Network evolution models governed by branching processes

Thesis for the Degree of Doctor of Philosophy (PhD)

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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Informatics, University of Debrecen in order to obtain a PhD Degree in Informatics at Debrecen University.

The results published in the thesis are not reported in any other PhD theses.

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Hereby I confirm that Attila Barta candidate conducted his studies with my supervision within the Theoretical foundation and applications of information technology and stochastic systems Doctoral Programme of the Doctoral School of Informatics between 2018 and 2022. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.
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I support the acceptance of the thesis.

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Introduction

Network theory is one of the most current fields of science nowadays. The actual challenges of our life require the analysis of the dynamics of different relationships. An appropriate model of such kind of phenomenon can be represented by a random graph. In this case, the nodes of the network are vertices and the links are edges of the graph. Therefore the mathematical background of network theory is graph theory, most of all, it starts from the early works of Erdős and Rényi [1]. In the original definition of the Erdős-Rényi graph, the number of vertices is fixed and in each time step they pair independently uniformly at random. It was shown that in the early stage of the evolution process, mostly tree components occur in our random graph, but when the number of edges extends half of the number of vertices a giant component appears. In the Erdős-Rényi-Gilbert graph, the number of vertices is fixed and they pair independently with some fixed probability. However, in certain papers it was empirically illustrated that real-life networks work differently [2]. The book of Barabási [3] is a nice summary of these kinds of empirical studies.

In their fundamental paper [2], Barabási and Albert proposed the preferential attachment method to describe the evolution of random networks. First, they list several real-world networks such as the collaboration graph of movie actors, the WWW, the electric power grid, and the citation patterns of scientific publications having power-law degree distributions. In their model, every timestep a new vertex with m edges is added to the network so that the probability that the new vertex is connected to an old vertex is proportional to the degree of the old vertex. Then they give a short argument and simulation results for the power law degree distribution. We have to mention that the Barabási-Albert model is not the only one with a preferential attachment mechanism, e.g. the Yule model was studied chronologically the earliest [4]. Such models are summarized in [5].

In [6] a careful study of the preferential attachment graph evolution process is presented. The authors give detailed mathematical proof for the asymptotic degree distribution. In [7] the following version of the preferential attachment tree was studied. At each step a new vertex is added to the existing tree and the new vertex is connected to one of the old vertices with a single edge. The other endpoint of the edge is chosen randomly so that a vertex with degree k is chosen with probability proportional to $k+\beta$, with $\beta > -1$. The author finds the limiting degree distribution by martingale methods. In [8] the authors introduce a generalization of the original preferential attachment model. During the evolution of the graph either a new vertex is born or two old vertices are connected with a new edge. The choice of the old vertex can be both uniform and according to the degrees. The authors obtain the limiting degree distribution.

In [9] and [10] the authors introduce and analyse a random graph evolution model which describes the interactions of 3 units. So, besides vertices and edges, triangles also take part in the evolution of the graph. In the model vertices, edges and triangles have their weights which give the numbers of their interactions. Like in [8], both preferential attachment rule and uniform choice are applied during the evolution of the graph. However, instead of the degrees, the weights are considered at the preferential attachment rule. The asymptotic degree distribution is obtained. To obtain the results, the authors use martingale methods. In [11] an extension of the model of [9] and [10] is considered. An interaction of N vertices is described by an N-clique. The weight of a clique is the number of its interactions. The evolution is a combination of the preferential attachment and the uniform choice. The asymptotic behaviour of the graph is studied by martingale methods. Scale-free properties both for the degrees and the weights of vertices are proved. It is obtained that any exponent in $(2, \infty)$ can be achieved. In [12] further generalization of the model is studied. In [13] the authors introduce the so-called PA-class which is a common framework to study several preferential attachment models. They obtain theorems for the limiting power-law degree distribution and the clustering coefficient.

We mention that in [14] the Erdős–Rényi graph, the configuration model and the preferential attachment graph were studied when the population was split into two types. The mathematical tool of the analysis in [14] is the theory of multi-type branching processes.

In contrast to discrete-time network evolution models, continuous-time models can handle overlaps between different generations, making real networks more realistic. However, they can also be seen as a tool for describing discrete-time networks because of their more tractable, mathematically closed form. There are several continuous-time network evolution models. Here, we list only some papers using continuous time branching processes. In [15] the theory of continuous time branching processes was used to obtain asymptotic results for certain random trees. The authors consider a tree which grows randomly in discrete time. Their model is a generalization of the well-known preferential attachment random tree. They introduce a weight function $w : \mathbb{N} \to \mathbb{R}_+$. At each time a new vertex is born and it is connected to a randomly chosen vertex of the existing tree. The probability that the vertex x is chosen to this end, is proportional to $w(\deg(x))$. If w is linear, then the model was formerly analysed carefully by [6] and [7]. In [15] the asymptotic distribution of the degree sequence and the asymptotic distribution of the subtree under a randomly selected vertex are obtained. For the proof, an appropriate continuous time branching process is introduced. If we observe the continuous time branching process at its jumping times, then we obtain the random tree. Then known results of the general branching processes (see [16], [17], [18]) imply the results.

Recently, in [19] multi-type preferential attachment trees were studied. In [19] the results of [20] on multi-type continuous time branching processes were applied to describe the evolution of the network.

In this thesis, we study two new network evolution models. Our models are generalizations of the one studied by Móri and Rokob [21]. The structure and the rules of the evolution of our models were inspired both by some everyday experiences and deep scientific results on motifs. On the one hand, we had in our mind activities and structures based on personal connections of the actors and where teams of some persons are important. Thus, we considered the friendship, the recruitment of party members and cooperation among party members, the recruitment and cooperation of volunteers, cooperation among scientists, informal connections among the employees of a company, etc. In these cases, the network consists of relatively small teams, a person can be a member of several teams at the same time, new teams can be born, and they can die, a newcomer can join the network if he/she joins an existing team.

On the second hand, our models are supported by the theory of motifs and their applications for real life networks. Here, we list only a few papers on this topic.

In [22] the authors used network motifs: 'patterns of interconnections occurring in complex networks at numbers that are significantly higher than those in randomized networks'. They developed an algorithm for detecting network motifs and found motifs with three or four vertices in biological and technological networks.

In [23] the authors analyse the local structure of several networks such as protein signaling, developmental genetic networks, power grids, protein-structure networks, World Wide Web links, social networks, and word-adjacency networks. For the study, they used motifs on three or four vertices. In [24] the authors found the numbers of all 3- and 4-node subgraphs, in both directed and non-directed geometric networks. In [25] a method for the identification of all ordered 3-node substructures and the visualization of their significance profile are offered.

Therefore, we wanted to study networks that consist of small substructures, a node can be a member of several substructures at the same time, new substructures can be born and they can die, a new node can join to the network if it joins to an existing substructure.

Concerning the mathematical tools, we follow the line of Móri and Rokob [21], where connections of two units were described by edges and the evolution of the edges was governed by a continuous time branching process.

The structure of our thesis is the following. In Chapter 1 we describe a new network evolution model with 3-interactions. This chapter is based on our papers [26] and [27]. In Section 1 the precise definition of our model is given. In Section 2 the general results on our model are presented. These are the survival function of a triangle (Theorem 2.1), the mean offspring number of a triangle (Corollary 2.1), the joint generating function of the birth process and the offspring number (Theorem 2.2) and the probability of the extinction (Theorem 2.3). In Section 3 asymptotic theorems on the number of triangles (Theorem 3.1), the number of vertices (Theorem 3.2) and the number of edges (Theorem 3.3) are proved. All of them have magnitude $e^{\alpha t}$ on the event of non-extinction, where α is the Malthusian parameter. To prove Theorems 3.1, 3.2 and 3.3, we used the underlying branching process counted with certain random characteristics and applied the asymptotic theorems of [17]. We also obtained asymptotic results for the degree of a fixed vertex. To this end we introduced a new branching process and again used general limit theorems of [17] to this new branching process counted with certain random characteristics. In Section 4 we present some simulation results supporting our theorems. The proofs are based on known general results of continuous-time branching processes. The main ideas of our proofs are similar to the method used in [21], but the analysis of our more complex model needed more complicated reasoning.

In Chapter 2 we describe a new network evolution model with 2- and 3-interactions. This chapter is based on papers [28] and [29]. In Section 6 a detailed description of our model is given. In Section 7 the general results are presented. These are the survival functions of an edge and of a triangle (Theorem 7.1), the mean offspring number of an edge and of a triangle (Corollary 7.1), the Perron root and the Malthusian parameter. As usual, we obtain only implicit expression for the Malthusian parameter, but our expression is simple and numerically tractable. In Section 8, asymptotic theorems on the number of edges and triangles (Theorem 8.1) are proved. Both of them have magnitude $e^{\alpha t}$ on the event of non-extinction, where α is the Malthusian parameter. To prove Theorem 8.1, we use the underlying multitype branching process counted with certain random characteristics and apply the asymptotic theorems of [20]. In Section 9 the generating functions are calculated. Using the generating functions, the probability of extinction are studied. In Section 10 the asymptotic behaviour of the degree of a fixed vertex is considered. Here, we again apply the asymptotic theorems of [20] but with other characteristics than in Section 8. In Section 11 we present some simulation results supporting our theorems. Our figures and tables show that the values obtained by simulation fit well to the theoretical results. The proofs are based on known general results of multi-type continuous-time branching processes. In Appendix A we summarize some known facts on branching processes while in Appendix B we show known results on multitype branching processes which we used during our proofs.

Chapter 1

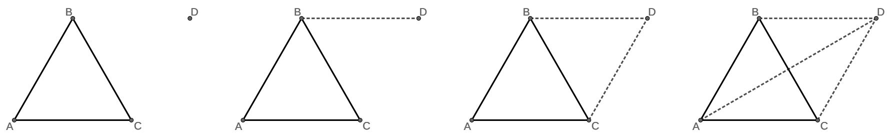
The 3-interaction model

In this chapter we describe our new results on the 3-interaction model. They were published in papers [26, 27].

1 Model description

We shall study the following evolving random graph model. At the initial time $t = 0$ we start with a single triangle. We call it the ancestor triangle. This ancestor triangle produces offspring triangles. Then these offspring triangles also produce their offspring triangles, and so on. Every triangle, including the ancestor, has its own birth process, which is a Poisson process with rate 1. Let $\Pi(t)$, $t \geq 0$, denote a generic Poisson process with rate 1. We assume that during the evolution of the model, the reproduction processes of the triangles are independent copies of the following generic reproduction mechanism.

For any fixed triangle the reproduction is the following. Let us denote the reproduction process by ξ and the birth times corresponding to the fixed triangle by $\tau_1, \tau_2, \ldots$. Here ξ is a point process and, as usual, $\xi(t)$ denotes the total number of children triangles of the given triangle up to time t, where $\xi(0) = 0$. However, at a birth time not only new triangles can be created but other ingredients can be added to the graph. At every birth time τ_i, a new vertex is added to the graph which can be connected to our fixed triangle with j edges ($j = 0, 1, 2, 3$). Let p_j denote the probability that the new vertex will be connected to j vertices of our fixed triangle. The vertices to be connected to the new vertex are chosen uniformly at random. It follows from the definition of the above evolution process that at each birth step we always add 1 new vertex, add $0, 1, 2$ or 3 new edges to the graph and the possible number of the new triangles is $0, 1$ or 3. Figure 1.1 shows the four cases of the evolution. The triangle ABC is the parent triangle and D is the new vertex. The figure shows that it can join to the parent triangle with 0, 1, 2 or 3 new edges. E.g. the rightmost part of the figure shows the case when there are 3 new edges and there are 3 children triangles: ABD, BCD and CAD.



p_0 p_1 p_2 p_3

Figure 1.1: The four cases of the evolution

Let us denote by $\varepsilon_1, \varepsilon_2, \ldots$ the litter sizes belonging to the birth times $\tau_1, \tau_2, \ldots$. That is, the generic triangle bears ε_i children triangles at the ith birth event. Then $\varepsilon_1, \varepsilon_2, \ldots$ are independent identically distributed discrete random variables with distribution $\mathbb{P}(\varepsilon_i = j) = q_j, j \geq 0$. In our model the distribution of the litter size ε_i is given by

$$\mathbb{P}(\varepsilon_i = 0) = q_0 = p_0 + p_1,\ \mathbb{P}(\varepsilon_i = 1) = q_1 = p_2,\ \mathbb{P}(\varepsilon_i = 3) = q_3 = p_3,$$

$$\mathbb{P}(\varepsilon_i = j) = q_j = 0, \quad \text{if} \quad j \notin \{0, 1, 3\}.$$

Throughout the chapter, we assume that $p_0 + p_1 < 1$, because otherwise there were no reproduction of the triangles. We assume that the litter sizes $\varepsilon_1, \varepsilon_2, \ldots$ are independent of the birth times $\tau_1, \tau_2, \ldots$, too.

Let the finite, non-negative random variable λ be the life-length of the individual (i.e. of the triangle). We assume that the reproduction terminates at the death of the individual, therefore $\xi(t) = \xi(\lambda)$ for $t > \lambda$. Then the reproduction process of a triangle can be given by

$$\xi(t) = \sum_{\tau_i \leq t \wedge \lambda} \varepsilon_i = S_{\Pi(t \wedge \lambda)}, \tag{1.1}$$

where $\Pi(t)$ is the Poisson process, $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ gives the total number of offspring before the $(n+1)$th birth event and $x \wedge y$ denotes the minimum of $\{x, y\}$.

The survival function of a triangle's life-length. Let $L(t)$ denote the distribution function of λ. Then the survival function of a triangle's life-length is

$$1 - L(t) = \mathbb{P}(\lambda > t \mid \xi(u), 0 \leq u \leq t) = \exp\left(-\int_0^t l(u)\, du\right), \tag{1.2}$$

where $l(t)$ is the hazard rate of the life-length λ. We assume that the hazard rate depends on the number of offspring, so that

$$l(t) = b + c\xi(t) \tag{1.3}$$

with non-negative constants b and c.

We have to mention that we do not delete the triangle when it dies, because its vertices and edges can belong to other triangles, too. So we consider a dead triangle as an inactive triangle not producing new offspring.

2 General results

In this section the general results on our model are presented. These are the survival function of a triangle (Theorem 2.1), the mean offspring number of a triangle (Corollary 2.1), the joint generating function of the birth process and the offspring number (Theorem 2.2) and the probability of the extinction (Theorem 2.3).

The survival function. First we calculate $L(t)$.

Remark 2.1. Let $t > 0$ and assume that $\Pi(t) = k$. Then the first k birth events happened before time t. Therefore the birth times $\tau_1, \dots, \tau_k$ and the corresponding litter sizes $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ are known, and so $\xi(u)$ is also known for $u < t$. Therefore, using (1.3), a simple calculation shows that the survival function of an individual (i.e. a triangle) is

$$1 - L(t) = \exp\left(-\int_0^t l(u)\,du\right) = \exp\left(-\left(bt + c\int_0^t \xi(u)\,du\right)\right) =$$

$$= \exp\left(-\left(bt + ctS_k - c\left(\varepsilon_1\tau_1 + \dots + \varepsilon_k\tau_k\right)\right)\right).$$

Theorem 2.1. *The survival function is*

$$1 - L(t) = \exp\left(-t(b+1) + \left(tq_0 + \frac{3q_1\left(1 - e^{-ct}\right) + q_3\left(1 - e^{-3ct}\right)}{3c}\right)\right). \tag{2.1}$$

Proof. By Remark 2.1, we have

$$\mathbb{P}\left(\lambda > t | \Pi(t) = k, \tau_1, \dots, \tau_k, \varepsilon_1, \dots, \varepsilon_k\right) =$$
$$= \exp\left(-\left(bt + ctS_k - c\left(\varepsilon_1\tau_1 + \dots + \varepsilon_k\tau_k\right)\right)\right).$$

Let $(U_1^*, \dots, U_k^*)$ be an ordered sample of size k from uniform distribution on $[0, 1]$. Then the joint conditional distribution of the birth times $\tau_1, \dots, \tau_k$ given $\Pi(t) = k$, coincides with the distribution of $(tU_1^*, \ldots, tU_k^*)$. Therefore

$$\mathbb{P}(\lambda > t|\Pi(t) = k) = \mathbb{E}\exp\left(-\left(bt + ct\sum_{i=1}^k \varepsilon_i\left(1 - \frac{\tau_i}{t}\right)\right)\right) =$$

$$= \mathbb{E}\exp\left(-bt + ct\sum_{i=1}^k \varepsilon_i\left(U_i^* - 1\right)\right),$$

because $\tau_i = tU_i^*$. The litter sizes $\varepsilon_1, \ldots, \varepsilon_k$ are independent identically distributed random variables which are independent of $U_1^*, \ldots, U_k^*$, too. Hence

$$\mathbb{P}(\lambda > t|\Pi(t) = k) = \mathbb{E}\exp\left(-bt + ct\sum_{i=1}^k \varepsilon_i\left(U_i - 1\right)\right) =$$

$$= e^{-bt}\mathbb{E}\prod_{i=1}^k e^{ct\varepsilon_i(U_i-1)} = e^{-bt}\left(\mathbb{E}_{\varepsilon_i}\left(\mathbb{E}_{U_i}\left(e^{ct\varepsilon_i U_i}\right)e^{-ct\varepsilon_i}\right)\right)^k =$$

$$= e^{-bt}\left(q_0 + \sum_{j=1}^\infty q_j\frac{e^{ctj} - 1}{ctj}e^{-ctj}\right)^k = e^{-bt}\left(q_0 + \sum_{j=1}^\infty q_j\frac{1 - e^{-ctj}}{ctj}\right)^k =$$

$$= e^{-bt}\left(q_0 + \frac{3q_1\left(1 - e^{-ct}\right) + q_3\left(1 - e^{-3ct}\right)}{3ct}\right)^k,$$

where we applied that U_i is uniformly distributed. Using this and the total probability theorem, we get

$$\mathbb{P}(\lambda > t) = \sum_{k=0}^\infty \mathbb{P}(\Pi(t) = k)\,\mathbb{P}(\lambda > t|\Pi(t) = k) =$$

$$= \sum_{k=0}^\infty \frac{t^k}{k!}e^{-t}e^{-bt}\left(q_0 + \frac{3q_1\left(1 - e^{-ct}\right) + q_3\left(1 - e^{-3ct}\right)}{3ct}\right)^k =$$

$$= e^{-t(b+1)}\sum_{k=0}^\infty \frac{1}{k!}\left(tq_0 + \frac{3q_1\left(1 - e^{-ct}\right) + q_3\left(1 - e^{-3ct}\right)}{3c}\right)^k =$$

$$= e^{-t(b+1)}e^{\left(tq_0 + \frac{3q_1\left(1 - e^{-ct}\right) + q_3\left(1 - e^{-3ct}\right)}{3c}\right)}.$$

□

The mean offspring number of a triangle. Let us denote by $\mu(t) = \mathbb{E}\xi(t)$ the expectation of the number of offspring of a triangle until time t.

Corollary 2.1. *For any $t \geq 0$ we have*

$$\mu(t) = \frac{q_1 + 3q_3}{c} \int_0^{1-e^{-ct}} (1-u)^{\frac{b+1-q_0}{c}-1} e^{\frac{u}{3c}\left(q_3 u^2 - 3q_3 u + 3(q_1+q_3)\right)} du \tag{2.2}$$

and

$$\mathbb{E}\lambda = \frac{1}{c} \int_0^1 (1-u)^{\frac{b+1-q_0}{c}-1} e^{\frac{u}{3c}\left(q_3 u^2 - 3q_3 u + 3(q_1+q_3)\right)} du. \tag{2.3}$$

$0 < \mathbb{E}\lambda < \infty$ *because* $b \geq 0$ *and* $q_0 < 1$.

Proof. By (1.1), we have $\mu(t) = \mathbb{E}S_{\Pi(t\wedge\lambda)} = \mathbb{E}\left(\varepsilon_1 + \cdots + \varepsilon_{\Pi(t\wedge\lambda)}\right)$. Using Wald's identity, the average number of children is

$$\mu(t) = \mathbb{E}\xi(t) = \mathbb{E}S_{\Pi(t\wedge\lambda)} = \mathbb{E}(\varepsilon_1)\,\mathbb{E}(\Pi(t\wedge\lambda)). \tag{2.4}$$

Using that Π is a Poisson process with rate 1 and $t \wedge \lambda$ is bounded for any t, from (2.4) we obtain that the average number of children is

$$\mu(t) = \mathbb{E}(\varepsilon_1)\,\mathbb{E}(\Pi(t\wedge\lambda)) =$$

$$= (q_1 + 3q_3) \int_0^t 1 - L(s)\,ds = (q_1 + 3q_3) \int_0^t \mathbb{P}(\lambda > s)\,ds. \tag{2.5}$$

Applying (2.1) and using the substitution $u = 1 - e^{-cs}$, we obtain

$$\int_0^t \mathbb{P}(\lambda > s)\,ds = \int_0^t e^{s(q_0-b-1)} e^{\frac{3q_1\left(1-e^{-cs}\right)+q_3\left(1-e^{-3cs}\right)}{3c}} ds =$$

$$= \frac{1}{c} \int_0^{1-e^{-ct}} (1-u)^{\frac{b+1-q_0}{c}-1} e^{\frac{u}{3c}\left(q_3 u^2 - 3q_3 u + 3(q_1+q_3)\right)} du. \tag{2.6}$$

So we obtained (2.2). Moreover, with $t \to \infty$, we have $\mathbb{E}\lambda = \int_0^\infty \mathbb{P}(\lambda > s)\,ds$. So (2.3) follows from (2.6). □

We see that $\mu(0) = 0 < 1$ and $\mu(t) < \infty$ for all t, so $\mathbb{P}(y_t < \infty, \forall t) = 1$, where y_t is the number of triangles that have been born up to time t, see Theorem (6.2.2) of [16].

The joint generating function of $\Pi(\lambda)$ and $\xi(\lambda)$. Let $w_{i,j} = \mathbb{P}(\Pi(\lambda) = i, \xi(\lambda) = j)$. We can see that $w_{i,j} = \mathbb{P}(\tau_i \leq \lambda < \tau_{i+1}, \xi(\tau_i) = j)$. So $w_{i,j}$ is the probability that the ith birth event is the last one which happened before death and the total number of offspring up to time τ_i is equal to j.

Now consider the sequence $u_{i,j} = \mathbb{P}(\tau_i \leq \lambda, \xi(\tau_i) = j)$. At each birth step, the total number of offspring of an individual can be changed by $0, 1$ or 3. Let $\xi(\tau_{i-1}) = m$ and assume for a while that τ_i and τ_{i-1} are fixed. Then using (1.2) and (1.3) for the hazard rate, short calculation gives that for fixed τ_i and τ_{i-1} we have

$$\mathbb{P}(\lambda > \tau_i | \lambda > \tau_{i-1}, \tau_{i-1}, \tau_i) = \exp(-(b+cm)(\tau_i - \tau_{i-1})).$$

However, the increment $(\tau_i - \tau_{i-1})$ is exponential with parameter 1, therefore

$$\mathbb{P}(\lambda > \tau_i | \lambda > \tau_{i-1}) = \mathbb{E}_{\tau_i - \tau_{i-1}} \exp(-(b+cm)(\tau_i - \tau_{i-1})) = \frac{1}{1+b+cm}. \tag{2.7}$$

Using these and the total probability theorem, we can give the following recursion for $u_{i,j}$.

$$\begin{aligned} u_{i,j} = \mathbb{P}(\tau_{i-1} \le \lambda, \xi(\tau_{i-1}) = j)\, q_0 \frac{1}{1+b+cj} + \\ + \mathbb{P}(\tau_{i-1} \le \lambda, \xi(\tau_{i-1}) = j-1)\, q_1 \frac{1}{1+b+c(j-1)} + \\ + \mathbb{P}(\tau_{i-1} \le \lambda, \xi(\tau_{i-1}) = j-3)\, q_3 \frac{1}{1+b+c(j-3)} = \end{aligned}$$

$$= u_{i-1,j} \frac{q_0}{1+b+cj} + u_{i-1,j-1} \frac{q_1}{1+b+c(j-1)} + u_{i-1,j-3} \frac{q_3}{1+b+c(j-3)}. \tag{2.8}$$

Now we can see that

$$\begin{aligned} w_{i,j} &= \mathbb{P}(\tau_i \le \lambda < \tau_{i+1}, \xi(\tau_i) = j) = \\ &= \mathbb{P}(\lambda < \tau_{i+1} | \tau_i \le \lambda, \xi(\tau_i) = j)\, \mathbb{P}(\tau_i \le \lambda, \xi(\tau_i) = j) = \frac{b+cj}{1+b+cj} u_{i,j}, \end{aligned}$$

where, by (2.7), $\frac{b+cj}{1+b+cj}$ is the probability that the individual dies before the next birth event.

Let $v_{i,j} = \dfrac{w_{i,j}}{b+cj} = \dfrac{u_{i,j}}{1+b+cj}$, $i = 1, 2, \dots, j = 0, 1, \dots, 3i$. Then from (2.8), we can obtain the following recursion for the sequence $v_{i,j}$,

$$(1+b+cj)\, v_{i,j} = v_{i-1,j} q_0 + v_{i-1,j-1} q_1 + v_{i-1,j-3} q_3, \tag{2.9}$$

where from $\tau_0 = 0$ comes that the initial values are

$$v_{0,0} = \frac{1}{1+b} \text{ and } v_{0,j} = 0 \text{ for } j \ne 0. \tag{2.10}$$

Now we will determine the generating function $G(x, y)$ of the sequence $v_{i,j}$,

$i = 0, 1, \ldots, j = 0, 1, \ldots, 3i$. We have

$$G(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{3i} v_{i,j} x^i y^j.$$

First, multiplying with $x^i y^j$ and then taking the sum of both sides of (2.9), we obtain

$$\sum_{i=1}^{\infty} \sum_{j=0}^{3i} v_{i,j} x^i y^j (1 + b + cj) = q_0 \sum_{i=1}^{\infty} \sum_{j=0}^{3i} v_{i-1,j} x^i y^j + \\ + q_1 \sum_{i=1}^{\infty} \sum_{j=0}^{3i} v_{i-1,j-1} x^i y^j + q_3 \sum_{i=1}^{\infty} \sum_{j=0}^{3i} v_{i-1,j-3} x^i y^j,$$

where $v_{0,j}, j = 0, 1, \ldots$ is given by (2.10) and define $v_{i,j} = 0$ if $j < 0$. From this equation, we get

$$(1+b)\left(G(x,y) - \frac{1}{1+b}\right) + ycG_y^{'}(x,y) = \\ = q_0 x G(x,y) + q_1 xy G(x,y) + q_3 xy^3 G(x,y). \quad (2.11)$$

Let $h(t) = G(x, ty)$. By the recursion (2.9)-(2.10) with $j = 0$, we get $v_{i,0} = \frac{1}{1+b}\left(\frac{q_0}{1+b}\right)^i$, and therefore

$$h(0) = G(x,0) = \sum_{i=0}^{\infty} v_{i,0} x^i = \frac{1}{1+b-q_0 x}.$$

Now, substituting y with ty in (2.11), we can obtain the following first order differential equation:

$$h^{'}(t) + h(t)\frac{1}{ct}\left((1+b) - q_0 x - q_1 txy - q_3 t^3 xy^3\right) = \frac{1}{ct} \quad (2.12)$$

with initial value condition

$$h(0) = \frac{1}{1+b-q_0 x}. \quad (2.13)$$

The solution of the above initial value problem (2.12)-(2.13) is

$$h(t) = t^{\frac{-(1+b)+q_0 x}{c}} e^{\frac{q_1 xy}{c} t + \frac{q_3 xy^3}{3c} t^3} \frac{1}{c} \int_0^t s^{\frac{1+b-q_0 x}{c} - 1} e^{-\left(\frac{q_1 xy}{c} s + \frac{q_3 xy^3}{3c} s^3\right)} ds.$$

Substituting $t = 1$, we obtain that

$$G(x,y) = h(1) = e^{\frac{q_1xy}{c} + \frac{q_3xy^3}{3c}} \frac{1}{c} \int_0^1 s^{\frac{1+b-q_0x}{c}-1} e^{-\left(\frac{q_1xy}{c}s + \frac{q_3xy^3}{3c}s^3\right)} ds.$$

Moreover, substituting $u = 1 - s$ into the above integral, we get

$$G(x,y) = \frac{1}{c} \int_0^1 (1-u)^{\frac{1+b-q_0x}{c}-1} e^{\left(\frac{q_1xy+q_3xy^3}{c}u - \frac{q_3xy^3}{c}u^2 + \frac{q_3xy^3}{3c}u^3\right)} du. \tag{2.14}$$

Theorem 2.2. *The joint generating function of* $\Pi(\lambda)$ *and* $\xi(\lambda)$ *is*

$$g_{\Pi,\xi}(x,y) = 1 + \frac{q_0x + q_1xy + q_3xy^3 - 1}{c} \times$$
$$\times \int_0^1 (1-u)^{\frac{1+b-q_0x}{c}-1} e^{\left(\frac{q_1xy+q_3xy^3}{c}u - \frac{q_3xy^3}{c}u^2 + \frac{q_3xy^3}{3c}u^3\right)} du, \tag{2.15}$$

where $-1 \le x, y \le 1$.

Proof. Using that $\mathbb{P}(\Pi(\lambda) = i, \xi(\lambda) = j) = w_{i,j} = v_{i,j}(b + cj)$, by (2.11) we have

$$g_{\Pi,\xi}(x,y) = \mathbb{E}\left(x^{\Pi(\lambda)}y^{\xi(\lambda)}\right) = \sum\nolimits_{i=0}^{\infty} \sum\nolimits_{j=0}^{3i} \mathbb{P}(\Pi(\lambda) = i, \xi(\lambda) = j)\, x^i y^j =$$

$$= b \sum\nolimits_{i=0}^{\infty} \sum\nolimits_{j=0}^{3i} v_{i,j} x^i y^j + cy \sum\nolimits_{i=0}^{\infty} \sum\nolimits_{j=1}^{3i} v_{i,j} x^i y^{j-1} j =$$
$$= bG(x,y) + cyG'_y(x,y) =$$
$$= G(x,y)\left(q_0x + q_1xy + q_3xy^3 - 1\right) + 1.$$

From this and (2.14), we obtain (2.15). □

The probability of extinction. The reproduction process $\xi(t)$ gives the number of offspring of an individual up to time t. With $t \to \infty$, we denote the total number of offspring of an individual with $\xi(\infty)$. Therefore, as we have seen it in the proof of Corollary 2.1, the expected offspring number of a triangle is

$$\mu(\infty) = \mathbb{E}\xi(\infty) = \mathbb{E}(\varepsilon_1)\,\mathbb{E}(\lambda \wedge \infty) = (q_1 + 3q_3)\,\mathbb{E}(\lambda) =$$
$$= (q_1 + 3q_3) \frac{1}{c} \int_0^1 (1-u)^{\frac{b+1-q_0}{c}-1} e^{\frac{u}{3c}\left(q_3u^2 - 3q_3u + 3(q_1+q_3)\right)} du. \tag{2.16}$$

To determine the extinction probability of the process, we consider the following embedded Galton-Watson process. At time $t = 0$, the 0th generation of the Galton-

Watson process consists of a single triangle which is our ancestor triangle. The first generation consists of all offspring triangles of the ancestor triangle. The offspring of the individuals (triangles) in the nth generation form the $(n+1)$th generation. The extinction of our original process is the same as the extinction of this embedded Galton-Watson process. Therefore, by Theorems (2.3.1) and (6.5.1) of [16], if $\mu(\infty) \leq 1$, then the probability of extinction of the process is equal to 1 (because in our model the case when the offspring number is precisely equal to 1 is not possible). Such basic results on branching processes also can be found in Chapter 1 of [18].

Consider the following equation:

$$1 = \frac{q_1 + q_3\left(y^2 + y + 1\right)}{c} \int_0^1 (1-u)^{\frac{1+b-q_0}{c}-1} e^{\left(\frac{q_1 y + q_3 y^3}{c} u - \frac{q_3 y^3}{c} u^2 + \frac{q_3 y^3}{3c} u^3\right)} du. \tag{2.17}$$

Theorem 2.3. *If $\mu(\infty) > 1$, then the probability of the extinction of the triangles is the smallest non-negative solution of equation* (2.17).

Proof. By Theorems (2.3.1) and (6.5.1) of [16], if $\mu(\infty) > 1$, then the extinction probability is the smallest non-negative root of the equation $g_\xi(y) = y$, where g_ξ is the generating function of $\xi(\lambda)\,(=\xi(\infty))$.

Using Theorem 2.2, we obtain

$$y = g_\xi(y) = g_{\Pi,\xi}(1, y) = 1 + \frac{q_0 + q_1 y + q_3 y^3 - 1}{c} \times$$
$$\times \int_0^1 (1-u)^{\frac{1+b-q_0}{c}-1} e^{\left(\frac{q_1 y + q_3 y^3}{c} u - \frac{q_3 y^3}{c} u^2 + \frac{q_3 y^3}{3c} u^3\right)} du.$$

Rearranging the above equation, we see

$$y - 1 = \frac{q_1(y-1) + q_3(y-1)\left(y^2 + y + 1\right)}{c} \times$$
$$\times \int_0^1 (1-u)^{\frac{1+b-q_0}{c}-1} e^{\left(\frac{q_1 y + q_3 y^3}{c} u - \frac{q_3 y^3}{c} u^2 + \frac{q_3 y^3}{3c} u^3\right)} du.$$

Dividing both sides by $y-1$, we obtain equation (2.17). $\square$

3 Asymptotic theorems on the number of triangles, edges, vertices and degrees

In this section asymptotic theorems on the number of triangles (Theorem 3.1), the number of vertices (Theorem 3.2) and the number of edges (Theorem 3.3) are proved. All of them have magnitude $e^{\alpha t}$ on the event of non-extinction, where α is the Malthusian parameter. To prove Theorems 3.1, 3.2 and 3.3, we used the underlying branching process counted with certain random characteristics and applied the asymptotic theorems presented in Appendix A. We also obtained asymptotic results for the degree of a fixed vertex.

Assume that $\mu(\infty) > 1$ that is, the branching process is supercritical. Then the Malthusian parameter α is the only positive solution of the equation

$$\int_0^\infty e^{-\alpha t}\mu(dt) = 1. \tag{3.1}$$

The asymptotic behaviour of the number of triangles. By (2.5) we have

$$\mu(t) = (q_1 + 3q_3)\int_0^t \mathbb{P}(\lambda > s)\, ds.$$

Therefore, in our model, by (3.1) and using Theorem 2.1 we have

$$1 = \int_0^\infty e^{-\alpha t}\mu(dt) = (q_1 + 3q_3)\int_0^\infty e^{-\alpha t}\mathbb{P}(\lambda > t)\, dt =$$

$$= (q_1 + 3q_3)\int_0^\infty e^{-(\alpha+(b+1))t} e^{\left(tq_0 + \frac{3q_1\left(1-e^{-ct}\right)+q_3\left(1-e^{-3ct}\right)}{3c}\right)} dt. \tag{3.2}$$

Substituting $u = 1 - e^{-ct}$ in the above integral, we obtain the following form of equation (3.1)

$$1 = \frac{(q_1 + 3q_3)}{c}\int_0^1 (1-u)^{\frac{\alpha+(b+1)}{c} - \frac{q_0}{c} - 1} e^{\frac{3q_1 u + q_3 u\left(u^2 - 3u + 3\right)}{3c}} du. \tag{3.3}$$

Lemma 3.1. *If $\mu(\infty) > 1$, then the Malthusian parameter α is the only positive solution of equation (3.3). The only positive solution α of equation (3.3) satisfies*

$$q_1 + 3q_3 - b - 1 < \alpha < q_1 + 3q_3 - b. \tag{3.4}$$

Proof. Here (3.2) implies that

$$1 \leq (q_1 + 3q_3) \int_0^\infty e^{-(\alpha+b)t} dt \quad \text{and} \quad 1 \geq (q_1 + 3q_3) \int_0^\infty e^{-(\alpha+(b+1))t} dt.$$

Form these inequalities we can obtain (3.4). ☐

Let us denote by $Z(t)$ the number of triangles alive at time t.

Theorem 3.1. *Let α be the solution of* (3.3). *Then we have*

$$\lim_{t\to\infty} e^{-\alpha t} Z(t) = Y_\infty m_\infty \tag{3.5}$$

almost surely and in L^1, where the random variable Y_∞ is non-negative and it is positive on the event of non-extinction. Moreover,

$$m_\infty = \frac{1}{(q_1 + 3q_3)^2 \int_0^\infty te^{-\alpha t}(1 - L(t))\, dt}. \tag{3.6}$$

Proof. We check the conditions of Proposition 12.1 in Appendix A. First calculate the quantity ${}_\alpha\xi(\infty)$ from (12.1).

$${}_\alpha\xi(\infty) = \int_0^\infty e^{-\alpha t} \xi(dt), \tag{3.7}$$

where ξ denotes the reproduction process of the ancestor. At each birth-step the maximal number of new offspring is 3, therefore we have

$${}_\alpha\xi(\infty) = \sum\nolimits_{\tau_i \leq \lambda} \varepsilon_i e^{-\alpha\tau_i} \leq 3 \sum\nolimits_{\tau_i \leq \lambda} e^{-\alpha\tau_i} \leq 3 \sum\nolimits_{i=1}^\infty e^{-\alpha\tau_i} = 3M.$$

In the Poisson process $\Pi(t)$ the distribution of the interarrival time $(\tau_i - \tau_{i-1})$ is exponential with rate 1, therefore τ_i has Γ-distribution $\mathit{\Gamma}(i, 1)$. Using this, we have

$$\mathbb{E}(3M) = 3 \sum\nolimits_{i=1}^\infty \mathbb{E}\left(e^{-\alpha\tau_i}\right) = 3 \sum\nolimits_{i=1}^\infty \frac{1}{(1+\alpha)^i} = \frac{3}{\alpha}. \tag{3.8}$$

Let us denote by η_i the interarrival time $\tau_i - \tau_{i-1}$. Let η_0 be an exponentially distributed random variable with rate 1 which is independent of M. Then

$$e^{-\alpha\eta_0}(1 + M) = e^{-\alpha\eta_0} + e^{-\alpha\eta_0} \sum\nolimits_{i=1}^\infty e^{-\alpha(\eta_1+\cdots+\eta_i)} = \sum\nolimits_{i=0}^\infty e^{-\alpha(\eta_0+\eta_1+\cdots+\eta_i)}.$$

Therefore the distribution of $e^{-\alpha\eta_0}(1 + M)$ coincides with the distribution of M.

Therefore, using (3.8), we have

$$\mathbb{E}M^2 = \mathbb{E}\left(e^{-\alpha\eta_0}\left(1+M\right)\right)^2 = \frac{1}{1+2\alpha}\left(1+\frac{2}{\alpha}+\mathbb{E}M^2\right).$$

From this, we get

$$\mathbb{E}M^2 = \frac{\alpha+2}{2\alpha^2} < \infty$$

and

$$\mathbb{E}\left(3M\right)^2 = \frac{9\left(\alpha+2\right)}{2\alpha^2} < \infty.$$

Therefore

$$\mathbb{E}\left[{}_\alpha\xi(\infty)\log^+{}_\alpha\xi(\infty)\right] \le \mathbb{E}(3M)^2 < \infty$$

holds. So condition (v) of Proposition 12.1 is satisfied. Moreover, with $\Phi\left(t\right) = I\left\{0 \le t < \lambda\right\}$ conditions $(i)-(iii)$ of Proposition 12.1 are also satisfied. We see that μ is not lattice and the existence of the positive Malthusian parameter is assumed. So conditions (a) and (b) of the Appendix A are satisfied.

If we show that $\int_0^\infty t^2e^{-\alpha t}\mu(dt) < \infty$, then conditions (c) and (iv) of the Appendix A will be proved. Now, from equations (2.5) and (2.6)

$$\int_0^\infty t^2e^{-\alpha t}\mu(dt) = \mathbb{E}\left(\varepsilon_1\right)\int_0^\infty t^2e^{-\alpha t}(1-L(t))dt =$$

$$= \mathbb{E}\left(\varepsilon_1\right)\int_0^\infty t^2e^{-\alpha t}e^{t(q_0-b-1)}e^{\frac{3q_1\left(1-e^{-ct}\right)+q_3\left(1-e^{-3ct}\right)}{3c}}dt \le C\int_0^\infty t^2e^{-\gamma t}dt < \infty$$

because $\gamma = \alpha - (q_0 - b - 1) > 0$.

Applying Proposition 12.1, we have

$$\lim_{t\to\infty} e^{-\alpha t}Z\left(t\right) = Y_\infty m_\infty$$

almost surely and in L^1. Here the random variable $Y_\infty \ge 0$ is non-negative and it positive on the event of non-extinction, it has expectation 1 and it does not depend on the choice of Φ. Moreover

$$m_\infty = m_\infty^\Phi = \frac{\int_0^\infty e^{-\alpha t}\left(1-L\left(t\right)\right)dt}{\int_0^\infty te^{-\alpha t}\mu\left(dt\right)} = \frac{1}{\left(q_1+3q_3\right)^2\int_0^\infty te^{-\alpha t}\left(1-L\left(t\right)\right)dt}, \quad (3.9)$$

where we applied (2.5) and the fact that α is the Malthusian parameter. □

Remark 3.1. If we consider the number of all triangles being born until time t, then for this T process we should use function $\Phi_T(t) = I\{0 \le t\}$.

Then $\lim_{t\to\infty} e^{-\alpha t}T(t) = Y_\infty m_\infty^{\Phi_T}$ almost surely and in L^1, where

$$m_\infty^{\Phi_T} = \frac{\int_0^\infty e^{-\alpha t}dt}{\int_0^\infty te^{-\alpha t}\mu(dt)} = \frac{1}{\alpha(q_1+3q_3)\int_0^\infty te^{-\alpha t}(1-L(t))\,dt}.$$

We also see that

$$\frac{T(t)}{Z(t)} \to \frac{m_\infty^{\Phi_T}}{m_\infty} = \frac{q_1+3q_3}{\alpha} > 1$$

because of (3.4).

The asymptotic behaviour of the number of vertices. Let us denote by $V(t)$ the total number of vertices being born up to time t.

Theorem 3.2. *$e^{-\alpha t}V(t)$ converges almost surely and*

$$\frac{V(t)}{Z(t)} \to \frac{1}{\alpha}$$

as $t\to\infty$ almost surely on the event of non-extinction.

Proof. At each birth step a new vertex is added to the graph, therefore the total number of vertices at time t is $V(t) = 3 + Z^\Phi(t)$, where $\Phi(t) = \Pi(t\wedge\lambda)$. We can see that conditions $(i)-(ii)$ of Proposition 12.1 are satisfied. Moreover, as we have seen it in the proof of Corollary 2.1,

$$\mathbb{E}\sup_t \Phi(t) = \mathbb{E}\Pi(\lambda) = \mathbb{E}(\lambda) = \int_0^\infty (1-L(s))\,ds \le C\int_0^1 (1-u)^{-1+\delta}du < \infty.$$

Here we applied Corollary 2.1 and $\delta > 0$ can be chosen because of the condition $p_0+p_1 < 1$. So condition (iii) of Proposition 12.1 is also satisfied. We have already seen in the proof of Theorem 3.1, that condition (12.2) is satisfied. By Proposition 12.1, we have

$$\lim_{t\to\infty} e^{-\alpha t}Z^\Phi(t) = Y_\infty m_\infty^\Phi$$

almost surely and in L^1. Here the random variable $Y_\infty \ge 0$ and the denominator of m_∞^Φ do not depend on the choice of Φ. The numerator of m_∞^Φ is

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\Phi(t)\,dt = \int_0^\infty e^{-\alpha t}\int_0^t 1-L(s)\,dsdt =$$

$$= \int_0^\infty (1-L(s))\int_s^\infty e^{-\alpha t}dtds = \int_0^\infty \frac{1}{\alpha}(1-L(s))\,e^{-\alpha s}ds.$$

From this equality and from (3.9) we obtain

$$\frac{V(t)}{Z(t)} = \frac{e^{-\alpha t}\left(3 + Z^{\Phi}(t)\right)}{e^{-\alpha t} Z(t)} \to \frac{Y_\infty m_\infty^{\Phi}}{Y_\infty m_\infty} = \frac{\frac{\int_0^\infty \frac{1}{\alpha}(1-L(s))e^{-\alpha s}ds}{\int_0^\infty te^{-\alpha t}\mu(dt)}}{\frac{\int_0^\infty e^{-\alpha t}(1-L(t))dt}{\int_0^\infty te^{-\alpha t}\mu(dt)}} = \frac{1}{\alpha}$$

as $t \to \infty$ almost surely on the event of non-extinction. □

The asymptotic behaviour of the number of edges. Let us denote by $W(t)$ the number of edges being born up to time t. Introduce the following random variables. Let γ_i denotes the number of new edges at birth time τ_i. Then $\gamma_1, \gamma_2 \dots$ are independent identically distributed random variables with distribution $\mathbb{P}(\gamma_i = j) = p_j, j = 0, 1, 2, 3$. Then $\mathbb{E}\gamma_1 = \sum_{j=0}^{3} jp_j$.

Theorem 3.3. *$e^{-\alpha t}W(t)$ converges almost surely and*

$$\frac{W(t)}{Z(t)} \to \frac{\mathbb{E}\gamma_1}{\alpha}$$

as $t \to \infty$ almost surely on the event of non-extinction.

Proof. Let $\Phi(t) = \gamma_1 + \dots + \gamma_{\Pi(t \wedge \lambda)}$. Then the number of edges at time t is given by

$$W(t) = 3 + Z^{\Phi}(t),$$

where $Z^{\Phi}(t) = \sum_e \Phi_e(t - \sigma_e)$ and the sum is taken for each individual e, where σ_e is the birth time of the individual e. Using that the non-negative random variable λ is finite, and using Wald's identity, we obtain

$$\mathbb{E}\sup_t \Phi(t) = \mathbb{E}\left(\gamma_1 + \dots + \gamma_{\Pi(\lambda)}\right) = \mathbb{E}\gamma_1 \mathbb{E}\Pi(\lambda) = \mathbb{E}\gamma_1 \mathbb{E}(\lambda) =$$

$$= \mathbb{E}\gamma_1 \int_0^\infty 1 - L(s)\, ds < \infty,$$

as in the proof of Theorem 3.2. So we can see that conditions $(i) - (iii)$ of Proposition 12.1 are satisfied. We have already seen in the proof of Theorem 3.1 that condition (12.2) is also satisfied. Therefore, applying Proposition 12.1, we have

$$\lim_{t\to\infty} e^{-\alpha t} Z^{\Phi}(t) = Y_\infty m_\infty^{\Phi}$$

almost surely and in L^1. Here the random variable $Y_\infty \geq 0$ and the denominator of m_∞^Φ do not depend on the choice of Φ. The numerator of m_∞^Φ is

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\Phi\left(t\right)dt = \int_0^\infty e^{-\alpha t}\mathbb{E}\gamma_1 \int_0^t 1 - L\left(s\right)dsdt = \int_0^\infty \frac{\mathbb{E}\gamma_1}{\alpha}\left(1 - L\left(s\right)\right)e^{-\alpha s}ds.$$

Therefore

$$\frac{W\left(t\right)}{Z\left(t\right)} = \frac{e^{-\alpha t}\left(3 + Z^\Phi\left(t\right)\right)}{e^{-\alpha t}Z\left(t\right)} \to \frac{Y_\infty m_\infty^\Phi}{Y_\infty m_\infty} = \frac{\frac{\int_0^\infty \frac{\mathbb{E}\gamma_1}{\alpha}(1-L(s))e^{-\alpha s}ds}{\int_0^\infty te^{-\alpha t}\mu(dt)}}{\frac{\int_0^\infty e^{-\alpha t}(1-L(t))dt}{\int_0^\infty te^{-\alpha t}\mu(dt)}} = \frac{\mathbb{E}\gamma_1}{\alpha}$$

as $t \to \infty$ almost everywhere on the event of non-extinction. □

Remark 3.2. Theorems 3.2 and 3.3 imply that the ratio of the number of edges and the number of vertices satisfy

$$\frac{W\left(t\right)}{V\left(t\right)} = \mathbb{E}\gamma_1$$

as $t \to \infty$ almost surely on the event of non-extinction. The meaning of this relation is obvious, as at one step one vertex is born with γ_1 edges.

The asymptotic behaviour of the degree of a fixed vertex. We can see that a newly born vertex can have 0, 1, 2 or 3 edges.

First we consider that our newly born vertex has 2 edges. Fix this vertex. Then precisely one triangle contains this fixed vertex. In this paragraph we shall call it as the ‘parent’ triangle. Then we distinguish those children triangles of the ‘parent’ triangle, which contribute to the degree of our fixed vertex. That is, we call a child triangle of the ‘parent’ triangle a “good child” if it contains our fixed vertex. Then the distribution of the number of “good children” at a reproduction event of the ‘parent’ triangle is

$$\mathbb{P}(\tilde{\varepsilon} = 0) = p_0 + p_1 + \frac{1}{3}p_2, \quad \mathbb{P}(\tilde{\varepsilon} = 1) = \frac{2}{3}p_2, \quad \mathbb{P}(\tilde{\varepsilon} = 2) = p_3.$$

Any “good child” contributes to the degree of our fixed vertex in two ways. When a “good child” is born, then it adds one new edge to our fixed vertex. Moreover, if at a reproduction time the “good child” produces a vertex with a single edge, then it is connected to our fixed edge with probability 1/3.

So first we have to consider the reproduction process of the “good child” which is the following

$$\tilde{\xi}(t) = \tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \cdots + \tilde{\varepsilon}_{\Pi(t\wedge\lambda)},$$

where $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \ldots$ are independent copies of $\tilde{\varepsilon}$. The mean offspring number in the case of "good children" is

$$\tilde{\mu}(t) = \mathbb{E}\tilde{\xi}(t) = \mathbb{E}\left(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \cdots + \tilde{\varepsilon}_{\Pi(t\wedge\lambda)}\right) = \mathbb{E}(\tilde{\varepsilon}_1)\,\mathbb{E}(\Pi(t\wedge\lambda)) =$$

$$= \left(\frac{2}{3}(p_2 + 3p_3)\right)\int_0^t (1 - L(s))\,ds = \frac{2}{3}\mu(t).$$

The reproduction process of the "good children" is supercritical if

$$1 < \tilde{\mu}(\infty) = \frac{2}{3}\mu(\infty).$$

In the following we assume supercriticality. The Malthusian parameter $\tilde{\alpha}$ of this process is the only positive solution of the equation

$$1 = \int_0^\infty e^{-\tilde{\alpha}t}\tilde{\mu}(dt) = \frac{2}{3}\int_0^\infty e^{-\tilde{\alpha}t}\mu(dt).$$

Denote by $C(t)$ the number of "good children" at time t. To check the conditions of Proposition 12.1 consider the quantity $\tilde{M} = \int_0^\infty e^{-\tilde{\alpha}t}\tilde{\xi}(dt)$. Using the same method as in the proof of Theorem 3.1, we can prove that $\mathbb{E}\left[\tilde{M}\log^+\tilde{M}\right] < \infty$. We can check condition $\int_0^\infty t^2 e^{-\tilde{\alpha}t}\tilde{\mu}(dt) < \infty$ similarly as in the proof of Theorem 3.1. So we can apply Proposition 12.1. Therefore we have almost surely

$$\lim_{t\to\infty} e^{-\tilde{\alpha}t}C(t) = \tilde{Y}_\infty \frac{\int_0^\infty e^{-\tilde{\alpha}t}(1 - L(t))dt}{\int_0^\infty te^{-\tilde{\alpha}t}\tilde{\mu}(dt)} =$$

$$= \tilde{Y}_\infty \frac{1}{\left(\frac{2}{3}\right)^2 (p_2 + 3p_3)^2 \int_0^\infty te^{-\tilde{\alpha}t}(1 - L(t))dt}, \tag{3.10}$$

where $\tilde{Y}_\infty$ is positive on the event of non-extinction of the "good children".

Now consider the case when the "good child" produces a vertex with a single edge. Then it is connected to our fixed edge with probability 1/3. So the number of these single edges is

$$\Phi(t) = \varrho_1 + \cdots + \varrho_{\Pi(t\wedge\lambda)},$$

where the above random variables are independent with distribution $\mathbb{P}(\varrho_i = 1) = p_1/3$ and $\mathbb{P}(\varrho_i = 0) = 1 - p_1/3$. Now $\mathbb{E}\Phi(t) = \mathbb{E}\varrho_1\mathbb{E}(t\wedge\lambda) = \frac{1}{3}p_1(1 - L(t))$. So this kind of contribution to the degree is $\tilde{Z}^\Phi$ and, by Proposition 12.1, we have almost surely

$$\lim_{t\to\infty} e^{-\tilde{\alpha}t}\tilde{Z}^\Phi(t) = \tilde{Y}_\infty \frac{\int_0^\infty e^{-\tilde{\alpha}t}\frac{1}{3}p_1(1 - L(t))dt}{\int_0^\infty te^{-\tilde{\alpha}t}\tilde{\mu}(dt)}. \tag{3.11}$$

Now denote by $F(t)$ the degree of the fixed vertex at time t. Then adding equations (3.10) and (3.11), we obtain the following. Almost surely

$$\lim_{t\to\infty} e^{-\tilde{\alpha}t}F(t) = \tilde{Y}_\infty \frac{1+\frac{1}{3}p_1}{\left(\frac{2}{3}\right)^2 (p_2+3p_3)^2 \int_0^\infty te^{-\tilde{\alpha}t}(1-L(t))dt}, \tag{3.12}$$

where $\tilde{Y}_\infty$ is positive on the event of non-extinction of the degree process.

Above the extinction of the degree process means that the degree of the vertex does not increase after a certain time, that is, reproduction process of the "good children" dies out. The probability of this kind of extinction is the smallest non-negative root of the equation

$$\tilde{G}(x) = x, \tag{3.13}$$

where $\tilde{G}(x)$ is the generator function of $\tilde{\xi}(\lambda)$. As $\tilde{\xi}(t) = \tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \cdots + \tilde{\varepsilon}_{\Pi(t\wedge\lambda)}$,

$$\tilde{G}(x) = h_{\Pi(\lambda)}\left(h_{\tilde{\varepsilon}}(x)\right),$$

where $h_{\Pi(\lambda)}$ is the generator function of $\Pi(\lambda)$ and $h_{\tilde{\varepsilon}}$ is the generator function of $\tilde{\varepsilon}$. Now

$$h_{\tilde{\varepsilon}}(x) = p_0 + p_1 + \frac{1}{3}p_2 + \frac{2}{3}p_2 x + p_3 x^2$$

and, by Theorem 2.2,

$$h_{\Pi(\lambda)}(x) = g_{\Pi,\xi}(x,1) =$$

$$= 1 + \frac{x-1}{c}\int_0^1 (1-u)^{\frac{1+b-(p_0+p_1)x}{c}-1} e^{\left(\frac{(p_2+p_3)x}{c}u - \frac{p_3 x}{c}u^2 + \frac{p_3 x}{3c}u^3\right)} du.$$

So, if the newly born vertex has 2 edges, then the limit of its degree process is given by (3.12), and the probability of extinction is the smallest non-negative root of the equation (3.13).

If the newly born vertex has 0 or one edge, then it will not get any new edge.

If the newly born vertex has 3 edges, then for its degree process $\hat{F}(t)$ we have

$$\lim_{t\to\infty} e^{-\tilde{\alpha}t}\hat{F}(t) = \left(\tilde{Y}_{1\infty} + \tilde{Y}_{2\infty} + \tilde{Y}_{3\infty}\right) \frac{1+\frac{1}{3}p_1}{\left(\frac{2}{3}\right)^2 (p_2+3p_3)^2 \int_0^\infty te^{-\tilde{\alpha}t}(1-L(t))dt}$$

almost surely, where $\tilde{Y}_{1\infty}, \tilde{Y}_{2\infty}, \tilde{Y}_{3\infty}$ are independent copies of $\tilde{Y}_\infty$. In this case the probability of extinction is x^3, where x is the smallest non-negative root of the equation (3.13).

4 Numerical and simulation results

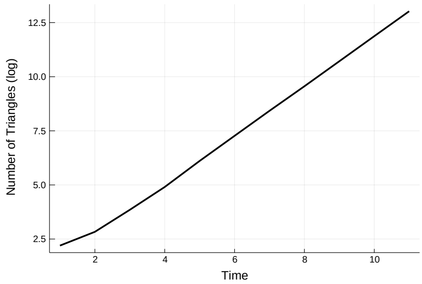
To get a closer look on the theoretical results, we made some simulations about them. We generated our code in Julia language. We chose Julia, because of the great implementation of priority queues. The simulation time of our code was significantly faster in Julia than in other programming languages. We handled the main objects (the triangles) of our model as arrays with 3 elements. The elements were the indices of the edges that formed an individual for the process. We put all triangles in a priority queue with the priority of it's birth time, because we can pop out the element with the lowest priority. After we've got the triangle with the lowest birth time, we can handle its birth process with the predefined b, c, q_1, q_3 parameters. In the birth process we generated an exponential time step for the next birth step of our triangle. After that we checked if the triangle is still alive by calculating the survival function. If the triangle is dead, we move to the next one. If it is alive, then we generate 1 or 3 new triangles and put them in the priority queue with the calculated birth time priorities. After it we moved to the next birth event. The pseudocode of the birth process is seen at Algorithm 1.

We made several simulation experiments. Here we show only some typical results. For the above demonstration we used the parameter set $b = 0.2,\ c = 0.2,\ p_0 = 0.05,\ p_1 = 0.05,\ p_2 = q_1 = 0.6,\ p_3 = q_3 = 0.3$. On Figure 1.2a, Process 1 shows the number of triangles. According to Theorem 3.1 it has asymptotic rate $e^{-\alpha t}$. Therefore we put logarithmic scale on the vertical axis so the function $Z(t)$ is a straight line for large values of t. On the figure one can see that the shape of the curve is close to a straight line, so it supports our Theorem 3.1.

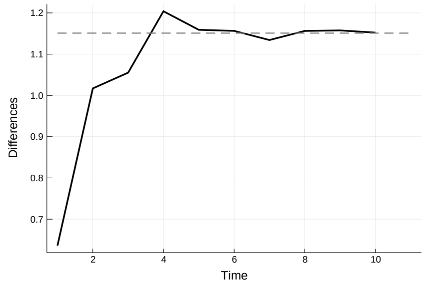
Then we checked the value of the Malthusian parameter α. We can find it in two ways. On the one hand, the slope of the line Process 1 is α for large values of the time. This slope can be approximated by the differences of the function. So on Figure 1.2b we present these differences (solid line). On the other hand, α can be calculated numerically from equation (3.3). This α value is shown of Figure 1.2b by a horizontal dashed line. The fit of the differences to α can be seen for large values of t.

To get a closer look on the Malthusian parameter α we fixed 5 parameter sets. Then we calculated α form equation (3.3) for each case. Then for each of the parameter sets we simulated our process $Z(t)$ five times. Then we calculated the differences of $\log Z(t)$ wich should be good approximations of α according to Theorem 3.1. In Table 1.1 $\widehat{\alpha_1},\ \widehat{\alpha_2},\ \widehat{\alpha_3},\ \widehat{\alpha_4},\ \widehat{\alpha_5}$ show the values of these approximations for large t. One can see that each $\widehat{\alpha_i}$ is close to the corresponding α.

We calculated numerically the probability of extinction from equation (2.17). It is shown in the column 'Numerical' of Table 1.2. In the column 'Simulation' the rela-



(a) The number of triangles in the evolving graph



(b) The differences of the number of triangles

Figure 1.2: Simulation results for $b = 0.2,\ c = 0.2,\ q_1 = 0.6,\ q_3 = 0.3$

b	c	q_1	q_3	α	$\widehat{\alpha_1}$	$\widehat{\alpha_2}$	$\widehat{\alpha_3}$	$\widehat{\alpha_4}$	$\widehat{\alpha_5}$
0.2	0.4	0.7	0.1	0.5628	0.5651	0.5730	0.5701	0.5611	0.5594
0.2	0.4	0.8	0.1	0.6531	0.6537	0.6497	0.6570	0.6510	0.6589
0.4	0.4	0.8	0.1	0.4531	0.4503	0.4519	0.4584	0.4541	0.4524
0.4	0.4	0.7	0.2	0.6545	0.6533	0.6517	0.6548	0.6534	0.6574
0.4	0.4	0.6	0.3	0.8535	0.8519	0.8489	0.8559	0.8547	0.8566

Table 1.1: α from equation (3.3) and $\widehat{\alpha_i}$ from simulations

tive frequency of the extinction is shown using our computer experiment. For each parameter sets, we simulated 10^4 processes and counted the number of extinctions occured. The value of the relative frequency is close to the corresponding value of the probability in each case. So Table 1.2 supports the result of Theorem 2.3.

b	c	q_1	q_3	**Simulation**	**Numerical**
0.0	0.2	0.4	0.4	0.0	0.0
0.1	0.2	0.4	0.4	0.1304	0.1282
0.1	0.2	0.5	0.4	0.1165	0.1158
0.1	0.2	0.5	0.5	0.1097	0.1025
0.2	0.2	0.5	0.4	0.2227	0.2180
0.2	0.2	0.6	0.4	0.2038	0.2002
0.3	0.3	0.5	0.4	0.3231	0.3185
0.4	0.4	0.5	0.4	0.3966	0.4020

Table 1.2: The relative frequency and the probability of the extinction of the triangles

To investigate how our process approximate α for time t, we simulated around 500 independent processes with the same $b = 0.2,\ c = 0.2,\ p_0 = 0.05,\ p_1 = 0.05,\ p_2 = q_1 = 0.6,\ p_3 = q_3 = 0.3$ parameters and same running time. Then we checked the differences of the last two values in the number of triangles that we simulated and made a histogram, seen in Figure 1.3. From equation (3.3) we obtained that the value of α is 0.3365. We see that the values of the differences are close to α.

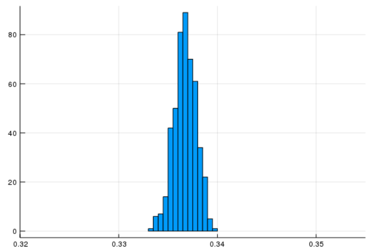
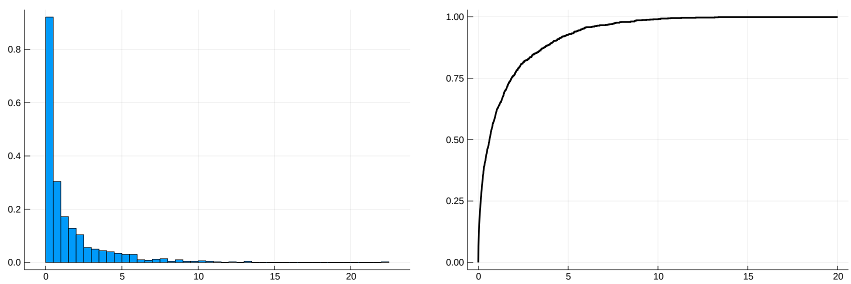


Figure 1.3: Histogram of differences

To get some information about the random variable $Y_\infty m_\infty$ represented in Theorem 3.1, we calculated the $Z(t)e^{-\alpha t}$ value for 1000 independent processes for the same t time and same $q_1 = 0.3,\ q_3 = 0.6,\ b = 0.2,\ c = 0.2$ parameterset. On Figure 1.4 we represent the histogram and the empirical cumulative distribution function calculated from the simulation. The Kolmogorov-Smirnov test gave us a p value 0.6713 for the gamma distribution.



(a) Histogram of $Z(t)e^{-\alpha t}$ (b) ECDF

Figure 1.4: Simulation results for $Z(t)e^{-\alpha t}$

Algorithm 1 Birth process of a triangle

```
procedure BIRTH PROCESS
    Y ← non-empty Priority Queue
    b, c, q_1, q_3 ← parameters of the survival function
    x ← dequeue Y
    if x is a new triangle then
        t_0 ← the birth time of x in the whole process
        t ← 0, lifetime of x
        l ← 1, life variable
        while l = 1 do
            t ← t + Exp(1)
            p ← the calculated survival function
            if p > Uni(0, 1) then
                p_0 ← Uni(0, 1)
                if p_0 < q_1 then
                    take a new triangle with t_0 + t birth time to Y
                    offspring number is 1 at birth time t
                else if p_0 > 1 − q_3 then
                    take three new triangles with t_0 + t birth times to Y
                    offspring number is 3 at birth time t
            else
                l ← 0
                take t as the death time of x to Y
```

Chapter 2

The 2- and 3-interaction model

In this chapter we describe our new results on our 2- amd 3- intercation model. They were published in paper [28] and [29].

6 Model description

We study the following network evolution model. At the initial time $t = 0$ the network consists of one single object, this object can be either an edge or a triangle. This object is called the ancestor. During the evolution, this ancestor object produces offspring objects, which can be either edges or triangles. Then, these offspring objects produce their offspring objects and so on. The reproduction times of any fixed object, including the ancestor, are the occurrences in its own Poisson process with rate 1.

From the theory of branching processes, we apply the following usual assumptions. That is we suppose that the reproduction processes of different objects are independent. Moreover, we assume that the reproduction processes of the edges are independent copies of the reproduction process of the generic edge. Similarly, the reproduction processes of the triangles are independent copies of the reproduction process of the generic triangle.

First, we explain the evolution of the generic edge. A Poisson process $\Pi_2(t)$ with parameter 1 gives its reproduction times. At any jumping time of this Poisson process, a new vertex appears and it is connected to the generic edge with one or two edges. The probability that this new vertex is connected to the generic edge by one new edge is r_1, where $0 \leq r_1 \leq 1$. The other end point of this new edge is chosen from the two vertices of the generic edge uniformly at random. We see that in this case the generic edge produces always one new edge. The other case is that when the new vertex is connected to both vertices of the generic edge. Its probability is $r_2 = 1 - r_1$. In this second case the offspring of the generic edge is a triangle consisting of the generic edge and the two new edges. We emphasize that in this last case the generic edge itself and the new triangle will produce offspring, but the two new edges are not substantive parts of the reproduction process, so they alone will not produce offspring.

The reproduction process of the generic triangle is similar. The Poisson process with rate 1 corresponding to the generic triangle is denoted by $\Pi_3(t), t \geq 0$. The jumping times of $\Pi_3(t)$ are the birth times of the generic triangle. At every birth time a new vertex is born and it joins to the existing graph so that it is connected to our generic triangle with 1, 2 or 3 edges. Denote by p_j $(j = 1, 2, 3)$ the probability that the new vertex is connected to j vertices of our generic triangle. The vertices of the generic triangle to be connected to the new vertex are chosen uniformly at random.

By the above definition of the evolution process, at each birth step we add precisely 1 new vertex. When the new vertex is connected to one vertex of the generic triangle, the generic triangle gives birth to one new edge. This event has probability p_1. However, in the remaining two cases we count only the new triangles and not the new edges. When the new edge is connected to the generic triangle by two edges, these two edges and one edge of the generic triangle form a new triangle. Therefore, with probability p_2, the generic triangle produces one child triangle. When the new edge is connected to the generic triangle by three edges, these edges and the edges of the generic triangle form three new triangles. Thus, with probability p_3, the generic triangle produces three children triangles.

Any edge is called a type 2 object, and any triangle is called a type 3 object. We use subscript 2 for edges and subscript 3 for triangles. Thus, we denote by $\xi_{i,j}(t)$ the number of type j offspring of the type i generic object up to time t $(i, j = 2, 3)$. Recall that $\xi_{i,j}$, $i, j = 2, 3$, are point processes. Then

$$\xi_2(t) = \xi_{2,2}(t) + \xi_{2,3}(t) \tag{6.1}$$

gives the total number of offspring (that is both edges and triangles) of the generic edge up to time t. We can also see that

$$\xi_3(t) = \xi_{3,2}(t) + \xi_{3,3}(t) \tag{6.2}$$

is the number of all offspring (edges or triangles) of the generic triangle up to time t.

We denote by $\tau_3(1), \tau_3(2), \ldots$ the birth times of the generic triangle, and we denote by $\varepsilon_3(1), \varepsilon_3(2), \ldots$ the corresponding total litter sizes. That is, at the ith birth event, the generic triangle bears $\varepsilon_3(i)$ children being either triangles or edges. The discrete random variables $\varepsilon_3(1), \varepsilon_3(2), \ldots$ are independent and identically distributed having distribution $\mathbb{P}(\varepsilon_3(i) = j) = q_j, j \geq 1$. By the above evolution process, we have

$$\mathbb{P}(\varepsilon_3(i) = 1) = q_1 = p_1 + p_2, \ \mathbb{P}(\varepsilon_3(i) = 3) = q_3 = p_3,$$

$$\mathbb{P}(\varepsilon_3(i) = j) = q_j = 0, \quad \text{if} \quad j \notin \{1, 3\}.$$

We assume that the litter sizes are independent of the birth times.

Let λ_3 be the life-length of the generic triangle. It is a finite, non-negative random variable. We assume that the reproduction terminates at the death of the individual. Therefore, $\xi_3(t) = \xi_3(\lambda_3)$ for $t > \lambda_3$. Then, the reproduction process of a triangle can be formulated as

$$\xi_3(t) = \sum_{\tau_3(i) \leq t \wedge \lambda_3} \varepsilon_3(i) = S_3(\Pi_3(t \wedge \lambda_3)), \tag{6.3}$$

where $\Pi_3(t)$ is the Poisson process, $S_3(n) = \varepsilon_3(1) + \cdots + \varepsilon_3(n)$ gives the total number of offspring of the generic triangle before the $(n+1)$th birth event and by $x \wedge y$ we denote the minimum of $\{x, y\}$.

The survival function of the life-length. Let $L_3(t)$ denote the distribution function of the triangle's life-length λ_3. Then, the survival function of λ_3 is

$$1 - L_3(t) = \mathbb{P}(\lambda_3 > t \mid \xi_3(u), 0 \leq u \leq t) = \exp\left(-\int_0^t l_3(u)\, du\right), \tag{6.4}$$

where $l_3(t)$ is the hazard rate of the life-length λ_3. We suppose that the hazard rate depends on the total number of offspring, so that

$$l_3(t) = b + c\xi_3(t) \tag{6.5}$$

with fixed positive constants b and c.

Let λ_2 be the life-length of the generic edge. Then, $\xi_2(t) = \xi_2(\lambda_2)$ for $t > \lambda_2$. As the edge always gives birth to one offspring (which can be an edge or a triangle); therefore,

$$\xi_2(t) = \Pi_2(t \wedge \lambda_2) \tag{6.6}$$

is the total number of offspring of the generic edge, where $\Pi_2(t)$ is the Poisson process.

We denote by $L_2(t)$ the distribution function of λ_2. Then, the survival function of the life-length of an edge is

$$1 - L_2(t) = \mathbb{P}(\lambda_2 > t \mid \xi_2(u), 0 \leq u \leq t) = \exp\left(-\int_0^t l_2(u)\, du\right), \qquad (6.7)$$

where l_2 is the hazard rate of the life-length λ_2. We suppose that l_2 is of the form $l_2(t) = b + c\xi_2(t)$, with the same constants as in (6.5).

We emphasize that we do not delete any edge or any triangle when it dies, because its ingredients can belong to other triangles or edges, too. Thus, dead triangles and edges will be considered as inactive objects not producing new offspring.

In Figure 2.1, an example is shown for our graph evolution model. For a clear view it contains only three birth steps after the initial time $t = 0$. The nodes of the ancestor are highlighted by red. The edges are labelled with the birth times t. The following objects appear in Figure 2.1, which are described by the labels of their nodes:

- (1-2-3): is a triangle, the ancestor with birth time $t = 0$,
- (1-2-3-4): represents three triangles, i.e., the offspring of (1-2-3) at its first reproduction time $t = 0.571$,
- (1-5): an edge, offspring of (1-2-3) with birth time $t = 0.847$,
- (1-5-6): a triangle, offspring of (1-5) with birth time $t = 1.06$.

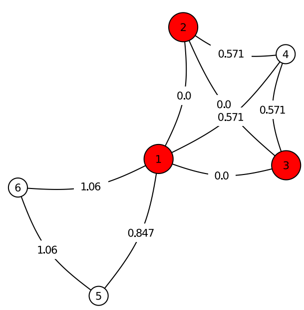
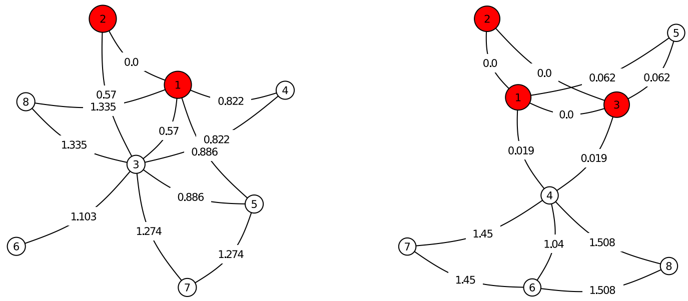


Figure 2.1: Example of the graph evolution model with parameter set:
$r_1 = 0.1,\ p_1 = 0.4,\ p_2 = 0.2,\ b = 0.1,\ c = 0.1.$

Two more examples are shown in Figure 2.2 with different parameters. In Figure 2.2a the ancestor is an edge, while in Figure 2.2b the ancestor is a triangle.



(a) $r_1 = 0.8$, $p_1 = 0.2$, $p_2 = 0.5$, $b = 0.2$, $c = 0.1$

(b) $r_1 = 0.2$, $p_1 = 0.3$, $p_2 = 0.5$, $b = 0.2$, $c = 0.2$

Figure 2.2: Examples of the graph evolution model with two different parameter sets

7 General results

In this section the general results are presented. These are the survival functions of an edge and of a triangle (Theorem 7.1), the mean offspring number of an edge and of a triangle (Corollary 7.1), the Perron root and the Malthusian parameter. As usual, we obtain only implicit expression for the Malthusian parameter, but our expression is simple and numerically tractable.

The survival functions.

Theorem 7.1. *The survival function for a triangle is*

$$\mathbb{P}\left(\lambda_3 > t\right) = e^{-t(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-ct}\right)+p_3\left(1-e^{-3ct}\right)}{3c}}. \tag{7.1}$$

The survival function for an edge is

$$\mathbb{P}\left(\lambda_2 > t\right) = e^{-t(b+1)} e^{\frac{1-e^{-ct}}{c}}. \tag{7.2}$$

Proof. At the first part of the proof we omit subscripts 2 and 3, because the calculations are the same for edges and triangles. Let $t > 0$ and assume that $\Pi(t) = k$. Then, the first k birth events happened before time t. Thus, the birth times $\tau(1), \tau(2), \ldots, \tau(k)$ and the corresponding litter sizes $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(k)$ are known. Therefore, the reproduction process $\xi(u)$ is also known for $u < t$. By (6.5), a simple calculation shows that the survival function of an object is

$$1 - L(t) = \exp\left(-\int_0^t l(u)\,du\right) = \exp\left(-\left(bt + c\int_0^t \xi(u)\,du\right)\right) =$$

$$= \exp\left(-\left(bt + ctS(k) - c\left(\varepsilon(1)\tau(1) + \cdots + \varepsilon(k)\tau(k)\right)\right)\right).$$

Then

$$\mathbb{P}\left(\lambda > t | \Pi(t) = k, \tau(1), \ldots, \tau(k), \varepsilon(1), \ldots, \varepsilon(k)\right) =$$

$$= \exp\left(-\left(bt + ctS(k) - c\left(\varepsilon(1)\tau(1) + \cdots + \varepsilon(k)\tau(k)\right)\right)\right).$$

Let $(U_1^*, \ldots, U_k^*)$ be an ordered sample of size k from uniform distribution on $[0, 1]$. Then, the joint conditional distribution of the birth times $\tau(1), \ldots, \tau(k)$ given $\Pi(t) = k$, coincides with the distribution of $(tU_1^*, \ldots, tU_k^*)$. Therefore

$$\mathbb{P}\left(\lambda > t | \Pi(t) = k\right) = \mathbb{E}\exp\left(-\left(bt + ct\sum_{i=1}^{k}\varepsilon(i)\left(1 - \frac{\tau(i)}{t}\right)\right)\right) =$$

$$= \mathbb{E}\exp\left(-bt + ct\sum_{i=1}^{k}\varepsilon(i)\left(U_i^* - 1\right)\right),$$

because $\tau(i) = tU_i^*$. The litter sizes $\varepsilon(1), \ldots, \varepsilon(k)$ are independent identically distributed random variables, which are independent also of $U_1^*, \ldots, U_k^*$. Hence

$$\mathbb{P}\left(\lambda > t | \Pi(t) = k\right) = \mathbb{E}\exp\left(-bt + ct\sum_{i=1}^{k}\varepsilon(i)\left(U_i - 1\right)\right) =$$

$$= e^{-bt}\mathbb{E}\prod_{i=1}^{k} e^{ct\varepsilon(i)(U_i - 1)} = e^{-bt}\left(\mathbb{E}_{\varepsilon(i)}\left(\mathbb{E}_{U_i}\left(e^{ct\varepsilon(i)U_i}\right)e^{-ct\varepsilon(i)}\right)\right)^k =$$

$$= e^{-bt}\left(\sum_{j=1}^{\infty} q_j \frac{e^{ctj} - 1}{ctj} e^{-ctj}\right)^k = e^{-bt}\left(\sum_{j=1}^{\infty} q_j \frac{1 - e^{-ctj}}{ctj}\right)^k,$$

where we applied that U_i is uniformly distributed. Using this and the total probability theorem, we find

$$\mathbb{P}\left(\lambda > t\right) = \sum_{k=0}^{\infty}\mathbb{P}\left(\Pi(t) = k\right)\mathbb{P}\left(\lambda > t | \Pi(t) = k\right) =$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} e^{-t} e^{-bt} \left(\sum_{j=1}^{\infty} q_j \frac{1-e^{-ctj}}{ctj} \right)^k =$$

$$= e^{-(b+1)t} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{j=1}^{\infty} q_j \frac{1-e^{-ctj}}{cj} \right)^k =$$

$$= e^{-(b+1)t} e^{\sum_{j=1}^{\infty} q_j \frac{1-e^{-ctj}}{cj}}.$$

Therefore, the survival function for a triangle is

$$\mathbb{P}\left(\lambda_3 > t\right) = e^{-t(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-ct}\right)+p_3\left(1-e^{-3ct}\right)}{3c}}.$$

Finally, the survival function for an edge is

$$\mathbb{P}\left(\lambda_2 > t\right) = e^{-t(b+1)} e^{\frac{1-e^{-ct}}{c}}.$$

□

The mean offspring number. Let us denote by $m_{i,j}(t) = \mathbb{E}\xi_{i,j}(t)$ the expectation of the number of type j offspring of a type i mother until time t.

Corollary 7.1. *For any $t \geq 0$, we have*

$$m_{2,2}(t) = r_1 F(t), \qquad m_{2,3}(t) = r_2 F(t), \tag{7.3}$$

where

$$F(t) = \int_0^t \left(1 - L_2(s)\right) ds = \int_0^t e^{-(b+1)s} e^{\frac{1-e^{-cs}}{c}} ds = \frac{1}{c} \int_0^{1-e^{-ct}} (1-u)^{\frac{b+1}{c}-1} e^{\frac{u}{c}} du.$$

$$\mathbb{E}\lambda_2 = \frac{1}{c} \int_0^1 (1-u)^{\frac{b+1}{c}-1} e^{\frac{u}{c}} du. \tag{7.4}$$

For any $t \geq 0$, we have

$$m_{3,2}(t) = p_1 G(t), \qquad m_{3,3}(t) = (p_2 + 3p_3) G(t), \tag{7.5}$$

where

$$G(t) = \int_0^t \left(1 - L_3(s)\right) ds = \int_0^t e^{-s(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-cs}\right)+p_3\left(1-e^{-3cs}\right)}{3c}} ds =$$

$$= \frac{1}{c}\int_0^{1-e^{-ct}} (1-u)^{\frac{b+1}{c}-1} e^{\frac{u}{3c}\left(p_3u^2-3p_3u+3\right)}du.$$

$$\mathbb{E}\lambda_3 = \frac{1}{c}\int_0^{1} (1-u)^{\frac{b+1}{c}-1} e^{\frac{u}{3c}\left(p_3u^2-3p_3u+3\right)}du. \tag{7.6}$$

$0 < \mathbb{E}\lambda_2, \mathbb{E}\lambda_3 < \infty$ *because* $b \geq 0$.

Proof. We have

$$m_{i,j}(t) = \mathbb{E}\xi_{i,j}(t) = \mathbb{E}\left(\varepsilon_{i,j}(1) + \varepsilon_{i,j}(2) + \cdots + \varepsilon_{i,j}\left(\Pi\left(t \wedge \lambda_i\right)\right)\right),$$

where $\varepsilon_{i,j}(k)$ is the number of type j offspring of a type i mother at her kth birth event. Using Wald's identity, the average number of children is

$$m_{i,j}(t) = \mathbb{E}\left(\varepsilon_{i,j}(1)\right)\mathbb{E}\left(\Pi\left(t \wedge \lambda_i\right)\right). \tag{7.7}$$

Using that Π is a Poisson process with rate 1, and $t \wedge \lambda$ is bounded for any t, from (7.7), we obtain that the average number of children is

$$m_{i,j}(t) = \mathbb{E}\left(\varepsilon_{i,j}(1)\right)\mathbb{E}\left(\Pi\left(t \wedge \lambda_i\right)\right) = \mathbb{E}\left(\varepsilon_{i,j}(1)\right)\int_0^t \left(1 - L_i(s)\right)ds. \tag{7.8}$$

Now, consider $m_{2,2}(t)$. Applying (7.2) and using the substitution $u = 1 - e^{-cs}$, we obtain

$$m_{2,2}(t) = r_1\int_0^t e^{-(b+1)s} e^{\frac{1-e^{-cs}}{c}}ds = \frac{r_1}{c}\int_0^{1-e^{-ct}} (1-u)^{\frac{b+1}{c}-1} e^{\frac{u}{c}}du. \tag{7.9}$$

If we write r_2 instead of r_1, then we obtain $m_{2,3}(t)$. Thus, we obtained (7.3). Moreover, with $t \to \infty$, we have $\mathbb{E}\lambda_2 = \int_0^\infty \mathbb{P}\left(\lambda_2 > s\right)ds$. Thus, (7.4) follows from (7.9).

Now, we turn to $m_{3,3}(t)$. Applying (7.1), and using the substitution $u = 1 - e^{-cs}$, we obtain ,

$$\int_0^t \mathbb{P}\left(\lambda_3 > s\right)ds = \int_0^t e^{-s(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-cs}\right)+p_3\left(1-e^{-3cs}\right)}{3c}}ds =$$

$$= \frac{1}{c}\int_0^{1-e^{-ct}} (1-u)^{\frac{b+1}{c}-1} e^{\frac{u}{3c}\left(p_3u^2-3p_3u+3(p_1+p_2+p_3)\right)}du. \tag{7.10}$$

As $\mathbb{E}\left(\varepsilon_{3,3}(1)\right) = p_2 + 3p_3$, so from (7.8) we obtain $m_{3,3}(t)$. Using that $\mathbb{E}\left(\varepsilon_{3,2}(1)\right) = p_1$, we obtain $m_{3,2}(t)$. Thus, we obtained (7.5). Moreover, we have $\mathbb{E}\lambda_3 =$

$\int_0^\infty \mathbb{P}(\lambda_3 > s)\, ds$. Thus, (7.6) follows from (7.10) with $t \to \infty$. □

Let

$$m_{i,j}^*(\kappa) = \int_0^\infty e^{-\kappa t} m_{i,j}(dt), \qquad i, j = 2, 3,$$

be the Laplace transform of $m_{i,j}$.

Proposition 7.1. *For any $\kappa \geq 0$, we have*

$$m_{2,2}^*(\kappa) = r_1 A(\kappa), \qquad m_{2,3}^*(\kappa) = r_2 A(\kappa), \tag{7.11}$$

where

$$A(\kappa) = \int_0^\infty e^{-\kappa s} e^{-(b+1)s} e^{\frac{1-e^{-cs}}{c}} ds = \frac{1}{c}\int_0^1 (1-u)^{\frac{\kappa+b+1}{c}-1} e^{\frac{u}{c}} du. \tag{7.12}$$

For any $\kappa \geq 0$, we have

$$m_{3,2}^*(\kappa) = p_1 B(\kappa), \qquad m_{3,3}^*(\kappa) = (p_2 + 3p_3) B(\kappa), \tag{7.13}$$

where

$$B(\kappa) = \int_0^\infty e^{-\kappa s} e^{-s(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-cs}\right)+p_3\left(1-e^{-3cs}\right)}{3c}} ds =$$

$$= \frac{1}{c}\int_0^1 (1-u)^{\frac{\kappa+b+1}{c}-1} e^{\frac{u}{3c}\left(p_3 u^2 - 3p_3 u + 3\right)} du.$$

Proof. Apply the definition of $m_{i,j}^*(\kappa)$, Corollary 7.1 and substitution $u = 1 - e^{-cs}$. □

The Perron root and the Malthusian parameter. Let

$$M(\kappa) = \begin{pmatrix} m_{2,2}^*(\kappa) & m_{2,3}^*(\kappa) \\ m_{3,2}^*(\kappa) & m_{3,3}^*(\kappa) \end{pmatrix} \tag{7.14}$$

be the matrix of the Laplace transforms. Direct calculation gives that the characteristic roots of $M(\kappa)$ are

$$\varrho_{1,2}(\kappa) =$$

$$= \frac{(p_2+3p_3)B(\kappa) + r_1 A(\kappa) \pm \sqrt{((p_2+3p_3)B(\kappa) - r_1 A(\kappa))^2 + 4p_1 B(\kappa) r_2 A(\kappa)}}{2}. \tag{7.15}$$

The greater of the values $\varrho_1(\kappa)$ and $\varrho_2(\kappa)$ is called the Perron root, so

$$\varrho(\kappa) = \varrho_1(\kappa) =$$
$$= \frac{(p_2+3p_3)B(\kappa) + r_1A(\kappa) + \sqrt{((p_2+3p_3)B(\kappa) - r_1A(\kappa))^2 + 4p_1B(\kappa)r_2A(\kappa)}}{2} \tag{7.16}$$

is the Perron root.

We assume that our process is supercritical; that is,

$$\varrho(0) > 1. \tag{7.17}$$

For supercriticality, condition

$$\max\{(p_2+3p_3)B(0), r_1A(0)\} > 1$$

is sufficient.

That value of κ for which the Perron root is equal to 1 is called the Malthusian parameter. Thus, using the usual notation in the theory of branching processes, α is the Malthusian parameter if $\varrho(\alpha) = 1$. In this chapter, we assume the existence of the Malthusian parameter. From relation $\varrho(\alpha) = 1$ and (7.16), we obtain that the Malthusian α satisfies the equation

$$r_1A(\alpha)(p_2+3p_3)B(\alpha) - (r_1A(\alpha) + (p_2+3p_3)B(\alpha)) = r_2A(\alpha)p_1B(\alpha) - 1. \tag{7.18}$$

Later, we use the eigenvectors of $M(\alpha)$. To this end, let α be the Malthusian parameter, and let $(v_2, v_3)^\top$ be the right eigenvector of $M(\alpha)$ corresponding to eigenvalue 1 and satisfying condition $v_2 + v_3 = 1$. Then, direct calculation shows that

$$v_2 = \frac{(r_1-1)A(\alpha)}{(2r_1-1)A(\alpha)-1}, \qquad v_3 = \frac{r_1A(\alpha)-1}{(2r_1-1)A(\alpha)-1}. \tag{7.19}$$

Again, let α be the Malthusian parameter and let $(u_2, u_3)^\top$ be the left eigenvector of $M(\alpha)$ satisfying condition $u_2v_2 + u_3v_3 = 1$. Direct calculation shows that

$$u_2 = \frac{p_1B(\alpha)\left((2r_1-1)A(\alpha)-1\right)}{p_1B(\alpha)(r_1-1)A(\alpha) - (r_1A(\alpha)-1)^2},$$
$$u_3 = \frac{(1-r_1A(\alpha))\left((2r_1-1)A(\alpha)-1\right)}{p_1B(\alpha)(r_1-1)A(\alpha) - (r_1A(\alpha)-1)^2}. \tag{7.20}$$

8 Asymptotic theorems on the number of triangles and edges

In this section asymptotic theorems on the number of edges and triangles (Theorem 8.1) are proved. Both of them have magnitude $e^{\alpha t}$ on the event of non-extinction, where α is the Malthusian parameter. To prove Theorem 8.1, we use the underlying multitype branching process counted with certain random characteristics and apply the asymptotic theorems of [20].

We use Proposition 13.1 from Appendix B. So we should check the conditions given in Appendix B. For condition (a) from Appendix B, we should guarantee that not all measures $m_{i,j}$ are concentrated on a lattice. By Corollary 7.1, these measures are absolutely continuous, and thus it is satisfied.

Concerning condition (b1), we underline that we suppose the existence of a positive Malthusian parameter α. To this end, in this section, we assume that (7.18) has a finite positive solution α. We can check numerically the existence of this value. For (b2), we assume (7.17). Condition (c) from Appendix B will be checked later in the proofs of the results together with other conditions related to it.

Now, we analyse condition (d). We can see from Corollary 7.1 that $F(\infty)$ and $G(\infty)$ are positive. Thus, we can concentrate on parameters r_i and p_i. If $r_2 = p_1 = 0$, then (d) is not satisfied; however, in this case, one can study separately the process of edges (it grows at any birth time by 1), and the process of triangles (this is described in [26]). If $r_1 = 0$ and $p_2 + p_3 = 0$, then (d) is not satisfied, and the evolution process is an alternating one. If either $r_2 = 0$ or $p_1 = 0$, then (d) is not satisfied.

To guarantee condition (d), in this section, we assume that $0 \leq r_1 < 1$, $0 < p_1 \leq 1$, and it is excluded that both $r_1 = 0$ and $p_1 = 1$ are satisfied at the same time. In this case, condition (d) from Appendix B is satisfied.

The denominator in the limit theorem. In the following theorem, we need the next formulae. In Appendix B, we see that the denominator of m_∞^Φ in the limiting expression is independent of Φ, and it is

$$\sum_{l,j=1}^{p} u_l v_j \int_0^\infty t e^{-\alpha t} m_{l,j}\,(dt).$$

It can be written in the form (and considering our two-dimensional case)

$$D(\alpha) = \sum_{l,j=2}^{3} u_l v_j \left(-m^*_{l,j}(\alpha)\right)'. \tag{8.1}$$

Here, u_i and v_i are from Equations (7.19) and (7.20). Moreover, by Corollary 7.1 or by Proposition 7.1, we have that

$$\left(-m^*_{2,2}(\alpha)\right)' = r_1\left(-A'(\alpha)\right), \qquad \left(-m^*_{2,3}(\alpha)\right)' = r_2\left(-A'(\alpha)\right), \tag{8.2}$$

$$\left(-m^*_{3,2}(\alpha)\right)' = p_1\left(-B'(\alpha)\right), \qquad \left(-m^*_{3,3}(\alpha)\right)' = (p_2+3p_3)\left(-B'(\alpha)\right), \tag{8.3}$$

where

$$-A'(\alpha) = \int_0^\infty s e^{-\alpha s} e^{-(b+1)s} e^{\frac{1-e^{-cs}}{c}} ds = -\frac{1}{c^2}\int_0^1 \ln(1-u)\,(1-u)^{\frac{\alpha+b+1}{c}-1} e^{\frac{u}{c}} du, \tag{8.4}$$

$$-B'(\alpha) = \int_0^\infty s e^{-\alpha s} e^{-s(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-cs}\right)+p_3\left(1-e^{-3cs}\right)}{3c}} ds = \tag{8.5}$$

$$= -\frac{1}{c^2}\int_0^1 \ln(1-u)\,(1-u)^{\frac{\alpha+b+1}{c}-1} e^{\frac{u}{3c}\left(p_3u^2-3p_3u+3\right)} du.$$

Now, we turn to the number of edges and triangles. Recall that an edge is a type 2, and a triangle is a type 3 object.

Theorem 8.1. *Assume that* (7.17) *is satisfied and* (7.18) *has a finite positive solution* α. *Assume that* $0 \le r_1 < 1$, $0 < p_1 \le 1$ *and it is excluded that both* $r_1 = 0$ *and* $p_1 = 1$ *are satisfied at the same time.*

Let ${}_iE(t)$ *denote the number of all edges being born up to time* t *if the ancestor of the population was a type* i *object,* $i = 2, 3$. *Then*

$$\lim_{t\to\infty} e^{-\alpha t}\,{}_iE(t) = {}_iW \frac{v_i u_2}{\alpha D(\alpha)} \tag{8.6}$$

almost surely for $i = 2, 3$.

Let ${}_i\hat{E}(t)$ *denote the number of all edges present at time* t *if the ancestor of the population was a type* i *object,* $i = 2, 3$. *Then*

$$\lim_{t\to\infty} e^{-\alpha t}\,{}_i\hat{E}(t) = {}_iW \frac{v_i u_2 A(\alpha)}{D(\alpha)} \tag{8.7}$$

almost surely for $i = 2, 3$.

Let ${}_iT(t)$ *denote the number of all triangles being born up to time* t *if the ancestor of the population was a type* i *object,* $i = 2, 3$. *Then*

$$\lim_{t\to\infty} e^{-\alpha t}\,{}_iT(t) = {}_iW \frac{v_i u_3}{\alpha D(\alpha)} \tag{8.8}$$

almost surely for $i = 2, 3$.

Let ${}_i\hat{T}(t)$ denote the number of all triangles present at time t if the ancestor of the population was a type i object, $i = 2, 3$. Then,

$$\lim_{t\to\infty} e^{-\alpha t}\, {}_i\hat{T}(t) = {}_iW \frac{v_i u_3 B(\alpha)}{D(\alpha)} \tag{8.9}$$

almost surely for $i = 2, 3$.

The quantities ${}_2W$ and ${}_3W$ are a.s. non-negative, $\mathbb{E}({}_2W) = \mathbb{E}({}_3W) = 1$, ${}_2W$ and ${}_3W$ are a.s. positive on the event of survival.

Proof. We apply Proposition 13.1. To obtain condition (13.8), it is enough to show that

$$\mathbb{E}\left[{}_\alpha\xi_i(\infty) \log^+ {}_\alpha\xi_i(\infty)\right] < \infty, \qquad i = 2, 3, \tag{8.10}$$

where

$${}_\alpha\xi_i(\infty) = \int_0^\infty e^{-\alpha t} \xi_i\,(dt)\,, \qquad i = 2, 3, \tag{8.11}$$

and

$$\xi_i(t) = \xi_{i,2}(t) + \xi_{i,3}(t), \qquad i = 2, 3. \tag{8.12}$$

If $i = 2$, then $\xi_2(t)$ is the birth process of an edge, and the children can be both edges and triangles. Therefore, at each birth, there is one child. Therefore,

$${}_\alpha\xi_2(\infty) = \int_0^\infty e^{-\alpha t} \xi_2\,(dt) = \sum_{\tau(i)\le\lambda_2} 1e^{-\alpha\tau(i)} \le \sum_{i=1}^{\infty} 1e^{-\alpha\tau(i)} = M,$$

where $\tau(1), \tau(2), \ldots$ are the jumps of the Poisson process Π_2. In the Poisson process $\Pi_2\,(t)$ the distribution of the interarrival time $(\tau(i) - \tau(i-1))$ is exponential with rate 1. Therefore, $\tau(i)$ has Γ-distribution $\mathit{\Gamma}\,(i, 1)$. Using this, we have

$$\mathbb{E}(M) = \sum_{i=1}^{\infty} \mathbb{E}\left(e^{-\alpha\tau(i)}\right) = \sum_{i=1}^{\infty} \frac{1}{(1+\alpha)^i} = \frac{1}{\alpha}. \tag{8.13}$$

Let us denote by η_i the interarrival time $\tau(i) - \tau(i-1)$. Let η_0 be an exponentially distributed random variable with rate 1 that is independent of M. Then,

$$e^{-\alpha\eta_0}\,(1+M) = e^{-\alpha\eta_0} + e^{-\alpha\eta_0} \sum_{i=1}^{\infty} e^{-\alpha(\eta_1+\cdots+\eta_i)} = \sum_{i=0}^{\infty} e^{-\alpha(\eta_0+\eta_1+\cdots+\eta_i)}.$$

Therefore, the distribution of $e^{-\alpha\eta_0}\,(1+M)$ coincides with the distribution of M.

Therefore, using (8.13), we have

$$\mathbb{E}M^2 = \mathbb{E}\left(e^{-\alpha\eta_0}(1+M)\right)^2 = \frac{1}{1+2\alpha}\left(1+\frac{2}{\alpha}+\mathbb{E}M^2\right).$$

From this, we find

$$\mathbb{E}M^2 = \frac{\alpha+2}{2\alpha^2} < \infty.$$

Thus, (8.10) is true for $i=2$.

If $i = 3$, then $\xi_3(t)$ is the birth process of a triangle and the children can be both edges and triangles. Therefore, at each birth there are at most three children. Therefore,

$$_\alpha\xi_3(\infty) = \int_0^\infty e^{-\alpha t}\xi_3(dt) = \sum_{\tau(i)\leq\lambda_3} \varepsilon(i)e^{-\alpha\tau(i)} \leq 3\sum_{i=1}^{\infty} 1e^{-\alpha\tau(i)} = 3M,$$

where $\tau(1), \tau(2), \ldots$ are the jumps of the Poisson process Π_3. By the above calculation $\mathbb{E}M^2 < \infty$, so (8.10) is true for $i = 3$.

If we show that $\int_0^\infty t^2 e^{-\alpha t} m_{i,j}(dt) < \infty$, for $i, j = 2, 3$, then conditions (c) and (iv) of Section 13 will be proved. Now, for $i = 2$ and $j = 2, 3$, we have from Corollary 2.1

$$\int_0^\infty t^2 e^{-\alpha t} m_{2,j}(dt) \leq \max\{r_1, r_2\}\int_0^\infty t^2 e^{-\alpha t}e^{-t(b+1)}e^{\frac{1-e^{-ct}}{c}}dt \leq$$
$$\leq \int_0^\infty t^2 e^{-t(\alpha+b+1-1)}dt < \infty$$

because $\alpha + b > 0$.

For $i = 3$ and $j = 2, 3$, we have from Corollary 7.1

$$\int_0^\infty t^2 e^{-\alpha t} m_{3,j}(dt) \leq$$
$$\leq \max\{p_1, p_2+3p_3\}\int_0^\infty t^2 e^{-\alpha t}e^{-t(b+1)}e^{(p_1+p_2)\frac{1-e^{-ct}}{c}+p_3\frac{1-e^{-3ct}}{3c}}dt \leq$$
$$\leq \int_0^\infty t^2 e^{-t(\alpha+b+1-1)}dt < \infty$$

Thus, conditions (c) and (iv) of Section 13 are proved.

Now, turn to the number of edges.

To obtain (8.6), let $\Phi_x(t) = 1$ if x is an edge, and $\Phi_x(t) = 0$ if x is a triangle.

Therefore, $\mathbb{E}\Phi_2(t) = 1$ and $\mathbb{E}\Phi_3(t) = 0$. Conditions $(i) - (ii) - (iii)$ and (v) of Section 13 are satisfied. Thus, (13.6) and (13.7) imply (8.6).

To obtain (8.7), let $\Phi_x(t) = 1$ if x is an edge and it is present at t, and $\Phi_x(t) = 0$ if x is a triangle. Therefore, $\mathbb{E}\Phi_2(t) = 1 - L_2(t)$ and $\mathbb{E}\Phi_3(t) = 0$. Conditions $(i) - (ii) - (iii)$ and (v) of Section 13 are satisfied. Now,

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\Phi_2(t)\,dt = \int_0^\infty e^{-\alpha t}(1 - L_2(t))dt = A(\alpha).$$

Thus, (13.6) and (13.7) imply (8.7).

Now, we turn to the number of triangles.

To obtain (8.8), let $\Phi_x(t) = 0$ if x is an edge, and $\Phi_x(t) = 1$ if x is a triangle. Therefore, $\mathbb{E}\Phi_2(t) = 0$ and $\mathbb{E}\Phi_3(t) = 1$. Conditions $(i) - (ii) - (iii)$ and (v) of Section 13 are satisfied. Thus, (13.6) and (13.7) imply (8.8).

To obtain (8.9), let $\Phi_x(t) = 0$ if x is an edge, and $\Phi_x(t) = 1$ if x is a triangle, and it is present at t. Therefore, $\mathbb{E}\Phi_2(t) = 0$ and $\mathbb{E}\Phi_3(t) = 1 - L_3(t)$. Conditions $(i) - (ii) - (iii)$ and (v) of Section 13 are satisfied. Now,

$$\int_0^\infty e^{-\alpha t}\mathbb{E}\Phi_3(t)\,dt = \int_0^\infty e^{-\alpha t}(1 - L_3(t))dt = B(\alpha).$$

Thus, (13.6) and (13.7) imply (8.9). □

Remark 8.1. If we let $r_1 = 1$, we get back the asymptotic results of the trianlges in the 3-interaction model, presented in Section 3.

9 Generating functions and the probability of extinction

In this section the generating functions are calculated. Using the generating functions, the probability of extinction are studied.

The joint generating function of $\Pi_2(\lambda_2)$, $\xi_{22}(\lambda_2)$ and $\xi_{23}(\lambda_2)$. Recall that Π_2 is the Poisson process describing the reproduction times of the generic edge and λ_2 is its life length. Thus,

$$w_{i,j,k} = \mathbb{P}\left(\Pi_2(\lambda_2) = i, \xi_{22}(\lambda_2) = j, \xi_{23}(\lambda_2) = k\right)$$

is the joint distribution of the offspring size of the generic edge during its whole life and its last reproduction time. We have

$$w_{i,j,k} = \mathbb{P}\left(\tau_i \le \lambda_2 < \tau_{i+1}, \xi_{22}\left(\tau_i\right) = j, \xi_{23}\left(\tau_i\right) = k\right),$$

where τ_i is the ith jumping time of the Poisson process Π_2. Thus, it again shows that $w_{i,j,k}$ is the probability that the ith birth event is the last one that occurred before death, and the total numbers of the two types of offspring up to time τ_i are equal to j and k, respectively.

Now, consider the sequence

$$u_{i,j,k} = \mathbb{P}\left(\tau_i \le \lambda_2, \xi_{22}\left(\tau_i\right) = j, \xi_{23}\left(\tau_i\right) = k\right).$$

Let $\xi_2\left(\tau_{i-1}\right) = m$ and assume for a while that τ_i and τ_{i-1} are fixed. Then, using (6.4) and (6.5) for the hazard rate, we can calculate that, for fixed τ_i and τ_{i-1},

$$\mathbb{P}\left(\lambda_2 \ge \tau_i | \lambda_2 \ge \tau_{i-1}, \tau_{i-1}, \tau_i\right) = \exp\left(-\left(b + cm\right)\left(\tau_i - \tau_{i-1}\right)\right).$$

We know that the increment $\left(\tau_i - \tau_{i-1}\right)$ is exponential with parameter 1; therefore,

$$\mathbb{P}\left(\lambda_2 \ge \tau_i | \lambda_2 \ge \tau_{i-1}\right) = \mathbb{E}_{\tau_i - \tau_{i-1}} \exp\left(-\left(b + cm\right)\left(\tau_i - \tau_{i-1}\right)\right) = \frac{1}{1 + b + cm}. \quad (9.1)$$

At each birth step, the new individual can be either an edge or a triangle. Therefore, using the above calculations, the total probability theorem, and the independence of the type of the newly born individual and (Π_2, λ_2), we have the following recursion for $u_{i,j,k}$.

$$u_{i,j,k} = u_{i-1,j-1,k} \frac{r_1}{1 + b + c(j + k - 1)} + u_{i-1,j,k-1} \frac{r_2}{1 + b + c\left(j + k - 1\right)}. \quad (9.2)$$

Now, by the definition of $w_{i,j,k}$, we can see that

$$\begin{aligned} w_{i,j,k} &= \mathbb{P}\left(\tau_i \le \lambda_2 < \tau_{i+1}, \xi_{22}\left(\tau_i\right) = j, \xi_{23}\left(\tau_i\right) = k\right) = \\ &\mathbb{P}\left(\lambda_2 < \tau_{i+1} | \tau_i \le \lambda_2, \xi_{22}\left(\tau_i\right) = j, \xi_{23}\left(\tau_i\right) = k\right) \mathbb{P}\left(\tau_i \le \lambda_2, \xi_{22}\left(\tau_i\right) = j, \xi_{23}\left(\tau_i\right) = k\right) \\ &= \frac{b + c(j + k)}{1 + b + c(j + k)} u_{i,j,k}, \end{aligned}$$

where by (9.1), $\frac{b+c(j+k)}{1+b+c(j+k)}$ is the probability that the generic individual dies before the next birth event.

Let $v_{i,j,k} = \dfrac{w_{i,j,k}}{b + c(j + k)} = \dfrac{u_{i,j,k}}{1 + b + c(j + k)}$. Then, from (9.2), we obtain the following recursion for the sequence $v_{i,j,k}$

$$(1+b+c(j+k))\, v_{i,j,k} = v_{i-1,j-1,k} r_1 + v_{i-1,j,k-1} r_2, \tag{9.3}$$

where the initial values are

$$v_{0,0,0} = \frac{1}{1+b} \text{ and } v_{0,j,k} = 0 \text{ for } j \neq 0 \text{ or } k \neq 0. \tag{9.4}$$

Now, we calculate the generating function $G(x,y,z)$ of the sequence $v_{i,j,k}$. We have

$$G(x,y,z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_{i,j,k} x^i y^j z^k.$$

First, multiplying with $x^i y^j z^k$ and then taking the sum of both sides of (9.3), we obtain

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_{i,j,k} x^i y^j z^k (1+b+cj+ck) =$$
$$= r_1 xy \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_{i-1,j-1,k} x^{i-1} y^{j-1} z^k + r_2 xz \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_{i-1,j,k-1} x^{i-1} y^j z^{k-1},$$

where $v_{0,j,k}, j = 0,1,\ldots, k = 0,1,\ldots$ is given by (9.4), and we define $v_{i,j,k} = 0$ if $j < 0$ or $k < 0$. From this equation, we find

$$(1+b)\left(G(x,y,z) - \frac{1}{1+b}\right) + ycG_y^{'}(x,y,z) + zcG_z^{'}(x,y,z) =$$
$$= r_1 xyG(x,y,z) + r_2 xzG(x,y,z). \tag{9.5}$$

Let $h(t) = G(x,ty,tz)$. Now, substituting y with ty, z with tz in (9.5), we can obtain the following linear differential equation.

$$h^{'}(t) + h(t)\left(\frac{1+b}{ct} - \frac{r_1 xy + r_2 xz}{c}\right) = \frac{1}{ct} \tag{9.6}$$

with the initial value condition

$$h(0) = \frac{1}{1+b}. \tag{9.7}$$

Now, we can use the well-known method for linear differential equations. We obtain that the solution of the initial value problem (9.6) and (9.7) is

$$h(t) = t^{-\frac{1+b}{c}} e^{\frac{r_1xy+r_2xz}{c}t} \frac{1}{c} \int_0^t s^{\frac{1+b-c}{c}} e^{-\frac{r_1xy+r_2xz}{c}s} ds.$$

With $t = 1$, we obtain that

$$G(x,y,z) = h(1) = \frac{1}{c} \int_0^1 s^{\frac{1+b-c}{c}} e^{\frac{r_1xy+r_2xz}{c}(1-s)} ds.$$

We need the generating function of $w_{i,j,k} = v_{i,j,k}(b + c(j+k))$. It is

$$H(x,y,z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_{i,j,k}(b + c(j+k)) x^i y^j z^k =$$

$$= bG(x,y,z) + cyG'_y(x,y,z) + czG'_z(x,y,z). \tag{9.8}$$

From here, we obtain

Proposition 9.1. *The joint generating function of* $\Pi_2(\lambda_2)$, $\xi_{22}(\lambda_2)$ *and* $\xi_{23}(\lambda_2)$ *is*

$$H(x,y,z) = e^{\frac{r_1xy+r_2xz}{c}} \frac{1}{c} \int_0^1 s^{\frac{1+b-c}{c}} e^{-\frac{r_1xy+r_2xz}{c}s} \left[b + (r_1xy + r_2xz)(1-s)\right] ds, \tag{9.9}$$

where $-1 \le x, y, z \le 1$.

Corollary 9.1. *The generating function of the total offspring distribution of the generic edge is*

$$f_2(y,z) = H(1,y,z) = e^{\frac{r_1y+r_2z}{c}} \frac{1}{c} \int_0^1 s^{\frac{1+b-c}{c}} e^{-\frac{r_1y+r_2z}{c}s} \left[b + (r_1y + r_2z)(1-s)\right] ds. \tag{9.10}$$

The joint generating function of $\Pi_3(\lambda_3)$**,** $\xi_{32}(\lambda_3)$ **and** $\xi_{33}(\lambda_3)$**.** Here, we study the offspring of a triangle. To distinguish the notation of this subsection and the previous subsection, but avoid too many subscripts, we use bar. Thus, here $\overline{w}_{i,j,k}$, $\overline{u}_{i,j,k}$, $\overline{v}_{i,j,k}$, $\overline{G}(x,y,z)$ and $\overline{H}(x,y,z)$ denote quantities relating offspring of the generic triangle. Recall that Π_3 is the Poisson process describing the reproduction times of the generic triangle and λ_3 is the life length of the triangle. Thus,

$$\overline{w}_{i,j,k} = \mathbb{P}(\Pi_3(\lambda_3) = i, \xi_{32}(\lambda_3) = j, \xi_{33}(\lambda_3) = k)$$

is the joint distribution of the offspring size of the generic triangle during its whole life and its last reproduction time. We have

$$\overline{w}_{i,j,k} = \mathbb{P}\left(\tau_i \leq \lambda_3 < \tau_{i+1}, \xi_{32}\left(\tau_i\right) = j, \xi_{33}\left(\tau_i\right) = k\right),$$

where τ_i is the ith jumping time of the Poisson process Π_3. Thus, we again show that $\overline{w}_{i,j,k}$ is the probability that the ith birth event is the last one that happened before death, and the total numbers of the two types of offspring up to time τ_i are equal to j and k, respectively.

Let

$$\overline{u}_{i,j,k} = \mathbb{P}\left(\tau_i \leq \lambda_3, \xi_{32}\left(\tau_i\right) = j, \xi_{33}\left(\tau_i\right) = k\right).$$

Let $\xi_3\left(\tau_{i-1}\right) = m$, and assume for a while that τ_i and τ_{i-1} are fixed. Then, using (6.4) and (6.5) for the hazard rate, we can calculate that, for fixed τ_i and τ_{i-1},

$$\mathbb{P}\left(\lambda_3 \geq \tau_i | \lambda_3 \geq \tau_{i-1}\right) = \exp\left(-\left(b + cm\right)\left(\tau_i - \tau_{i-1}\right)\right).$$

We know that the increment $\left(\tau_i - \tau_{i-1}\right)$ is exponential with parameter 1; therefore,

$$\mathbb{P}\left(\lambda_3 \geq \tau_i | \lambda_3 \geq \tau_{i-1}\right) = \mathbb{E}_{\tau_i - \tau_{i-1}} \exp\left(-\left(b + cm\right)\left(\tau_i - \tau_{i-1}\right)\right) = \frac{1}{1 + b + cm}. \tag{9.11}$$

At each birth step, the new individual can be either an edge or a triangle. Therefore, using the above calculations, the total probability theorem, and the independence of the type of the newly born individual and (Π_3, λ_3), we have the following recursion for $\overline{u}_{i,j,k}$.

$$\begin{aligned}\overline{u}_{i,j,k} = \overline{u}_{i-1,j-1,k} \frac{p_1}{1 + b + c(j + k - 1)} +& \\ + \overline{u}_{i-1,j,k-1} \frac{p_2}{1 + b + c\left(j + k - 1\right)} &+ \overline{u}_{i-1,j,k-3} \frac{p_3}{1 + b + c\left(j + k - 3\right)}.\end{aligned} \tag{9.12}$$

Now, by the definition of $\overline{w}_{i,j,k}$, we can see that

$$\begin{aligned}&\overline{w}_{i,j,k} = \mathbb{P}\left(\tau_i \leq \lambda_3 < \tau_{i+1}, \xi_{32}\left(\tau_i\right) = j, \xi_{33}\left(\tau_i\right) = k\right) = \\ &\mathbb{P}\left(\lambda_3 < \tau_{i+1} | \tau_i \leq \lambda_3, \xi_{32}\left(\tau_i\right) = j, \xi_{33}\left(\tau_i\right) = k\right) \mathbb{P}\left(\tau_i \leq \lambda_3, \xi_{32}\left(\tau_i\right) = j, \xi_{33}\left(\tau_i\right) = k\right) \\ &= \frac{b + c(j + k)}{1 + b + c(j + k)} \overline{u}_{i,j,k},\end{aligned}$$

where by (9.11), $\frac{b+c(j+k)}{1+b+c(j+k)}$ is the probability that the generic individual dies before the next birth event.

Now, let $\overline{v}_{i,j,k} = \dfrac{\overline{w}_{i,j,k}}{b + c(j + k)} = \dfrac{\overline{u}_{i,j,k}}{1 + b + c(j + k)}$. Then, from (9.12), we obtain the following recursion for the sequence $\overline{v}_{i,j,k}$

$$(1+b+c(j+k))\,\overline{v}_{i,j,k} = \overline{v}_{i-1,j-1,k}p_1 + \overline{v}_{i-1,j,k-1}p_2 + \overline{v}_{i-1,j,k-3}p_3, \tag{9.13}$$

where the initial values are

$$\overline{v}_{0,0,0} = \frac{1}{1+b} \text{ and } \overline{v}_{0,j,k} = 0 \text{ for } j \neq 0 \text{ or } k \neq 0. \tag{9.14}$$

Now, we calculate the generating function $\overline{G}\,(x,y,z)$ of the sequence $\overline{v}_{i,j,k}$. We have

$$\overline{G}\,(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \overline{v}_{i,j,k}x^i y^j z^k.$$

First, multiplying with $x^i y^j z^k$ and then taking the sum of both sides of (9.13), we obtain

$$\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \overline{v}_{i,j,k}x^i y^j z^k\,(1+b+cj+ck) = p_1xy\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \overline{v}_{i-1,j-1,k}x^{i-1}y^{j-1}z^k$$
$$+p_2xz\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \overline{v}_{i-1,j,k-1}x^{i-1}y^j z^{k-1} + p_3xz^3\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \overline{v}_{i-1,j,k-3}x^{i-1}y^j z^{k-3},$$

where $\overline{v}_{0,j,k}, j = 0,1,\dots$ and $k = 0,1,\dots$ is given by (9.14) and we define $\overline{v}_{i,j,k} = 0$ if $j < 0$ or $k < 0$. From this equation, we find

$$(1+b)\left(\overline{G}\,(x,y,z) - \frac{1}{1+b}\right) + yc\overline{G}_y^{'}\,(x,y,z) + zc\overline{G}_z^{'}\,(x,y,z) =$$
$$= p_1xy\overline{G}\,(x,y,z) + p_2xz\overline{G}\,(x,y,z) + p_3xz^3\overline{G}\,(x,y,z)\,. \tag{9.15}$$

Let $\overline{h}\,(t) = \overline{G}\,(x,ty,tz)$. Now, substituting y with ty, z with tz in (9.15), we can obtain the following linear differential equation.

$$\overline{h}^{'}\,(t) + \overline{h}\,(t)\left(\frac{1+b}{ct} - \frac{p_1xy + p_2xz + p_3xz^3t^2}{c}\right) = \frac{1}{ct} \tag{9.16}$$

with the initial value condition

$$\overline{h}\,(0) = \frac{1}{1+b}. \tag{9.17}$$

One can see that the solution of the initial value problem (9.16) and (9.17) is

$$\overline{h}\left(t\right)=t^{-\frac{1+b}{c}}e^{\frac{p_1xy+p_2xz}{c}t+\frac{p_3xz^3}{3c}t^3}\frac{1}{c}\int_0^t s^{\frac{1+b-c}{c}}e^{-\frac{p_1xy+p_2xz}{c}s-\frac{p_3xz^3}{3c}s^3}ds.$$

With $t=1$, we obtain that

$$\overline{G}\left(x,y,z\right)=\overline{h}\left(1\right)=\frac{1}{c}\int_0^1 s^{\frac{1+b-c}{c}}e^{\frac{p_1xy+p_2xz}{c}(1-s)+\frac{p_3xz^3}{3c}(1-s^3)}ds.$$

Therefore, the generating function of $\overline{w}_{i,j,k}=\overline{v}_{i,j,k}(b+c(j+k))$ is

$$\begin{aligned}\overline{H}\left(x,y,z\right)&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\overline{v}_{i,j,k}(b+c(j+k))x^iy^jz^k=\\&=b\overline{G}\left(x,y,z\right)+cy\overline{G}_y'\left(x,y,z\right)+cz\overline{G}_z'\left(x,y,z\right).\end{aligned}\tag{9.18}$$

From here, we obtain

Proposition 9.2. *The joint generating function of* $\Pi_3(\lambda_3)$, $\xi_{32}(\lambda_3)$ *and* $\xi_{33}(\lambda_3)$ *is*

$$\begin{aligned}\overline{H}\left(x,y,z\right)=\frac{1}{c}\int_0^1 &s^{\frac{1+b-c}{c}}e^{\frac{p_1xy+p_2xz}{c}(1-s)+\frac{p_3xz^3}{3c}(1-s^3)}\\&\cdot\left[b+(p_1xy+p_2xz)(1-s)+p_3xz^3(1-s^3)\right]ds,\end{aligned}\tag{9.19}$$

where $-1\le x,y,z\le 1$.

Corollary 9.2. *The generating function of the total offspring distribution of the generic triangle is*

$$\begin{aligned}f_3(y,z)=\overline{H}\left(1,y,z\right)=e^{\frac{p_1y+p_2z}{c}+\frac{p_3z^3}{3c}}\frac{1}{c}\int_0^1 &s^{\frac{1+b-c}{c}}e^{-\frac{p_1y+p_2z}{c}s-\frac{p_3z^3}{3c}s^3}\\&\cdot\left[b+(p_1y+p_2z)(1-s)+p_3z^3(1-s^3)\right]ds.\end{aligned}\tag{9.20}$$

The probability of extinction. In Theorem 9.1, we give the probability of extinction. To determine the extinction probability of the process, we consider the well-known embedded multi type Galton–Watson process. At time $t=0$, the 0th generation of the Galton–Watson process consists of a single individual, i.e., the ancestor. The first generation consists of all offspring of the ancestor. The offspring of the individuals of the nth generation form the $(n+1)$th generation. Under some assumptions, the extinction of our original process has the same probability as the extinction of this embedded Galton–Watson process. The reproduction process $\xi_{i,j}\left(t\right)$ gives the number of type j offspring of an ancestor of type i up to time t.

With $t \to \infty$, we obtain that the total number of offspring is $\xi_{i,j}(\infty)$. Therefore, Corollary 7.1 gives us the 2×2 matrix of the expected total offspring number as

$$\mathbb{M} = (m_{i,j}(\infty))_{i,j=2}^{3} .$$

Actually, $m_{i,j}(\infty)$ is the expected offspring number of the embedded Galton–Watson process.

Let s_2 and s_3 denote the probability of extinction of our process when the ancestor is an edge, resp. triangle.

Theorem 9.1. *Assume that $0 \leq r_1 < 1$, $0 < p_1 \leq 1$ and it is excluded that both $r_1 = 0$ and $p_1 = 1$ are satisfied at the same time. Let ϱ be the Perron–Frobenius root of $\mathbb{M}$. If $\varrho \leq 1$, then $s_2 = s_3 = 1$. If $\varrho > 1$, then $s_2 < 1$ and $s_3 < 1$. In any case, (s_2, s_3) is the smallest non-negative solution of the vector equation*

$$(s_2, s_3) = (f_2(s_2, s_3), f_3(s_2, s_3)) ,$$

such that, for any other non-negative solution (s_2^*, s_3^*), we have that $s_i \leq s_i^*$, $i = 2, 3$. The functions f_2 and f_3 are given in Corollaries 9.1 and 9.2.

Proof. We apply Theorem 7.1 in Chapter 1 of [30]. By Corollary 7.1, $m_{i,j}(0) = 0$ and $m_{i,j}(t)$ is finite for any i, j. Therefore, by Theorem 7.1 in Chapter 3 of [30], the extinction of our original process has the same probability as the extinction of the embedded Galton–Watson process. Thus, we can apply Theorem 7.1 in Chapter 1 of [30]. Here, $\mathbb{M}$ is the matrix of the expected offspring numbers of the embedded Galton–Watson process. Now, $\mathbb{M}$ is positively regular because we assume that $0 \leq r_1 < 1$, $0 < p_1 \leq 1$ and it is excluded, that both $r_1 = 0$ and $p_1 = 1$ are satisfied at the same time. Thus, our result follows from Theorem 7.1 in Chapter 1 of [30]. □

10 The asymptotic behaviour of the degree of a fixed vertex

In this section the asymptotic behaviour of the degree of a fixed vertex is considered. Here, we again apply the asymptotic theorems of [20] but with other characteristics than in Section 8.

The process of the 'good children'. To describe the degree of a fixed vertex, we introduce a new branching process that we call the process of 'good children'. This process contains those objects that contribute to the degree of the fixed vertex.

We can see that a newly born vertex can have 1 or 2 edges if its parent is an edge object and 1, 2 or 3 edges if its parent is a triangle object.

First, we consider the case when the newly born vertex has one edge, and thus, at the beginning, it belongs to an edge object. In this paragraph, we call this edge the 'parent' edge. We fix the newly born vertex. Then, we distinguish those children objects of the 'parent' edge, which contribute to the degree of our fixed vertex. We call a child object of the 'parent' edge a 'good child' if it contains our fixed vertex. We can see that only the 'good children' and their 'good children' offspring can contribute to the degree of the fixed vertex. Then, the distribution of the number of 'good children' at a reproduction event of the 'parent' edge is

$$\mathbb{P}(\tilde{\varepsilon}_{22}=0)=1-\frac{1}{2}r_1,\quad \mathbb{P}(\tilde{\varepsilon}_{22}=1)=\frac{1}{2}r_1,\quad \mathbb{P}(\tilde{\varepsilon}_{23}=0)=1-r_2,\quad \mathbb{P}(\tilde{\varepsilon}_{23}=1)=r_2,$$

where $\tilde{\varepsilon}_{22}$ denotes the number of edge type 'good children' and $\tilde{\varepsilon}_{23}$ denotes the triangle type 'good children'. We have to consider the reproduction process of the 'good child', which is the following

$$\tilde{\xi}_{2,2}(t)=\tilde{\varepsilon}_{22}(1)+\tilde{\varepsilon}_{22}(2)+\cdots+\tilde{\varepsilon}_{22}(\Pi(t\wedge\lambda_2)), \tag{10.1}$$

$$\tilde{\xi}_{2,3}(t)=\tilde{\varepsilon}_{23}(1)+\tilde{\varepsilon}_{23}(2)+\cdots+\tilde{\varepsilon}_{23}(\Pi(t\wedge\lambda_2)), \tag{10.2}$$

where $\tilde{\xi}_{2,2}(t)$ denotes the number of all edge type 'good children', and $\tilde{\xi}_{2,3}(t)$ denotes the number of all triangle type 'good children' born by the 'parent' edge, $\tilde{\varepsilon}_{22}(1),\tilde{\varepsilon}_{22}(2),\ldots$ are i.i.d. copies of $\tilde{\varepsilon}_{22}$ and $\tilde{\varepsilon}_{23}(1),\tilde{\varepsilon}_{23}(2),\ldots$ are i.i.d. copies of $\tilde{\varepsilon}_{23}$. Using Corollary 7.1, we see that the mean values of the number of edge type and triangle type 'good children' are

$$\tilde{m}_{2,2}(t)=\mathbb{E}\tilde{\xi}_{2,2}(t)=\mathbb{E}(\tilde{\varepsilon}_{22})\mathbb{E}(\Pi(t\wedge\lambda_2))=\frac{1}{2}r_1F(t)=\frac{1}{2}m_{2,2}(t),$$

$$\tilde{m}_{2,3}(t)=\mathbb{E}\tilde{\xi}_{2,3}(t)=\mathbb{E}(\tilde{\varepsilon}_{23})\mathbb{E}(\Pi(t\wedge\lambda_2))=r_2F(t)=m_{2,3}(t).$$

Now, consider the second case where the newly born vertex has two edges, and thus the 'parent' object is a single triangle. Let $\tilde{\varepsilon}_{32}$ and $\tilde{\varepsilon}_{33}$ denote the number of edge, resp. triangle type 'good children' of the 'parent' triangle. The distribution of the number of 'good children' will be the following

$$\mathbb{P}(\tilde{\varepsilon}_{32}=0)=1-\frac{1}{3}p_1,\quad \mathbb{P}(\tilde{\varepsilon}_{32}=1)=\frac{1}{3}p_1,$$

$$\mathbb{P}(\tilde{\varepsilon}_{33}=0)=1-\frac{2}{3}p_2-p_3,\quad \mathbb{P}(\tilde{\varepsilon}_{33}=1)=\frac{2}{3}p_2,\quad \mathbb{P}(\tilde{\varepsilon}_{33}=2)=p_3.$$

Let $\tilde{\xi}_{3,2}(t)$ denote the number of all edge type 'good children', and $\tilde{\xi}_{3,3}(t)$ denote the number of all triangle type 'good children' born by the 'parent' triangle. We obtain from Corollary 7.1 that

$$\tilde{m}_{3,2}(t) = \mathbb{E}\tilde{\xi}_{3,2}(t) = \mathbb{E}(\tilde{\varepsilon}_{32})\mathbb{E}(\Pi(t \wedge \lambda_3)) = \frac{1}{3}p_1 G(t) = \frac{1}{3}m_{3,2}(t),$$

$$\tilde{m}_{3,3}(t) = \mathbb{E}\tilde{\xi}_{3,3}(t) = \mathbb{E}(\tilde{\varepsilon}_{33})\mathbb{E}(\Pi(t \wedge \lambda_3)) = \frac{2}{3}(p_2 + 3p_3)G(t) = \frac{2}{3}m_{3,3}(t).$$

Therefore, from Proposition 7.1, it is easily seen that the Laplace transforms of the average number of offspring are

$$\tilde{m}^*_{2,2}(\kappa) = \frac{1}{2}r_1 A(\kappa), \quad \tilde{m}^*_{2,3}(\kappa) = r_2 A(\kappa), \quad \tilde{m}^*_{3,2}(\kappa) = \frac{1}{3}p_1 B(\kappa),$$

$$\tilde{m}^*_{3,3}(\kappa) = \frac{2}{3}(p_2 + 3p_3)B(\kappa).$$

Let

$$\tilde{M}(\kappa) = \begin{pmatrix} \tilde{m}^*_{2,2}(\kappa) & \tilde{m}^*_{2,3}(\kappa) \\ \tilde{m}^*_{3,2}(\kappa) & \tilde{m}^*_{3,3}(\kappa) \end{pmatrix}$$

be the matrix of the previous Laplace transforms. The Perron root that is the largest eigenvalue of $\tilde{M}(\kappa)$ is

$$\tilde{\varrho}(\kappa) = \frac{\frac{2}{3}(p_2 + 3p_3)B(\kappa) + \frac{1}{2}r_1 A(\kappa) + \sqrt{\left(\frac{2}{3}(p_2 + 3p_3)B(\kappa) - \frac{1}{2}r_1 A(\kappa)\right)^2 + \frac{4}{3}p_1 B(\kappa) r_2 A(\kappa)}}{2}. \tag{10.3}$$

In the following, we assume supercriticality of the 'good children' process; that is, we suppose that $\tilde{\varrho}(0) > 1$. We can see that the reproduction process of the 'good children' is supercritical if

$$\max\left\{\frac{1}{2}r_1 A(0), \frac{2}{3}(p_2 + 3p_3)B(0)\right\} > 1.$$

We assume the existence of finite and positive Malthusian parameter of the 'good children' process. Thus, let $\tilde{\alpha}$ be the Malthusian parameter; it satisfies equation $\tilde{\varrho}(\tilde{\alpha}) = 1$. From this equation and from (10.3), we see that $\tilde{\alpha}$ is the solution of

$$\frac{1}{3}\left(r_1(p_2 + 3p_3) - r_2 p_1\right) A(\tilde{\alpha})B(\tilde{\alpha}) - \frac{1}{2}r_1 A(\tilde{\alpha}) - \frac{2}{3}(p_2 + 3p_3)B(\tilde{\alpha}) + 1 = 0. \tag{10.4}$$

Let $(\tilde{v}_2, \tilde{v}_3)^\top$ denote the right eigenvector of $\tilde{M}(\tilde{\alpha})$ corresponding to the eigenvalue 1, and let $(\tilde{u}_2, \tilde{u}_3)^\top$ be the left eigenvector with the conditions $\tilde{v}_2 + \tilde{v}_3 = 1$ and $\tilde{v}_2\tilde{u}_2 + \tilde{v}_3\tilde{u}_3 = 1$. Direct calculations show that

$$\tilde{v}_2 = \frac{(1-r_1)A(\tilde{\alpha})}{\left(1-\frac{3}{2}r_1\right)A(\tilde{\alpha})+1}, \qquad \tilde{v}_3 = \frac{1-\frac{1}{2}r_1A(\tilde{\alpha})}{\left(1-\frac{3}{2}r_1\right)A(\tilde{\alpha})+1},$$

$$\tilde{u}_2 = \frac{\left((1-\frac{3}{2}r_1)A(\tilde{\alpha})+1\right)\frac{1}{3}p_1B(\tilde{\alpha})}{\frac{1}{3}r_2A(\tilde{\alpha})p_1B(\tilde{\alpha})+\left(\frac{1}{2}r_1A(\tilde{\alpha})-1\right)^2},$$

$$\tilde{u}_3 = \frac{\left((\frac{3}{2}r_1-1)A(\tilde{\alpha})-1\right)\left(\frac{1}{2}r_1A(\tilde{\alpha})-1\right)}{\frac{1}{3}r_2A(\tilde{\alpha})p_1B(\tilde{\alpha})+\left(\frac{1}{2}r_1A(\tilde{\alpha})-1\right)^2}.$$

Limit results for the degree. We have already mentioned that the 'good children' and only they can contribute to the degree of the fixed vertex. Thus, its degree is equal to the initial degree plus the number of 'good children'. Let ${}_2\tilde{C}(t)$ be the degree of a fixed vertex at time t after its birth in the case when the vertex belongs to an edge at its birth. Similarly, ${}_3\tilde{C}(t)$ is its degree in the case when the vertex belongs to triangle at its birth. Up to an additive constant, ${}_i\tilde{C}(t)$ is the number of 'good children' offspring of an i type 'parent' object at time t. It is the sum of the number of edge type 'good children' ${}_i\tilde{E}(t)$ and the triangle type 'good children' ${}_i\tilde{T}(t)$. To apply Proposition 13.1, we can use the same method as in Theorem 8.1. Thus, for the edges, we can again use the random characteristic $\Phi_x(t) = 1$ if x is an edge and $\Phi_x(t) = 0$ if x is a triangle, but the underlying process is the process of 'good children'. This is similar for triangles.

Therefore, we have almost surely

$$\lim_{t\to\infty} e^{-\tilde{\alpha}t}{}_i\tilde{C}(t) = \lim_{t\to\infty} e^{-\tilde{\alpha}t}\left({}_i\tilde{E}(t) + {}_i\tilde{T}(t)\right) = {}_i\tilde{W}\frac{\tilde{v}_i(\tilde{u}_2+\tilde{u}_3)}{\tilde{\alpha}\tilde{D}(\tilde{\alpha})},$$

for $i = 2, 3$, where ${}_2\tilde{W}$ and ${}_3\tilde{W}$ are positive on the event of non-extinction of the 'good children'.

The last case is when the newly born vertex has three edges. Then, three triangles contribute to the degree of that vertex. Let ${}_3\tilde{\tilde{C}}(t)$ be the degree of this vertex. Then, ${}_3\tilde{\tilde{C}}(t)$ is the sum if 'good' offspring of three triangles. Thus, almost surely,

$$\lim_{t\to\infty} e^{-\tilde{\alpha}t}{}_3\tilde{\tilde{C}}(t) = ({}_3\tilde{W}_1 + {}_3\tilde{W}_2 + {}_3\tilde{W}_3)\frac{\tilde{v}_3(\tilde{u}_2+\tilde{u}_3)}{\tilde{\alpha}\tilde{D}(\tilde{\alpha})},$$

where ${}_3\tilde{W}_1, {}_3\tilde{W}_2, {}_3\tilde{W}_3$ are independent copies of ${}_3\tilde{W}$.

Checking the conditions of Proposition 13.1 for the 'good children' pro-

cess. To complete the previous reasoning, we should check the conditions of Proposition 13.1. First, we find the the denominator in the limit theorem that is we calculate $\tilde{D}$. By Section 13, we see that

$$\tilde{D}(\tilde{\alpha}) = \sum_{l,j=2}^{3} \tilde{u}_l \tilde{v}_j \left(-\tilde{m}^*_{l,j}(\tilde{\alpha})\right)'. \tag{10.5}$$

Here, $\tilde{u}_i$ and $\tilde{v}_i$ are the eigenvectors. Moreover,

$$\left(-\tilde{m}^*_{2,2}(\tilde{\alpha})\right)' = \frac{r_1}{2}\left(-A'(\tilde{\alpha})\right), \qquad \left(-\tilde{m}^*_{2,3}(\tilde{\alpha})\right)' = r_2\left(-A'(\tilde{\alpha})\right), \tag{10.6}$$

$$\left(-\tilde{m}^*_{3,2}(\tilde{\alpha})\right)' = \frac{p_1}{3}\left(-B'(\tilde{\alpha})\right), \quad \left(-\tilde{m}^*_{3,3}(\tilde{\alpha})\right)' = \tfrac{2}{3}(p_2+3p_3)\left(-B'(\tilde{\alpha})\right), \tag{10.7}$$

where $\tilde{\alpha}$ is the Malthusian parameter in the process of 'good children' and A', B' denotes the derivatives given in (8.4) and (8.5).

Condition (a) of Proposition 13.1 is true because the measures $\tilde{m}_{i,j}$ are non-lattice as they are absolutely continuous. For condition (b1), we assume the existence of a positive Malthusian parameter. That is, we assume that (10.4) has a finite and positive solution $\tilde{\alpha}$. Condition (b2) is true, because we assume that $\tilde{\varrho}(0) > 1$. Condition (c) is a consequence of Section 8, because $\tilde{m}_{i,j}(t)$ has shape $cm_{i,j}$, where c is positive number.

To guarantee condition (d), in this section, we assume that $0 \le r_1 < 1$, $0 < p_1 \le 1$, and it is excluded that both $r_1 = 0$ and $p_1 = 1$ are satisfied at the same time. Conditions (i)-(ii)-(iii) and (v) are true because of the shape of Φ. Conditions (iv) and (vi) are consequences of $\tilde{\xi}_{i,j}(t) \le \xi_{i,j}(t)$ as one can see from the proof of Theorem 8.1.

The extinction of the degree process. The extinction of the degree process means that the degree of the vertex does not increase after a certain time, that is, the reproduction process of the 'good children' dies out. The probability of this kind of extinction is the smallest non-negative root $(\tilde{s}_2, \tilde{s}_3)$ of the equation

$$(\tilde{s}_2, \tilde{s}_3) = \left(\tilde{f}_2(\tilde{s}_2, \tilde{s}_3), \tilde{f}_3(\tilde{s}_2, \tilde{s}_3)\right),$$

where $\tilde{f}_2$ and $\tilde{f}_3$ are the generating functions of the total 'good children' distribution of an edge, resp. a triangle. Now, by (10.1) and (10.2),

$$\tilde{f}_2(y,z) = h_{\Pi_2(\lambda_2)}\left(h_{\tilde{\varepsilon}_{2,2},\tilde{\varepsilon}_{2,3}}(y,z)\right),$$

where $h_{\Pi_2(\lambda_2)}$ is the generating function of $\Pi_2(\lambda_2)$, and $h_{\tilde{\varepsilon}_{2,2},\tilde{\varepsilon}_{2,3}}$ is the joint generating function of $\tilde{\varepsilon}_{2,2}$ and $\tilde{\varepsilon}_{2,3}$. Here, by (9.9),

$$h_{\Pi_2(\lambda_2)}(x) = H(x,1,1) = \frac{1}{c}\int_0^1 s^{\frac{1+b-c}{c}} e^{\frac{(r_1+r_2)x}{c}(1-s)}[b+(r_1+r_2)x(1-s)]ds.$$

By direct calculation,

$$h_{\tilde{\varepsilon}_{2,2},\tilde{\varepsilon}_{2,3}}(y,z) = \frac{1}{2}r_1 + \frac{1}{2}r_1 y + r_2 z.$$

Similarly,

$$\tilde{f}_3(y,z) = h_{\Pi_3(\lambda_3)}\left(h_{\tilde{\varepsilon}_{3,2},\tilde{\varepsilon}_{3,3}}(y,z)\right),$$

where by (9.19), the generating function of $\Pi_3(\lambda_3)$ is

$$h_{\Pi_3(\lambda_3)}(x) = \overline{H}(x,1,1) =$$
$$\frac{1}{c}\int_0^1 s^{\frac{1+b-c}{c}} e^{\frac{(p_1+p_2)x}{c}(1-s)+\frac{p_3 x}{3c}(1-s^3)}\left[b+(p_1x+p_2x)(1-s)+p_3x(1-s^3)\right]ds.$$

Moreover, the joint generating function of $\tilde{\varepsilon}_{3,2}$ and $\tilde{\varepsilon}_{3,3}$ is

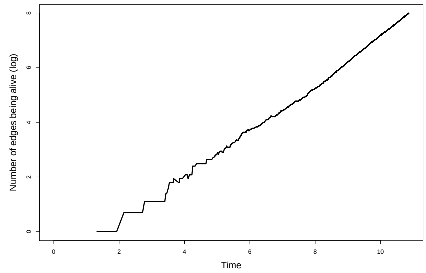
$$h_{\tilde{\varepsilon}_{3,2},\tilde{\varepsilon}_{3,3}}(y,z) = \frac{2}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_1 y + \frac{2}{3}p_2 z + p_3 z^2.$$

11 Simulations

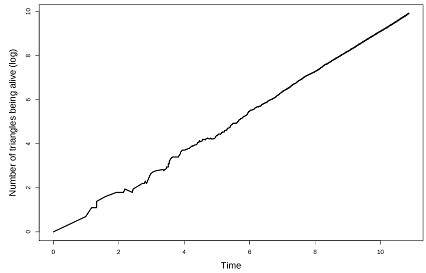
In this section, we provide some empirical results for our asymptotic theorems. We generated our process in the programming language Julia. We needed an environment, where the priority queues were highly applicable. Using this structure, the running time was reasonable. A more detailed explanation of the algorithm can be found in [27].

According to Theorem 8.1, for large t, the graphs of the numbers of edges and triangles are approximately straight lines on the logarithmic scale. To obtain empirical evidence of our Theorem 8.1, we investigated the slope of the simulated number of edges and triangles being born and being present up to time t on the logarithmic scale. The initial instability of the single processes (Figure 2.3) motivated us to exclude the first few observations from the calculations, but the lack of them was not relevant, because the asymptotic properties can be observed in the later stage of the processes.

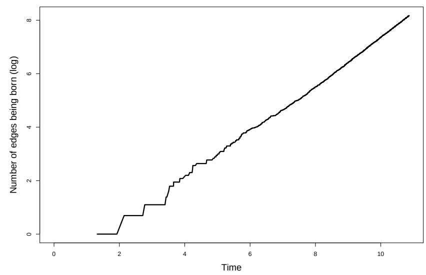
For each parameter set, we stored the mentioned measurements only in integer time steps, and then we took the average of 100 simulated processes. In Figure 2.4,



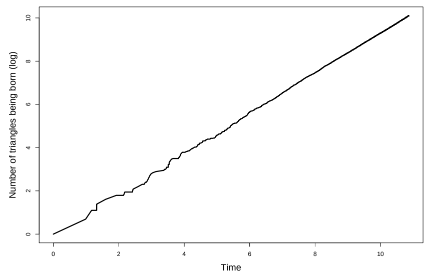
(**a**) Number of edges present



(**b**) Number of triangles present



(**c**) Number of edges being born



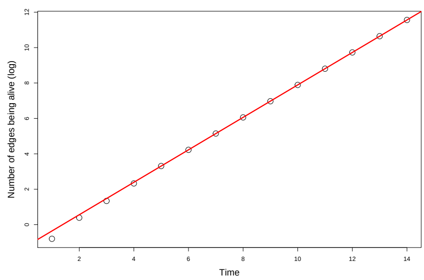
(**d**) Number of triangles being born

Figure 2.3: Measurements of a single process on a logarithmic scale.

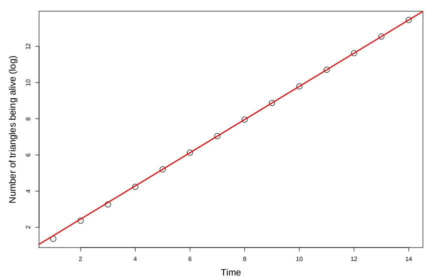
an example is shown for a specific parameter set ($r_1 = 0.1$, $p_1 = 0.2$, $p_3 = 0.6$, $b = 0.25$, $c = 0.25$). The values of the averages are plotted by dots. In each case, we fitted a regression line (plotted by continuous red line) to the last 9 values. We can see that the fit is perfect, thus, supporting our theorem.

Our main goal was to obtain a 95% confidence interval for the slope of the linear regression line, as that was our simulated approximation of the Malthusian parameter α. Table 2.1 contains the boundaries of the 95% confidence intervals for α. The columns labelled with 2.5% and 97.5% refer to the lower and the upper bounds obtained from simulations, while the column of $\hat{\alpha}$ refers to the numerical solution of Equation (7.18).

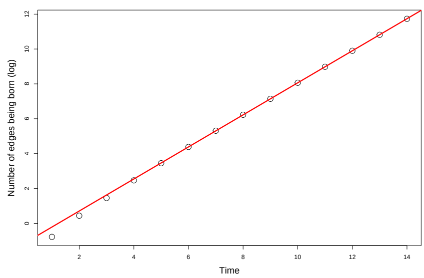
For each fixed parameter set $\{r_1,\ p_1,\ p_2,\ b,\ c\}$, we present the confidence intervals calculated from the number of edges being born (E) resp. being present ($\tilde{E}$) and from the number of triangles being born (T) resp. being present ($\tilde{T}$) up to time $t = 14$. The confidence intervals containing the numerical Malthusian parameter $\hat{\alpha}$ are highlighted with the $*$ symbol. We see that any confidence interval is narrow, and it either contains $\hat{\alpha}$, or $\hat{\alpha}$ is very close to the interval. These results show that the approximation is good for moderate values of t.



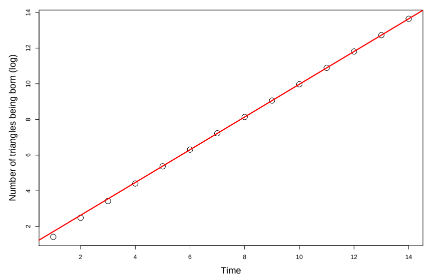
(**a**) Number of edges that exist



(**b**) Number of triangles that exist



(**c**) Number of edges being born



(**d**) Number of triangles being born

Figure 2.4: The average of 100 processes generated by the same parameter set and the regression line.

Finally, we present some simulation results for Theorem 9.1, that is, for the probability of extinction of the evolution process. We made the following computer experiment for any fixed parameter set $\{r_1,\ p_1,\ p_2,\ b,\ c\}$ and for type 2 and type 3 ancestors. We started to generate the process. If this process reached 2^{10} birth steps, then we stopped it and considered it as a non-extinct process. Otherwise, when the process did not reach 2^{10} birth steps, then the process died out. Applying the above method, we generated 10^5 processes for each parameter sets and counted the relative frequencies of the processes being extinct.

In Table 2.2, we show some of the results. Column Ancestor contains the type of the ancestor. In the column Numeric we show the numeric solution of the non-linear equation in Theorem 9.1. We used Julia's trust region method. Column Simulation contains the relative frequencies extracted from the simulations. The simulation results slightly underestimate the numeric values. This is reasonable because we stopped all processes at a fixed time.

Table 2.1: The 95% confidence intervals for α.

	$\boldsymbol{r_1}$	$\boldsymbol{p_1}$	$\boldsymbol{p_2}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\hat{\boldsymbol{\alpha}}$	**2.5%**	**97.5%**
E	0.1	0.5	0.5	0.2	0.4	**0.5394**	**0.5393 ***	**0.5443 ***
T							**0.5390 ***	**0.5440 ***
$\tilde{E}$							0.5410	0.5453
$\tilde{T}$							0.5395	0.5444
E	0.1	0.2	0.6	0.25	0.25	**0.9133**	**0.9130 ***	**0.9141 ***
T							0.9134	0.9142
$\tilde{E}$							**0.9133 ***	**0.9141 ***
$\tilde{T}$							**0.9133 ***	**0.9148 ***
E	0.1	0.2	0.6	0.45	0.35	**0.6622**	**0.6585 ***	**0.6659 ***
T							**0.6606 ***	**0.6648 ***
$\tilde{E}$							**0.6608 ***	**0.6647 ***
$\tilde{T}$							**0.6597 ***	**0.6638 ***

Table 2.2: Comparison of the numeric values of the extinction probabilities and their relative frequencies from 10^5 repetitions.

$\boldsymbol{r_1}$	$\boldsymbol{p_1}$	$\boldsymbol{p_2}$	$\boldsymbol{b}$	$\boldsymbol{c}$	**Ancestor**	**Numeric**	**Simulation**
0.1	0.2	0.6	0.8	0.8	2	0.9095	0.9053
					3	0.8855	0.8805
0.2	0.3	0.6	0.7	0.7	2	0.9247	0.9184
					3	0.9141	0.9070
0.3	0.3	0.5	0.6	0.6	2	0.7371	0.7207
					3	0.6896	0.6834

Summary

In this PhD thesis, we described the mathematical construction of two new network evolution models. The dynamic of the evolutions was provided by branching processes, where the units of the evolution were certain substructures of the graph, namely the different types of cliques.

In the Introduction, we mentioned the background literature for our models. From the discrete case results, we drove through the continuous case models, then we considered the possible applications.

Chapter 1 was based on the new results of our articles [26, 27]. Here we defined a new continuous-time network evolution model, where the interactions were based on the 3-cliques, i.e. the triangles. In the initial time only one triangle, the ancestor is given. This ancestor attracts new incomers, where these objects can join by 0, 1, 2, or 3 new edges. The connections with 2 and 3 edges form 1 and 3 triangles with probabilities q_1 and q_3 respectively. In the other two cases the offspring is not capable of reproduction, with q_0 probability in total. An arbitrary triangle, just like the ancestor tringle has its own reproduction process. An object's mean offspring number was defined by the $\mu(t)$ quantity at time t. The death of a triangle, i.e. the end of their reproduction phase is given by the $l(t) = b + c\xi(t)$ hazard rate, where b, c are non-negative constants and $\xi(t)$ is the number of offspring at time t. In the asymptotic results the Malthusian parameter α determines the increment of the number of triangles $Z(t)$, the number of vertices $V(t)$ and the number of edges $W(t)$.

After describing the mathematical construction, our main results were the following:

Let $\mu(\infty) > 1$.

1. The probability of the extinction of the triangles is the smallest non-negative solution of equation

$$1=\frac{q_1+q_3\left(y^2+y+1\right)}{c}\int_0^1(1-u)^{\frac{1+b-q_0}{c}-1}e^{\left(\frac{q_1y+q_3y^3}{c}u-\frac{q_3y^3}{c}u^2+\frac{q_3y^3}{3c}u^3\right)}du.$$

2. $$\lim_{t\to\infty}e^{-\alpha t}Z\left(t\right)=Y_\infty m_\infty$$

almost surely and in L^1, where the random variable Y_∞ is non-negative and it is positive on the event of non-extinction, it has expectation 1. Moreover,

$$m_\infty=\frac{1}{\left(q_1+3q_3\right)^2\int_0^\infty te^{-\alpha t}\left(1-L\left(t\right)\right)dt}.$$

3. $e^{-\alpha t}V(t)$ converges almost surely and

$$\frac{V\left(t\right)}{Z\left(t\right)}\to\frac{1}{\alpha}$$

as $t\to\infty$ almost surely on the event of non-extinction.

4. $e^{-\alpha t}W(t)$ converges almost surely and

$$\frac{W\left(t\right)}{Z\left(t\right)}\to\frac{\mathbb{E}\gamma_1}{\alpha}$$

as $t\to\infty$ almost surely on the event of non-extinction.

The results assume that our graph evolution model is super-critical, so we investigate the non-trivial case when the probability of extinction is less than 1. In this case we gave a formula for the probability of extinction that can be approximated numerically. The further results are on the asymptotical behavior of number of triangles, number of vertices and number of edges. Similar results are applied on the degree of a fixed vertex. To give an empirical evidence of our theorems, we presented some simulation results according to them.

Chapter 2 was based on the new results of our articles [28, 29]. Here we generalized our previously presented model for 2 types of objects. A new vertex can join to an old edge either with one or with two edges. Similarly, a new vertex can join to a triangle with 1, 2 or 3 edges. Therefore unlike in the previous model, here not only the triangles are capable for reproduction, but also the edges and both of them can reproduce the other. An edge can give birth to an edge with r_1 and a triangle with r_2 probabilities, while a triangle can give birth to an edge with p_1, a triangle with

p_2 and three triangles with p_3 probabilities. The hazard rate is the linear function of the number of offspring with constants b, c, just like in our previous model. Let $m_{ij}(t)$ be the expected number of j type offspring of an i type ancestor. Let $\mathbb{M}$ denote the matrix of the $m^*_{ij}(t)$ Laplace transforms of $m_{ij}(t)$ functions.

5. Denote by s_2 the probability of the extinctions if the ancestor is an edge, and by s_3 if the ancestor is a triangle. Assume that $0 \le r_1 < 1$, $0 < p_1 \le 1$ and it is excluded that both $r_1 = 0$ and $p_1 = 1$ are satisfied at the same time. Let ϱ be the Perron–Frobenius root of $\mathbb{M}$. If $\varrho \le 1$, then $s_2 = s_3 = 1$. If $\varrho > 1$, then $s_2 < 1$ and $s_3 < 1$. In any case, (s_2, s_3) is the smallest non-negative solution of the vector equation

$$(s_2, s_3) = \left(f_2(s_2, s_3), f_3(s_2, s_3)\right),$$

where f_2 and f_3 are the generating functions of the offspring distributions of an edge, resp. a triangle.

Assume that our process is super-critical and α is the Malthusian parameter. Assume that $0 \le r_1 < 1$, $0 < p_1 \le 1$ and it is excluded that both $r_1 = 0$ and $p_1 = 1$ are satisfied at the same time. In the following results the quantities ${}_2W$ and ${}_3W$ are a.s. non-negative, $\mathbb{E}({}_2W) = \mathbb{E}({}_3W) = 1$, ${}_2W$ and ${}_3W$ are a.s. positive on the event of survival. $A(\alpha)$ and $B(\alpha)$ are given by the Laplace transforms, v and u denote the right and left eigenvectors of $\mathbb{M}$, and

$$D(\alpha) = \sum_{l,j=2}^{3} u_l v_j \left(-m^*_{l,j}(\alpha)\right)'.$$

6. Let ${}_iE(t)$ denote the number of all edges being born up to time t if the ancestor of the population was a type i object, $i = 2, 3$. Then

$$\lim_{t\to\infty} e^{-\alpha t} {}_iE(t) = {}_iW \frac{v_i u_2}{\alpha D(\alpha)}$$

almost surely for $i = 2, 3$.

7. Let ${}_i\hat{E}(t)$ denote the number of all edges present at time t if the ancestor of the population was a type i object, $i = 2, 3$. Then

$$\lim_{t\to\infty} e^{-\alpha t} {}_i\hat{E}(t) = {}_iW \frac{v_i u_2 A(\alpha)}{D(\alpha)}$$

almost surely for $i = 2, 3$.

8. Let ${}_iT(t)$ denote the number of all triangles being born up to time t if the ancestor of the population was a type i object, $i = 2, 3$. Then

$$\lim_{t\to\infty} e^{-\alpha t}{}_iT(t) = {}_iW \frac{v_i u_3}{\alpha D(\alpha)}$$

almost surely for $i = 2, 3$.

9. Let ${}_i\hat{T}(t)$ denote the number of all triangles present at time t if the ancestor of the population was a type i object, $i = 2, 3$. Then,

$$\lim_{t\to\infty} e^{-\alpha t}{}_i\hat{T}(t) = {}_iW \frac{v_i u_3 B(\alpha)}{D(\alpha)}$$

almost surely for $i = 2, 3$.

The fifth result reflects on the extinction of the edges and triangles. The further results describe the asymptotic behavior of the edges and triangles being born and being alive at time t. Similar results are applied on the degree of a fixed vertex. To give an empirical evidence of our theorems some simulation results were shown related to them.

Összefoglaló

A doktori értekezésben két új típusú hálózatfejlődési modell matematikai konstrukcióját ismertettük. A modellek dinamikai alapját a folytonos idejű elágazó folyamatok szolgáltatták, melyekben a fejlődésben résztvevő egységek a gráf alegységei, amik a mi esetünkben a különböző klikkek. A Bevezetésben felsoroltuk azokat az irodalmi előzményeket, amelyek a mi modellünk alapjául szolgáltak. A klasszikus diszkrét idejű eredményektől eljutottunk a folytonos idejű modellekig, majd felmértük a lehetséges alkalmazási területeket.

Az 1. Fejezet a [26, 27] cikkek alapján íródott. Ebben definiáltunk egy újfajta folytonos idejű gráffejlődési modellt, amelyben a 3-klikkek, azaz a háromszögek az alapegységek. Ezek mindegyike 3 egyed együttműködését jelenti. A kezdeti időpontban csak egyetlen háromszögünk, az ős az ami adott. Az ős képes új csúcsokat bevonzani a hálózatba, amelyek 0, 1, 2 vagy 3 éllel tudnak csatlakozni hozzá. A 2 és 3 éllel való csatlakozás esetén 1, illetve 3 új szaporodóképes háromszög születik, mindez q_1, illetve q_3 valószínűségekkel. A maradék két esetben az utódok nem lesznek szaporodóképesek, összesen q_0 valószínűséggel. Az utódháromszögek az ős háromszöghöz hasonlóan rendelkeznek a saját születési folyamataikkal. Egy szaporodóképes egyed átlagos születéseinek a számát $\mu(t)$ jelöli a t időpillanatban. A háromszögek haláluk, azaz a szaporodóképes fázisuk végét a $l(t) = b + c\xi(t)$ kockázati ráta határozza meg, ahol b, c nem-negatív konstansok, illetve $\xi(t)$ a t időpillanatig megszületett utódoknak a száma. A hálózat aszimptotikájára nézve az α Malthusi paraméter határozta meg mind a háromszögek $Z(t)$ számának, a csúcsok $V(t)$ számának és az élek $W(t)$ számának a növekményét.

A matematikai konstrukció megadása után a legfőbb eredményeink a következőek voltak:

Legyen $\mu(\infty) > 1$.

1. A háromszögek kihalásának a valószínűsége a legkisebb nem-negatív megoldása a következő egyenletnek:

$$1 = \frac{q_1 + q_3\left(y^2 + y + 1\right)}{c} \int_0^1 (1-u)^{\frac{1+b-q_0}{c}-1} e^{\left(\frac{q_1 y + q_3 y^3}{c} u - \frac{q_3 y^3}{c} u^2 + \frac{q_3 y^3}{3c} u^3\right)} du.$$

2. Legyen α a Malthusi paraméter. Ekkor

$$\lim_{t\to\infty} e^{-\alpha t} Z\left(t\right) = Y_\infty m_\infty$$

majdnem biztosan és L^1-ben, ahol az Y_∞ valószínűségi változó nem-negatív és pozitív a nem-kihalás eseménye felett, 1 várható értékű. Továbbá,

$$m_\infty = \frac{1}{\left(q_1 + 3q_3\right)^2 \int_0^\infty t e^{-\alpha t}\left(1 - L\left(t\right)\right) dt}.$$

3. $e^{-\alpha t} V(t)$ majdnem biztosan konvergál és

$$\frac{V\left(t\right)}{Z\left(t\right)} \to \frac{1}{\alpha}$$

$t \to \infty$ esetén majdnem biztosan a nem-kihalás eseménye felett.

4. $e^{-\alpha t} W(t)$ majdnem biztosan konvergál és

$$\frac{W\left(t\right)}{Z\left(t\right)} \to \frac{\mathbb{E}\gamma_1}{\alpha}$$

$t \to \infty$ esetén majdnem biztosan a nem-kihalás eseménye felett.

A eredményeink feltételezik, hogy a gráffejlődési modellünk szuperkritikus, azaz azt a nem-triviális esetet vizsgálja, amikor a folyamat kihalásának valószínűsége kisebb mint 1. Ekkor a paraméterektől függően megadtunk egy numerikusan kezelhető alakot a kihalás valószínűségére. A további eredmények rendre a háromszögek, a csúcsok és az élek aszimptotikus viselkedésére adnak eredményt. Hasonló eredmények megadhatóak egy adott csúcs fokszámára. A tételek empirikus szemléltetéséhez néhány hozzájuk kapcsolódó szimulációs eredményt prezentáltunk.

A 2. Fejezet a [28, 29] cikkek alapján íródott, amelyekben az előző fejezetben prezentált modellt általánosítottuk két típusra. Az előzővel ellentétben itt már nem csak a háromszögek képesek szaporodni, hanem az élek is, és mindkét típusú objektum képes a másikat is produkálni. Egy él minden egyes reprodukciós időpontjában egy új csúcs csatlakozik az élhez 1 vagy 2 éllel. Egy háromszög egy reprodukciós időpontjában egy új csúcs csatlakozik a háromszöghöz 1, 2 vagy 3 éllel. Ezek alapján egy él r_1 valószínűséggel él, r_2 valószínűséggel pedig háromszög utódot tud szülni, míg egy háromszög p_1 valószínűséggel élt, p_2 valószínűséggel egy háromszöget és p_3 valószínűséggel három háromszöget képes szülni. A kockázati ráta az utódok számának lineáris transzformáltja b és c nem-negatív konstansokkal, ugyanúgy mint az előző modellünkben. Legyen $m_{ij}(t)$ az i típusú ős j típusú utódai átlagos száma a t ideig. Legyen $\mathbb{M}$ ezek $m_{ij}^*(t)$ Laplace-transzformáltjainak mátrixa.

5. Jelölje a kihalás valószínűségét s_2, ha az ős egy él, és s_3, ha az ős egy háromszög. Tegyük fel, hogy $0 \leq r_1 < 1$, $0 < p_1 \leq 1$ és a $r_1 = 0$, $p_1 = 1$ feltételek közül legfeljebb az egyik teljesül. Legyen ϱ a Perron–Frobenius gyöke $\mathbb{M}$-nek. Ha $\varrho \leq 1$, akkor $s_2 = s_3 = 1$. Ha $\varrho > 1$, akkor $s_2 < 1$ és $s_3 < 1$. Bármely esetben (s_2, s_3) a legkisebb nem-negatív megoldása az alábbi vektor-egyenletnek:

$$(s_2, s_3) = (f_2(s_2, s_3), f_3(s_2, s_3)),$$

ahol f_2 és f_3 az élek, illetve a háromszögek utódeloszlásainak generátorfüggvénye.

Tegyük fel, hogy a folyamat szuperkritikus és α a Malthusi paraméter. Tegyük fel, hogy $0 \leq r_1 < 1$, $0 < p_1 \leq 1$ és a $r_1 = 0$, $p_1 = 1$ feltételek közül legfeljebb az egyik teljesül. A következő eredményekben ${}_2W$ és ${}_3W$ mennyiségek m.m. nem-negatívak, $\mathbb{E}({}_2W) = \mathbb{E}({}_3W) = 1$, ${}_2W$ és ${}_3W$ m.m. pozitívak a túlélés eseménye mellett. $A(\alpha)$ és $B(\alpha)$ a Laplace-transzformáltak által meghatározottak, v és u jelölik $\mathbb{M}$ jobb és bal oldali sajátvektorait, és

$$D(\alpha) = \sum_{l,j=2}^{3} u_l v_j \left(-m_{l,j}^*(\alpha)\right)'.$$

6. Jelölje ${}_iE(t)$ a t ideig megszületett élek számát, azon esetben mikor az ős i-típusú, $i = 2, 3$. Ekkor

$$\lim_{t\to\infty} e^{-\alpha t}{}_iE(t) = {}_iW \frac{v_i u_2}{\alpha D(\alpha)}$$

majdnem biztosan $i = 2, 3$ esetén.

7. Jelölje ${}_i\hat{E}(t)$ a t időpillanatban életben lévő élek számát, azon esetben mikor az ős i-típusú, $i = 2, 3$. Ekkor

$$\lim_{t\to\infty} e^{-\alpha t}{}_i\hat{E}(t) = {}_iW \frac{v_i u_2 A(\alpha)}{D(\alpha)}$$

majdnem biztosan $i = 2, 3$ esetén.

8. Jelölje ${}_iT(t)$ a t ideig megszületett háromszögek számát, azon esetben mikor az ős i-típusú, $i = 2, 3$. Ekkor
$$\lim_{t\to\infty} e^{-\alpha t}{}_iT(t) = {}_iW \frac{v_i u_3}{\alpha D(\alpha)}$$
majdnem biztosan $i = 2, 3$ esetén.

9. Jelölje ${}_i\hat{T}(t)$ a t időpillanatban életben lévő háromszögek számát, azon esetben mikor az ős i-típusú, $i = 2, 3$.
$$\lim_{t\to\infty} e^{-\alpha t}{}_i\hat{T}(t) = {}_iW \frac{v_i u_3 B(\alpha)}{D(\alpha)}$$
majdnem biztosan $i = 2, 3$ esetén.

Az ötödik eredmény az élek, illetve háromszögek kihalásának valószínűségét adja meg. A további eredmények az élek és háromszögek aszimptotikus viselkedését írják le az összesen megszületett egyedszám, illetve az éppen életben lévő egyedek számára tekintettel. Hasonló eredmények megadhatóak egy adott csúcs fokszámára. A tételek empirikus vizsgálatához néhány hozzájuk kapcsolódó szimulációs eredményt mutattunk.

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Appendices

In this thesis, we use known results of the theory of continuous-time branching processes. The single type general Crump–Mode–Jagers branching processes have been described e.g., in [16, 31, 17]. The general multi-type branching processes have been studied, e.g., in [30, 32, 20]. Here we present some of these results.

12 Appendix A

Consider the following general Crump-Mode-Jagers process. This process is determined by the reproduction process $\xi(t)$, $t \geq 0$, and the life-time distribution $L(t) = \mathbb{P}(\lambda \leq t)$. The random point process $\xi(t)$ is determined by the birth events and the numbers of offspring whilst the life-time λ is a non-negative random variable which is not necessarily independent from the reproduction. Let us denote by $\tau_1, \tau_2, \ldots$ the time points of the birth events and by $\varepsilon_1, \varepsilon_2, \ldots$ the corresponding litter sizes. Then the reproduction point process of the generic individual is $\xi(t) = \sum_{\tau_i \leq t} \varepsilon_i$ giving the number of offspring up to time t. The process starts at time $t = 0$ with one individual called the ancestor. When a child is born, then it starts its own reproduction process, and so on. The birth time of the individual e is denoted by σ_e.

Let us denote by $\mu(t)$ the expected reproduction which can be described using the reproduction function by $\mu(t) = \mathbb{E}\xi(t)$.

Let $\Phi(t)$ be a random function which describes a certain aspect of the life history of the individual. It is usually assumed that $\Phi(t) = 0$ for $t \leq 0$. Then $\Phi(t)$ is called a random characteristic. The behaviour of the individual e is described by $(\xi_e, \lambda_e, \Phi_e)$. These triplets are independent copies of the generic triplet (ξ, λ, Φ). Let us define the branching process $Z^{\Phi}(t)$ counted by the characteristic Φ as

$$Z^{\Phi}(t) = \sum\nolimits_e \Phi_e(t - \sigma_e),$$

where we summarize for all individuals e.

The following facts are well-known, see [16], [31] or [17]. We assume the following basic conditions.

(a) μ as a measure is not concentrated on any lattice.

(b) There exists a positive Malthusian parameter α, that is, a finite positive solution of the equation

$$\int_0^\infty e^{-\alpha t}\mu\,(dt) = 1.$$

(c) The first moment of $e^{-\alpha t}\mu\,(dt)$ is finite, that is,

$$\int_0^\infty te^{-\alpha t}\mu\,(dt) < \infty.$$

Let

$$_\alpha\xi(\infty) = \int_0^\infty e^{-\alpha t}\xi\,(dt)\,. \tag{12.1}$$

Proposition 12.1. *Let α be the Malthusian parameter. Assume that the random characteristic Φ satisfies the following conditions:*

(i) $\Phi\,(t) \geq 0$,

(ii) the trajectories of Φ belong to the Skorohod space D, i.e. they do not have discontinuities of the second kind,

(iii) $\mathbb{E}\,(\sup\Phi\,(t)) < \infty$.

Assume also

(iv) for some $\varepsilon > 0$

$$\int_0^\infty t(\log^+ t)^{1+\varepsilon}e^{-\alpha t}\mu\,(dt) < \infty,$$

(v) suppose that

$$\mathbb{E}\left[{}_\alpha\xi(\infty)\log^+{}_\alpha\xi(\infty)\right] < \infty. \tag{12.2}$$

Then

$$\lim_{t\to\infty} e^{-\alpha t}Z^\Phi\,(t) = Y_\infty m_\infty^\Phi \tag{12.3}$$

almost surely and in L^1, where

$$m_\infty^\Phi = \frac{\int_0^\infty e^{-\alpha t}\mathbb{E}\Phi\,(t)\,dt}{\int_0^\infty te^{-\alpha t}\mu\,(dt)},$$

Y_∞ is an a.s. non-negative random variable, which is a.s. positive on the event of non-extinction, $\mathbb{E}Y_\infty = 1$, and it does not depend on the choice of Φ.

In particular, for the number $Z(t)$ of individuals alive at time t we have

$$\lim_{t\to\infty} e^{-\alpha t} Z(t) = Y_\infty \frac{\int_0^\infty e^{-\alpha t}(1 - L(t))\, dt}{\int_0^\infty t e^{-\alpha t} \mu(dt)} \tag{12.4}$$

almost surely and in L^1, where $L(t) = \mathbb{P}(\lambda \le t)$ is the distribution function of the life length.

Concerning the proof we just remark, that Theorem 5.4 of [17] implies the almost sure convergence because Condition 5.1 of [17] follows from our condition (iv) and Condition 5.2 of [17] follows from our condition (iii). Because of Corollary 3.3 of [17], our condition (v) implies convergence in L^1 and that $\mathbb{E}Y_\infty = 1$. Corollary 2.5 of [17] shows that Y_∞ is an a.s. non-negative random variable. To obtain that Y_∞ is a.s. positive on the event of non-extinction, we can apply Proposition 1.1 of [17] and the fact that the event of non-extinction is a.s. the same as the event $\{Z(t) \to \infty\}$. This later fact follows from Theorem (6.5.2) of [16] which can be applied because we assume the existence of a positive Malthusian parameter.

13 Appendix B

Here, we give a short description of the general multi-type branching processes based on [20]. The individuals of this process can be of p different types, which we denote by $1, 2, \dots, p$. Any individual x is described by the quantities λ_x, ξ_x, Φ_x, $\Psi_x, \dots$. The quantities $\lambda_x, \xi_x, \Phi_x, \Psi_x, \dots$ are independent copies of the quantities $\lambda, \xi, \Phi, \Psi, \dots$. Thus, we should give the definition of $\lambda, \xi, \Phi, \Psi, \dots$, which we consider as the quantities corresponding to the generic individual.

The lifetime λ is a non-negative random variable which is not necessarily independent from the reproduction. The lifetime distribution is $L(t) = \mathbb{P}(\lambda \leq t)$. The reproduction process is $\xi_i(t) = (\xi_{i,1}(t), \dots, \xi_{i,p}(t))$, $t \geq 0$. Here, the random point process $\xi_{i,j}$ describes the births of type j offspring of a type i mother. $\xi_{i,j}(t)$ gives the number of type j offspring of a type i mother up to time t. $\xi_{i,j}$ is determined by the birth events and the numbers of offspring. The process starts at time $t = 0$ with one individual called the ancestor and denoted by x_0. When a child is born, it starts its own reproduction process and so on. The birth time of the individual x is denoted by σ_x.

Let $\Phi(t)$ be a non-negative random function that describes a certain aspect of the life history of the individual. It is usually assumed that $\Phi(t) = 0$ for $t \leq 0$. Then, $\Phi(t)$ is called a random characteristic. Let $\Psi(t)$ be another random characteristic. Thus, the behaviour of the individual x is described by $\xi_x, \lambda_x, \Phi_x, \Psi_x, \dots$.

Let us define the branching process ${}_{x_0}Z^{\Phi}(t)$ counted by the characteristic Φ as

$$ {}_{x_0}Z^{\Phi}(t) = \sum_x \Phi_x\left(t - {}_{x_0}\sigma_x\right), $$

where we summarize for all individuals x. Here, the left subscript x_0 of Z and of the birth time σ_x is important, because it denotes that the process starts with ancestor x_0 and the type of x_0 has influence for the evolution of the population.

Let us denote by $m_{i,j}(t)$ the reproduction function, which is the expected reproduction number $m_{i,j}(t) = \mathbb{E}\xi_{i,j}(t)$.

The following facts are well-known (see [20] or [32]).

We assume the following basic conditions in this section.

(a) Not all of the measures $m_{i,j}$ are concentrated on a lattice.

Let

$$ m^*_{i,j}(\kappa) = \int_0^\infty e^{-\kappa t} m_{i,j}(dt), \qquad i, j = 1, \dots, p, $$

be the Laplace transform of $m_{i,j}$. Let $M(\kappa)$ be the matrix

$$M(\kappa) = \left(m^*_{i,j}(\kappa)\right)_{i,j=1}^{p}.$$

(b1) There exists a positive Malthusian parameter α that is a finite positive value so that $M(\alpha)$ has finite entries only, and the Perron–Frobenius root of $M(\alpha)$ is equal to 1. Here, the Perron–Frobenius root is the largest eigenvalue of the matrix. Let $(v_1,\ldots,v_p)^\top$ be the right positive eigenvector and $(u_1,\ldots,u_p)^\top$ the left positive eigenvector of $M(\alpha)$ corresponding to the Perron–Frobenius root. We normalize them as $\sum_{i=1}^p v_i = 1$ and $\sum_{i=1}^p u_i v_i = 1$.

(b2) The matrix $(m_{i,j}(\infty))_{i,j=1}^p$ has an infinite entry, or all of them are finite, and its Perron–Frobenius root is greater than 1.

(c) The first moment of $e^{-\alpha t} m_{i,j}\,(dt)$ is finite and positive; that is,

$$0 < \int_0^\infty t e^{-\alpha t} m_{i,j}\,(dt) < \infty, \qquad i,j = 1,\ldots,p.$$

(d) There exists a finite positive integer K so that all elements of the Kth power of the matrix $(m_{i,j}\,(\infty))_{i,j=1}^p$ are positive.

Let

$$_{\alpha}\xi_{i,j}(\infty) = \int_0^\infty e^{-\alpha t} \xi_{i,j}\,(dt)\,. \tag{13.5}$$

Proposition 13.1. *Let α be the Malthusian parameter. Assume that the random characteristic Φ satisfies the following conditions:*

(i) $\Phi\,(t) \geq 0$,

(ii) The trajectories of Φ belong to the Skorohod space D, i.e., they do not have discontinuities of the second kind,

(iii) $\mathbb{E}\,(\sup_t \Phi\,(t)) < \infty$.

Assume also

(iv) for some $\varepsilon > 0$

$$\int_0^\infty t(\log(1+t))^{1+\varepsilon} e^{-\alpha t} m_{i,j}\,(dt) < \infty, \qquad i,j = 1,\ldots,p$$

and

(v) for some $\varepsilon > 0$

$$\mathbb{E} \sup_{t \geq 0} \left\{ \max \left\{ t(\log(1+t))^{1+\varepsilon}, 1 \right\} e^{-\alpha t} \Phi(t) \right\} < \infty$$

for any ancestor.

Then,

$$\lim_{t \to \infty} e^{-\alpha t} {}_{x_0}Z^{\Phi}(t) = {}_{x_0}Y_{\infty} v_i m_{\infty}^{\Phi} \tag{13.6}$$

is likely, where i *is the type of* x_0*,*

$$m_{\infty}^{\Phi} = \frac{\sum_{j=1}^{p} u_j \int_0^{\infty} e^{-\alpha t} \mathbb{E} \Phi_j(t)\, dt}{\sum_{l,j=1}^{p} u_l v_j \int_0^{\infty} t e^{-\alpha t} m_{l,j}(dt)}, \tag{13.7}$$

${}_{x_0}Y_{\infty}$ *is an a.s. non-negative random variable depending on the type of the ancestor* x_0 *but not depending on the choice of* Φ*.*

If, in addition, we assume that

(vi)

$$\mathbb{E} \left[{}_{\alpha}\xi_{i,j}(\infty) \log^{+} {}_{\alpha}\xi_{i,j}(\infty) \right] < \infty, \qquad i, j = 1, \ldots, p, \tag{13.8}$$

then $\mathbb{E}\left({}_{x_0}Y_{\infty}\right) = 1$*,* ${}_{x_0}Y_{\infty}$ *is positive with positive probability, and* ${}_{x_0}Y_{\infty}$ *is a.s. positive on the survival set.*

The proof is a simple consequence of Theorem 2.4 and Proposition 4.1 of [20].

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List of publications related to the dissertation

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