-1}, x_j, x_{j+1}; f \rfloor > -\infty$ and $\lfloor x_0, x_i, x_{n+1}; f \rfloor < +\infty$. The first inequality implies, for all $j \in \{1, \dots, n\}$, that

$$\min\{f(x_{j-1}), -f(x_j), f(x_{j+1})\} > -\infty.$$

In view of $n \ge 2$, the set $\{1, \dots, n\}$ contains at least two elements, therefore, for all $j \in \{1, \dots, n\}$, we get that $f(x_j) \in \mathbb{R}$ and $\min\{f(x_0), f(x_{n+1})\} > -\infty$. Thus, $f(x_i) \in \mathbb{R}$ and hence the inequality $\lfloor x_0, x_i, x_{n+1}; f \rfloor < +\infty$ yields $\max\{f(x_0), f(x_{n+1})\} < +\infty$, which proves that, for all $j \in \{0, \dots, n+1\}$, we have $f(x_j) \in \mathbb{R}$. We also note that, in this case, the first inequality is a consequence of [**23**, Corollary 1]. $\square$

Now we are able to define lower and upper M-convexity of extended real valued functions. For a fixed strict mean $M : I^2_{\le} \to \mathbb{R}$, we say that the function $f : I \to \overline{\mathbb{R}}$ is *lower M-convex on I* if

$$\lfloor x, M(x, y), y; f \rfloor \ge 0, \qquad \big((x, y) \in I^2_{<}\big) \tag{33}$$

holds. On the other hand, the function f is called *upper M-convex on I* provided that

$$\lceil x, M(x, y), y; f \rceil \ge 0 \tag{34}$$

holds on the same domain.

Note that, due to the property (32), if f is lower M-convex, then it is also upper M-convex.

The *lower* and *upper M-concavity* of functions can be also interpreted, namely we may consider (33) and (34) with the reverse inequality. It is easy to verify, that these definitions are equivalent to the upper and lower M-convexity of the function $-f$, respectively.

The next statement clarifies an essential difference between lower and upper M-convexity.

LEMMA 2.1. *Let $M : I^2_{\le} \to \mathbb{R}$ be a strict mean and $f : I \to \overline{\mathbb{R}}$. Then the following statements hold.*

(a) The function f is lower M-convex if and only if $f(u) > -\infty$ for all $u \in I$ and for all $(x, y) \in I^2_{<}$, the inequalities $f(M(x, y)) < +\infty$ and

$$f(M(x,y)) \le \frac{y - M(x,y)}{y - x} f(x) + \frac{M(x,y) - x}{y - x} f(y) \tag{35}$$

hold.

(b) The function f is upper M-convex if and only if, for all $(x, y) \in I^2_{<}$, the inequality

$$f(M(x,y)) \le \frac{y - M(x,y)}{y - x} f(x) \dot{+} \frac{M(x,y) - x}{y - x} f(y) \tag{36}$$

holds.

PROOF. First we prove the statement (b). Suppose that f is upper M-convex and let $(x, y) \in I^2_{<}$ be any element. For the brevity, denote the value $M(x, y)$ by p. The upper M-convexity of f means that we have $\lceil x, p, y; f \rceil \ge 0$. Due to *(i)* of Proposition 1, this inequality is equivalent to

$$\frac{f(p)}{(p - x)(y - p)} \le \frac{f(x)}{(p - x)(y - x)} \dot{+} \frac{f(y)}{(x - y)(p - y)}. \tag{37}$$

Using that $(p - x)(y - p)$ is positive, we obtain that (36) is valid for (x, y), which was arbitrarily chosen.

To prove the reverse implication of (b), suppose that (36) holds on the domain indicated. Then (37) is also valid and, in view of *(i)* of Proposition 1, this implies (36).

Now we prove the statement (a). Suppose that f is lower M-convex, let $(x, y) \in I^2_{<}$ be arbitrary, and let again $p := M(x, y)$. Then $\lfloor x, p, y; f \rfloor \ge 0$ holds. Based on *(ii)* of Proposition 1, it follows that $-\infty < \min\{f(x), -f(p), f(y)\}$ and that

$$\frac{f(p)}{(p - x)(y - p)} \le \frac{f(x)}{(p - x)(y - x)} \underset{\cdot}{+} \frac{f(y)}{(x - y)(p - y)}. \tag{38}$$

Thus, for all $u \in I$, we get $-\infty < f(u)$ and, by the positivity of the product $(p - x)(y - p)$, the inequality (38) is equivalent to the inequalities (35) and $f(p) < +\infty$.

Finally, to prove the reversed implication of the statement (a), suppose that $f(M(x,y)) < +\infty$ and (35) hold for all pairs $(x,y) \in I^2_<$, furthermore we have $-\infty < f(u)$ for all $u \in I$. Then (38) is also valid and, in view of (ii) in Proposition 1, this implies (35). □

2.2. The lower and upper convexity class

In the sequel, we are going to formulate Jensen type theorems concerning this extended concept of convexity. To do this, similarly to the standard case, we introduce the related „parameter families", which, in our case, instead of real numbers, will contain strict means.

For a given function $f : I \to \overline{\mathbb{R}}$, define

$$\underline{\mathcal{M}}_f := \{M : I^2_\leq \to \mathbb{R} \mid f \text{ is lower } M\text{-convex on } I\}$$

and

$$\overline{\mathcal{M}}_f := \{M : I^2_\leq \to \mathbb{R} \mid f \text{ is upper } M\text{-convex on } I\}.$$

Note that, due to the strictness of the means in the definition, unlike the standard case, the above families can be also empty. The following proposition is about a certain algebraic closedness property of $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$.

PROPOSITION 4. *Let $f : I \to \overline{\mathbb{R}}$ be any function and let $\mathcal{M} \in \{\underline{\mathcal{M}}_f, \overline{\mathcal{M}}_f\}$. Then the following statements hold.*

(a) If $M, N_1, N_2 \in \mathcal{M}$ with $N_1 < N_2$ on the set $I^2_<$, then $M \circ (N_1, N_2) \in \mathcal{M}$.

(b) If $M, N \in \mathcal{M}$, then the compositions $M \circ (\min, N)$ and $M \circ (N, \max)$ also belong to the family $\mathcal{M}$.

We note that the statement (b) is not a direct consequence of (a), because the means $\min$ and $\max$ are not strict, and hence they do not belong to the family $\mathcal{M}$.

PROOF. We prove the statements for the family $\overline{\mathcal{M}}_f$ only. The proof in the other case is completely analogous and also based on Lemma 2.1.

Let $(x,y) \in I^2_<$ be arbitrarily fixed, furthermore define $p_1 := N_1(x,y)$ and $p_2 := N_2(x,y)$. Obviously, under our conditions, it follows that $p_1 < p_2$. Using these notations, in view of Lemma 2.1, it is sufficient to show, that

$$f(M(p_1,p_2)) \leq \frac{y - M(p_1,p_2)}{y-x} f(x) \dot{+} \frac{M(p_1,p_2) - x}{y-x} f(y), \tag{39}$$

holds. Applying the upper M-convexity, and then the upper N_1- and N_2-convexity of f, we have the following calculation.

$$\begin{aligned}
f(M(p_1,p_2)) &\le \frac{p_2 - M(p_1,p_2)}{p_2 - p_1} f(p_1) \dot{+} \frac{M(p_1,p_2) - p_1}{p_2 - p_1} f(p_2) \\
&= \frac{p_2 - M(p_1,p_2)}{p_2 - p_1} f(N_1(x,y)) \dot{+} \frac{M(p_1,p_2) - p_1}{p_2 - p_1} f(N_2(x,y)) \\
&\le \frac{p_2 - M(p_1,p_2)}{p_2 - p_1} \Big(\frac{y - p_1}{y - x} f(x) \dot{+} \frac{p_1 - x}{y - x} f(y) \Big) \\
&\quad \dot{+} \frac{M(p_1,p_2) - p_1}{p_2 - p_1} \Big(\frac{y - p_2}{y - x} f(x) \dot{+} \frac{p_2 - x}{y - x} f(y) \Big) \\
&= \frac{y - M(p_1,p_2)}{y - x} f(x) \dot{+} \frac{M(p_1,p_2) - x}{y - x} f(y).
\end{aligned}$$

Thus the inequality (39) is satisfied, which means that (a) is true.

A completely similar calculation shows that the statement (b) is also valid. □

As a consequence, we get that the separately continuous subfamily of $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$ has only accumulation points with respect to the pointwise convergence.

COROLLARY 2.2. *Let $f : I \to \overline{\mathbb{R}}$ be any function, define*

$$\underline{\mathcal{M}}^*_f := \{M \in \underline{\mathcal{M}}_f \mid M \textit{ is separately continuous in both variables}\},$$
$$\overline{\mathcal{M}}^*_f := \{M \in \overline{\mathcal{M}}_f \mid M \textit{ is separately continuous in both variables}\},$$

and, finally, let $\mathcal{M}^* \in \{\underline{\mathcal{M}}^*_f, \overline{\mathcal{M}}^*_f\}$. Then $\mathcal{M}^*$ has no isolated points with respect to the pointwise convergence, more precisely, for all $M \in \mathcal{M}^*$, there exist sequences of means $(L_n), (U_n) \subseteq \mathcal{M}^*$ such that $L_n < M < U_n$ whenever $n \in \mathbb{N}$, furthermore $L_n \to M$ and $U_n \to M$ pointwise on the set $I^2_<$ as $n \to \infty$.

PROOF. We prove the statement only for the class $\underline{\mathcal{M}}^*_f$. Let $M \in \underline{\mathcal{M}}^*_f$ be an arbitrarily but fixed mean. We construct only the lower sequence (U_n), because the existence of (L_n) can be proved similarly.

Let $U_0 = \max$ furthermore, for $n \in \mathbb{N}$, let U_n be defined by the composition $M \circ (M, U_{n-1})$. Firstly we show that the sequence (U_n) belongs to $\underline{\mathcal{M}}^*_f$. To see this, we prove, by induction, that $M < U_n < U_{n-1}$ for all $n \in \mathbb{N}$ on $I^2_<$.

Let $(x,y) \in I^2_<$ be any point. For $n=1$, using that M is a strict mean, we get

$$\begin{aligned} U_1(x,y) &= M(M(x,y), U_0(x,y)) \\ &= M(M(x,y), y) \in]M(x,y), y[=]M(x,y), U_0(x,y)[\,. \end{aligned}$$

Assume that the inequalities $M < U_n < U_{n-1}$ hold on $I^2_<$ for some $n \in \mathbb{N}$. Using this, for $n+1$, we obtain that

$$U_{n+1}(x,y) = M(M(x,y), U_n(x,y)) \in]M(x,y), U_n(x,y)[\,.$$

Hence $M(x,y) < U_{n+1}(x,y) < U_n(x,y)$ follows for all $(x,y) \in I^2_<$, which completes the proof of the induction. Then, Proposition 4 yields that $(U_n) \subseteq \underline{\mathcal{M}}_f$. Moreover, by its definition, U_n is a strict mean and it is separately continuous in both variables for all $n \in \mathbb{N}$, hence $(U_n) \subseteq \underline{\mathcal{M}}^*_f$ also holds.

In the second step we show, that $U_n \downarrow M$ pointwise on $I^2_<$ as $n \to \infty$. Let $(x,y) \in I^2_<$ be arbitrarily fixed again. Obviously, the sequence $(U_n(x,y)) \subseteq]x,y[$ has to be convergent, because it is monotone decreasing and bounded from below by $M(x,y)$. Denote the limit $\lim_{n\to\infty} U_n(x,y)$ by $U^*(x,y)$ which, clearly, cannot be smaller than $M(x,y)$. Upon taking the limit $n \to \infty$ in the identity

$$U_n(x,y) = M(M(x,y), U_n(x,y)),$$

we obtain that

$$U^*(x,y) = M(M(x,y), U^*(x,y)).$$

The inequality $M(x,y) < U^*(x,y)$ would contradict the strictness of M, therefore, $U^*(x,y) = M(x,y)$ must be valid. □

The following theorem is one of our main results. Roughly speaking, it states that $\underline{\mathcal{M}}_f$ is closed under deriving the descendants.

THEOREM 2.3. *Let $f : I \to \overline{\mathbb{R}}$ be any function, $n \geq 2$, furthermore $M_1, \dots, M_n \in \underline{\mathcal{M}}_f$ be continuous means. Then $\mathbf{D}_i(M_1, \dots, M_n) \subseteq \underline{\mathcal{M}}_f$ for all $i \in \{1, \dots, n\}$.*

PROOF. Let $i \in \{1, \dots, n\}$ and $N \in \mathbf{D}_i(M_1, \dots, M_n)$ be arbitrarily fixed. We have already seen that, under our conditions, N is a strict mean. If $(x,y) \in I^2_<$, then there exists $k \in \{1, \dots, n\}$ and $(\xi_1, \dots, \xi_n) \in \Phi_{(x,y)}$ such that $N(x,y) = \xi_k$. Furthermore, with $\xi_0 := x$ and $\xi_{n+1} := y$, we have

$$M_j(\xi_{j-1}, \xi_{j+1}) = \xi_j, \qquad (j \in \{1, \dots, n\}).$$

Using this and, for all $j \in \{1, \dots, n\}$, the lower M_j-convexity of the function f, we obtain that

$$0 \leq \lfloor \xi_{j-1}, \xi_j, \xi_{j+1}; f \rfloor, \qquad (j \in \{1, \dots, n\}).$$

Now, applying the Extended Chain Inequality, we get that

$$0 \le \min_{1\le j\le n} \lfloor \xi_{j-1}, \xi_j, \xi_{j+1}; f\rfloor \le \lfloor x, \xi_k, y; f\rfloor = \lfloor x, N(x,y), y; f\rfloor.$$

By the definition, this means that f is lower N-convex, or equivalently, we must have $N \in \underline{\mathcal{M}}_f$. ☐

In view of the results about the descendants of Matkowski means obtained in the previous chapter, Theorem 2.3 has several consequences for $\underline{\mathcal{M}}_f$, provided that it contains Matkowski means.

COROLLARY 2.4. *Let $f : I \to \overline{\mathbb{R}}$, $n \ge 2$, $s_1, \dots, s_n \in]0,1[$, and finally $h : I \to \mathbb{R}$ be a continuous, strictly increasing function. Assume further that $\mathcal{M}^{(s_i h,\, (1-s_i)h)} \in \underline{\mathcal{M}}_f$ for all $i \in \{1,\dots,n\}$. Then, for all $i \in \{1,\dots,n\}$, the Matkowski mean $\mathcal{M}^{(\sigma_i h,\, (1-\sigma_i)h)}$ also belongs to the family $\underline{\mathcal{M}}_f$, where the weight σ_i is defined as in* (28) *for all $i \in \{1,\dots,n\}$.*

PROOF. For $(x,y) \in I^2_<$, define the mapping $\varphi_{(x,y)} : [x,y]^n_\le \to \mathbb{R}^n$ as in Theorem 1.10. In view of this theorem, it follows that, for all $(x,y) \in I^2_<$, the fixed point set $\Phi_{(x,y)}$ equals with the singleton $\{(\xi_1,\dots,\xi_n)\}$, where we have $\xi_i = \mathcal{M}^{(\sigma_i h,\, (1-\sigma_i)h)}(x,y)$ for all $i \in \{1,\dots,n\}$. Thus, for $i \in \{1,\dots,n\}$, the function $\mathcal{M}^{(\sigma_i h,\, (1-\sigma_i)h)}$ is the i^{th} descendant of the n-tuple of means $(\mathcal{M}^{(s_1 h,\, (1-s_1)h)}, \dots, \mathcal{M}^{(s_n h,\, (1-s_n)h)})$. Therefore, due to Theorem 2.3, we obtain that $\mathcal{M}^{(\sigma_i h,\, (1-\sigma_i)h)} \in \underline{\mathcal{M}}_f$ for all $i \in \{1,\dots,n\}$. ☐

COROLLARY 2.5. *Let $n \ge 2$, $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ be continuous, strictly increasing functions and $f : I \to \overline{\mathbb{R}}$. Set further $h_0 := h_n := 0$ and assume that there exists $j \in \{1,\dots,n\}$ such that, for all $i \in \{1,\dots,n\}$, the mean M_i defined by*

$$M_i := \begin{cases} \mathcal{M}^{(p+h_{i-1},\, h_i)} & \text{if } i \in \{1,\dots,j-1\}, \\ \mathcal{M}^{(p+h_{j-1},\, h_j+q)} & \text{if } i = j, \\ \mathcal{M}^{(h_{i-1},\, h_i+q)} & \text{if } i \in \{j+1,\dots,n\} \end{cases} \tag{40}$$

is contained in $\underline{\mathcal{M}}_f$. Then $N_1, \dots, N_n \in \underline{\mathcal{M}}_f$, where, for all $(x,y) \in I^2_\le$,

$$N_i(x,y) = \begin{cases} \mathcal{M}^{(p,\, h_i)}(x, N_{i+1}(x,y)) & \text{if } i \in \{1,\dots,j-1\}, \\ \mathcal{M}^{(p,\, q)}(x,y) & \text{if } i = j \\ \mathcal{M}^{(h_{i-1},\, q)}(N_{i-1}(x,y), y) & \text{if } i \in \{j+1,\dots,n\}. \end{cases}$$

PROOF. The method of the proof is same as that of Corollary 2.4. For a given pair $(x,y) \in I^2_<$, define the mapping $\varphi_{(x,y)}$ as in (19) by the using the means $M_1, \dots, M_n$ defined in (40). Due to Theorem 1.11, it follows that, for all $(x,y) \in I^2_<$, the fixed point set $\Phi_{(x,y)}$ is the singleton $\{(\xi_1,\dots,\xi_n)\}$, where we have

$$\xi_j := \mathcal{M}^{(p,q)}(x,y) \quad \text{and} \quad \xi_i := \begin{cases} \mathcal{M}^{(p,h_i)}(x,\xi_{i+1}) & \text{if } i \in \{1,\dots,j-1\}, \\ \mathcal{M}^{(h_{i-1},q)}(\xi_{i-1},y) & \text{if } i \in \{j+1,\dots,n\}. \end{cases}$$

Thus, for $i \in \{1,\dots,n\}$, the function $N_i : I^2_\le \to \mathbb{R}$, $N_i(x,y) := \xi_i$ is the i^{th} descendant of the n-tuple

$$(\mathcal{M}^{(p+h_{i-1},h_i)},\dots,\mathcal{M}^{(p+h_{j-1},h_j+q)},\dots,\mathcal{M}^{(h_{i-1},h_i+q)}).$$

Hence, by Theorem 2.3, it follows that $N_i \in \underline{\mathcal{M}}_f$ for all $i \in \{1,\dots,n\}$. $\square$

2.3. The class of asymmetrically t-convex functions

In this section we restrict our attention to a special subfamily of $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$. First, for a given $t \in [0,1]$, denote $\mathcal{A}_t : I \times I \to \mathbb{R}$ the t-weighted arithmetic mean on I.

For a given extended real valued function $f : I \to \overline{\mathbb{R}}$ consider the sets $\underline{\mathcal{AC}}_f$ and $\overline{\mathcal{AC}}_f$ defined by

$$\underline{\mathcal{AC}}_f := \{t \in]0,1[\mid f \text{ is lower } \mathcal{A}_t\text{-convex on } I\}$$

and

$$\overline{\mathcal{AC}}_f := \{t \in]0,1[\mid f \text{ is upper } \mathcal{A}_t\text{-convex on } I\}.$$

If f is real-valued, then, clearly, these two sets are the same. Therefore, in this case, we will simply denote them by $\mathcal{AC}_f$. Note that, by the definitions, both sets can be empty. On the other hand, these sets can be easily identified with the subfamily of weighted arithmetic means in $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$ respectively. More precisely, $t \in \underline{\mathcal{AC}}_f$ and $s \in \overline{\mathcal{AC}}_f$ if and only if $\mathcal{A}_t|_{I^2_\le} \in \underline{\mathcal{M}}_f$ and $\mathcal{A}_s|_{I^2_\le} \in \overline{\mathcal{M}}_f$. The motivation for our investigations is the well known result due to N. Kuhn [**15**], which was mentioned in the Introduction. In view of our new notations, the theorem states that for a given function $f : I \to \mathbb{R}$, the intersection

$$\mathcal{C}_f = \mathcal{AC}_f \cap (1 - \mathcal{AC}_f) \tag{41}$$

is either empty or it can be written in the form $F \cap]0,1[$ for some suitable subfield $F \subseteq \mathbb{R}$. The following results are about some algebraical properties of the sets $\underline{\mathcal{AC}}_f$ and $\overline{\mathcal{AC}}_f$.

THEOREM 2.6. *Let $f : I \to \overline{\mathbb{R}}$ be any function and $\mathcal{AC} \in \{\underline{\mathcal{AC}}_f, \overline{\mathcal{AC}}_f\}$. Then the following statements hold.*

(1) If $t, s_1, s_2 \in \mathcal{AC}$ with $s_1 < s_2$, then $ts_2 + (1-t)s_1 \in \mathcal{AC}$.

(2) If $t, s \in \mathcal{AC}$, then ts and $1-(1-t)(1-s)$ also belong to $\mathcal{AC}$.

(3) The set $\mathcal{AC}$ is dense in the open unit interval, provided that it is nonempty.

PROOF. We verify only the statements about $\underline{\mathcal{AC}}_f$. The proof for $\overline{\mathcal{AC}}_f$ is analogous.

Let $t, s_1, s_2 \in \underline{\mathcal{AC}}_f$ with $s_1 < s_2$ be any parameters. Then the means $\mathcal{A}_t, \mathcal{A}_{s_1}$ and $\mathcal{A}_{s_2}$ belong to $\underline{\mathcal{M}}_f$ and, because of $s_1 < s_2$, we have $\mathcal{A}_{s_2} < \mathcal{A}_{s_1}$ on $I^2_<$. Using Proposition 4 for $M := \mathcal{A}_t$, $N_1 := \mathcal{A}_{s_2}$ and $N_2 := \mathcal{A}_{s_1}$, we obtain that $\mathcal{A}_t \circ (\mathcal{A}_{s_2}, \mathcal{A}_{s_1}) \in \underline{\mathcal{M}}_f$. On the other hand, for $(x, y) \in I^2_<$, we have

$$\begin{aligned}
\mathcal{A}_t \circ (\mathcal{A}_{s_2}, \mathcal{A}_{s_1})(x, y) &= \mathcal{A}_t(\mathcal{A}_{s_2}(x, y), \mathcal{A}_{s_1}(x, y)) \\
&= \mathcal{A}_t(s_2x + (1-s_2)y, s_1x + (1-s_1)y) \\
&= (ts_2 + (1-t)s_1)x + (1-(ts_2 + (1-t)s_1))y \\
&= \mathcal{A}_{ts_2+(1-t)s_1}(x, y).
\end{aligned}$$

Consequently $ts_2 + (1-t)s_1 \in \underline{\mathcal{AC}}_f$, which proves the statement (1).

To prove (2), observe that, under our notation, $\min = \mathcal{A}_1$ and $\max = \mathcal{A}_0$ on $I^2_\leq$. Thus, according to the second statement of Proposition 4, the means $\mathcal{A}_t \circ (\mathcal{A}_1, \mathcal{A}_s)$ and $\mathcal{A}_t \circ (\mathcal{A}_s, \mathcal{A}_0)$ belong to $\underline{\mathcal{M}}_f$. Then the same calculation yields that $1-(1-t)(1-s)$ and ts belong to $\underline{\mathcal{AC}}_f$, respectively.

To verify (3), assume that $\underline{\mathcal{AC}}_f$ is nonempty and indirectly suppose that $\underline{\mathcal{AC}}_f$ is not dense in $]0,1[$, that is there exist $\alpha < \beta$ in $[0,1]$ such that $\underline{\mathcal{AC}}_f \cap]\alpha, \beta[$ is empty. We may also assume that the interval $]\alpha, \beta[$ is maximal, or equivalently, for all $\varepsilon > 0$, the intersection $\underline{\mathcal{AC}}_f \cap]\alpha - \varepsilon, \beta + \varepsilon[$ is nonempty. Observe that, due to the second assertion of the theorem, it easily follows that $0 < \alpha$ and $\beta < 1$. Indeed, if $t \in \underline{\mathcal{AC}}_f$ is arbitrary, then, due to the fact that $\underline{\mathcal{AC}}_f$ is closed under the multiplication, for all $k \in \mathbb{N}$, the value t^k belongs to $\underline{\mathcal{AC}}_f$. Thus any open neighborhood of zero contains an element from $\underline{\mathcal{AC}}_f$, which means that $0 < \alpha$. Similarly, using the closedness of $\underline{\mathcal{AC}}_f$ under the operation $(t, s) \longmapsto 1-(1-t)(1-s)$, we get that $\beta < 1$. Thus we obtained that $[\alpha, \beta] \subseteq]0,1[$. Now, let $t \in \underline{\mathcal{AC}}_f$ be arbitrarily fixed and $(r_n), (s_n) \subseteq \underline{\mathcal{AC}}_f$ be sequences such that $r_n \nearrow \alpha$ and $s_n \searrow \beta$ as $n \to \infty$. Then, in view of the first statement of the theorem, $ts_n + (1-t)r_n \in \underline{\mathcal{AC}}_f$ for all $n \in \mathbb{N}$ and $ts_n + (1-t)r_n \to t\beta + (1-t)\alpha \in]\alpha, \beta[$ as $n \to \infty$. Therefore, for sufficiently large n, we get that $ts_n + (1-t)r_n \in]\alpha, \beta[$, which contradicts that $\underline{\mathcal{AC}}_f \cap]\alpha, \beta[$ is empty. Hence $\underline{\mathcal{AC}}_f$ must be dense in $]0,1[$. $\square$

COROLLARY 2.7. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \overline{\mathbb{R}}$, $n \geq 2$ and $s_1, \dots, s_n \in \underline{\mathcal{AC}}_f$. Then $\sigma_i \in \underline{\mathcal{AC}}_f$ for all $i \in \{1, \dots, n\}$, where*

$$\sigma_i := \Big(\sum_{j=i}^{n} \prod_{k=1}^{j} \frac{s_k}{1 - s_k} \Big) \Big(\sum_{j=0}^{n} \prod_{k=1}^{j} \frac{s_k}{1 - s_k} \Big)^{-1}. \tag{42}$$

PROOF. Apply Corollary 2.4 under $h := \mathrm{id}$. □

COROLLARY 2.8. *For a function $f : I \to \overline{\mathbb{R}}$ the following statements hold.*

(1) If $1/2 \in \underline{\mathcal{AC}}_f$ then $\mathbb{Q} \cap]0, 1[\subseteq \underline{\mathcal{AC}}_f$.

(2) If $\ell/m \in \underline{\mathcal{AC}}_f$ for some $\ell, m \in \mathbb{N}$ with $\ell < m$ and $2\ell \neq m$, then, for all $n \geq 2$ and for all $i \in \{1, \dots, n\}$, the fraction

$$r_i := \frac{\ell^{n+1} - \ell^i (m - \ell)^{n+1-i}}{\ell^{n+1} - (m - \ell)^{n+1}}$$

belongs to $\underline{\mathcal{AC}}_f$.

PROOF. To prove (1), assume that $1/2 \in \underline{\mathcal{AC}}_f$ and let $p, q \in \mathbb{N}$ be arbitrarily fixed numbers such that $q > 1$ and $p < q$. For $q = 2$, the statement (1) is trivial, thus we may assume that $q > 2$. Now set $n := q - 1$ and $i_0 := q - p$. Then $n \geq 2$ and $i_0 \in \{1, \dots, n\}$. Thus, using Corollary 2.7 for $s_1 := \dots = s_n := 1/2$, we get that

$$\sigma_{i_0} = \frac{n - i_0 + 1}{n + 1} = \frac{q - 1 - (q - p) + 1}{q - 1 + 1} = \frac{p}{q}.$$

This means that $\mathbb{Q} \cap]0, 1[\subseteq \underline{\mathcal{AC}}_f$.

To prove (2), assume that $\ell/m \in \underline{\mathcal{AC}}_f$ for some $\ell, m \in \mathbb{N}$, where $\ell < m$ and $2\ell \neq m$. Let further $n \geq 2$ be arbitrarily fixed and set $s_1 := \dots = s_n := \ell/m$. Then a simple calculation yields that $\sigma_i = r_i$ for all $i \in \{1, \dots, n\}$. Due to Corollary 2.7, we get that $r_i \in \underline{\mathcal{AC}}_f$ for all $i \in \{1, \dots, n\}$. □

2.4. Counterpart of Kuhn's Theorem

Now, we turn to the main difference between the standard and asymmetrical upper convexity of functions.

We recall that the t-convexity of a real valued function implies its Jensen-convexity provided that t is different from 0 and 1. One can ask, what can we state about such an implication if we turn to the asymmetric notion, that is, if we require the validity of

$$f(tx + (1 - t)y) \leq t f(x) + (1 - t) f(y) \tag{43}$$

only for pairs $(x, y) \in I \times I$ with $x < y$. This problem was first formulated by Zsolt Páles and we have only partial results. Michał Lewicki and Andrzej Olbryś in [**16**, Example 3.1.] obtained the following example.

PROPOSITION 5. *For a given transcendental number $t \in [0, 1]$, there exists a function $d_0 : \mathbb{R} \to \mathbb{R}$ which, for all $x, y \in \mathbb{R}$ with $x < y$, fulfills* (43) *with a sharp inequality, furthermore we also have*

$$(44) \quad d_0((1-t)x + ty) > (1-t)d_0(x) + d_0 f(y), \qquad (x, y \in \mathbb{R},\ x < y).$$

Shortly, there exists a *strictly asymmetrically t-convex* real valued function defined on $\mathbb{R}$, which is *strictly asymmetrically $(1-t)$-concave* provided that t is not an algebraic number.

To perform the proof, we need the notion of algebraic derivations. We say that a function $d : \mathbb{R} \to \mathbb{R}$ is an *algebraic derivation* if, for all $x, y \in \mathbb{R}$, we have

$$d(x + y) = d(x) + d(y) \qquad \text{and} \qquad d(xy) = xd(y) + d(x)y,$$

that is, d is *additive* and fulfills the *Leibniz rule*, respectively. It can be shown that any algebraic derivation vanishes on the field of algebraic numbers, furthermore, for all transcendental number $\lambda \in \mathbb{R}$, there exists an algebraic derivation which does not vanish at λ.

PROOF OF PROPOSITION 5. Let $t \in [0, 1]$ be a fixed transcendental parameter and $d_0 : \mathbb{R} \to \mathbb{R}$ be an algebraic derivation such that $d_0(t) > 0$. If $x, y \in \mathbb{R}$ are any points with $x < y$ then the validity of the sharp version of (43) with $f = d_0$ is equivalent to the validity of

$$d_0(t)(x - y) < 0,$$

which is obviously true. Using this, we also have

$$0 < -d_0(t)(x - y) = \big(d_0(1) - d_0(t)\big)(x - y) = d_0(1 - t)(x - y),$$

which proves that (44) holds too. □

Now we show that, for certain rational numbers t, there exists an upper $\mathcal{A}_t$-convex extended real valued function, which is not upper $\mathcal{A}_{1-t}$-convex. Such a function cannot be t-convex. Having this example and the above result of Lewicki and Olbryś, it is still an open problem if there exists a *real-valued* function with the same property.

To construct our function, let us define the sets $\mathbb{Q}_0$ and $\mathbb{Q}_1$ by

$$\mathbb{Q}_0 := \left\{ \frac{2k}{2n-1} \,\middle|\, k \in \mathbb{Z},\ n \in \mathbb{N} \right\} \quad \text{and} \quad \mathbb{Q}_1 := \left\{ \frac{2k-1}{2n-1} \,\middle|\, k \in \mathbb{Z},\ n \in \mathbb{N} \right\}.$$

It is easy to see, that the sets $\mathbb{Q}_0$ and $\mathbb{Q}_1$ are disjoint and that we have the inclusions

$$\begin{aligned} &\mathbb{Q}_0 + \mathbb{Q}_0 \subseteq \mathbb{Q}_0, \qquad &&\mathbb{Q}_0 + \mathbb{Q}_1 \subseteq \mathbb{Q}_1, \qquad &&\mathbb{Q}_1 + \mathbb{Q}_1 \subseteq \mathbb{Q}_0,\\ &\mathbb{Q}_0\mathbb{Q}_0 \subseteq \mathbb{Q}_0, &&\mathbb{Q}_0\mathbb{Q}_1 \subseteq \mathbb{Q}_0, &&\mathbb{Q}_1\mathbb{Q}_1 \subseteq \mathbb{Q}_1. \end{aligned} \tag{45}$$

THEOREM 2.9. *Let $I \subseteq \mathbb{R}$ be any subinterval with $a := \sup I \in I \cap \mathbb{Q}_1$, $C : I \to \mathbb{R}$ be any convex function, and define $f : I \to \overline{\mathbb{R}}$ by*

$$f(x) := \begin{cases} C(x) & \text{if } x \in (I \cap \mathbb{Q}_0) \cup \{a\}, \\ +\infty & \text{if } x \in I \setminus (\mathbb{Q}_0 \cup \{a\}). \end{cases} \tag{46}$$

Then, for all $t \in]0,1[\cap \mathbb{Q}_1$, the function f is upper $\mathcal{A}_t$-convex but it is not upper $\mathcal{A}_{1-t}$-convex.

PROOF. Let $x, y \in I$ with $x < y$ and $t \in]0,1[\cap \mathbb{Q}_1$ be arbitrarily fixed. Then $1 - t \in \mathbb{Q}_0$. We need to check that (34) is satisfied with $\mathcal{A}_t$ for the function f. This is equivalent to the validity of the inequality

$$f(tx + (1-t)y) \le tf(x) \dot{+} (1-t)f(y). \tag{47}$$

If $\max\{f(x), f(y)\} = +\infty$, then the right hand side of (47) is equal to $+\infty$, thus, we can suppose that the right hand side is finite, that is $f(x) = C(x)$ and $f(y) = C(y)$. Now we have $x \in \mathbb{Q}_0$ and $y \in \mathbb{Q}_0 \cup \mathbb{Q}_1$. Then, using (45), it follows that $tx + (1-t)y \in \mathbb{Q}_0$. Therefore, applying the convexity of C, we get

$$\begin{aligned} f(tx + (1-t)y) &= C(tx + (1-t)y) \\ &\le tC(x) + (1-t)C(y) = tf(x) \dot{+} (1-t)f(y). \end{aligned}$$

This proves that f is upper $\mathcal{A}_t$-convex for all $t \in]0,1[\cap \mathbb{Q}_1$.

To show that f is not upper $\mathcal{A}_{1-t}$-convex, let $y := a \in \mathbb{Q}_1$ and let $x \in I \cap \mathbb{Q}_0$ be an arbitrary point. It follows from (45) that the convex combination $(1-t)x + ty$ belongs to $\mathbb{Q}_1$ and it is also different from a. Therefore we have $f((1-t)x + ty) = +\infty$ and $(1-t)f(x) \dot{+} tf(y) = (1-t)C(x) + tC(y) \in \mathbb{R}$, which means that (47) cannot be satisfied. □

COROLLARY 2.10. *Keeping the above notation and conditions, $\overline{\mathcal{AC}}_f$ is not closed under addition, consequently it cannot be written as an intersection of $]0,1[$ and a proper subfield of $\mathbb{R}$.*

PROOF. For arbitrarily fixed parameters $s, t \in]0,1[\cap \mathbb{Q}_1 \subseteq \overline{\mathcal{AC}}_f$ with $s + t < 1$, in view of (45), the sum $s + t$ belongs to $\mathbb{Q}_0$. To prove that $s + t \notin \overline{\mathcal{AC}}_f$, we construct $x < y$ in I such that

$$f((s+t)x + (1-(s+t))y) > (s+t)f(x) + (1-(s+t))f(y). \tag{48}$$

Let $x \in I \cap \mathbb{Q}_0$ be arbitrarily fixed and set $y := a$. Then, using again (45), the convex combination $u := (s+t)x + (1-(s+t))y$ belongs to $I \cap \mathbb{Q}_1$ and it is also different from a. Consequently, $f(u)$ must be $+\infty$, on the other hand

$$(s+t)f(x) + (1-(s+t))f(y) = (s+t)h(x) + (1-(s+t))h(y) \in \mathbb{R}.$$

This means that (48) is satisfied. $\square$

Introduction to the Second Part

In the second part of the dissertation we are going to investigate the phenomenon mentioned in II. We recall that a function $f : D \to \mathbb{R}$, defined on a convex subset D of a linear space X, is called *midpoint convex* or *Jensen convex* if

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2}, \qquad (x,y \in D). \tag{49}$$

In view of Theorem 0.1, the validity of (49) is equivalent to the validity of

$$f\Big(\frac{x_1+\cdots+x_n}{n}\Big) \le \frac{f(x_1)+\cdots+f(x_n)}{n}, \qquad (x_1,\dots,x_n \in D). \tag{50}$$

for any fixed $n \in \mathbb{N}$. Now we would like to focus only the derivation of the inequality (49) from (50). Obviously, it is enough to prove that having (50) for some fixed $n \in \mathbb{N}$ with $n \ge 2$, it implies the validity of

$$f\Big(\frac{x_1+\cdots+x_{n-1}}{n-1}\Big) \le \frac{f(x_1)+\cdots+f(x_{n-1})}{n-1}, \quad (x_1,\dots,x_{n-1} \in D).$$

To prove this, let $x_1,\dots,x_{n-1} \in D$ be any points and define y as the arithmetic mean of them, that is,

$$y := \frac{1}{n-1}(x_1+\cdots+x_{n-1}) = \mathcal{A}_{\frac{1}{n-1}}(x_1,\dots,x_{n-1}).$$

Then, one can easily observe that y satisfies the equality

$$\mathcal{A}_{\frac{1}{n}}(x_1,\dots,x_{n-1},y) = y \tag{51}$$

on the set conv$\{x_1,\dots,x_{n-1}\}$. Now, using the definition of y, the inequality (50), and, finally, the fact that y solves (51), we obtain that

$$\begin{aligned} f\Big(\frac{x_1+\cdots+x_{n-1}}{n-1}\Big) &= f(y) = f\Big(\frac{x_1+\cdots+x_{n-1}+y}{n}\Big) \\ &\le \frac{f(x_1)+\cdots+f(x_{n-1})+f(y)}{n} \\ &= \frac{f(x_1)+\cdots+f(x_{n-1})+f\big(\frac{x_1+\cdots+x_{n-1}}{n-1}\big)}{n} \end{aligned}$$

holds. Multiplying both sides by n, subtracting $f(\frac{x_1+\cdots+x_{n-1}}{n-1})$ from both sides, and dividing the inequality so obtained by $n-1$, we get that

$$f\Big(\frac{x_1+\cdots+x_{n-1}}{n-1}\Big) \leq \frac{f(x_1)+\cdots+f(x_{n-1})}{n-1}.$$

In view of this short calculation, the reducibility of the Jensen inequality strongly depends on the fact that (51) has a unique solution, which is nothing else, but the $n-1$-variable arithmetic mean of the points in question.

Now let us turn to the generalized problem. The main idea is to replace the two appearance of the arithmetic mean in the inequality (50) by arbitrary n-variable means $M : D^n \to D$ and $N : I^n \to I$, and to consider functions $f : D \to I$ satisfying

$$(52) \qquad f\big(M(x_1,\dots,x_n)\big) \leq N\big(f(x_1),\dots,f(x_n)\big), \quad (x_1,\dots,x_n \in D).$$

According to the previous calculation, first we have to investigate the reducibility of the means M and N. This will be interpreted using a fixed point equation, which, in the case of the arithmetic means, goes back to the equation (51). Finally, the type of the solution will be also important for us, namely, that the solution remain in the same class.

Our main aim is to describe general sufficient conditions under which, for $k \in \{1,\dots,n\}$, a k-variable convexity property can be deduced from the inequality (52). This requires the construction of k-variable means which are the reductions of M and N, respectively. The construction and computation of the k-variable reductions will be elaborated in the class of Daróczy means mentioned in Section 1.1. of the first chapter. Then, it will be also described related to the class of generalized deviation means, which was introduced in the paper [**12**] and which provides a broad class of means for the vector valued setting. We also demonstrate how generalized deviation means can be derived as solutions of convex minimum problems. Finally, we consider and establish the reducibility property of Hölder–Minkowski type inequalities under natural assumptions.

CHAPTER 3

Reducibility of means and convexity properties

3.1. Reducibility of mean values

To avoid the long computations and to make our results more compact, let us introduce some notations. For a nonempty set S and for a positive integer $n \in \mathbb{N}$, we will identify the elements of the Cartesian product S^n with the set of all functions mapping $\mathbb{N}_n$ to S, that is, with the set $S^{\mathbb{N}_n} := \{x : \mathbb{N}_n \to S\}$. If $x \in S^n$, then $x(i)$ will simply denoted by x_i for all $i \in \mathbb{N}_n$.

Now we are able to introduce the main notation of Chapter 3. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and $\chi : \mathbb{N}_k \to \mathbb{N}_n$ be an injective function. For $x \in S^k$ and $y \in S$, the symbol $(x|\chi)(y)$ will stand for the element of S^n defined by

$$(x|\chi)(y)_i := \begin{cases} y & \text{if } i \in \mathbb{N}_n \setminus \chi(\mathbb{N}_k), \\ x_j & \text{if } i \in \chi(\mathbb{N}_k) \text{ and } i = \chi(j). \end{cases}$$

To understand the above notation, consider the following example. Let $n \in \mathbb{N}$ with $n \geq 2$, $k := n - 1$, and let $\chi := \mathrm{id}\,|_{\{1,\dots,n-1\}}$. Then, for any vector $x = (x_1, \dots, x_{n-1}) \in D^{n-1}$ and $y \in D$, we have

$$(x|\chi)(y) = (x_1, \dots, x_{n-1}, y) \in D^n.$$

Observe that this vector came up in the argument of the arithmetic mean in the equation (51).

To see a much simpler example, set $n := 5$, $k := 3$, and let us define the function $\chi : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$ by

$$\chi(1) := 2, \qquad \chi(2) := 5, \qquad \text{and} \qquad \chi(3) := 1.$$

Then, for $x = (x_1, x_2, x_3) \in D^3$ and $y \in D$, we have

$$(x|\chi)(y) = (x_3, x_1, y, y, x_2) \in D^5.$$

In what follows, we define the notions of continuity and reduction of a mean $M : D^n \to X$ with respect to a given injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$.

We say that a mean $M : D^n \to X$ is χ-*continuous* if, for any $x \in D^k$, the mapping $m_{x,M} : \mathrm{conv}(x(\mathbb{N}_k)) \to X$ defined as

$$m_{x,M}(y) := M\big((x|\chi)(y)\big) \tag{53}$$

is continuous on conv$(x(\mathbb{N}_k))$.

The mean $M : D^n \to X$ will be called χ-*reducible* if there exists a mean $K : D^k \to X$ such that, for all $x \in D^k$, the vector $y = K(x)$ is a solution of the equation

$$M\big((x|\chi)(y)\big) = y. \tag{54}$$

In this case, the mean K will be called a χ-*reduction of* M. If for all $x \in D^k$, the equation (54) has a unique solution $y \in \text{conv}(x(N_k))$, that is, if K is uniquely determined, then we say that M is a *uniquely* χ-*reducible mean*, furthermore, the mean K will be called the χ-*reduction of* M and will be denoted by M_χ.

To make the notion of reducibility more clear, let us turn back our previous example, where $(x|\chi)(y)$ was the vector $(x_1, \dots, x_{n-1}, y)$ for all $(x_1, \dots, x_{n-1}) \in D^{n-1}$ and $y \in D$. Replacing the mean M by the arithmetic mean $\mathcal{A}_{\frac{1}{n}}$ in (54), the equation (54) goes back to the form (51).

Here we also note that, in general, if an n-variable symmetric mean is reducible for some injective function mapping $\mathbb{N}_k$ to $\mathbb{N}_n$, then it is also reducible with respect to any injective $\mathbb{N}_n$-valued function defined on the set $\mathbb{N}_k$.

The next theorem establishes a crucial connection among the notions of χ-reducibility and χ-continuity.

THEOREM 3.1. *The mean $M : D^n \to X$ is χ-reducible provided that it is χ-continuous.*

PROOF. Let $x \in D^k$ be arbitrarily fixed and define the function $m_{x,M} :$ conv$(x(\mathbb{N}_k)) \to X$ as in (53). Obviously, the target set of $m_{x,M}$ is conv$(x(\mathbb{N}_k))$, and, because of the χ-continuity of the mean M, the function $m_{x,M}$ is continuous on the compact convex set conv$(x(\mathbb{N}_k))$. Thus, due to the Brouwer Fixed Point Theorem, the fixed point set

$$\text{Fix}(m_{x,M}) := \{y \in \text{conv}(x(\mathbb{N}_k)) \mid m_{x,M}(y) = y\}$$

is nonempty. Define now $K(x)$ to be any element of Fix$(m_{x,M})$. Then, for all $x \in D^k$, the vector $y = K(x)$ will be a solution of (54), meaning that K is a χ-reduction of M. □

For the setting of unique χ-reducibility, we shall need the following useful lemma.

LEMMA 3.2. *Let $I \subseteq \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and $\chi : \mathbb{N}_k \to \mathbb{N}_n$ be an injective function. Assume that the χ-continuous mean $M : I^n \to \mathbb{R}$ is uniquely χ-reducible. Then, for all $x \in I^k$ and for all*

$y \in J_x := [\min(x), \max(x)]$, *we have*

$$\operatorname{sgn}\big(m_{x,M}(y) - y\big) = \operatorname{sgn}(M_\chi(x) - y). \tag{55}$$

PROOF. Let $x \in I^k$ be arbitrarily fixed. If $\min(x) = \max(x)$, then the statement is obvious, thus we may assume that $\min(x) < \max(x)$. For the sake of brevity, define

$$\mu_{x,M}(y) := m_{x,M}(y) - y$$

for $y \in J_x$. Then, due to the definition of the χ-reduction of means, we have $\mu_{x,M}(y) = 0$ for $y \in J_x$ if and only if $y = M_\chi(x)$.

First assume that $M_\chi(x)$ belongs to the interior of J_x. Because of the mean-property of M, obviously, we have $\mu_{x,M}(\max(x)) < 0 < \mu_{x,M}(\min(x))$. Then, because of the uniqueness of the zero of $\mu_{x,M}$ and of the χ-continuity of M on the interval J_x, it immediately follows that $\mu_{x,M}$ must be strictly positive on the subinterval $[\min(x), M_\chi(x)[$, and it must be strictly negative on the subinterval $]M_\chi(x), \max(x)]$.

On the other hand, if either $M_\chi(x) = \min(x)$ or $M_\chi(x) = \max(x)$, then a similar argument shows that the function $\mu_{x,M}$ is strictly positive on the interval $J_x \setminus \{\min(x)\}$ or it is strictly negative on the entire interval $J_x \setminus \{\max(x)\}$, respectively, which finishes the proof. □

3.2. Reducibility of special mean values

Before we turn the most general setting, we demonstrate the reducibility of some easy to use mean values. The following notation will be very useful. For a nonempty set S and for $u = (u_1, \dots, u_n) \in S^n$ let u_χ stand for the k-tuple $(u_{\chi_1}, \dots, u_{\chi_k}) \in S^k$.

Concerning the χ-reduction of a *functionally weighted arithmetic mean*, which is a very special deviation mean, we have the following result. Roughly speaking, the functionally weighted arithmetic mean is uniquely χ-reducible, for all injective function χ, and the χ-reduction is a functionally weighted arithmetic mean again, where the weight functions are determined by the members of the image of χ.

PROPOSITION 6. *Let $\omega : D \to \mathbb{R}_+^n$ be a weight function. Then we have*

$$\mathcal{A}_\chi^\omega = \mathcal{A}^{\omega_\chi}.$$

PROOF. For the mean $M = \mathcal{A}^\omega$ and for $x \in D^k$, the equation (54) can be rewritten as

$$\frac{\omega_{\chi_1}(x_1)x_1 + \cdots + \omega_{\chi_k}(x_k)x_k + \big(\sum_{i \notin \chi(\mathbb{N}_k)} \omega_i(y)\big)y}{\omega_{\chi_1}(x_1) + \cdots + \omega_{\chi_k}(x_k) + \sum_{i \notin \chi(\mathbb{N}_k)} \omega_i(y)} = y.$$

It immediately follows that the unique solution y of this equation is of the form

$$y = \frac{\omega_{\chi_1}(x_1)x_1 + \cdots + \omega_{\chi_k}(x_1)x_k}{\omega_{\chi_1}(x_1) + \cdots + \omega_{\chi_k}(x_k)} = \mathcal{A}^{\omega_\chi}(x),$$

which proves that $\mathcal{A}^{\omega}_{\chi}(x) = \mathcal{A}^{\omega_\chi}(x)$. □

As we will see later, similar result remains true in the class of standard deviation functions. Instead of proving this directly, first we are going to extend the notion of deviation means and then formulate the theorem concerning this class. Before that, we present a uniquely reducible mean, where the reduction is of the same form, and which does not belong to the class of standard deviation means.

Let $s, t \geq 0$ be fixed real numbers and, for $n \in \mathbb{N}$, define

$$\Lambda_n^{(s,t)}(x) := \frac{1}{s+n+t}\Bigg(s\min(x) + \sum_{\xi \in x(\mathbb{N}_n)} \xi + t\max(x)\Bigg), \qquad (x \in I^n).$$

It is easy to see that the above expression indeed defines an n-variable symmetric mean on I. If $s = t = 0$, then it gives back the n-variable arithmetic mean, and $\Lambda_n^{(s,t)}$ is not a deviation mean whenever $s + t > 0$.

PROPOSITION 7. *Let $s, t \geq 0$ be fixed numbers, $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $\chi : \mathbb{N}_k \to \mathbb{N}_n$ be any injective function. Then $\Lambda_n^{(s,t)}$ is uniquely χ-reducible and its χ reduction is $\Lambda_k^{(s,t)}$.*

PROOF. If $s = t = 0$, then our statement is a direct consequence of Proposition 6, hence we assume that one of them is different from zero. Let $x \in I^k$ be any point. Our mean is χ-continuous, consequently, in view of Theorem 3.1, it must be χ-reducible. Thus let $y \in \operatorname{conv}(x(\mathbb{N}_k))$ be a solution of the equation (54). Then, for $M = \Lambda_n^{(s,t)}$, the equation (54) can be written in the form

$$\frac{1}{s+n+t}\Bigg(s\min\big(x(\mathbb{N}_k) \cup \{y\}\big) + \sum_{\xi \in x(\mathbb{N}_k)} \xi + (n-k)y \\ + t\max\big(x(\mathbb{N}_k) \cup \{y\}\big)\Bigg) = y.$$

Because of the inclusion $y \in \operatorname{conv}(x(\mathbb{N}_k))$, the singleton $\{y\}$ can be omitted in the argument of the minimum and the maximum. Thus we obtain that

$$s\min(x) + \sum_{\xi \in x(\mathbb{N}_k)} \xi + (n-k)y + t\max(x) = (s+n+t)y,$$

from where y can be expressed as

$$y = \frac{1}{s+k+t}\left(s\min(x) + \sum_{\xi\in x(\mathbb{N}_k)} \xi + t\max(x)\right) = \Lambda_k^{(s,t)}(x).$$

This finishes the proof. □

3.3. Extension of Daróczy means and their reducibility

In the sequel, let X be a *Hausdorff topological vector space* over $\mathbb{R}$. For an arbitrary nonempty subset $S \subseteq X$, let S^* denote the the space of all continuous linear functionals defined on the linear hull of $S - S$. In what follows, we shall extend the notion of deviation function and deviation mean to convex subsets of linear spaces.

Let $D \subseteq X$ be a nonempty convex set. We say that a mapping $E : D \times D \to D^*$ is a *generalized deviation function* if it satisfies the following two properties:

(GE1) $E(u, u) = 0$ for all $u \in D$, and

(GE2) for all fixed $u \in D$, the function $v \mapsto -E(u, v)$ is continuous and *strictly monotone* on D, that is

$$(E(u, v) - E(u, w))(v - w) < 0, \qquad (u, v, w \in D \text{ with } v \neq w).$$

The class of generalized deviation functions defined on D will be denoted by $\mathbf{E}(D)$. Observe that the properties (1) and (2) imply that, for a generalized deviation $E \in \mathbf{E}(D)$, we always have

$$E(u, v)(u - v) > 0, \qquad (u, v \in D,\ u \neq v). \tag{56}$$

Now, using a finite collection of generalized deviations, we can define means on the convex set D. In contrast to the definition of deviation means (that are defined on real intervals), the notion of generalized deviation mean will be defined by a system of inequalities.

Let $E = (E_1, \dots, E_n) \in \mathbf{E}(D)^n$. For $x \in D^n$, we say that the vector $y \in \mathrm{conv}(x(\mathbb{N}_n))$ is the *generalized E-deviation mean* of x if

$$(E_1(x_1, y) + \cdots + E_n(x_n, y))(x_i - y) \leq 0, \qquad (i \in \mathbb{N}_n). \tag{57}$$

If $y \in \mathrm{conv}(x(\mathbb{N}_n))$ exists and unique, then it will be denoted by $\mathcal{D}^E(x)$.

The next theorem states that the notion of generalized E-deviation mean is well-defined.

THEOREM 3.3. *Let $n \in \mathbb{N}$ and $E \in \mathbf{E}(D)^n$. Then, for all $x \in D^n$, there uniquely exists $y \in \mathrm{conv}(x(\mathbb{N}_n))$ such that the inequality* (57) *holds.*

PROOF. Let $x \in D^n$ be an arbitrarily fixed vector and, for the brevity, denote the compact convex set $\mathrm{conv}(x(\mathbb{N}_n))$ by C_x, finally, let us define the function $\mathcal{E}_{E,x} : D \to D^*$ by (4). Then, by the defining properties of generalized deviations, the function $-\mathcal{E}_{E,x}$ is continuous and strictly monotone. Observe, that the real valued mapping $\phi : C_x \times C_x \to \mathbb{R}$, given by

$$\phi(u,v) := \mathcal{E}_{E,x}(u)(v-u),$$

is continuous in its first variable, and, in view of the linearity of $\mathcal{E}_{E,x}(u)(\cdot)$ for any fixed $u \in C_x$, it is affine, that is, it is convex and concave simultaneously in its second variable. Thus, due to the Ky Fan Variational Inequality Theorem, there exists $y \in C_x$, such that

$$\begin{aligned}\sup_{v\in C_x} \mathcal{E}_{E,x}(y)(v-y) = \sup_{v\in C_x} \phi(y,v) &\le \sup_{w\in C_x} \phi(w,w)\\ &= \sup_{w\in C_x} \mathcal{E}_{E,x}(w)(w-w) = 0.\end{aligned}$$

Thus, for every $v \in C_x$, in particular, for every $v \in \{x_1,\dots,x_n\}$, we have

$$\mathcal{E}_{E,x}(y)(v-y) \le 0.$$

This proves the existence of $y \in \mathrm{conv}(x(\mathbb{N}_n))$ satisfying (57).

To prove the uniqueness, assume, indirectly, that there exist $y \ne z$ in $\mathrm{conv}(x(\mathbb{N}_n))$ satisfying (57). Then, for all $i \in \mathbb{N}_n$, we have

$$\mathcal{E}_{E,x}(y)(x_i-y) \le 0 \qquad \text{and} \qquad \mathcal{E}_{E,x}(z)(x_i-z) \le 0. \tag{58}$$

The vectors y and z belong to the convex hull of $x(\mathbb{N}_n)$, therefore there exist convex combination coefficients $\lambda_1,\dots,\lambda_n \ge 0$ with $\lambda_1+\cdots+\lambda_n = 1$ and $\mu_1,\dots,\mu_n \ge 0$ with $\mu_1+\cdots+\mu_n = 1$ such that

$$y = \lambda_1x_1+\cdots+\lambda_nx_n \qquad \text{and} \qquad z = \mu_1x_1+\cdots+\mu_nx_n.$$

Multiplying the first and the second inequalities in (58) by μ_i and λ_i, respectively, and then adding up the inequalities so obtained, we get

$$\mathcal{E}_{E,x}(y)(z-y) \le 0 \qquad \text{and} \qquad \mathcal{E}_{E,x}(z)(y-z) \le 0.$$

The sum of these two inequalities can be written as

$$\big(\mathcal{E}_{E,x}(y) - \mathcal{E}_{E,x}(z)\big)(y-z) \ge 0. \tag{59}$$

On the other hand, using the strict monotonicity of $(-\mathcal{E}_{E,x})$, we obtain that

$$(\mathcal{E}_{E,x}(y) - \mathcal{E}_{E,x}(z))(y-z) < 0,$$

which contradicts (59). This proves that the vector $y \in \mathrm{conv}(x(\mathbb{N}_n))$, satisfying the inequality (57), has to be uniquely determined. □

It is obvious that if X is the real line and $D \subseteq \mathbb{R}$ is an interval, then $D^* \equiv \mathbb{R}$, furthermore the notion of generalized deviation functions and generalized deviation means reduces to that of deviation functions and deviation means, respectively.

To see the statement about the means, let $n \in \mathbb{N}$, $E \in \mathbf{E}(D)^n$, and $x \in D^n$ be arbitrary, and assume that $\min(x) < \max(x)$. We need to show that the value $y \in D$ is the solution of the equation (3) in D if and only if it is the solution of the system of inequalities (57) in $\operatorname{conv}(x(\mathbb{N}_n)) = [\min(x), \max(x)]$.

If the value $y \in D$ is the solution of (3), or, equivalently, it is the E-deviation mean of x, then, the inequalities $\mathcal{E}_{E,x}(\min(x)) \geq 0 \geq \mathcal{E}_{E,x}(\max(x))$ show that $y \in [\min(x), \max(x)]$ and it trivially satisfies the inequalities of (57), that is, the vector y is the generalized E-deviation mean of x.

Conversely, assume that $y \in [\min(x), \max(x)]$ is the generalized E-deviation mean of x, which means, it is the solution of the system (57). Then, in particular, we have

$$\mathcal{E}_{E,x}(y) \cdot (\min(x) - y) \leq 0 \quad \text{and} \quad \mathcal{E}_{E,x}(y) \cdot (\max(x) - y) \leq 0. \tag{60}$$

If y were one of the endpoints of the interval $[\min(x), \max(x)]$, say $y = \min(x)$, then $y < \max(x)$, therefore the second inequality yields that $\mathcal{E}_{E,x}(y) \leq 0$. On the other hand, $y \leq x_i$ for all indices $i \in \mathbb{N}_n$, and, for at least one index $j \in \mathbb{N}_n$, we have that $y < x_j$. Thus, for all $i \in \mathbb{N}_n$, the inequalities $E_i(x_i, y) \geq 0$ and $E_j(x_j, y) > 0$ hold. This implies that $\mathcal{E}_{E,x}(y) > 0$. The contradiction so obtained shows that y must be greater than $\min(x)$. Similarly, y must be lesser than $\max(x)$. Therefore, the two inequalities in (60) result that $\mathcal{E}_{E,x}(y)$ is nonnegative and also non-positive. Consequently, we must have $\mathcal{E}_{E,x}(y) = 0$, that is, y is the E-deviation mean of x.

Now we formulate the main theorem of Chapter 3.

THEOREM 3.4. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $E \in \mathbf{E}(D)^n$. Then the generalized E-deviation mean $\mathcal{D}^E : D^n \to D$ is reducible with respect to any injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$. Furthermore, the χ-reduction of $\mathcal{D}^E$ is uniquely determined, namely we have*

$$\mathcal{D}^E_\chi(x) = \mathcal{D}^{E_\chi}(x), \qquad (x \in D^k).$$

PROOF. Let $x \in D^k$ be arbitrarily fixed and denote the value $\mathcal{D}^{E_\chi}(x)$ by y_0. The property (1) of generalized deviations provides that

$$E_i\big((x|\chi)(y)_i, y\big) = \begin{cases} 0 & \text{if } i \in \mathbb{N}_n \setminus \chi(\mathbb{N}_k), \\ E_i(x_j, y) & \text{if } i \in \chi(\mathbb{N}_k) \text{ and } i = \chi(j). \end{cases}$$

Therefore,

$$\begin{aligned}\mathcal{E}_{E,(x|\chi)(y)}(y) &= E_1\big((x|\chi)(y)_1, y\big) + \cdots + E_n\big((x|\chi)(y)_n, y\big)\\ &= E_{\chi 1}(x_1, y) + \cdots + E_{\chi k}(x_k, y) = \mathcal{E}_{E_\chi, x}(y).\end{aligned}$$

In view of the definition of χ-reducibility, we need to show that $y = y_0$ is the unique solution of the equation $\mathcal{D}^E\big((x|\chi)(y)\big) = y$ in $\operatorname{conv}(x(\mathbb{N}_k))$, that is, $y = y_0$ is the unique solution of the system of inequalities

$$\mathcal{E}_{E,(x|\chi)(y)}(y)\big((x|\chi)(y)_i - y\big) = \mathcal{E}_{E_\chi,x}(y)\big((x|\chi)(y)_i - y\big) \le 0, \quad (i \in \mathbb{N}_n).$$

The inequalities automatically hold when $i \in \mathbb{N}_n \setminus \chi(\mathbb{N}_k)$, because in these cases we always have $(x|\chi)(y)_i = y$, therefore the above system is equivalent to

$$\mathcal{E}_{E_\chi,x}(y)(x_i - y) \le 0, \qquad (i \in \mathbb{N}_k). \tag{61}$$

In view of Theorem 3.3, the system of inequalities in (61) is uniquely solvable in $\operatorname{conv}(x(\mathbb{N}_k))$ and its solution y equals $y_0 = \mathcal{D}^{E_\chi}(x)$, which was to be proved. □

3.4. Characterization of standard Daróczy means

In the theorem below, we construct the large class of generalized deviations in terms families of relatively Gâteaux differentiable strictly convex functions. As a consequence of such a representation, generalized deviation means can be viewed as the unique minimizers of certain strictly convex functions.

Given an arbitrary set $S \subseteq X$, a point $u \in S$ is called a *relative algebraic interior point of* S if, for all $v \in S$, the set $\{t \in \mathbb{R} \mid tv + (1-t)u \in S\}$ is a right neighborhood of 0 in $\mathbb{R}$. The set S is said to be *relatively algebraically open* if every point of S is its relative algebraic interior point.

A function $f : S \to \mathbb{R}$ is called *relatively Gâteaux differentiable* at a relatively algebraically interior point u of S if there exists a continuous linear functional $f'(u) \in S^*$ such that, for all $v \in S$,

$$\lim_{t\to 0^+} \frac{f(u + t(v-u)) - f(u)}{t} = f'(u)(v-u). \tag{62}$$

The notion of Gâteaux differentiability with respect to a subspace of X (in our case, with respect to the linear span of $S - S$), was considered in the paper [**33**].

We need the following auxiliary result, which is the adaptation of some well-known theorems about convex functions to our setting.

THEOREM 3.5. *Let $D \subseteq X$ be a convex set and $f : D \to \mathbb{R}$ be a relatively Gâteaux differentiable function on D. Then the following statements hold.*

(1) D is relatively algebraically open and, for every $u \in D$, the relative Gâteaux derivative $f'(u)$ is uniquely determined.

(2) The function f is convex if and only if

$$f(v) \geq f(u) + f'(u)(v-u), \qquad (u, v \in D), \tag{63}$$

and f is strictly convex if and only if this inequality is strict whenever $u \neq v$.

(3) The function f is convex if and only if its Gâteaux derivative f' is monotone, that is,

$$(f'(u) - f'(v))(u-v) \geq 0, \qquad (u, v \in D), \tag{64}$$

and f is strictly convex if and only if this inequality is strict whenever $u \neq v$.

(4) If $S \subseteq D$ is a nonempty convex set and f attains its minimum at $u \in S$ on the set S, then

$$f'(u)(v-u) \geq 0, \qquad (v \in S), \tag{65}$$

Conversely, if f is convex and (65) *holds for some $u \in S$, then f attains its minimum at u on the set S.*

PROOF. Let u be arbitrarily fixed in D. Then, because of the convexity of D, for all $v \in D$, we have $[0,1] \subseteq \{t \in \mathbb{R} \mid tv + (1-t)u \in S\}$, which shows that u is a relative algebraic interior point of D. Assume that $f'(u)$ is not uniquely determined, that is, there exists $\varphi, \psi \in D^*$ such that, for all $v \in D$,

$$\lim_{t \to 0^+} \frac{f(u + t(v-u)) - f(u)}{t} = \varphi(v-u) = \psi(v-u).$$

Then, $(\varphi - \psi)(v-u) = 0$ for all $v \in D$. Now, let $h \in D - D$ be arbitrary. Then there exist $v, w \in D$ such that $h = v - w$, hence

$$(\varphi - \psi)(h) = (\varphi - \psi)(v-u) - (\varphi - \psi)(w-u) = 0.$$

Therefore, $\varphi - \psi$ vanishes on the linear span of $D - D$, showing that $\varphi = \psi$.

To prove (2), assume that f is convex. Then, for all $u, v \in D$, the map $t \mapsto \frac{1}{t}(f(u + t(v-u)) - f(u))$ is nondecreasing, hence

$$\begin{aligned} f(v) - f(u) &= \frac{f(u + 1(v-u)) - f(u)}{1} \\ &\geq \lim_{t \to 0} \frac{f(u + t(v-u)) - f(u)}{t} = f'(u)(v-u), \end{aligned}$$

which gives (63). If f is strictly convex and $u \neq v$, then $t \mapsto \frac{1}{t}(f(u+t(v-u))-f(u))$ is strictly increasing, which results that (63) holds with strict inequality.

For the converse, assume (63), and let $u, v \in D$ and $t \in [0,1]$ be arbitrary. Then, based on (63), we get

$$
\begin{aligned}
f(u) &\geq f(tu+(1-t)v) + f'(tu+(1-t)v)(u-(tu+(1-t)v)) \\
&= f(tu+(1-t)v) + (1-t)f'(tu+(1-t)v)(u-v), \\
f(v) &\geq f(tu+(1-t)v) + f'(tu+(1-t)v)(v-(tu+(1-t)v)) \\
&= f(tu+(1-t)v) + tf'(tu+(1-t)v)(v-u).
\end{aligned} \tag{66}
$$

Multiplying the first inequality by t, the second one by $(1-t)$, and adding up the inequalities so obtained side by side, we get

$$tf(u) + (1-t)f(v) \geq f(tu+(1-t)v),$$

which proves the convexity of f. If (63) holds with strict inequality for $u \neq v$ and $x \neq y$, then the inequalities in (66) are strict for $t \notin \{0,1\}$, hence we obtain the strict convexity of f.

To prove the second assertion, assume again that f is convex. Then (63) holds, thus, applying this inequality twice, we obtain that

$$f(v) \geq f(u) + f'(u)(v-u) \qquad \text{and} \qquad f(u) \geq f(v) + f'(v)(u-v)$$

for all $u, v \in D$. Adding up these inequalities side by side, it results that (64) is valid. If f is strictly convex and $u \neq v$, then (63) is strict, which yields that (64) is also strict.

Conversely, assume (64) and, for $u, v \in D$, define the function $f_{u,v} : [0,1] \to \mathbb{R}$ by

$$f_{u,v}(t) := f(tu+(1-t)v).$$

Observe that $f_{u,v}$ is differentiable on $[0,1]$, furthermore the derivative $f'_{u,v} := \frac{d}{dt} f_{u,v}$ is nondecreasing. Indeed, a short calculation shows that

$$f'_{u,v}(t) = \lim_{\tau \to t} \frac{f_{u,v}(\tau) - f_{u,v}(t)}{\tau - t} = f'(tu+(1-t)v)(u-v).$$

Now let $t, s \in [0,1]$ such that $t \neq s$. Then, due to (64), we have

$$
\begin{aligned}
0 &\leq (t-s)(f'(tu+(1-t)v) - f'(su+(1-s)v))(u-v) \\
&= (t-s)(f'_{u,v}(t) - f'_{u,v}(s)),
\end{aligned}
$$

which implies that $f'_{u,v}$ is nondecreasing. We obtained that $f_{u,v}$ is convex for any fixed $u, v \in D$.

Finally, let $u, v \in D$ and $t \in [0, 1]$ be arbitrarily fixed. Then we have the calculation

$$\begin{aligned} f(tu + (1-t)v) &= f_{u,v}(t) = f_{u,v}(t \cdot 1 + (1-t) \cdot 0) \\ &\leq t f_{u,v}(1) + (1-t) f_{u,v}(0) = t f(u) + (1-t) f(v), \end{aligned}$$

consequently f is convex.

For the third statement, let $S \subseteq D$ be a nonempty convex set and assume that f attains its minimum on S at the point $u \in S$. Then, for all $t \in [0, 1]$ and $v \in S$, we have that $f(u + t(v - u)) \geq f(u)$. Hence, in view of formula (62), we get that $f'(u)(v - u) \geq 0$ for all $v \in S$.

Now assume that f is convex and, for some $u \in S$, (65) holds. Then, applying (63) for $u, v \in S$, we get

$$f(v) \geq f(u) + f'(u)(v - u) \geq f(u).$$

This proves that f attains its minimum on S at the point $u \in S$. $\square$

To formulate the next theorem, let $\boldsymbol{F}(D)$ denote the class of real functions $F : D \times D \to \mathbb{R}$ which possess the following property.

(F) For any fixed $u \in D$, the function $F_u := F(u, \cdot)$ is relatively Gâteaux differentiable and strictly convex on D, furthermore the derivative F_u' vanishes at u.

THEOREM 3.6. *Assume that $D \subseteq X$ is a convex set and let $F \in \boldsymbol{F}(D)$. Then the function $E_F : D \times D \to D^*$, defined by*

$$E_F(u, v) = -F_u'(v), \tag{67}$$

is a generalized deviation. Furthermore, if $n \in \mathbb{N}$, $F \in \boldsymbol{F}(D)^n$ and $E_F = (E_{F_1}, \ldots, E_{F_n})$, then, for $x \in D^n$, the equality $y = \mathcal{D}^{E_F}(x)$ holds if and only if y is the unique minimizer over $\mathrm{conv}(x(\mathbb{N}_n))$ *of the function $\mathcal{F}_{F,x} : D \to \mathbb{R}$ defined by*

$$\mathcal{F}_{F,x}(v) := F_1(x_1, v) + \cdots + F_n(x_n, v). \tag{68}$$

Conversely, if X is the real line and D is an open subinterval, then, for all deviations $E \in \mathbf{E}(D)$, there exists a function $F \in \boldsymbol{F}(D)$ such that, for all $u \in D$,

$$F_u'(v) = -E(u, v), \qquad (v \in D) \tag{69}$$

is satisfied.

PROOF. First let $F \in \boldsymbol{F}(D)$ and define the function $E_F : D \times D \to D^*$ as in (67). We are going to show that E_F is a generalized deviation. It only suffices to verify the strict monotonicity of $-E_F$ in its second variable. To do this, let $u, v, w \in D$ such that $v \neq w$. According to the property (F) of

F, the function F_u is strictly convex on its domain, or equivalently, based on Theorem 3.5, we have that

$$0 < (F_u'(v) - F_u'(w))(v - w) = -(E_F(u, v) - E_F(u, w))(v - w).$$

Consequently, the function $-E_F(u, \cdot)$ is strictly monotone on D.

Now let $n \in \mathbb{N}$, $F \in \boldsymbol{F}(D)^n$, $E_F = (E_{F_1}, \dots, E_{F_n})$ and let $x \in D^n$ be arbitrarily fixed. The function $\mathcal{F}_{F,x} : D \to \mathbb{R}$, defined in (68), is continuous on the convex, compact set $\operatorname{conv}(x(\mathbb{N}_n))$, thus there exists a point $y \in \operatorname{conv}(x(\mathbb{N}_n))$, which minimizes $\mathcal{F}_{F,x}$ on the set $\operatorname{conv}(x(\mathbb{N}_n))$. Moreover, because of the strict convexity of $\mathcal{F}_{F,x}$, the minimizer y has to be unique. Thus, based on the last statement of Theorem 3.5, for all $v \in \operatorname{conv}(x(\mathbb{N}_n))$, we have

$$0 \le \mathcal{F}_{F,x}'(y)(v - y) = -(E_{F_1}(x_1, y) + \dots + E_{F_n}(x_n, y))(v - y).$$

In particular, this inequality holds also for all $v \in \{x_1, \dots, x_n\}$. Because of the uniqueness of the generalized E_F-deviation mean of x (cf. Theorem 3.3), we must have $y = \mathcal{D}^{E_F}(x)$.

Conversely, if

$$(E_{F_1}(x_1, y) + \dots + E_{F_n}(x_n, y))(v - y) \le 0$$

for all $v \in \{x_1, \dots, x_n\}$, then this inequality is also valid for all $v \in \operatorname{conv}(x(\mathbb{N}_n))$. Hence, for all $v \in \operatorname{conv}(x(\mathbb{N}_n))$,

$$\mathcal{F}_{F,x}'(y)(v - y) \ge 0.$$

In view of the reversed implication in the last statement of Theorem 3.5, this implies that y is the minimizer of the function $\mathcal{F}_{F,x}$ over the set $\operatorname{conv}(x(\mathbb{N}_n))$.

Let finally X be the real line and $D \subseteq \mathbb{R}$ be a subinterval, furthermore let $E \in \mathbf{E}(D)$ be a deviation and define the function $F : D \times D \to \mathbb{R}$ by the formula

$$F(u, v) := -\int_u^v E(u, t)\, dt, \qquad (u, v \in D). \tag{70}$$

For all $u \in D$, the function $t \mapsto E(u, t)$ is continuous on D, thus, due to the Fundamental Theorem of Calculus, F_u is continuously differentiable on D, and (69) holds. The strict decreasingness of E in its second variable implies that F_u' is a strictly monotone and hence F_u is strictly convex. Obviously we also have that $F_u'(u) = -E(u, u) = 0$ for all $u \in D$. □

The following result offers the construction of families of strictly convex functions in terms of two single variable functions. We recall that the unit ball of a normed space $(X, \|\cdot\|)$ is called strictly convex if $\|x\| = \|y\| = 1$ and $x \ne y$ implies that $\|tx + (1 - t)y\| < 1$ for all $t \in]0, 1[$. Observe that the strict convexity of the unit ball does not imply that the norm is a strictly convex function, moreover, by the positive homogeneity, any norm cannot be strictly convex.

PROPOSITION 8. *Let $(X, \|\cdot\|)$ be a normed space, assume that the unit ball is strictly convex and the norm is Gâteaux differentiable on $X \setminus \{0\}$. Let further $D \subseteq X$ be a convex set and $\omega : D \to \mathbb{R}_+$. Then the function $F : D \times D \to \mathbb{R}$, defined by*

$$F(u,v) := \omega(u)\|v-u\|^2, \qquad (u,v \in D), \tag{71}$$

satisfies property (F).

PROOF. Let $u \in D$ be fixed. To show that $F_u := F(u,\cdot)$ is strictly convex, let $v, w \in D$ with $v \neq w$ and $t \in]0,1[$. Now, we distinguish two cases.

First assume that the vectors $v-u$ and $w-u$ are not parallel, that is, there is no $t \in [0,1]$ such that $t(v-u) = (1-t)(w-u)$. Then non of them is zero, furthermore $x := \frac{v-u}{\|v-u\|}$ and $y := \frac{w-u}{\|w-u\|}$ are distinct unit vectors. Therefore, by the strict convexity of the unit ball, we have that $\|sx + (1-s)y\| < 1$ for all $s \in]0,1[$. Now, by also using the convexity of the square function, we get

$$\begin{aligned}
&F_u(tv+(1-t)w) = \omega(u)\|tv+(1-t)w-u\|^2\\
&\quad= \omega(u)\|t(v-u)+(1-t)(w-u)\|^2\\
&\quad= \omega(u)\big(t\|v-u\|+(1-t)\|w-u\|\big)^2\cdot\\
&\quad\left\|\frac{t\|v-u\|}{t\|v-u\|+(1-t)\|w-u\|}x + \frac{(1-t)\|w-u\|}{t\|v-u\|+(1-t)\|w-u\|}y\right\|^2\\
&\quad< \omega(u)\big(t\|v-u\|+(1-t)\|w-u\|\big)^2\\
&\quad\le \omega(u)\big(t\|v-u\|^2+(1-t)\|w-u\|^2\big) = tF_u(v)+(1-t)F_u(w).
\end{aligned}$$

Secondly, assume that $v-u$ and $w-u$ are parallel vectors. Then, the relation $v \neq w$ implies that $\|v-u\| \neq \|w-u\|$. Thus, by the subadditivity and the positive homogeneity of the norm, and by the strict convexity of the square function, we get that

$$\begin{aligned}
F_u(tv+(1-t)w) &= \omega(u)\|t(v-u)+(1-t)(w-u)\|^2\\
&\le \omega(u)\big(t\|v-u\|+(1-t)\|w-u\|\big)^2\\
&< \omega(u)\big(t\|v-u\|^2+(1-t)\|w-u\|^2\big)\\
&= tF_u(v)+(1-t)F_u(w).
\end{aligned}$$

To check the Gâteaux differentiability, denote $p(x) := \|x\|$ and let $v \in D\setminus\{u\}$ and $h \in X$. Then

$$\begin{aligned}
F_u'(v)(h) &= \lim_{t\to 0^+} \frac{F_u(v+th) - F_u(v)}{t} \\
&= \lim_{t\to 0^+} \frac{\omega(u)\|v+th-u\|^2 - \omega(u)\|v-u\|^2}{t} \\
&= \omega(u) \lim_{t\to 0^+} \big(\|v+th-u\| + \|v-u\|\big) \frac{p(v-u+th) - p(v-u)}{t} \\
&= 2\omega(u)\|v-u\|p'(v-u)(h).
\end{aligned}$$

Therefore, for $u \neq v$, we get $F_u'(v) = 2\omega(u)\|v-u\|p'(v-u)$.

On the other hand, for $v = u$, we have

$$\begin{aligned}
F_u'(u)(h) &= \lim_{t\to 0^+} \frac{F_u(u+th) - F_u(u)}{t} \\
&= \lim_{t\to 0^+} \frac{\omega(u)\|th\|^2 - \omega(u)\|0\|^2}{t} \\
&= \omega(u) \lim_{t\to 0^+} \frac{t^2\|h\|^2}{t} = 0,
\end{aligned}$$

which proves that $F_u'(u)$ is identically zero. This completes the proof of property (F). $\square$

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{R}$, $D \subseteq X$ be a nonempty convex set, and $\omega : D \to \mathbb{R}_+$. Then, by the previous result, the function $F : D \times D \to \mathbb{R}$, defined by (71), belongs to $\boldsymbol{F}(D)$ and, for all $u, v \in D$, we have

$$\begin{aligned}
E_F(u,v)(h) &= -F_u'(v)(h) \\
&= -2\omega(u)\|v-u\|p'(v-u)(h) \\
&= 2\omega(u)\langle u-v, h\rangle
\end{aligned} \tag{72}$$

for all $h \in X$. Now we can explicitly compute the generalized deviation mean generated by such generalized deviations. Let $n \in \mathbb{N}$, $\omega_1, \ldots, \omega_n : D \to \mathbb{R}_+$ and $F_1, \ldots, F_n : D \times D \to \mathbb{R}$ be functions, defined as in (71) using the weight functions $\omega_1, \ldots, \omega_n$, respectively, furthermore let $E_F := (E_{F_1}, \ldots, E_{F_n})$. Then

$$\mathcal{D}^{E_F}(x) = \frac{\omega_1(x_1)x_1 + \cdots + \omega_n(x_n)x_n}{\omega_1(x_1) + \cdots + \omega_n(x_n)} = \mathcal{A}^\omega(x), \qquad (x \in D^n).$$

Indeed, for $x \in D^n$ and $h \in X$, with the notation $y := \mathcal{A}^\omega(x) \in \operatorname{conv}(x(\mathbb{N}_n))$, we have

$$\begin{aligned}(E_{F_1}(x_1,y) + \cdots + E_{F_n}(x_n,y))(h) \\ &= 2(\omega_1(x_1)\langle x_1 - y, h\rangle + \cdots + \omega_n(x_n)\langle x_n - y, h\rangle) \\ &= 2\langle \omega_1(x_1)x_1 + \cdots + \omega_n(x_n)x_n - (\omega_1(x_1) + \cdots + \omega_n(x_n))y, h\rangle = 0.\end{aligned}$$

In particular, this equality holds also for $h \in \{x_1, \dots, x_n\} - y$, thus we must have $y = \mathcal{D}^{E_F}(x)$.

On the other hand, by Theorem 3.6, the vector $y = \mathcal{A}^\omega(x)$ is the unique minimizer of the function

$$\begin{aligned}\mathcal{F}_{F,x}(v) :&= F_1(x_1, v) + \cdots + F_n(x_n, v) \\ &= \omega_1(x_1)\|x_1 - v\|^2 + \cdots + \omega_n(x_n)\|x_n - v\|^2,\end{aligned}$$

that is, y is *the weighted least square approximant* of the elements $x_1, \dots, x_n \in D$.

3.5. Reducible inequalities involving means

In this section we consider convexity properties, comparison and Hölder–Minkowski type inequalities and establish their reducibility. Let $D \subseteq X$ be a nonempty convex set, $n \in \mathbb{N}$ and let $M : D^n \to X$ and $N : \mathbb{R}^n \to \mathbb{R}$ be means. We say that a function $f : D \to \mathbb{R}$ is *convex with respect to the pair of means* (M, N) *on* D or, shortly, that f is (M, N)*-convex on* D if

$$(f \circ M)(x) \le N(f \circ x), \qquad (x \in D^n), \tag{73}$$

that is, if we have

$$f\big(M(x_1, \dots, x_n)\big) \le N\big(f(x_1), \dots, f(x_n)\big), \qquad (x_1, \dots, x_n \in D).$$

THEOREM 3.7. *Let $D \subseteq X$ be a nonempty convex set, $I \subseteq \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $\chi : \mathbb{N}_k \to \mathbb{N}_n$ be an injective function. Let further $M : D^n \to X$ and $N : I^n \to \mathbb{R}$ be means such that M is χ-reducible and N is χ-continuous and uniquely χ-reducible. If a function $f : D \to I$ is (M, N)-convex, then it is also (K, N_χ)-convex for all χ-reduction $K : D^k \to X$ of the mean M.*

PROOF. Let $f : D \to I$ be an (M, N)-convex function, $K : D^n \to X$ be any χ-reduction of M and let $x \in D^k$ be arbitrarily fixed. Denote $y := K(x)$. Then, because of the definition of y, we have $M\big((x|\chi)(y)\big) = y$. Using this, the (M, N)-convexity of f, and the notation (53), we obtain that

$$f(y) = (f \circ M)\big((x|\chi)(y)\big) \le N\big(f\big((x|\chi)(y)\big)\big) = m_{f\circ x, N}(f(y)),$$

which is equivalent to the inequality

$$0 \leq m_{f\circ x,N}(f(y)) - f(y).$$

Due to the χ-continuity and to the unique χ-reducibility of N, using Lemma 3.2, it immediately follows that $f(y) \leq N_\chi(f \circ x)$ holds, that is

$$(f \circ K)(x) \leq N_\chi(f \circ x).$$

Consequently, f is (K, N_χ)-convex on its domain. □

The subsequent corollaries immediately follow from the theorem above, from Proposition 6 and from Theorem 3.4.

COROLLARY 3.8. *Let $D \subseteq X$ be a nonempty convex set, $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. Let further $\omega : D \to \mathbb{R}_+^n$ and $E : I \times I \to \mathbb{R}^n$ such that E_i is a deviation for all $i \in \mathbb{N}_n$. If a function $f : D \to I$ satisfies the n-variable inequality*

$$f\big(\mathcal{A}^\omega(x_1, \ldots, x_n)\big) \leq \mathcal{D}^E\big(f(x_1), \ldots, f(x_n)\big), \qquad (x_1, \ldots, x_n \in D),$$

then, for all $k \in \mathbb{N}_n$ and for all injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$, it also satisfies the k-variable inequality

$$f\big(\mathcal{A}^{\omega_\chi}(x_1, \ldots, x_k)\big) \leq \mathcal{D}^{E_\chi}\big(f(x_1), \ldots, f(x_k)\big), \qquad (x_1, \ldots, x_k \in D).$$

COROLLARY 3.9. *Let $D \subseteq X$ be a nonempty convex set, $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. Let further $G : D \times D \to (D^*)^n$ and $E : I \times I \to \mathbb{R}^n$ such that G_i is a generalized deviation and E_i is a deviation for all $i \in \mathbb{N}_n$. If a function $f : D \to I$ satisfies the n-variable inequality*

$$f\big(\mathcal{D}^G(x_1, \ldots, x_n)\big) \leq \mathcal{D}^E\big(f(x_1), \ldots, f(x_n)\big), \qquad (x_1, \ldots, x_n \in D),$$

then, for all $k \in \mathbb{N}_n$ and for all injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$, it also satisfies the k-variable inequality

$$f\big(\mathcal{D}^{G_\chi}(x_1, \ldots, x_k)\big) \leq \mathcal{D}^{E_\chi}\big(f(x_1), \ldots, f(x_k)\big), \qquad (x_1, \ldots, x_k \in D).$$

REMARK. Obviously, if, for all $i \in \mathbb{N}_n$, we have $\omega_i = 1$ and $E_i(u, v) := u - v$ for all $u, v \in I$ in Corollary 3.8, or if X is an inner product space, and, for all $i \in \mathbb{N}_n$, we have $G_i(x, y)(\cdot) := \langle x - y, \cdot\rangle$ and $E_i(u, v) := u - v$ for all $x, y \in D$ and for all $u, v \in I$, respectively, in Corollary 3.9, then, in both cases, we get back the reducibility of the Jensen inequality.

In particular, by applying the previous corollary to the function $f(x) = x$, we immediately obtain the following consequence for the comparison of deviation means.

COROLLARY 3.10. *Let $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. Let further $G, E : I \times I \to \mathbb{R}^n$ such that G_i and E_i are deviations for all $i \in \mathbb{N}_n$. If the n-variable inequality*

$$\mathcal{D}^G(x_1, \dots, x_n) \le \mathcal{D}^E(x_1, \dots, x_n), \qquad (x_1, \dots, x_n \in D)$$

holds, then, for all $k \in \mathbb{N}_n$ and for all injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$, we also have the k-variable inequality

$$\mathcal{D}^{G_\chi}(x_1, \dots, x_k) \le \mathcal{D}^{E_\chi}(x_1, \dots, x_k), \qquad (x_1, \dots, x_k \in D).$$

The following result establishes the reducibility of an abstract Hölder–Minkowski type inequality.

THEOREM 3.11. *Let $X_1, \dots, X_\ell$ be real Hausdorff topological linear spaces, let $D_1 \subseteq X_1, \dots, D_\ell \subseteq X_\ell$ be nonempty convex sets and $I \subseteq \mathbb{R}$ be an interval. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $\chi : N_k \to \mathbb{N}_n$ be an injective function. Let $N_1 : D_1^n \to X_1, \dots, N_\ell : D_\ell^n \to X_\ell$ be χ-reducible means and let $M : I^n \to \mathbb{R}$ be a χ-continuous, uniquely χ-reducible mean. If a function $f : D_1 \times \dots \times D_\ell \to I$ satisfies the $n \cdot \ell$-variable inequality*

$$M\big(f(x^1, \dots, x^\ell)\big) \le f\big(N_1(x^1), \dots, N_\ell(x^\ell)\big) \tag{74}$$

for all $x^1 \in D_1^n, \dots, x^\ell \in D_\ell^n$, then, for any χ-reductions $K_1 : D_1^k \to X_1, \dots, K_\ell : D_\ell^k \to X_\ell$ of $N_1, \dots, N_\ell$, respectively, it also fulfills the $k \cdot \ell$-variable inequality

$$M_\chi\big(f(x^1, \dots, x^\ell)\big) \le f\big(K_1(x^1), \dots, K_\ell(x^\ell)\big) \tag{75}$$

for all $x^1 \in D_1^k, \dots, x^\ell \in D_\ell^k$, where, for $m \in \mathbb{N}$ and $x^1 \in D_1^m, \dots, x^\ell \in D_\ell^m$, we denote

$$f(x^1, \dots, x^\ell) := (f(x_1^1, \dots, x_1^\ell), \dots, f(x_m^1, \dots, x_m^\ell)).$$

PROOF. Let $x^1 \in D_1^k, \dots, x^\ell \in D_\ell^k$ be arbitrarily fixed, $K_1 : D_1^k \to X_1, \dots, K_\ell : D_\ell^k \to X_\ell$ be any χ-reduction of $N_1, \dots, N_\ell$, respectively, denote $u_1 := K_1(x^1), \dots, u_\ell := K_\ell(x^\ell)$, finally let $u := (u_1, \dots, u_\ell)$. Using inequality (74), we get

$$\begin{aligned} M\big((f(x^1, \dots, x^\ell)|\chi)(f(u))\big) &\le f\big(N_1((x^1|\chi)(u_1)), \dots, N_\ell((x^\ell|\chi)(u_\ell))\big) \\ &= f\big(K_1(x^1), \dots, K_\ell(x^\ell)\big) = f(u), \end{aligned}$$

that is, the inequality

$$m_{f(x^1,\dots,x^\ell),M}(f(u)) - f(u) = M\big((f(x^1, \dots, x^\ell)|\chi)(f(u))\big) - f(u) \le 0$$

holds. The mean M is χ-continuous and uniquely χ-reducible, thus, using Lemma 3.2 for the vector $x := f(x^1, \dots, x^\ell)$ and for $y := f(u)$, we obtain that

$$M_\chi\big(f(x^1, \dots, x^\ell)\big) \le f(u) = f\big(K_1(x^1), \dots, K_\ell(x^\ell)\big),$$

which finishes the proof. $\square$

To derive various consequences of Theorem 3.11, one can specialize the means M and $N_1, \dots, N_\ell$ by letting them equal to weighted arithmetic mean or to a generalized deviation mean. Then the two choices $f(x^1, \dots, x^\ell) := x^1 + \dots + x^\ell$ and $f(x^1, \dots, x^\ell) := x^1 \cdots x^\ell$ result inequalities of Minkowski and of Hölder type, respectively.

Summary

In the sequel, I am going to sum up the main areas which are touched by my dissertation and, in parallel, I also describe the most important related results. The motivation of our investigations was served by the following essential result from the theory of standard convexity of real valued functions, which is originally due to Johan Jensen.

THEOREM. (Jensen, 1906) *Let X be a linear space and $D \subseteq X$ be a nonempty convex subset. Then the following statements are pairwise equivalent.*

(1) The function $f : D \to \mathbb{R}$ is Jensen convex.

(2) For any given positive integer $n \in \mathbb{N}$, the function $f : D \to \mathbb{R}$ fulfills the n-variable Jensen Inequality, that is, for all $x_1, \dots, x_n \in D$, we have

$$f\Big(\frac{x_1 + \cdots + x_n}{n}\Big) \le \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

(3) The function $f : D \to \mathbb{R}$ is rationally convex on D, that is, for all $r \in [0, 1] \cap \mathbb{Q}$ and for all $x, y \in D$, we have

$$f(rx + (1 - r)y) \le rf(x) + (1 - r)f(y).$$

Among others, this result has two crucial message for us. According to this, my dissertation can be divided in two main parts.

• **First Part.** Here we were concentrating on the connection between the statements (1) and (3). In view of this, having the inequality of the standard convexity with the weight $\frac{1}{2}$, that is, having the *Jensen inequality* for some real valued function, we can conclude its rational convexity. This immediately yields some crucial properties of the convexity parameter set of a Jensen convex function. Namely, it follows that is must be at least a countable *(cardinality property)* and dense *(topological property)* subset of $[0, 1]$, furthermore it contains the intersection of the field of rational numbers and $[0, 1]$ *(algebraic property)*. The full characterization of the convexity parameter set is due to Norbert Kuhn.

THEOREM. (Kuhn, 1984) *For any function* $f : I \to \mathbb{R}$, *the convexity parameter set* $\mathcal{C}_f$ *is either* $\{0,1\}$ *or it can be written as* $F \cap [0,1]$, *where* F *is the subfield of* $\mathbb{R}$ *generated by* $\mathcal{C}_f$.

In the first part of the dissertation, related to a generalized notion of convexity of extended real valued functions, we are going to formulate Kuhn type theorems. Now we turn to the detailed explanation of the different sections.

In *Capter 1.*, we explain the notion of mean values and the most important types of classes of means what we will need for our purposes. Then we formulate the main tools from linear algebra and fixed point theory what will be used in the further steps. Finally we state our main results about deriving new means from given ones and apply them for the class of Matkowski means.

The main notion of this chapter is *the descendant of a mean*. Theorem 1.7 provides that this notion is well-defined assuming, among others, that the original means are continuous.

The following theorem states that the descendants of a chain of weighted quasi-arithmetic means with the same generator function always exists, they are uniquely determined, and are also weighted quasi-arithmetic means. As one can see, the generator function is the same and the weights of the descendants can be directly calculated using the original weights.

THEOREM. *Let* $n \geq 2$, $s_1, \dots, s_n \in]0,1[$, *and* $h : I \to \mathbb{R}$ *be a continuous, strictly increasing function. For* $(x,y) \in I^2_<$, *define the function* $\varphi_{(x,y)} : [x,y]^n_\leq \to \mathbb{R}^n$ *as in* (19) *using the means* $M_i := \mathcal{M}^{(s_i h,(1-s_i)h)}$, *where* $i \in \{1,\dots,n\}$. *Then, for all* $(x,y) \in I^2_<$, *the fixed point set* $\Phi_{(x,y)}$ *is the singleton* $\big\{\big(\mathcal{M}^{(\sigma_1 h,\,(1-\sigma_1)h)}(x,y), \dots, \mathcal{M}^{(\sigma_n h,\,(1-\sigma_n)h)}(x,y)\big)\big\}$, *where*

$$\sigma_i := \left(\sum_{j=i}^{n}\prod_{k=1}^{j}\frac{s_k}{1-s_k}\right)\left(\sum_{j=0}^{n}\prod_{k=1}^{j}\frac{s_k}{1-s_k}\right)^{-1}, \quad (i \in \{1,\dots,n\}).$$

In general, having proper Matkowski means (that is, not necessarily weighted quasi-arithmetic means), the calculation of the descendants might be difficult. However, assuming some relations between the generator functions the task can be done using a two way recursion.

THEOREM. *Let* $n \geq 2$, $j \in \{1,\dots,n\}$ *and* $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ *be continuous, strictly increasing functions, furthermore set* $h_0 := h_n := 0$. *For*

$(x, y) \in I^2_<$, *define the mapping* $\varphi_{(x,y)} : [x, y]^n_\leq \to \mathbb{R}^n$ *by* (19)*, using the means*

$$M_i := \begin{cases} \mathcal{M}^{(p+h_{i-1},\, h_i)} & \text{if } i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(p+h_{i-1},\, h_i+q)} & \text{if } i = j, \\ \mathcal{M}^{(h_{i-1},\, h_i+q)} & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Then, for $(x, y) \in I^2_<$*, the fixed point set* $\Phi_{(x,y)}$ *defined by* (20) *is the singleton* $\{(\xi_1, \dots, \xi_n)\}$*, where* $\xi_j := \mathcal{M}^{(p,\, q)}(x, y)$ *and the rest of the coordinates are defined by the two-way recurrence*

$$\xi_i := \begin{cases} \mathcal{M}^{(p,\, h_i)}(x, \xi_{i+1}) & \text{if } i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(h_{i-1},\, q)}(\xi_{i-1}, y) & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

In *Chapter 2.*, we introduce and also characterize the concept of *lower and upper* M*-convex functions*, we apply our previous results, and investigate the algebraic and topological properties of their *generalized convexity classes*. Then, taking the special subclass of *asymmetrically* t*-convex functions*, we formulate also *the counterpart of Kuhn's Theorem*.

It follows from Kuhn's theorem that the standard convexity set is closed under taking convex combinations weighted with its elements. The following proposition generalizes this statement.

PROPOSITION. *Let* $f : I \to \overline{\mathbb{R}}$ *be any function and let* $\mathcal{M} \in \{\underline{\mathcal{M}}_f, \overline{\mathcal{M}}_f\}$*. Then the following statements hold.*

(a) If $M, N_1, N_2 \in \mathcal{M}$ *with* $N_1 < N_2$ *on the set* $I^2_<$*, then* $M \circ (N_1, N_2) \in \mathcal{M}$*.*

(b) If $M, N \in \mathcal{M}$*, then the compositions* $M \circ (\min, N)$ *and* $M \circ (N, \max)$ *also belong to the family* $\mathcal{M}$*.*

Similarly to the standard case, a topological property can be derived from the above proposition.

COROLLARY. *Let* $f : I \to \overline{\mathbb{R}}$ *be any function, define*

$$\underline{\mathcal{M}}^*_f := \{M \in \underline{\mathcal{M}}_f \mid M \text{ is separately continuous in both variables}\},$$

$$\overline{\mathcal{M}}^*_f := \{M \in \overline{\mathcal{M}}_f \mid M \text{ is separately continuous in both variables}\},$$

and, finally, let $\mathcal{M}^* \in \{\underline{\mathcal{M}}^*_f, \overline{\mathcal{M}}^*_f\}$*. Then* $\mathcal{M}^*$ *has no isolated points with respect to the pointwise convergence, more precisely, for all* $M \in \mathcal{M}^*$*, there exist sequences of means* $(L_n), (U_n) \subseteq \mathcal{M}^*$ *such that* $L_n < M < U_n$ *whenever* $n \in \mathbb{N}$*, furthermore* $L_n \to M$ *and* $U_n \to M$ *pointwise on the set* $I^2_<$ *as* $n \to \infty$*.*

Using the results earned in the previous sections, it can be proved that the lower convexity class is always closed under taking the descendants. This is not true in the case of upper convexity.

THEOREM. *Let $f : I \to \overline{\mathbb{R}}$ be any function, $n \geq 2$, furthermore $M_1, \dots, M_n \in \underline{\mathcal{M}}_f$ be continuous means. Then $\mathbf{D}_i(M_1, \dots, M_n) \subseteq \underline{\mathcal{M}}_f$ for all $i \in \{1, \dots, n\}$.*

Assuming that the lower convexity class contains certain type of Matkowski means, we get the following consequences.

COROLLARY. *Let $f : I \to \overline{\mathbb{R}}$, $n \geq 2$, $s_1, \dots, s_n \in]0, 1[$, and finally $h : I \to \mathbb{R}$ be a continuous, strictly increasing function. Assume further that $\mathcal{M}^{(s_i h,\, (1-s_i)h)} \in \underline{\mathcal{M}}_f$ for all $i \in \{1, \dots, n\}$. Then, for all $i \in \{1, \dots, n\}$, the Matkowski mean $\mathcal{M}^{(\sigma_i h,\, (1-\sigma_i)h)}$ also belongs to the family $\underline{\mathcal{M}}_f$, where the weight σ_i is defined as in* (28) *for all $i \in \{1, \dots, n\}$.*

COROLLARY. *Let $n \geq 2$, $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ be continuous, strictly increasing functions and $f : I \to \overline{\mathbb{R}}$. Set further $h_0 := h_n := 0$ and assume that there exists $j \in \{1, \dots, n\}$ such that, for all $i \in \{1, \dots, n\}$, the mean M_i defined by*

$$M_i := \begin{cases} \mathcal{M}^{(p+h_{i-1},\, h_i)} & \text{if } i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(p+h_{j-1},\, h_j+q)} & \text{if } i = j, \\ \mathcal{M}^{(h_{i-1},\, h_i+q)} & \text{if } i \in \{j+1, \dots, n\} \end{cases}$$

is contained in $\underline{\mathcal{M}}_f$. Then $N_1, \dots, N_n \in \underline{\mathcal{M}}_f$, where, for all $(x, y) \in I^2_{\leq}$,

$$N_i(x, y) = \begin{cases} \mathcal{M}^{(p,\, h_i)}(x, N_{i+1}(x, y)) & \text{if } i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(p,\, q)}(x, y) & \text{if } i = j \\ \mathcal{M}^{(h_{i-1},\, q)}(N_{i-1}(x, y), y) & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Turning to the notion of asymmetrical convexity, the related lower and upper convexity classes can be identified with suitable subsets of the open unit interval. In this case the statements about the algebraic and topological structure became more transparent. We obtain the special convexity property of the parameter set. It turns to be closed under the multiplication of its elements and we also get its density in $]0, 1[$.

THEOREM. *Let $f : I \to \overline{\mathbb{R}}$ be any function and $\mathcal{AC} \in \{\underline{\mathcal{AC}}_f, \overline{\mathcal{AC}}_f\}$. Then the following statements hold.*

(1) If $t, s_1, s_2 \in \mathcal{AC}$ with $s_1 < s_2$, then $ts_2 + (1-t)s_1 \in \mathcal{AC}$.

(2) If $t, s \in \mathcal{AC}$, then ts and $1 - (1-t)(1-s)$ also belong to $\mathcal{AC}$.

(3) *The set $\mathcal{AC}$ is dense in the open unit interval, provided that it is nonempty.*

Applying our general results obtained for Matkowski means, we earn the following corollaries.

COROLLARY. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \overline{\mathbb{R}}$, $n \geq 2$ and $s_1, \dots, s_n \in \underline{\mathcal{AC}}_f$. Then $\sigma_i \in \underline{\mathcal{AC}}_f$ for all $i \in \{1, \dots, n\}$, where*

$$\sigma_i := \Big(\sum_{j=i}^{n} \prod_{k=1}^{j} \frac{s_k}{1 - s_k} \Big) \Big(\sum_{j=0}^{n} \prod_{k=1}^{j} \frac{s_k}{1 - s_k} \Big)^{-1}.$$

COROLLARY. *For a function $f : I \to \overline{\mathbb{R}}$ the following statements hold.*

(1) *If $1/2 \in \underline{\mathcal{AC}}_f$ then $\mathbb{Q} \cap]0, 1[\subseteq \underline{\mathcal{AC}}_f$.*

(2) *If $\ell/m \in \underline{\mathcal{AC}}_f$ for some $\ell, m \in \mathbb{N}$ with $\ell < m$ and $2\ell \neq m$, then, for all $n \geq 2$ and for all $i \in \{1, \dots, n\}$, the fraction*

$$r_i := \frac{\ell^{n+1} - \ell^i (m - \ell)^{n+1-i}}{\ell^{n+1} - (m - \ell)^{n+1}}$$

belongs to $\underline{\mathcal{AC}}_f$.

Finally, we construct a proper *upper asymmetrically convex extended real valued function* whose parameter set contains rational (and hence algebraic) numbers, is dense in $[0, 1]$ but it fails to be an intersection of a field and the open unit interval. An other example having similar behavior was given by Lewicki and Olbryś concerning transcendental parameters and the real valued case. The existence of a real valued function with the same property under algebraic parameters forms still an open problem.

THEOREM. *Let $I \subseteq \mathbb{R}$ be a subinterval with $a := \sup I \in I \cap \mathbb{Q}_1$, $C : I \to \mathbb{R}$ be any convex function, and define $f : I \to \overline{\mathbb{R}}$ by*

$$f(x) := \begin{cases} C(x) & \text{if } x \in (I \cap \mathbb{Q}_0) \cup \{a\}, \\ +\infty & \text{if } x \in I \setminus (\mathbb{Q}_0 \cup \{a\}). \end{cases}$$

Then, for all $t \in]0, 1[\cap \mathbb{Q}_1$, the function f is upper $\mathcal{A}_t$-convex but it is not upper $\mathcal{A}_{1-t}$-convex.

COROLLARY. *Keeping the above notation and conditions, $\overline{\mathcal{AC}}_f$ is not closed under addition, consequently it cannot be written as an intersection of $]0, 1[$ and a proper subfield of $\mathbb{R}$.*

• **Second Part.** The second part of the dissertation generalizes and investigates the statement about the equivalence of (1) and (2). The original assertion provides that, having the Jensen inequality of n variables with fixed positive integer $n \geq 2$ for a real function, its Jensen convexity can be deduced. Obviously, the statement is interesting only when $n > 2$. In view of our terminology, this means that the n variable Jensen inequality is *reducible*. The calculation in the proof of the theorem shows that this depends strongly on *the reducibility of the mean in background*, namely of the arithmetic mean.

In *Chapter 3.*, we formulate precisely the notion of reducibility of general mean values. The main theorem of this part, which gives a sufficient condition for being reducible, is the following.

THEOREM. *The mean $M : D^n \to X$ is χ-reducible provided that it is χ-continuous.*

After this, generalizing widely the well-known means, we introduce the notion of *generalized deviation means* on topological vector spaces of Hausdorff type. These means naturally turned out to be reducible. Moreover, the reductions belong to the same class and the generators can be easily given using the original ones.

THEOREM. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $E \in \mathbf{E}(D)^n$. Then the generalized E-deviation mean $\mathcal{D}^E : D^n \to D$ is reducible with respect to any injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$. Furthermore, the χ-reduction of $\mathcal{D}^E$ is uniquely determined, namely we have*

$$\mathcal{D}^E_\chi(x) = \mathcal{D}^{E_\chi}(x), \qquad (x \in D^k).$$

In the *Section 3.4.* of *Chapter 3.*, we also characterize the generalized deviation means using relatively Gâteaux-differentiable strictly convex functions.

THEOREM. *Assume that $D \subseteq X$ is a convex set and let $F \in \boldsymbol{F}(D)$. Then the function $E_F : D \times D \to D^*$, defined by*

$$E_F(u, v) = -F'_u(v),$$

is a generalized deviation. Furthermore, if $n \in \mathbb{N}$, $F \in \boldsymbol{F}(D)^n$ and $E_F = (E_{F_1}, \ldots, E_{F_n})$, then, for $x \in D^n$, the equality $y = \mathcal{D}^{E_F}(x)$ holds if and only if y is the unique minimizer over $\operatorname{conv}(x(\mathbb{N}_n))$ *of the function $\mathcal{F}_{F,x} : D \to \mathbb{R}$ defined by*

$$\mathcal{F}_{F,x}(v) := F_1(x_1, v) + \cdots + F_n(x_n, v).$$

Conversely, if X is the real line and D is an open subinterval, then, for all deviations $E \in \mathbf{E}(D)$, there exists a function $F \in \boldsymbol{F}(D)$ such that, for all $u \in D$,

$$F'_u(v) = -E(u, v), \qquad (v \in D)$$

is satisfied.

In the very last section we are dealing with the reducibility of the notion of (M, N)-convexity of real functions, which is a self-evident generalization of the Jensen convexity; one can obtain it by replacing the arithmetic mean on the left hand side and on the right hand side by the general mean values M and N, respectively. As in the standard case, its reducibility depends strongly on the reducibility property of the mean values what are involved, namely of M and N.

THEOREM. *Let $D \subseteq X$ be a nonempty convex set, $I \subseteq \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $\chi : \mathbb{N}_k \to \mathbb{N}_n$ be an injective function. Let further $M : D^n \to X$ and $N : I^n \to \mathbb{R}$ be means such that M is χ-reducible and N is χ-continuous and uniquely χ-reducible. If a function $f : D \to I$ is (M, N)-convex, then it is also (K, N_χ)-convex for all χ-reduction $K : D^k \to X$ of the mean M.*

Applying this result, we immediately get the following consequences, which concern special mean values instead of general ones.

COROLLARY. *Let $D \subseteq X$ be a nonempty convex set, $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. Let further $\omega : D \to \mathbb{R}^n_+$ and $E : I \times I \to \mathbb{R}^n$ such that E_i is a deviation for all $i \in \mathbb{N}_n$. If a function $f : D \to I$ satisfies the n-variable inequality*

$$f\big(\mathcal{A}^{\omega}(x_1, \ldots, x_n)\big) \leq \mathcal{D}^{E}\big(f(x_1), \ldots, f(x_n)\big), \qquad (x_1, \ldots, x_n \in D),$$

then, for all $k \in \mathbb{N}_n$ and for all injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$, it also satisfies the k-variable inequality

$$f\big(\mathcal{A}^{\omega_\chi}(x_1, \ldots, x_k)\big) \leq \mathcal{D}^{E_\chi}\big(f(x_1), \ldots, f(x_k)\big), \qquad (x_1, \ldots, x_k \in D).$$

COROLLARY. *Let $D \subseteq X$ be a nonempty convex set, $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. Let further $G : D \times D \to (D^*)^n$ and $E : I \times I \to \mathbb{R}^n$ such that G_i is a generalized deviation and E_i is a deviation for all $i \in \mathbb{N}_n$. If a function $f : D \to I$ satisfies the n-variable inequality*

$$f\big(\mathcal{D}^{G}(x_1, \ldots, x_n)\big) \leq \mathcal{D}^{E}\big(f(x_1), \ldots, f(x_n)\big), \qquad (x_1, \ldots, x_n \in D),$$

then, for all $k \in \mathbb{N}_n$ and for all injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$, it also satisfies the k-variable inequality

$$f\big(\mathcal{D}^{G_\chi}(x_1, \ldots, x_k)\big) \leq \mathcal{D}^{E_\chi}\big(f(x_1), \ldots, f(x_k)\big), \qquad (x_1, \ldots, x_k \in D).$$

COROLLARY. *Let $I \subseteq \mathbb{R}$ be an interval and $n \in \mathbb{N}$. Let further $G, E : I \times I \to \mathbb{R}^n$ such that G_i and E_i are deviations for all $i \in \mathbb{N}_n$. If the n-variable* *inequality*

$$\mathcal{D}^G(x_1,\dots,x_n) \le \mathcal{D}^E(x_1,\dots,x_n), \qquad (x_1,\dots,x_n \in D)$$

holds, then, for all $k \in \mathbb{N}_n$ and for all injective function $\chi : \mathbb{N}_k \to \mathbb{N}_n$, we also have the k-variable inequality

$$\mathcal{D}^{G_\chi}(x_1,\dots,x_k) \le \mathcal{D}^{E_\chi}(x_1,\dots,x_k), \qquad (x_1,\dots,x_k \in D).$$

Finally we also establish the reducibility property of an abstract version of a Hölder-Minkowski type inequality.

THEOREM. *Let $X_1,\dots,X_\ell$ be real Hausdorff topological linear spaces, let $D_1 \subseteq X_1,\dots,D_\ell \subseteq X_\ell$ be nonempty convex sets and $I \subseteq \mathbb{R}$ be an interval. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_n$, and let $\chi : N_k \to \mathbb{N}_n$ be an injective function. Let $N_1 : D_1^n \to X_1,\dots,N_\ell : D_\ell^n \to X_\ell$ be χ-reducible means and let $M : I^n \to \mathbb{R}$ be a χ-continuous, uniquely χ-reducible mean. If a function $f : D_1 \times \cdots \times D_\ell \to I$ satisfies the $n \cdot \ell$-variable inequality*

$$M\big(f(x^1,\dots,x^\ell)\big) \le f\big(N_1(x^1),\dots,N_\ell(x^\ell)\big), \qquad (x^1 \in D_1^n,\dots,x^\ell \in D_\ell^n),$$

then, for any χ-reductions $K_1 : D_1^k \to X_1,\dots,K_\ell : D_\ell^k \to X_\ell$ of $N_1,\dots,N_\ell$, respectively, it also fulfills the $k \cdot \ell$-variable inequality

$$M_\chi\big(f(x^1,\dots,x^\ell)\big) \le f\big(K_1(x^1),\dots,K_\ell(x^\ell)\big), \qquad (x^1 \in D_1^k,\dots,x^\ell \in D_\ell^k),$$

where, for $m \in \mathbb{N}$ and $x^1 \in D_1^m,\dots,x^\ell \in D_\ell^m$, we denote

$$f(x^1,\dots,x^\ell) := (f(x_1^1,\dots,x_1^\ell),\dots,f(x_m^1,\dots,x_m^\ell)).$$

Összefoglaló (Summary in Hungarian)

Az alábbiakban néhány oldalon összefoglalom azokat a témaköröket, amelyekkel a disszertációm foglalkozik és, ezzel párhuzamosan, felsorom a kapcsolódó fontosabb eredményeket. A vizsgálatainkat Johan Jensen következő, a valós függvények konvexitási elméletében alapvetőnek számító eredménye motiválta.

TÉTEL. (Jensen, 1906) *Legyen X valós vektortér és $D \subseteq X$ nemüres, konvex részhalmaz. Ekkor az alábbi állítások páronként ekvivalensek.*

(1) Az $f : D \to \mathbb{R}$ függvény Jensen-konvex.

(2) Bármely rögzített $n \in \mathbb{N}$ esetén, az $f : D \to \mathbb{R}$ függvény eleget tesz az n-változós Jensen egyenlőtlenségnek, vagyis, bármely $x_1, \ldots, x_n \in D$ esetén, fennáll az

$$f\Big(\frac{x_1 + \cdots + x_n}{n}\Big) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

egyenlőtlenség.

(3) Az $f : D \to \mathbb{R}$ függvény racionálisan konvex az értelmezési tartományán, vagyis, bármely $r \in [0, 1] \cap \mathbb{Q}$ súly és $x, y \in D$ pontok esetén

$$f(rx + (1 - r)y) \leq rf(x) + (1 - r)f(y).$$

A fenti tételnek, többek között, két fontos üzenete van. Ennek megfelelően, témáját tekintve, a disszertáció is két nagyobb részre bontható.

• **Első rész.** Ebben a részben az (1) és (3) állítások közötti kapcsolatot emelném ki. Ennek értelmében, ha egy valós értékű függvény $\frac{1}{2}$ súllyal standard értelemben konvex, tehát Jensen konvex, akkor racionálisan is konvex. Ebből az állításból rögtön következik a Jensen konvex függvények konvexitási paraméterhalmazának több lényeges tulajdonsága. Nevezetesen, a konvexitási paraméterhalmaz egy legalább megszámlálható *(számossági tulajdonság)* sűrű *(topologikus tulajdonság)* részhalmaza a $[0, 1]$ intervallumnak, továbbá tartalmazza a racionális számok testének $[0, 1]$ intervalumba eső szeletét *(algebrai tulajdonság)*. A paraméterhalmaz jellemzése Norbert Kuhn nevéhez fűződik.

TÉTEL. (Kuhn, 1984) *Adott* $f : I \to \mathbb{R}$ *függvényre, a* $\mathcal{C}_f$ *konvexitási paraméterhalmaz vagy a triviális* $\{0,1\}$ *halmaz vagy felírható* $F \cap [0,1]$ *alakban, ahol* F *a legszűkebb* $\mathcal{C}_f$*-et tartalmazó részteste* $\mathbb{R}$*-nek.*

A disszertáció első részében Kuhn eredményéhez hasonló tételeket fogalmazunk meg egy, bővített valós értékű függvényekre bevezetett, általánosított konvexitási fogalom mellett. Most rátérünk az egyes alfejezetek részletesebb ismertetésére.

Az első alfejezetben definiáljuk a középérték fogalmát és a fontosabb középosztályokat, amelyekre a későbbiekben szükségünk lesz. Ezután megemlítünk héhány nélkülözhetetlen eszközt a lineáris algebrából és a fixponTelméletből. Végül megfogalmazzuk a közepek származtatására vonatkozó főbb eredményeinket és alkalmazzuk őket a Matkowski közepek osztályára.

Az alfejezet központi fogalma *egy közép leszármazottjainak osztálya*. Az 1.7 Tétel biztosítja, hogy jóldefiniált fogalomról van szó, amennyiben, néhány további feltétel mellett, feltesszük, hogy az eredeti közepek folytonosak voltak.

A következő tétel kimondja, hogy súlyozott kváziaritmetikai közepek esetén a származtatott közepek mindig léteznek, súlyozott kváziaritmetikaiak maradnak az eredeti generátorfüggvénnyel, és hogy az új súlyok a régiek segítségével egyértelműen számolhatóak.

TÉTEL. *Legyen* $n \geq 2$*,* $s_1, \dots, s_n \in]0,1[$ *és* $h : I \to \mathbb{R}$ *folytonos, szigorúan növő függvény. Adott* $(x,y) \in I^2_<$ *esetén, legyen* $\varphi_{(x,y)} : [x,y]^n_\leq \to \mathbb{R}^n$ *a* (19) *képlet alapján definiált függvény, ahol* $M_i := \mathcal{M}^{(s_i h,(1-s_i)h)}$*, ha* $i \in \{1,\dots,n\}$*. Ekkor, bármely* $(x,y) \in I^2_<$ *esetén, a* $\Phi_{(x,y)}$ *fixpontok halmaza megegyezik az* $\{(\mathcal{M}^{(\sigma_1 h,\,(1-\sigma_1)h)}(x,y), \dots, \mathcal{M}^{(\sigma_n h,\,(1-\sigma_n)h)}(x,y))\}$ *egyelemű halmazzal, ahol*

$$\sigma_i := \left(\sum_{j=i}^{n}\prod_{k=1}^{j}\frac{s_k}{1-s_k}\right)\left(\sum_{j=0}^{n}\prod_{k=1}^{j}\frac{s_k}{1-s_k}\right)^{-1}, \quad (i \in \{1,\dots,n\}).$$

Általános esetben, ha a közepeink nem feltétlenül súlyozott kváziaritmetikaiak, a leszármazottak számolása nehézkessé, sőt, esetenként akár lehetetlenné is válhat. Kiderül azonban, hogy ha a generátorfüggvények két előre megadott függvény speciális eltoltjai, akkor a leszármazottak egy kétirányú rekurzió segítségével könnyen leírhatók.

TÉTEL. *Legyen* $n \geq 2$*,* $j \in \{1,\dots,n\}$ *és* $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ *adott folytonos, szigorúan növő függvények, továbbá legyen* $h_0 := h_n := 0$*. Adott* $(x,y) \in I^2_<$ *pár esetén, legyen* $\varphi_{(x,y)} : [x,y]^n_\leq \to \mathbb{R}^n$ *az* (19) *szerint definiált* *függvény, úgy, hogy*

$$M_i := \begin{cases} \mathcal{M}^{(p+h_{i-1},\, h_i)}, & \text{ha } i \in \{1,\dots,j-1\}, \\ \mathcal{M}^{(p+h_{i-1},\, h_i+q)}, & \text{ha } i = j, \\ \mathcal{M}^{(h_{i-1},\, h_i+q)}, & \text{ha } i \in \{j+1,\dots,n\}. \end{cases}$$

Ekkor, bármely $(x,y) \in I^2_<$ esetén, a $\Phi_{(x,y)}$ fixpontok halmaza megegyezik a $\{(\xi_1,\dots,\xi_n)\}$ egyelemű halmazzal, ahol $\xi_j := \mathcal{M}^{(p,\,q)}(x,y)$, továbbá

$$\xi_i := \begin{cases} \mathcal{M}^{(p,\,h_i)}(x,\xi_{i+1}), & \text{ha } i \in \{1,\dots,j-1\}, \\ \mathcal{M}^{(h_{i-1},\,q)}(\xi_{i-1},y), & \text{ha } i \in \{j+1,\dots,n\}. \end{cases}$$

A második fejezetben definiáljuk és jellemezzük *bővített valós értékű függvények alsó- és felső konvexitását egy adott M középre vonatkozóan.* Bevezetjük a kapcsolódó *konvexitási osztályok*at és megvizsgáljuk algebrai és topologikus tulajdonságaikat. Végül, áttérve az *aszimmetrikus konvexitás* speciális esetére, megfogalmazzuk és bizonyítjuk *Kuhn tételének ellenpárját* is.

Kuhn tételéből következik, hogy a standard konvexitási paraméterhalmaz zárt a saját elemeivel súlyozott konvex kombinációk képzésére nézve. A következő állítás ezt az eredményt általánosítja.

ÁLLÍTÁS. *Legyen $f : I \to \overline{\mathbb{R}}$ adott függvény és $\mathcal{M} \in \{\underline{\mathcal{M}}_f, \overline{\mathcal{M}}_f\}$. Ekkor az alábbi állítások igazak.*

(a) Ha $M, N_1, N_2 \in \mathcal{M}$ és $N_1 < N_2$ az $I^2_<$ halmazon, akkor $M \circ (N_1,N_2) \in \mathcal{M}$.

(b) Ha $M, N \in \mathcal{M}$, akkor az $M \circ (\min, N)$ és $M \circ (N, \max)$ kompozíciók ismét az $\mathcal{M}$ osztályhoz tartoznak.

A standard esethez hasonlóan, topologikus tulajdonság származtatható a fenti eredményből.

KÖVETKEZMÉNY. *Legyen $f : I \to \overline{\mathbb{R}}$ esetén*

$$\underline{\mathcal{M}}^*_f := \{M \in \underline{\mathcal{M}}_f \mid M \text{ szeparáltan folytonos}\},$$

$$\overline{\mathcal{M}}^*_f := \{M \in \overline{\mathcal{M}}_f \mid M \text{ szeparáltan folytonos}\},$$

és $\mathcal{M}^* \in \{\underline{\mathcal{M}}^*_f, \overline{\mathcal{M}}^*_f\}$. Ekkor az $\mathcal{M}^*$ osztálynak, a pontonkénti konvergenciára nézve, nem létezik izolált pontja. Pontosabban fogalmazva, bármely $M \in \mathcal{M}^*$ közép esetén, léteznek közepeknek olyan $(L_n), (U_n) \subseteq \mathcal{M}^*$ sorozatai, hogy $L_n < M < U_n$, valahányszor $n \in \mathbb{N}$, továbbá $L_n \to M$ és $U_n \to M$ pontonként az $I^2_<$ halmazon, ha $n \to \infty$.

Az előzőekben elért eredményeket felhasználva, bizonyítható az alsó konvexitási osztály leszármazottakra való zártsága. Felső konvexitás esetén ez az állítás nem marad érvényben.

TÉTEL. *Legyen* $f : I \to \overline{\mathbb{R}}$ *adott függvény,* $n \geq 2$*, továbbá legyenek* $M_1, \dots, M_n \in \underline{\mathcal{M}}_f$ *folytonos közepek. Ekkor* $\mathbf{D}_i(M_1, \dots, M_n) \subseteq \underline{\mathcal{M}}_f$ *minden* $i \in \{1, \dots, n\}$ *esetén.*

Feltéve, hogy az alsó konvexitási halmaz speciális Matkowski közepeket is tartalmaz, az alábbi következmények vezethetők le.

KÖVETKEZMÉNY. *Legyen* $f : I \to \overline{\mathbb{R}}$*,* $n \geq 2$*,* $s_1, \dots, s_n \in]0, 1[$ *és, végül, legyen* $h : I \to \mathbb{R}$ *folytonos, szigorúan növő függvény. Tegyük fel, hogy* $\mathcal{M}^{(s_i h, (1-s_i)h)} \in \underline{\mathcal{M}}_f$ *minden* $i \in \{1, \dots, n\}$ *esetén. Ekkor, minden* $i \in \{1, \dots, n\}$ *indexre, a* $\mathcal{M}^{(\sigma_i h, (1-\sigma_i)h)}$ *Matkowski közép tagja az* $\underline{\mathcal{M}}_f$ *osztálynak, ahol a* σ_i *súly* (28) *módon számolható értelmezve minden* $i \in \{1, \dots, n\}$ *esetén.*

KÖVETKEZMÉNY. *Legyen* $n \geq 2$*,* $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ *folytonos, szigorúan növő függvények és* $f : I \to \overline{\mathbb{R}}$*. Legyen továbbá* $h_0 := h_n := 0$ *és tegyük fel, hogy létezik olyan* $j \in \{1, \dots, n\}$ *index, hogy, minden* $i \in \{1, \dots, n\}$ *esetén az*

$$M_i := \begin{cases} \mathcal{M}^{(p+h_{i-1}, h_i)}, & ha\ i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(p+h_{j-1}, h_j+q)}, & ha\ i = j, \\ \mathcal{M}^{(h_{i-1}, h_i+q)}, & ha\ i \in \{j+1, \dots, n\} \end{cases}$$

közép az $\underline{\mathcal{M}}_f$ *osztályhoz tartozik. Ekkor* $N_1, \dots, N_n \in \underline{\mathcal{M}}_f$*, ahol, bármely* $(x, y) \in I^2_{\leq}$ *pontra,*

$$N_i(x, y) = \begin{cases} \mathcal{M}^{(p, h_i)}(x, N_{i+1}(x, y)), & ha\ i \in \{1, \dots, j-1\}, \\ \mathcal{M}^{(p, q)}(x, y), & ha\ i = j \\ \mathcal{M}^{(h_{i-1}, q)}(N_{i-1}(x, y), y), & ha\ i \in \{j+1, \dots, n\}. \end{cases}$$

Áttérve az aszimmetrikus konvexitás fogalmára a felső és alsó konvexitási osztályok beazonosíthatóak a $[0, 1]$ intervallum valamilyen alkalmas részhalmazával. Ekkor az algebrai és topologikus tulajdonságokról szóló tételek még kifejezőbbek lesznek. Megkapjuk a paraméterhalmaz speciális konvexitását, a szorzásra való zártságát és a sűrűségi tulajdonságot is.

TÉTEL. *Legyen* $f : I \to \overline{\mathbb{R}}$ *és* $\mathcal{AC} \in \{\underline{\mathcal{AC}}_f, \overline{\mathcal{AC}}_f\}$.

(1) Bármely $t, s_1, s_2 \in \mathcal{AC}$ *esetén* $ts_2 + (1-t)s_1 \in \mathcal{AC}$ *feltéve, hogy* $s_1 < s_2$.

(2) *Bármely* $t, s \in \mathcal{AC}$ *esetén,* ts *és* $1-(1-t)(1-s)$ *eleme az* $\mathcal{AC}$ *halmaznak.*

(3) *Az* $\mathcal{AC}$ *paraméterhalmaz sűrű* $]0,1[$ *-ben feltéve, hogy nem üres.*

Alkalmazva a Matkowski közepekre nyert általános eredményeket, az alábbi következmények vezethetők le.

KÖVETKEZMÉNY. *Legyen* $I \subseteq \mathbb{R}$ *nemüres intervallum,* $f : I \to \overline{\mathbb{R}}$*,* $n \geq 2$ *és* $s_1, \dots, s_n \in \underline{\mathcal{AC}}_f$*. Ekkor* $\sigma_i \in \underline{\mathcal{AC}}_f$ *bármely* $i \in \{1, \dots, n\}$ *esetén, ahol*

$$\sigma_i := \left(\sum_{j=i}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k}\right)\left(\sum_{j=0}^{n} \prod_{k=1}^{j} \frac{s_k}{1-s_k}\right)^{-1}.$$

KÖVETKEZMÉNY. *Adott* $f : I \to \overline{\mathbb{R}}$ *függvényre az alábbi állítások igazak.*

(1) *Ha* $1/2 \in \underline{\mathcal{AC}}_f$*, akkor* $\mathbb{Q} \cap]0,1[\subseteq \underline{\mathcal{AC}}_f$*.*

(2) *Ha* $\ell/m \in \underline{\mathcal{AC}}_f$ *valamilyen* $\ell, m \in \mathbb{N}$ *pozitív egészek mellett úgy, hogy* $\ell < m$ *és* $2\ell \neq m$*, akkor, bármely* $n \geq 2$ *esetén és bármely* $i \in \{1, \dots, n\}$ *indexre, az*

$$r_i := \frac{\ell^{n+1} - \ell^i (m-\ell)^{n+1-i}}{\ell^{n+1} - (m-\ell)^{n+1}}$$

hányados az $\underline{\mathcal{AC}}_f$ *halmazhoz tartozik.*

Végül konstruálunk egy *felülről aszimmetrikusan konvex, bővített valós értékű függvényt* amelynél a kapcsolódó paraméterhalmaz tartalmaz racionális (és így algebrai) számokat, sűrű a $]0,1[$ intervallumban, de nem igaz rá a Kuhn tétel állítása, nevezetesen, nem írható fel a $]0,1[$ intervallum és valamilyen alkalmas résztest metszeteként. Egy hasonlóan viselkedő valós értékű függvényre sikerült példát adnia Lewicki és Olbryś lengyel matematikusoknak, ahol csak azt tudjuk, hogy a paraméterhalmaz tartalmaz transzcendens elemet. Olyan valós értékű függvény létezése, amely aszimmetrikusan t-konvex valamilyen algebrai t paraméterrel, de nem aszimmetrikusan $(1-t)$-konvex, máig nyitott probléma.

TÉTEL. *Legyen* $I \subseteq \mathbb{R}$ *olyan intervallum, amelyre* $a := \sup I \in I \cap \mathbb{Q}_1$*, legyen* $C : I \to \mathbb{R}$ *konvex függvény és* $f : I \to \overline{\mathbb{R}}$ *olyan, hogy*

$$f(x) := \begin{cases} C(x), & \text{ha } x \in (I \cap \mathbb{Q}_0) \cup \{a\}, \\ +\infty, & \text{ha } x \in I \setminus (\mathbb{Q}_0 \cup \{a\}). \end{cases}$$

Ekkor, bármely $t \in]0,1[\cap \mathbb{Q}_1$ *esetén, az* f *függvény felülről* $\mathcal{A}_t$*-konvex, de nem felülről* $\mathcal{A}_{1-t}$*-konvex.*

KÖVETKEZMÉNY. *A fenti jelöléseket és feltételeket megtartva,* $\overline{\mathcal{AC}}_f$ *nem zárt az összeadására, következésképpen nem írható fel a* $]0,1[$ *intervallum és* $\mathbb{R}$ *valamilyen résztestének metszeteként.*

• **Második rész.** Ebben a részben az (1) és (2) állítások ekvivalenciájáról szóló állítást általánosítjuk. Az eredeti állításból következik, hogy ha egy valós értékű függvény eleget tesz az n-változós Jensen egyenlőtlenségnek, valamilyen rögzített n mellett, akkor teljesíti a kétváltozós Jensen egyenlőtlenséget is. Nyilván, az állítás akkor érdekes, ha $n > 2$. A disszertációban ezt az *n-változós Jensen egyenlőtlenség redukálhatóságá*nak nevezzük. Az eredeti tétel bizonyításából kiderül, hogy ez a tulajdonság szorosan összefügg a háttérben lévő *közép redukálhatóságával*, ami esetünkben a számtani közép.

A harmadik fejezetben precízen definiáljuk adott közép redukálhatóságát. Az alábbi tétel egy elegendő feltételt fogalmaz meg.

TÉTEL. *Az* $M : D^n \to X$ *közép* χ*-redukálható, ha* χ*-folytonos.*

Ezt követően, messzemenően általánosítva a jól ismert középosztályokat, bevezetjük az *általánosított eltérésközép* fogalmát Hausdorff-féle topologikus vektortereken. Kiderül, hogy ezek a közepek természetes módon rendelkeznek a redukálhatósági tulajdonsággal, továbbá a redukált közepek generátorfüggvényei könnyen megadhatók az eredeti generátorok segítségével.

TÉTEL. *Legyen* $n \in \mathbb{N}$*,* $k \in \mathbb{N}_n$*, és* $E \in \mathbf{E}(D)^n$*. Ekkor a* $\mathcal{D}^E : D^n \to D$ *általánosított* E*-eltérésközép bármely injekítv* $\chi : \mathbb{N}_k \to \mathbb{N}_n$ *függvényre nézve redukálható. Továbbá a* $\mathcal{D}^E$ *közép* χ*-redukáltja egyértelműen meghatározott, nevezetesen*

$$\mathcal{D}^E_\chi(x) = \mathcal{D}^{E_\chi}(x), \qquad (x \in D^k).$$

A *3.4. fejezetben*, speciális Gâteaux-differenciálható konvex függvények segítségével, egy jellemzését adjuk az általánosított eltérésközepeknek.

TÉTEL. *Legyen* $D \subseteq X$ *konvex halmaz és* $F \in \boldsymbol{F}(D)$*. Ekkor az*

$$E_F(u, v) = -F'_u(v)$$

módon értelmezett $E_F : D \times D \to D^*$ *függvény egy általánosított eltérés. Továbbá, ha* $n \in \mathbb{N}$*,* $F \in \boldsymbol{F}(D)^n$ *és* $E_F = (E_{F_1}, \dots, E_{F_n})$*, akkor, bármely* $x \in D^n$ *esetén, az* $y = \mathcal{D}^{E_F}(x)$ *egyenlőség pontosan akkor teljesül, ha* y *egyértelmű minimumhelye az*

$$\mathcal{F}_{F,x}(v) := F_1(x_1, v) + \cdots + F_n(x_n, v)$$

módon értelmezett $\mathcal{F}_{F,x} : D \to \mathbb{R}$ *függvénynek a* $\operatorname{conv}(x(\mathbb{N}_n))$ *halmaz felett.*

Megfordítva, ha $X = \mathbb{R}$ és D egy nyílt intervallum, akkor, bármely $E \in \mathbf{E}(D)$ eltérés esetén, létezik $F \in \boldsymbol{F}(D)$ függvény, hogy, bármely $u \in D$ pontra,

$$F'_u(v) = -E(u,v), \qquad (v \in D)$$

teljesül.

A legutolsó fejezet valós függvények (M,N)-konvexitásának redukálhatóságával foglalkozik. Ez közvetlen általánosítása a standard Jensen egyenlőtlenségnek; megkapjuk, ha a bal, illetve a jobb oldalon lévő számtani közepet az M, illetve az N általános közepekkel helyettesítjük. of real functions, which is a self-evident generalization of the Jensen convexity. As in the standard case, its reducibility depends strongly on the reducibility property of the mean values what are involved, namely of M and N.

TÉTEL. *Legyen $D \subseteq X$ egy nemüres konvex halmaz, $I \subseteq \mathbb{R}$ egy intervallum, $n \in \mathbb{N}$, $k \in \mathbb{N}_n$ és legyen $\chi : \mathbb{N}_k \to \mathbb{N}_n$ injektív. Legyenek továbbá $M : D^n \to X$ és $N : I^n \to \mathbb{R}$ közepek úgy, hogy M χ-redukálható, N pedig χ-folytonos és egyértelműen χ-redukálható. Ha az $f : D \to I$ függvény (M,N)-konvex, akkor (K, N_χ)-konvex is az M közép minden $K : D^k \to X$ χ-redukáltjával.*

A fenti tételt alkalmazva speciális redukálható közepekkel, az alábbi állítások bizonyíthatók.

KÖVETKEZMÉNY. *Legyen $D \subseteq X$ nemüres konvex halmaz, $I \subseteq \mathbb{R}$ intervallum és $n \in \mathbb{N}$. Legyen továbbá $\omega : D \to \mathbb{R}^n_+$ és $E : I \times I \to \mathbb{R}^n$ úgy, hogy E_i egy eltérés minden $i \in \mathbb{N}_n$ index esetén. Ha az $f : D \to I$ függvény kielégíti az n-változós*

$$f\big(\mathcal{A}^\omega(x_1,\dots,x_n)\big) \le \mathcal{D}^E\big(f(x_1),\dots,f(x_n)\big), \qquad (x_1,\dots,x_n \in D),$$

egyenlőtlenséget, akkor, bármely $k \in \mathbb{N}_n$ és bármely $\chi : \mathbb{N}_k \to \mathbb{N}_n$ injektív függvény esetén, kielégíti a k-változós

$$f\big(\mathcal{A}^{\omega_\chi}(x_1,\dots,x_k)\big) \le \mathcal{D}^{E_\chi}\big(f(x_1),\dots,f(x_k)\big), \qquad (x_1,\dots,x_k \in D).$$

egyenlőtlenséget is.

KÖVETKEZMÉNY. *Legyen $D \subseteq X$ egy nemüres konvex halmaz, $I \subseteq \mathbb{R}$ intervallum és $n \in \mathbb{N}$. Legyen továbbá $G : D \times D \to (D^*)^n$ és $E : I \times I \to \mathbb{R}^n$ olyan, hogy G_i egy általánosított eltérés és E_i egy eltérés minden $i \in \mathbb{N}_n$ esetén. Ha az $f : D \to I$ függvény kielégíti az n-változós*

$$f\big(\mathcal{D}^G(x_1,\dots,x_n)\big) \le \mathcal{D}^E\big(f(x_1),\dots,f(x_n)\big), \qquad (x_1,\dots,x_n \in D),$$

egyenlőtlenséget, akkor, minden $k \in \mathbb{N}_n$ és minden $\chi : \mathbb{N}_k \to \mathbb{N}_n$ injektív függvény esetén, kielégíti a k-változós

$$f\big(\mathcal{D}^{G_\chi}(x_1, \dots, x_k)\big) \le \mathcal{D}^{E_\chi}\big(f(x_1), \dots, f(x_k)\big), \qquad (x_1, \dots, x_k \in D).$$

egyenlőtlenséget is.

A következő állítás szerint, ha két eltérésközép összehasonlítható, akkor a redukáltjaik is összehasonlíthatók.

KÖVETKEZMÉNY. *Legyen $I \subseteq \mathbb{R}$ intervallum és $n \in \mathbb{N}$. Legyen továbbá $G, E : I \times I \to \mathbb{R}^n$ olyan, hogy G_i és E_i eltérés minden $i \in \mathbb{N}_n$ esetén. Ha*

$$\mathcal{D}^G(x_1, \dots, x_n) \le \mathcal{D}^E(x_1, \dots, x_n), \qquad (x_1, \dots, x_n \in D),$$

akkor, bármely $k \in \mathbb{N}_n$ és bármely $\chi : \mathbb{N}_k \to \mathbb{N}_n$ injektív függvény esetén,

$$\mathcal{D}^{G_\chi}(x_1, \dots, x_k) \le \mathcal{D}^{E_\chi}(x_1, \dots, x_k), \qquad (x_1, \dots, x_k \in D).$$

is teljesül.

Végül, a fenti állítások bizonyításában szereplő technikák segítségével, Hölder-Minkowski-típusú egyenlőtlenségek redukálhatósági tétele is bizonyítható.

TÉTEL. *Legyenek $X_1, \dots, X_\ell$ Hausdorff-féle topologikus vektorterek, $D_1 \subseteq X_1, \dots, D_\ell \subseteq X_\ell$ nemüres konvex halmazok és $I \subseteq \mathbb{R}$ intervallum. Legyen $n \in \mathbb{N}$, $k \in \mathbb{N}_n$ és $\chi : N_k \to \mathbb{N}_n$ egy ijektív függvény. Legyenek végül $N_1 : D_1^n \to X_1, \dots, N_\ell : D_\ell^n \to X_\ell$ χ-redukálható közepek, $M : I^n \to \mathbb{R}$ pedig χ-folytonos és egyértelműen χ-redukálható. Ha az $f : D_1 \times \dots \times D_\ell \to I$ függvény kielégíti $n \cdot \ell$-változós*

$$M\big(f(x^1, \dots, x^\ell)\big) \le f\big(N_1(x^1), \dots, N_\ell(x^\ell)\big), \qquad (x^1 \in D_1^n, \dots, x^\ell \in D_\ell^n),$$

egyenlőtlenséget, akkor az $N_1, \dots, N_\ell$ közepek bármely $K_1 : D_1^k \to X_1, \dots, K_\ell : D_\ell^k \to X_\ell$ χ-redukáltja esetén, kielégíti a $k \cdot \ell$-változós

$$M_\chi\big(f(x^1, \dots, x^\ell)\big) \le f\big(K_1(x^1), \dots, K_\ell(x^\ell)\big), \qquad (x^1 \in D_1^k, \dots, x^\ell \in D_\ell^k),$$

egyenlőtlenséget is, ahol, adott $m \in \mathbb{N}$ és $x^1 \in D_1^m, \dots, x^\ell \in D_\ell^m$ elemekre,

$$f(x^1, \dots, x^\ell) := \big(f(x_1^1, \dots, x_1^\ell), \dots, f(x_m^1, \dots, x_m^\ell)\big).$$

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