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Convergence of Cesàro means with variable parameters of Walsh-Fourier series

Dissertation for the Degree of Doctor of Philosophy (PhD)

Anas Ahmad Mohammad Abu Joudeh

Supervisor: Prof. Dr. Gát György Tamás

UNIVERSITY OF DEBRECEN
Doctoral Council of Natural Sciences and Information Technology
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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences/Informatics at Debrecen University.
The results published in the thesis are not reported in any other PhD theses.

Debrecen, December 1, 2020.

. . . .
Anas Ahmad Abu Joudeh
signature of the candidate

Hereby I confirm that Anas Ahmad Mohammad Abu Joudeh candidate conducted his studies with my supervision within the Mathematics Doctoral Program of the Doctoral School of Natural Sciences and Information Technology between 2015 and 2020 The independent studies and research work of the candidate significantly contributed to the results published in the thesis.
I also declare that the results published in the thesis are not reported in any other theses.
I support the acceptance of the thesis.

Gát György Tamás

Debrecen, December 1, 2020.

. . . .
Gát György Tamás
signature of the supervisor

Convergence of Cesàro means with variable parameters of Walsh-Fourier series

Dissertation submitted in partial fulfilment of the requirements
for the doctoral (PhD) degree in Mathematics.

Written by: Anas Ahmad Mohammad Abu Joudeh certified Mathematician

The thesis was written in the framework of the Mathematical Analysis, Functional analysis program of the Doctoral School of Mathematical and computational Sciences of the University of Debrecen.

Dissertation advisor: Prof. Dr. Gát György Tamás.

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The official opponents of the dissertation:

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Finally, I would like to specifically thank my family My parents, Brothers and sisters, for their help and encouragement.

Contents

Introduction

Jean Baptiste Joseph Fourier began to work on the theory of heat and how it flows have.in 1822, he published book with tittle of Théorie Analytic de la Chaleur (The Analytic Theory of Heat).

A great deal of effort has been expended after that work, one of the most influential forms of mathematical ideas has been developed and developed, including Fourier theory and the so-called field of harmonic analysis. Since then, this subject has gained exceptional importance in both theoretical content and its enormous scope and great relevance everywhere in mathematics, science and engineering.Where an increasing number of mathematicians have adopted the point of view that the most appropriate setting for the development of the theory of Fourier analysis is furnished by the class of all locally compact groups.

On the theoretical side, Fourier series theory has gained a leading force in developing and improving mathematical analysis and studying the functions of real variables. Where one can also argue that set theory, inclusive the construction of the real numbers and the ideas notification of cardinality and accountability, it was developed because of Fourier theory. In the application segment, all signal processing processes today are based on Fourier's theory. Everything in mobile technology, including the principle and method of storing and transmitting images, depends on the Fourier series theory. In 1926 Kolmogoroff [5] gave the construction of an integrable function with everywhere divergent trigonometric Fourier series. That is, if we want to have some pointwise convergence result for each function belonging to the Lebesgue space L^1 then it is needed to use some summation method. The invention of Fejér [11] was to use the arithmetical means of the partial sums. Among others, he proved for continuous functions that these means converge to the function in the supreumum norm. One year later, Lebesgue proved the almost everywhere convergence of these so-called Fejér means to the function for each integrable function. That is, the behavior of the Fejér (or also called $(C, 1)$) means is better than the behavior of the partial sums in this point of view. This fact also justifies the investigation of various summation methods of Fourier series. Later on, we write about the (C, α) summation - which is a generalization of the Fejér summation - of Fourier series. The result of Lebesgue above for the (C, α) case ($\alpha > 0$) is due to M. Riesz [33]. For example, the dyadic group is the simplest but nontrivial model of the complete product of finite groups. Representing the characters of the dyadic group ordered in the Paley's sense, we obtain the Walsh system.

A new things of the generalization on the Walsh-Paley system is the Vilenkin system introduced by Vilenkin [37] in 1947. He used the set of all characters of the complete product of arbitrary cyclic groups to obtain the commutative case.

In Hungary a dyadic analysis team works leaded by Schipp having several results in this theory. For instance, they proved that the Paley theorem is true for an arbitrary Vilenkin group, i.e. the partial sums of the Vilenkin-Fourier series of a function in $L^p(G)$ $(1 < p < 1)$ converge in the appropriate norm to the function (Schipp [29], Simon [34]). And so from Canada Young [41].

The example above is not true for all cases if we take the complete product of arbitrary finite group (not necessarily commutative). These studies were appeared in [14] by Gát and Toledo first and they obtained not only negative results for this groups, because they also proved the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case.

This thesis comes to study in the Dyadic harmonic analysis. Moreover, I would also like to mention paper [18], in which Gát and Toledo discussed the norm convergence of Fejér means of integrable functions on noncommutative bounded Vilenkin groups. This area, which have roots lie in the physics of vibration, uses integration to decompose (integrable) functions into piecewise constant components by generating numbers (called Walsh-Fourier coefficients) and infinite series (Walsh-Fourier series). These numbers and series can be used to approximate and to characterize the original function. We are particularly interested and in problems of the convergence of Cesàro means (under varying parameters and two variable Walsh-Fourier series) and growth (how fast the partial sums or the Cesàro means of a Walsh- Fourier series grow?). Specific results can be obtained.

Dyadic Harmonic analysis has many applications. Using Walsh-Fourier series to approximate a given function makes it possible to transmit data efficiently (e.g. multiplexing), to filter data (e.g. remove noise from weak video signals), and for data compression (e.g. transmit hundreds of signals through a single fiber optic cable). Using Walsh-Fourier coefficients to characterize functions makes it possible to recognize patterns (e.g. read handwritten zipcodes). Walsh functions have also been used to design genetic algorithms, methods to optimize non-differentiable problems for which the standard approach via calculus will not work.

The thesis is organized as follows:

In *Chapter one*, we follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [32]) and others. The notion of the Hardy space $H(I)$ is introduced in the following way [32]. Set the definition of the nth ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$, the Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels, respectively and so for the Fejér or $(C, 1)$ means of f. The kernel of the (C, α_n) summability method will simple be called (C, α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$. Finally, we give an introduction to the two-dimensional Fourier coefficients, the rectangular partial sums of the two-dimensional Fourier series, the rectangular Dirichlet kernels and the (C, α_n) Cesàro-Marcinkiewicz means of integrable function f for two variables.

In *Chapter two*, a new result about almost everywhere convergence of Cesàro means with varying parameters of Walsh-Fourier series is given we prove the almost everywhere convergence of a subsequnce of the Cesàro (C, α_n) means of integrable functions $\sigma_{2^n}^{\alpha_{2^n}} f \to f$ for $f \in L^1(I)$, where I is the unit interval for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$.

In *Chapter three*, a new result about almost everywhere convergence of Cesàro means with varying parameters of Walsh-Fourier series is given We prove the almost everywhere convergence of the the Cesàro (C, α_n) means of integrable functions $\sigma_n^{\alpha_n} f \to f$, where $\mathbb{N}_{\alpha,K} \ni n \to \infty$ for $f \in L^1(I)$, where I is the unit interval for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$.

In *Chapter four*, i talk about a new result of almost everywhere convergence of Cesàro means of two variable Walsh-Fourier series with varying parameteres. We prove that the maximal operator of some (C, β_n) means of cubical partial sums of two variable Walsh-Fourier series of integrable functions is of weak type (L_1, L_1). Everywhere write in the dissertation L^1 or L_1 if you wish, but please only one them should occur. Moreover, the (C, β_n)-means $\sigma_{2^n}^{\beta_n} f$ of the function $f \in L_1$ converge a.e. to f for $f \in L^1(I^2)$, where I is the unit interval for some sequences $1 > \beta_n \searrow 0$.

It should finally be noted that most of the results obtained in this thesis have been published (or accepted for publication) in a series of articles: [6],[7]. This dissertation is based on the results of a two recently published papers in peer reviewed journals. It is worth mentioning that our paper entitled convergence of Cesàro means with varying parameters of Walsh-Fourier series which was published in Miskolc Mathematical Notes Journal has been cited twice by other researchers in the field. For instance, G. Gát and U. Goginava have cited it in their article entitled Maximal operators of Cesàro means with varying parameters of Walsh–Fourier series [20] and F. Weisz have cited it in their article entitled Cesàro and Riesz summability with varying parameters of multi-dimensional Walsh–Fourier series [40]. We expect this work to receive more attention by other researchers in the future.

Finally, the author would like to thank for all Professors who worked and made every efforts for this valuable ideas and Professor Dr. György Gát for this work and for his several advices and remarks to improve this work.

Chapter 1

Preliminaries

In this chapter, We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [32]) and others. The notion of the Hardy space $H(I)$ is introduced in the following way [32]. Set the definition of the nth ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$. the Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels, respectively and so for the Fejér or $(C, 1)$ means of f. the kernel of the summability method (C, α_n) and call it the (C, α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$. Finally, an introduction to the two-dimensional Fourier coefficients, the rectangular partial sums of the two-dimensional Fourier series, the rectangular Dirichlet kernels and the (C, α_a) Cesàro-Marcinkiewicz means of integrable function f for two variables.

1.1 The Standard Notions Of Dyadic Analysis

In this section, we follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [32]) and others. Denote by $\mathbb{N} := \{0, 1, ...\}, \mathbb{P} := \mathbb{N} \setminus \{0\}$, the set of natural numbers, the set of positive integers and $I := [0, 1)$ the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set $B(B \subset I)$.

Denote by $L^p(I)$ the usual Lebesgue spaces and $\|.\|_p$ the corresponding norms $(1 \leq p \leq \infty)$. Set

$$\mathcal{J} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\}$$

the set of dyadic intervals and for given $x \in I$ and let $I_n(x)$ denote the interval $I_n(x) \in \mathcal{J}$ of length 2^{-n} which contains $x\,(n \in \mathbb{N})$. Also use the notation $I_n := I_n(0)\,(n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number $(x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\})$ we choose the expansion which terminates in 0's.

The notion of the Hardy space $H(I)$ is introduced in the following way [32]. A function $a \in L^{\infty}(I)$ is called an atom, if either $a = 1$ or a has the following properties: *supp* $a \subseteq I_a$, $\|a\|_{\infty} \leq |I_a|^{-1}$, $\int_I a = 0$, for some $I_a \in \mathcal{J}$. We say that the function f belongs to H, if f can be represented as $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where a_i's are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$.

Definition 1.1. The nth ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$ is:

$$\omega_n(x) := \prod_{j=0}^{\infty} (-1)^{x_j n_j},$$

where $\mathbb{N} \ni n = \sum_{n=0}^{\infty} n_j 2^j\,(n_j \in \{0,1\}\,(j \in \mathbb{N}))$. It is known (see [23] or [36]) that for the elements of the system $(\omega_n, n \in \mathbb{N})$ we have the almost everywhere equality

$$\omega_n(x+y) = \omega_n(x)\omega_n(y),$$

where the operation $+$ is the so-called logical addition on I. That is, for any $x, y \in I$

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$

Definition 1.2. The Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels

Denote by

$$\hat{f}(n) := \int_I f \omega_n d\lambda, \quad D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n^1 := \frac{1}{n+1} \sum_{k=0}^{n} D_k.$$

Definition 1.3. The Fejér or $(C, 1)$ means of f

It is also known that the Fejér or $(C, 1)$ means of f is

$$\sigma_n^1 f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n^1(y + x) d\lambda(x)$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} \int_I f(x) D_k(y + x) d\lambda(x), \quad (n \in \mathbb{N}, y \in I).$$

It is known [32] that for $n \in \mathbb{N}, x \in I$ it holds

$$D_{2^n}(x) = \begin{cases} 2^n \text{ , if } x \in I_n \\ 0 \text{ , if } x \notin I_n \end{cases}$$

and also that

$$D_n(x) = \omega_n(x) \sum_{k=1}^{\infty} D_{2^k}(x) n_k (-1)^{x_k},$$

where $n = \sum_{i=1}^{\infty} n_i 2^i$, $n_i = \{0, 1\}$ $(i \in \mathbb{N})$.

Definition 1.4. The (C,α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1,-2,\dots\}$

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C,α_n) and call it the (C,α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1,-2,\dots\}$

$$K_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} D_k,$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)...(\alpha_n+n)}{k!}.$$

It is known [44] that $A_n^{\alpha_n} = \sum_{k=0}^{n} A_k^{\alpha_n - 1}, \;\; A_k^{\alpha_n} - A_{k+1}^{\alpha_n} = -\frac{\alpha_n A_k^{\alpha_n}}{k+1}$.

Definition 1.5. Cesàro means of integrable function f

The (C,α_n) Cesàro means of integrable function f is

$$\sigma_n^{\alpha_n} f(y) := \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} S_k f(y) = \int_I f(x) K_n^{\alpha_n}(y+x) d\lambda(x).$$

Definition 1.6. The two-dimensional Fourier coefficients

Now, for the two variable case we have for $x = \left(x^1, x^2\right), y = \left(y^1, y^2\right) \in I^2, \;\; n = (n_1, n_2) \in \mathbb{N}^2$ the two-dimensional Fourier coefficients

$$\hat{f}(n_1, n_2) := \int\limits_{I \times I} f(x^1, x^2) \omega_{n_1}(x^1) \omega_{n_2}(x^2) d\lambda(x^1, x^2).$$

Definition 1.7. Rectangular partial sums of the two-dimensional Fourier series

The rectangular partial sums of the two-dimensional Fourier are

$$S_{n_1,n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \omega_{k_1}(y^1) \omega_{k_2}(y^2).$$

Definition 1.8. Rectangular Dirichlet kernels.

The rectangular Dirichlet kernels are

$$D_{n_1,n_2}(z) := D_{n_1}(z^1)D_{n_2}(z^2) = \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \omega_{k_1}(z^1)\omega_{k_2}(z^2),$$

where $(z = (z^1, z^2) \in I^2)$.

Definition 1.9. Marcinkiewicz mean and kernel.

We have the n^{th} Marcinkiewicz mean and kernel

$$\sigma_n^1 f(y) := \frac{1}{n+1}\sum_{k=0}^{n} S_{j,j}f(y), \quad K_n^1(z) = \frac{1}{n+1}\sum_{j=0}^{n} D_{j,j}(z).$$

Thus, we get

$$\sigma_n^1 f(y^1, y^2) = \int_{I\times I} f(x^1, x^2)K_n^1(y^1 + x^1, y^2 + x^2)d\lambda(x^1, x^2).$$

Definition 1.10. The (C, α_n) kernel or the Cesàro-Marcinkiewicz kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C, α_n)-Marcinkiewicz and call it the (C, α_a) kernel or the Cesàro-Marcinkiewicz kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

$$K_n^{\alpha_n}(x_1, x_2) = \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1}D_{j,j}(x_1, x_2)$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n + 1)(\alpha_n + 2)...(\alpha_n + k)}{k!}.$$

Definition 1.11. The (C, α_n) Cesàro-Marcinkiewicz means of integrable function f for two variables

The (C, α_n) Cesàro-Marcinkiewicz means of integrable function f for two variables are

$$\sigma_n^{\alpha_n} f(y^1, y^2) = \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} S_{k,k} f(y^1, y^2)(x)$$

$$= \int_{I \times I} f(x^1, x^2) K_n^{\alpha_n}(y^1 + x^1, y^2 + x^2) d\lambda(x^1, x^2).$$

$$= \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} \int_{I \times I} A_{n-k}^{\alpha_n - 1} f(x^1, x^2) D_k(y^1 + x^1) D_k(y^2 + x^2) d\lambda(x^1, x^2).$$

Over all of the chapter discussing the generalized Marcinkiewicz-Cesàro means we suppose that monotone decreasing sequences (α_n) and (β_n) satisfy

$$\beta_n = \alpha_{2^n}, \quad \frac{\alpha_N}{A_N^{\alpha_N}} \log^{\delta}\left(1 + \frac{N}{n}\right) \leq C \frac{\alpha_n}{A_n^{\alpha_n}} \ (N \geq n, n, N \in \mathbb{P}) \tag{1.1}$$

for some $\delta > 1$ and for some positive constant C. We remark that from condition (1.1) it follows that sequence $(\frac{\alpha_n}{A_n^{\alpha_n}})$ is quasi monotone decreasing. That is, for some $C > 0$ we have $\frac{\alpha_N}{A_N^{\alpha_N}} \leq C \frac{\alpha_n}{A_n^{\alpha_n}}$ $(N \geq n, n, N \in \mathbb{P})$.

Chapter 2

ALMOST EVERYWHERE CONVERGENCE OF CESÀRO MEANS WITH VARYING PARAMETERS (C, α_{2^n})

2.1 Cesàro means of Fourier series with variable parameters (C, α_{2^n})

In this chapter, we introduced the notion of Cesàro means of Fourier series with variable parameters. We prove the almost everywhere convergence of a subsequnce of the Cesàro (C, α_n) means of integrable functions $\sigma_{2^n}^{\alpha_{2^n}} f \to f$ for $f \in L^1(I)$, for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. This theorem for the case of $\alpha \equiv 1$ and for the whole sequence $\sigma_n^{\alpha_n}$ was proved by Fine [9]. For the case of the $(C, 1)$ or Fejér means there are several generalizations known with respect to some orthonormal systems. One could mention the papers [13], [16], [45].

In [9] Fine proved the almost everywhere convergence $\sigma_n^{\alpha_n} f \longrightarrow f$ for all integrable function f with constant sequence $\alpha_n = \alpha > 0$. With respect to it was a question of Taibleson [36] open for a long time, that does the Fejér-Lebesgue theorem, that is the a.e. convergence $\sigma_n^1 f \longrightarrow f$ hold for all integrable function f with respect to the character system of the group of 2-adic integers. In 1997 Gát answered [16] this question in the affirmative. Zheng and Gát generalized this result [13], [45] for more general orthonormal systems. Thus, in the future these system could also be investigated in the point of view of varying parameter summability. In this chapter C denotes an absolute constant which may depend only on α. The introduction of (C, α_n) means due to Akhobadze investigated [1] the L^1-norm convergence of $\sigma_n^{\alpha_n} f \to f$ for the trigonometric system. In this chapter it is also supposed that $1 > \alpha_n > 0$ for all n.

The main aim of this chapter is to prove:

Theorem 2.1. *(Abu Joudeh and Gát [6]) Suppose that* $1 > \alpha_n > 0$. *Let* $f \in L^1(I)$. *Then we have the a.e convergence* $\sigma_{2^n}^{\alpha_{2n}} f \longrightarrow f$.

The method we use to prove Theorem 2.1 is to investigate the maximal operator $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_{2^n}^{\alpha_{2n}} f|$. We also prove that this operator is of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

Theorem 2.2. *(Abu Joudeh and Gát [6]) Suppose that* $1 > \alpha_n > 0$. *Let* $f \in H(I)$. *Then we have*

$$\|\sigma_*^\alpha f\|_1 \leq C\|f\|_H.$$

Moreover, the operator σ_*^α *is of type* (L^p, L^p) *for all* $1 < p \leq \infty$. *That is,*

$$\|\sigma_*^\alpha f\|_p \leq C_p\|f\|_p \text{ for all } 1 < p \leq \infty.$$

Basically, in order to prove Theorem 2.1 we verify that the maximal operator $\sigma_*^\alpha f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we get this, the investigation of kernel functions, and its maximal function on the unit interval I by making a hole around zero. To have the proof of Theorem 2.2 is the standard way after having the fact that $\sigma_*^\alpha f$ is of weak type (L^1, L^1). We need several Lemmas in the next section.

2.2 Proofs

Lemma 2.3. *([24]) For $j, n \in \mathbb{N}, j < 2^n$ we have*

$$D_{2^n-j}(x) = D_{2^n}(x) - \omega_{2^n-1}(x) D_j(x).$$

Lemma 2.3 can be found in [24] in a more genearal situation and so it is not a new one. In spite of this fact, in order to help to understand the behavior of the Walsh functions I decided to put here the proof of it.

Proof.

$$\begin{aligned} D_{2^n}(x) = \sum_{k=0}^{2^n-1} \omega_k(x) &= \sum_{k=0}^{2^n-j-1} \omega_k(x) + \sum_{k=2^n-j}^{2^n-1} \omega_k(x) \\ &= D_{2^n-j} + \sum_{k=2^n-j}^{2^n-1} \omega_k(x). \end{aligned}$$

We have to prove :

$$\sum_{k=2^n-j}^{2^n-1} \omega_k(x) = \omega_{2^n-1}(x) D_j(x).$$

For $k < j, k = k_{n-1}2^{n-1} + ... + k_1 2^1 + k_0$ we have

$$\begin{aligned} &\omega_{2^n-1}(x)\omega_k \\ &= \omega_{2^{n-1}+...+2^1+2^0}(x)\omega_{k_{n-1}2^{n-1}+...+k_0}(x) \\ &= \omega_{(1+k_{n-1}(mod\, 2))2^{n-1}+...+(1+k_0(mod\, 2))2^0}(x) \\ &= \omega_{(1-k_{n-1})2^{n-1}+...+(1-k_0)2^0}(x) \\ &= \omega_{2^{n-1}+2^{n-2}+...+2^0-(k_{n-1}2^{n-1}+...+k_0)}(x) = \omega_{2^n-1-k}(x). \end{aligned}$$

Thus,

$$\omega_{2^n-1}(x) D_j(x) = \omega_{2^n-1}(x) \sum_{k=0}^{j-1} \omega_k(x) = \sum_{k=0}^{j-1} \omega_{2^n-1-k}(x) = \sum_{k=2^n-j}^{2^n-1} \omega_k(x).$$

This completes the proof of Lemma 2.3. □

Introduce the following notations: for $n, j \in \mathbb{N}$ let $n_{(j)} := \sum_{i=0}^{j-1} n_i 2^i$, that is, $n_{(0)} = 0$, $n_{(1)} = n_0$ and for $2^B \leq n < 2^{B+1}$, let $|n| := B$, $n = n_{(B+1)}$. Moreover, introduce the following functions and operators for $n, a \in \mathbb{N}$ and $1 > \alpha_a > 0$

$$T_n^{\alpha_a} := \frac{1}{A_n^{\alpha_a}} \sum_{j=0}^{2^{|n|}-1} A_{n-j}^{\alpha_a - 1} D_j,$$

$$t_n^{\alpha_a} f(y) := \int_I f(x) T_n^{\alpha_a}(y+x) d\lambda(x).$$

Now, we need to prove the next Lemma which means that maximal operator $\sup_{n,a} |t_n^{\alpha_a}|$ is quasi-local. In this chapter it would have been possible to define operator $t_n^{\alpha_a}(n, a \in \mathbb{N})$ only for $n = a$, that is, $t_n^{\alpha_n}$. Because the main aim of this chapter is to discuss the behavior of $\sigma_n^{\alpha_n}$ where n is a power of two. But, in chapter 3 it will be needed to have Lemma 2.4. or more precisely method of its proof. That is, a result for operator $t_n^{\alpha_a}$, where both a and n are natural numbers but not necessarily the same.

Lemma 2.4. *(Abu Joudeh and Gát [6]) Let* $1 > \alpha_a > 0$, $f \in L^1(I)$ *such that supp* $f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ *for some dyadic interval* $I_k(u)$ $(a, k \in \mathbb{N}, u \in I)$*. Then we have*

$$\int_{I \setminus I_k(u)} \sup_{n,a \in \mathbb{N}} |t_n^{\alpha_a} f| d\lambda \leq C \|f\|_1.$$

.

Proof. It is easy to have that for $n < 2^k$ and $x \in I_k(u)$ we have $T_n^{\alpha_a}(y+x) = T_n^{\alpha_a}(y+u)$ and

$$\int_{I_k(u)} f(x) T_n^{\alpha_a}(y+x) d\lambda(x) = T_n^{\alpha_a}(y+u) \int_{I_k(u)} f(x) d\lambda(x) = 0.$$

Therefore,

$$\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} |t_n^{\alpha_a} f| d\lambda = \int_{I \setminus I_k(u)} \sup_{n \geq 2^k} |t_n^{\alpha_a} f| d\lambda.$$

Recall that $B = |n|$. Then

$$\begin{aligned} & A_n^{\alpha_a} T_n^{\alpha_a} \\ &= \sum_{j=0}^{2^B} A_{2^B+n_{(B)}-j}^{\alpha_a-1} D_j \\ &= \sum_{j=0}^{2^B} A_{n_{(B)}+j}^{\alpha_a-1} D_{2^B-j} \end{aligned}$$

By Lemma 2.3 we have

$$\begin{aligned} & A_n^{\alpha_a} T_n^{\alpha_a} \\ &= D_{2^B} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} - \omega_{2^B-1} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j . \end{aligned}$$

It is easy to have that $\frac{1}{A_n^{\alpha_a}} D_{2^B}(z) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} = 0$, for any $z \in I \backslash I_k$. This holds because $D_{2^B}(z) = 0$ for $B = |n| \geq k$ and $z \in I \setminus I_k$. By the help of the Abel transform we get:

$$\begin{aligned} & \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j \\ &= \sum_{j=0}^{2^B-1} (A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1}) \sum_{i=0}^{j} D_i + A_{n_{(B)}+2^B}^{\alpha_a-1} \sum_{i=0}^{2^B-1} D_i \\ &= (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1 \\ &= (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 \\ &+ (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1 \\ &=: I + II + III. \end{aligned}$$

Since for any $j < 2^k$ we have that the Fejér kernel $K_j^1(y+x)$ depends (with respect to x) only on coordinates $x_0, \ldots, x_{k-1}$, then

$$\int_{I_k(u)} f(x) K_j^1(y+x) d\lambda(x) = K_j^1(y+u) \int_{I_k(u)} f(x) d\lambda(x) = 0$$

gives $\int_{I_k(u)} f(x) I(y+x) d\lambda(x) = 0$. On the other hand,

$$\begin{aligned} |II| &\leq (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} |K_j^1| \\ &\leq \sup_{j \geq 2^k} |K_j^1| (1-\alpha_a) \sum_{j=0}^{n} A_j^{\alpha_a-1} = A_n^{\alpha_a}(1-\alpha_a) \sup_{j \geq 2^k} |K_j^1|. \end{aligned}$$

This by Lemma 3 in [13] gives

$$\int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} |II| d\lambda \leq \int_{I \setminus I_k} \sup_{j \geq 2^k} |K_j^1| d\lambda \leq C.$$

The situation with III is similar. Namely,

$$\frac{A_n^{\alpha_a-1} n}{A_n^{\alpha_a}} = \frac{\alpha_a \cdot n}{(\alpha_a + n)} \leq \alpha_a < 1.$$

So, just as in the case of II we apply Lemma 3 in [13]

$$\int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} |III| d\lambda \leq \int_{I \setminus I_k} \sup_{n \geq 2^k} |K_{2^{|n|}-1}^1| d\lambda \leq C.$$

Therefore, substituting $z = x + y \in I \setminus I_k$ (since $x \in I_k(u)$ and $y \in I \setminus I_k(u)$)

$$\begin{aligned} &\int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} |t_n^{\alpha_a} f| d\lambda \\ &= \int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \left| \int_{I_k(u)} f(x) T_n^{\alpha_a}(y+x) d\lambda(x) \right| d\lambda(y) \\ &\leq \int_{I \setminus I_k(u)} \int_{I_k(u)} |f(x)| \sup_{n \geq 2^k, a \in \mathbb{N}} \left(|II(y+x)| + |III(y+x)| \right) d\lambda(x) \\ &= \int_{I_k(u)} |f(x)| \int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \left(|II(z)| + |III(z)| \right) d\lambda(z) d\lambda(x) \\ &\leq C \int_{I_k(u)} |f(x)| d\lambda(x). \end{aligned}$$

So, just as in the case of II we apply Lemma 3 in [13]

$$\begin{aligned}
&\int_{I\setminus I_k} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} III d\lambda \\
&\leq \int_{I\setminus I_k} \sup_{n\geq 2^k, a\in\mathbb{N}} |K^1_{2^{|n|}-1}| d\lambda \\
&\leq C.
\end{aligned}$$

Therefore, substituting $z = x + y \in I \setminus I_k$ (since $x \in I_k(u)$ and $y \in I \setminus I_k(u)$)

This completes the proof of Lemma 2.4. □

A straightforward corollary of this lemma is:

Corollary 2.5. *(Abu Joudeh and Gát [6]) Let $1 > \alpha_n > 0$. Then we have $\|T_n^{\alpha_n}\|_1 \leq C$ for all natural number n, where C is some absolute constant.*

Proof. The proof is a straightforward consequence of some steps of the proof of Lemma 2.4. Let $B = |n|$.

$$\begin{aligned}
&\|A_n^{\alpha_n} T_n^{\alpha_n}\|_1 \leq \|D_{2^B}\|_1 \sum_{j=0}^{2^B-1} A^{\alpha_n-1}_{n_{(B)}+j} \\
&+ (1-\alpha_n) \sum_{j=0}^{2^B-1} A^{\alpha_n-1}_{n_{(B)}+j} \frac{j+1}{n_{(B)}+j+1} \|K^1_j\|_1 + A_n^{\alpha_n-1} 2^B \|K^1_{2^B-1}\|_1.
\end{aligned}$$

Then by $\|D_{2^B}\| = 1$, $\|K^1_j\|_1 \leq C$ we complete the proof of Corollary 2.5. □

If n is a power of two, say $n = 2^B$ then it is easy to see that $t_n^{\alpha_n} f = \sigma_{2^B}^{\alpha_{2^B}} f$. Also recall that $\sigma_*^\alpha f := \sup_B |t_{2^B}^{\alpha_{2^B}} f| = \sup_B |\sigma_{2^B}^{\alpha_{2^B}} f|$. That is, $\sigma_*^\alpha f \leq \sup_n |t_n^{\alpha_n} f|$.

Lemma 2.6. *(Abu Joudeh and Gát [6]) The operator σ_*^α is of type (L^∞, L^∞).*

Proof. if $f \in L^\infty$ we need to prove $\|t_*^\alpha f\|_\infty \leq C\|f\|_\infty$. By Corollary 2.5 we have

$$\begin{aligned}
\|\sigma_*^\alpha f\|_\infty &= ess \sup_{y\in I} (\sup_n | \int_I f(x) T_n^{\alpha_n}(y+x) d\lambda x|) \\
&= ess \sup_{y\in I} (\sup_n | \int_I f(y+x) T_n^{\alpha_n}(x) d\lambda x|) \\
&\le ess \sup_{y\in I} (\sup_n \int_I \|f\|_\infty |T_n^{\alpha_n}(x) d\lambda x|) \\
&\le ess \sup_{y\in I} (\sup_n \int_I \|f\|_\infty |T_n^{\alpha_n}(x) d\lambda x|) \\
&= \|f\|_\infty ess \sup_{y\in I} (\sup_n \int_I |T_n^{\alpha_n}(x)| d\lambda x) \le C \|f\|_\infty .
\end{aligned}$$

Hence σ_*^α is of type (L^∞, L^∞) type. This completes the proof of Lemma 2.6.

□

Now, we can prove the main tool in order to have Theorem 2.1.

Lemma 2.7. *(Abu Joudeh and Gát [6]) Let* $1 > \alpha_n > 0$. *The operator* σ_*^α *is of weak type* (L^1, L^1) *(*$\sigma_*^\alpha f := \sup_n |\sigma_{2^n}^{\alpha_{2^n}} f|$ *).*

Proof. We apply the Calderon-Zygmund decomposition lemma [45]. That is, let $f \in L^1$ and $\|f\|_1 < \lambda$ then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that

$$\|f_0\|_\infty \le C\lambda, \; \|f_0\|_1 \le C\|f\|_1$$

and $I^j = I_{k_j}(u^j)$ are disjoint dyadic intervals for which

$$\text{supp } f_j \subset I^j \; , \; \int_{I^j} f_j d\lambda = 0 \; , \; |F| \le \frac{C\|f_1\|}{\lambda}$$

$(u^j \in I \; , \; k_j \in \mathbb{N} \; , \; j \in P)$, where $F = \cup_{i=1}^{\infty} I^j$. By the σ-sublinearity of the maximal operator we have

$$\mu(\sigma_*^\alpha f > 2C\lambda) \le \mu(\sigma_*^\alpha f_0 > C\lambda) + \mu(\sigma_*^\alpha (\sum_{i=1}^{\infty} f_i) > C\lambda) := I + II \, .$$

Since $\|\sigma_*^\alpha f_0\|_\infty \le \|f_0\|_\infty \le C\lambda$ then we have $I = 0$. So,

$$\mu(\sigma_*^\alpha(\sum_{i=1}^\infty f_i) > C\lambda) \le |F| + \mu(\bar{F} \cap \{\sigma_*^\alpha(\sum_{i=1}^\infty f_i) > C\lambda\})$$

$$\le \frac{C\|f\|_1}{\lambda} + \frac{C}{\lambda}\sum_{i=1}^\infty \int_{\bar{I}^j} \sigma_*^\alpha f_j d\lambda =: \frac{C\|f\|_1}{\lambda} + \frac{C}{\lambda}\sum_{i=1}^\infty III_j$$

$$III_j := \int_{\bar{I}_j} \sigma_*^\alpha f_j d\lambda \le \int_{\bar{I}_{k_j}(u^j)} \sup_{n\in\mathbb{N}} \left| \int_{I_{k_j}(u^j)} f_j(x) T_n^{\alpha_n}(y+x) d\lambda(x) \right| d\lambda(y).$$

We investigate III_j by the help Lemma 2.4 : $III_j \le C\|f_j\|_1$. This completes the proof of Lemma 2.7. □

Proof of Theorem 2.1. (Abu Joudeh and Gát [6]) let P be a Walsh polynomial, where $P(x) = \sum_{i=0}^{2^k-1} c_i\omega_i$. Then for all natural number $n \ge 2^k$ we have that $S_n P \equiv P$. Consequently, the relation $\sigma_{2^n}^{\alpha_{2^n}} P \longrightarrow P$ holds everywhere.

Now Let $\epsilon, \delta > 0$, $f \in L^1$ Let P be a polynomial such that $\|f - P\|_{L^1} < \delta$ Then

$$\lambda(\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > \epsilon)$$

$$\le \lambda(\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}}(f-P)| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} P - P| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_n}|P-f| > \frac{\epsilon}{3})$$

$$\le \lambda(\sup_n |\sigma_{2^n}^{\alpha_{2^n}}(f-P)| > \frac{\epsilon}{3}) + 0 + \frac{3}{\epsilon}\|P-f\|_1 \le C\|P-f\|_1 \frac{3}{\epsilon} \le \frac{C}{\epsilon}\delta.$$

Because σ_*^α is of weak type (L^1, L^1). So for all $\delta > 0$ and consequently for arbitrary $\epsilon > 0$ we have

$$\lambda(\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > \epsilon) = 0.$$

By the set inclusion

$$\{\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > 0\} \subset \bigcup_{k=1}^\infty \{\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > \frac{1}{k}\}$$

and by the fact that the union of each member on the right side is a 0 measure set we have that the left side is also a 0 measure set. Thus,

$$\mu\{\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| > 0\} = 0$$

$$\overline{\lim_n}|\sigma_{2^n}^{\alpha_{2^n}} f - f| = 0 \quad a.e$$

$$\lim_n |\sigma_{2^n}^{\alpha_{2^n}} f - f| = 0 \quad a.e$$

$$\lim_n (\sigma_{2^n}^{\alpha_{2^n}} f - f) = 0 \quad a.e$$

$$\sigma_{2^n}^{\alpha_{2^n}} f \longrightarrow f \quad a.e$$

That is, the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. (Abu Joudeh and Gát [6]) Lemma 2.6. and Lemma 2.7 by the interpolation theorem of Marcinkiewicz [45] gives that the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. Let a be an atom ($a \neq 1$ can be supposed), supp $a \subset I_k(x)$, $\int_I a d\lambda = 0$ and $\|a\|_\infty \leq 2^k$ for some $k \in \mathbb{N}$ and $x \in I$. Then , $n < 2^k$ implies $S_n a = 0$ because $\int_{I_k(x)} a(t)d\lambda(t) = 0$ That is,

$$\sigma_*^\alpha a = \sup_{2^n \geq 2^k} |\sigma_{2^n}^{\alpha_{2^n}} f|.$$

By the help Lemma 2.4 it gives

$$\int_{I\setminus I_k(x)} \sigma_*^\alpha a \ d\lambda \leq \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \left| \int_{I_k(x)} a(y) T_n^{\alpha_n}(z+y) d\lambda(y) \right| d\lambda(z)$$
$$\leq C\|a\|_1 \leq C.$$

Since the operator σ_*^α is of type (L^2, L^2) (i.e $\|\sigma_*^\alpha f\|_2 \leq C\|f\|_2$ for all $f \in L^2(I)$), then we have

$$\|\sigma_*^\alpha a\|_1 = \int_{I\setminus I_a} \sigma_*^\alpha a + \int_{I_k(x)} \sigma_*^\alpha a$$
$$\leq C + |I_k(x)|^{\frac{1}{2}} \|\sigma_*^\alpha a\|_2 \leq C + C2^{\frac{-k}{2}} \|a\|_2 \leq C + C2^{\frac{-k}{2}} 2^{\frac{k}{2}} \leq C.$$

That is, $\|\sigma_*^\alpha a\|_1 \leq C$ and consequently the σ-sublinearity of σ_*^α gives

$$\|\sigma_*^\alpha f\|_1 \leq \sum_{i=0}^{\infty} |\lambda_i| \|\sigma_*^\alpha a_i\|_1 \leq C \sum_{i=0}^{\infty} |\lambda_i| \leq C\|f\|_H$$

for all $\sum_{i=0}^{\infty} \lambda_i a_i \in H$. That is , the operator σ_*^α is of type (H, L). That is, the proof of Theorem 2.2 is complete. □

Chapter 3

ALMOST EVERYWHERE CONVERGENCE OF CESÀRO MEANS WITH VARYING PARAMETERS (C, α_n)

3.1 Cesàro means of Fourier series with variable parameters (C, α_n).

In this chapter, we introduced the notion of Cesàro means of Fourier series with variable parameters. We prove the almost everywhere convergence of the Cesàro (C, α_n) means of integrable functions $\sigma_n^{\alpha_n} f \to f$, where $\mathbb{N}_{\alpha,K} \ni n \to \infty$ for $f \in L^1(I)$, where I is the unit interval for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. This theorem for constant sequences α that is, $\alpha \equiv \alpha_1$ was proved by Fine [9].

In [9] Fine proved the almost everywhere convergence $\sigma_n^{\alpha_n} f \longrightarrow f$ for all integrable function f with constant sequence $\alpha_n = \alpha_1 > 0$. On the rate of convergence of Cesàro means in this constant case see the paper of Fridli [10]. For the two-dimensional situation see the paper of Goginava [19].

Set two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$. For instance $P(n, 1) = n$. Also set for sequences $\alpha = (\alpha_n)$ and positive reals K the subset of natural numbers

$$\mathbb{N}_{\alpha,K} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.$$

We can easily remark that for a sequence α such that $1 > \alpha_n \geq \alpha_0 > 0$ we have $\mathbb{N}_{\alpha,K} = \mathbb{N}$ for some K depending only on α_0. We also remark that $2^n \in \mathbb{N}_{\alpha,K}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $K \geq 1$.

In this chapter C denotes an absolute constant and C_K another one which may depend only on K. The introduction of (C, α_n) means due to Akhobadze investigated [1] the behavior of the L^1-norm convergence of $\sigma_n^{\alpha_n} f \to f$ for the trigonometric system. In this chapter it is also supposed that $1 > \alpha_n > 0$ for all n.

The main aim of this chapter is to prove:

Theorem 3.1. *(Abu Joudeh and Gát [6]) Suppose that* $1 > \alpha_n > 0$. *Let* $f \in L^1(I)$. *Then we have the almost everywhere convergence* $\sigma_n^{\alpha_n} f \to f$ *provided that* $\mathbb{N}_{\alpha,K} \ni n \to \infty$. *for any fixed* $K > 0$

The method we use to prove Theorem 3.1 is to investigate the maximal operator $\sigma_*^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f|$. We also prove that this operator is a kind of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

Theorem 3.2. *(Abu Joudeh and Gát [6]) Suppose that* $1 > \alpha_n > 0$. *Let* $|f| \in H(I)$. *Then we have*

$$\|\sigma_*^{\alpha} f\|_1 \leq C_K \||f|\|_H.$$

Moreover, the operator σ_*^{α} *is of type* (L^p, L^p) *for all* $1 < p \leq \infty$. *That is,*

$$\|\sigma_*^{\alpha} f\|_p \leq C_{K,p} \|f\|_p$$

for all $1 < p \leq \infty$.

For the sequence $\alpha_n = 1$ Theorem 3.2 is due to Fujii [12]. For the more general but constant sequence $\alpha_n = \alpha_1$ see Weisz [38].

Basically, in order to prove Theorem 3.1 we verify that the maximal operator $\sigma_*^{\alpha} f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we get this, the investigation of kernel functions, and its maximal function on the unit interval I by making a hole around zero and some quasi locality issues (for the notion of quasi-locality see [32]). To have the proof of Theorem 3.2 is the standard way. We need several Lemmas in the next section.

3.2 Proofs

Recall **Lemma 2.3.** [24] That is,

$$For\ j,n\in\mathbb{N},\ j<2^n\ we\ have\ D_{2^n-j}(x)=D_{2^n}(x)-\omega_{2^n-1}(x)D_j(x).$$

Introduce the following notations: for $a,n,j\in\mathbb{N}$ let $n_{(j)}:=\sum_{i=0}^{j-1}n_i2^i$, that is, $n_{(0)}=0$, $n_{(1)}=n_0$ and for $2^B\leq n<2^{B+1}$, let $|n|:=B$, $n=n_{(B+1)}$. Moreover, introduce (and recall from Chapter 2) the following functions and operators for $n,a\in\mathbb{N}$ and $1>\alpha_a>0$

$$T_n^{\alpha_a}:=\frac{1}{A_n^{\alpha_a}}\sum_{j=0}^{2^{|n|}-1}A_{n-j}^{\alpha_a-1}D_j,$$

$$\tilde{T}_n^{\alpha_a}:=\frac{1}{A_n^{\alpha_a}}D_{2^B}\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}+(1-\alpha_a)\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}\frac{j+1}{n_{(B)}+j+1}\left|K_j^1\right|$$

$$+A_n^{\alpha_a-1}2^B\left|K_{2^B-1}^1\right|,$$

$$t_n^{\alpha_a}f(y):=\int_I f(x)T_n^{\alpha_a}(y+x)d\lambda(x),$$

$$\tilde{t}_n^{\alpha_a}f(y):=\int_I f(x)\tilde{T}_n^{\alpha_a}(y+x)d\lambda(x).$$

Now, we need to prove the next Lemma which means that maximal operator $\sup_{n,a}|\tilde{t}_n^{\alpha_a}|$ is quasi-local. This lemma together with the next one are the most important tools in the proof of the main results of this chapter.

Lemma 3.3. *(Abu Joudeh and Gát [6]) Let* $1>\alpha_a>0$, $f\in L^1(I)$ *such that supp* $f\subset I_k(u)$, $\int_{I_k(u)}fd\lambda=0$ *for some dyadic interval* $I_k(u)$*. Then we have*

$$\int_{I\setminus I_k(u)}\sup_{n,a\in\mathbb{N}}|\tilde{t}_n^{\alpha_a}f|d\lambda\leq C\|f\|_1.$$

Moreover, $|T_n^{\alpha_a}|\leq\tilde{T}_n^{\alpha_a}$.

Proof. It is easy to have that for $n<2^k$ and $x\in I_k(u)$ we have $\tilde{T}_n^{\alpha_a}(y+x)=\tilde{T}_n^{\alpha_a}(y+u)$ and

$$\int_{I_k(u)}f(x)\tilde{T}_n^{\alpha_a}(y+x)d\lambda(x)=\tilde{T}_n^{\alpha_a}(y+u)\int_{I_k(u)}f(x)d\lambda(x)=0.$$

Therefore,

$$\int_{I \setminus I_k(u)} \sup_{n,a \in \mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda = \int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda.$$

Recall that $B = |n|$. Then

$$A_n^{\alpha_a} T_n^{\alpha_a} = \sum_{j=0}^{2^B-1} A_{2^B+n_{(B)}-j}^{\alpha_a-1} D_j = \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{2^B-j}$$

By Lemma 2.3 and also on page 15, in the proof of Lemma 2.4 we have

$$A_n^{\alpha_a} T_n^{\alpha_a} = D_{2^B} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} - \omega_{2^B-1} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j.$$

It is easy to have that $\frac{1}{A_n^{\alpha_a}} D_{2^B}(z) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} = 0$, for any $z \in I \setminus I_k$. This holds because $D_{2^B}(z) = 0$ for $B = |n| \geq k$ and $z \in I \setminus I_k$. By the help of the Abel transform and by the steps of the proof of Lemma 2.4 (on page 15) we get:

$$\begin{aligned}
&= \left| (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1 \right| \\
&= \Bigg| (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 \\
&+ (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1 + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1 \Bigg| \\
&\leq (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left| K_j^1 \right| \\
&+ (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left| K_j^1 \right| + A_n^{\alpha_a-1} 2^B \left| K_{2^B-1}^1 \right| \\
&=: I + II + III.
\end{aligned}$$

These equalities above immediately proves inequality $|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}$.

Since for any $j < 2^k$ we have that the Fejér kernel $K_j^1(y+x)$ depends (with respect to x) only on coordinates $x_0, \ldots, x_{k-1}$, then $\int_{I_k(u)} f(x)|K_j^1(y+x)|d\lambda(x) = |K_j^1(y+u)| \int_{I_k(u)} f(x)d\lambda(x) = 0$ gives $\int_{I_k(u)} f(x)I(y+x)d\lambda(x) = 0$.

On the other hand,

$$
\begin{aligned}
II &= (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} |K_j^1| \\
&\leq \sup_{j\geq 2^k} |K_j^1| (1-\alpha_a) \sum_{j=0}^{n} A_j^{\alpha_a-1} = A_n^{\alpha_a}(1-\alpha_a) \sup_{j\geq 2^k} |K_j^1|.
\end{aligned}
$$

This by Lemma 3 in [13] just as in the proof of Lemma 2.4 gives

$$
\int_{I\setminus I_k} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} II d\lambda \leq \int_{I\setminus I_k} \sup_{j\geq 2^k} |K_j^1| d\lambda \leq C.
$$

The situation with III is similar. Namely, reciting the proof of Lemma 2.4 again we have

$$
\frac{A_n^{\alpha_a-1} n}{A_n^{\alpha_a}} < 1.
$$

So, just as in the case of II recall the corresponding part of the proof of Lemma 2.4

$$
\int_{I\setminus I_k} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} III d\lambda \leq C.
$$

Therefore, substituting $z = x + y \in I \setminus I_k$ (since $x \in I_k(u)$ and $y \in I \setminus I_k(u)$)

$$
\begin{aligned}
&\int_{I\setminus I_k(u)} \sup_{n\geq 2^k, a\in\mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda \\
&= \int_{I\setminus I_k(u)} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y+x) d\lambda(x) \right| d\lambda(y) \\
&\leq \int_{I\setminus I_k(u)} \int_{I_k(u)} |f(x)| \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} \left(II(y+x) + III(y+x) \right) d\lambda(x) \\
&= \int_{I_k(u)} |f(x)| \int_{I\setminus I_k} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} \left(II(z) + III(z) \right) d\lambda(z) d\lambda(x) \\
&\leq C \int_{I_k(u)} |f(x)| d\lambda(x).
\end{aligned}
$$

This completes the proof of Lemma 3.3. □

A straightforward corollary of this lemma is:

Corollary 3.4. *(Abu Joudeh and Gát [6]) Let* $1 > \alpha_a > 0$. *Then we have* $\|T_n^{\alpha_a}\|_1 \leq \|\tilde{T}_n^{\alpha_a}\|_1 \leq C$, $\|t_n^{\alpha_a} f\|_1, \|\tilde{t}_n^{\alpha_a} f\|_1 \leq C\|f\|_1$ *and* $\|t_n^{\alpha_a} g\|_\infty, \|\tilde{t}_n^{\alpha_a} g\|_\infty \leq C\|g\|_\infty$ *for all natural numbers* a, n, *where* C *is some absolute constant and* $f \in L^1, g \in L^\infty$. *That is, operators* $\tilde{t}_n^{\alpha_a}, t_n^{\alpha_a}$ *are of type* (L^1, L^1) *and* (L^∞, L^∞) *(uniformly in* n*).*

Proof. The proof is a straightforward consequence of Lemma 3.3 and an easy estimation below. Let $B = |n|$. Then

$$\left\| A_n^{\alpha_a} \tilde{T}_n^{\alpha_a} \right\|_1 \leq \|D_{2^B}\|_1 \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a - 1}$$

$$+ (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \|K_j^1\|_1 + A_n^{\alpha_a-1} 2^B \|K_{2^B-1}^1\|_1.$$

Then by $\|D_{2^B}\|_1 = 1, \|K_j^1\|_1 \leq C$ we complete the proof of Corollary 3.4. □

Lemma 3.5. *(Abu Joudeh and Gát [6]) Let* n, N *be any natural numbers and* $0 < \alpha < 1$. *Then we have*

$$\frac{A_n^\alpha}{A_N^\alpha} \leq 2\left(\frac{n+1}{N}\right)^\alpha.$$

Proof. By definition we have

$$\frac{A_n^\alpha}{A_N^\alpha} =$$

$$\left(1 - \frac{\alpha}{n+1+\alpha}\right) \cdots \left(1 - \frac{\alpha}{N+\alpha}\right) \leq \left(1 - \frac{\alpha}{n+2}\right) \cdots \left(1 - \frac{\alpha}{N+1}\right).$$

It is well-known that

$$\left(1 - \frac{\alpha}{i(n+1)+1}\right) \cdots \left(1 - \frac{\alpha}{(i+1)(n+1)}\right)$$

$$\leq \left(1 - \frac{\alpha}{(i+1)(n+1)}\right)^{n+1} \leq \left(e^{-1}\right)^{\frac{\alpha}{i+1}}.$$

for every $n \in \mathbb{N}$. This gives

$$
\begin{aligned}
&\left(1-\frac{\alpha}{n+2}\right)\cdots\left(1-\frac{\alpha}{N+1}\right) \le \left(e^{-1}\right)^{\alpha\sum_{i=2}^{\left\lfloor \frac{N}{n+1}\right\rfloor}\frac{1}{i}} \\
&\le \left(e^{-1}\right)^{\alpha \log_e \left\lfloor \frac{N}{n+1}\right\rfloor -1+c} \\
&\le 2\left(e^{-1}\right)^{\alpha \log_e \left(\frac{N}{n+1}\right)} = 2\left(\frac{n+1}{N}\right)^{\alpha}.
\end{aligned}
$$

where $c \approx 0.5772$ is the Euler-Mascheroni constant. This completes the proof of Lemma 3.5. $\square$

Recall that the two variable function $P(n,\alpha) = \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$ and $K \in \mathbb{R}$ determines the set of natural numbers

$$
\mathbb{N}_{\alpha,K} = \left\{ n \in \mathbb{N} : \frac{P(n,\alpha_n)}{n^{\alpha_n}} \le K \right\}.
$$

Let $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \ge 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \cdots + 2^{h_0}$. This means $n = n^{(s)}$. Define the following kernel function and operators

$$
\tilde{K}_n^{\alpha_n} := \tilde{T}_{n^{(s)}}^{\alpha_n} + \sum_{l=0}^{s}\left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \tilde{T}_{n^{l-1}}^{\alpha_n} \right)
$$

and

$$
\tilde{\sigma}_n^{\alpha_n} f := f * \tilde{K}_n^{\alpha_n}, \quad \tilde{\sigma}_*^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha,K}} |f * \tilde{K}_n^{\alpha}|.
$$

In the sequel we prove that maximal operator $\tilde{\sigma}_*^{\alpha} f$ is quasi-local. This is the very base of the proof of the main results of this chapter. That is, Theorem 3.1 and Theorem 3.2.

Lemma 3.6. *(Abu Joudeh and Gát [6]) Let $1 > \alpha_n > 0$, $f \in L^1(I)$ such that supp $f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$. Then we have*

$$
\int_{I \setminus I_k(u)} \tilde{\sigma}_*^{\alpha} f d\lambda \le C_K \|f\|_1,
$$

where constant C_K can depend only on K.

Proof. Recall that $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \geq 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \cdots + 2^{h_0}$. this can be found a couple of lines above. Use also the notation

$$\begin{aligned}
&\tilde{K}^{\alpha_n}_{n^{(s)}} \\
&= \tilde{T}^{\alpha_n}_{n^{(s)}} + \sum_{l=0}^{s} \left(\frac{A^{\alpha_n}_{n^{(l-1)}}}{A^{\alpha_n}_{n^{(s)}}} D_{2^{h_l}} + \frac{A^{\alpha_n}_{n^{(l-1)}}}{A^{\alpha_n}_{n^{(s)}}} \tilde{T}^{\alpha_n}_{n^{l-1}} \right) \\
&=: G_1 + G_2 + G_3.
\end{aligned}$$

Since $n^{(l-1)} < 2^{h_{(l-1)}+1}$, then by Lemma 3.5 we have

$$\frac{A^{\alpha_n}_{n^{(l-1)}}}{A^{\alpha_n}_{n^{(s)}}} \leq 2 \left(\frac{n^{(l-1)} + 1}{n^{(s)}} \right)^{\alpha_n} \leq 2 \frac{2^{\alpha_n (h_{l-1}+1)}}{2^{\alpha_n h_s}} \leq C \frac{2^{h_{l-1}\alpha_n}}{n^{\alpha_n}}.$$

That is, by the above written we also have

$$\begin{aligned}
&\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x) G_2(y+x) d\lambda(x) \right| d\lambda(y) \\
&\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \sum_{l=0}^{s} \frac{2^{h_{l-1}\alpha_n}}{n^{\alpha_n}} \left| \int_{I_k(u)} f(x) D_{2^{h_l}}(y+x) d\lambda(x) \right| d\lambda(y) = 0
\end{aligned}$$

since $f * D_{2^h} = 0$ for $h \leq k$ because of the $\mathcal{A}_k$ measurability of D_{2^h} and $\int f = 0$. Besides, for $h > k$ $D_{2^h}(y+x) = 0$ $(y + x \notin I_k)$.

As a result of these estimations above and by the help of Lemma 3.3, that is the quasi-locality of operator $\tilde{t}^{\alpha}_{*} = \sup_{n,a \in \mathbb{N}} |\tilde{t}^{\alpha_a}_n|$ we conclude

$$\begin{aligned}
&\int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x)(G_1(y+x) + G_3(y+x)) d\lambda(x) \right| d\lambda(y) \\
&\leq C_K \int_{I \setminus I_k(u)} \sup_{n,a \in \mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}^{\alpha_a}_n (y+x) d\lambda(x) \right| d\lambda(y) \\
&\leq C_K \|f\|_1.
\end{aligned}$$

This completes the proof of Lemma 3.6. □

Lemma 3.7. *(Abu Joudeh and Gát [6]) The operator $\tilde{\sigma}^{\alpha}_{*}$ is of type (L^∞, L^∞)* $(\tilde{\sigma}^{\alpha}_{*} f := \sup_{n \in \mathbb{N}_{\alpha,K}} |\tilde{\sigma}^{\alpha_n}_n f|)$.

Proof. By the help of the method of Lemma 3.6 and by Corollary 3.4 we have

$$\begin{aligned}
&\left\|\tilde{K}_n^{\alpha_n}\right\|_1 = \left\|\tilde{K}_{n^{(s)}}^{\alpha_n}\right\|_1 \\
&\leq \left\|\tilde{T}_{n^{(s)}}^{\alpha_n}\right\|_1 + \sum_{l=0}^{s}\left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}}\|D_{2^{h_l}}\|_1 + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}}\|\tilde{T}_{n^{l-1}}^{\alpha_n}\|_1\right) \\
&\leq C + C\sum_{l=0}^{s}\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \leq C_K
\end{aligned}$$

because $n \in \mathbb{N}_{\alpha,K}$. Hence $\tilde{\sigma}_*^\alpha$ is of type (L^∞, L^∞) (with constant C_K). This completes the proof of Lemma 3.7. □

Now, we can prove the main tool in order to have Theorem 3.1 for operator $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f|$.

Lemma 3.8. *(Abu Joudeh and Gát [6]) The operators $\tilde{\sigma}_*^\alpha$ and σ_*^α are of weak type (L^1, L^1).*

Proof. The steps of the first part of the proof are similar to those int the quite proof of Lemma 2.7. First, we prove Lemma 3.8 for operator $\tilde{\sigma}_*^\alpha$. We apply the Calderon-Zygmund decomposition lemma [32]. That is, let $f \in L^1$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that $\|f_0\|_\infty \leq C\delta$, $\|f_0\|_1 \leq C\|f\|_1$ and $I^j = I_{k_j}(u^j)$ are disjoint dyadic intervals for which

$$f_j \subset I^j \ , \ \int_{I^j} f_j d\lambda = 0 \ , \ |F| \leq \frac{C\|f_1\|}{\delta}$$

$(u^j \in I \ , \ k_j \in N \ , \ j \in P)$, where $F = \cup_{i=1}^{\infty} I^j$. By the σ-sublinearity of the maximal operator with an appropriate constant C_K we have

$$\lambda(\tilde{\sigma}_*^\alpha f > 2C_K\delta) \leq \lambda(\tilde{\sigma}_*^\alpha f_0 > C_K\delta) + \lambda(\tilde{\sigma}_*^\alpha(\sum_{i=1}^{\infty} f_i) > C_K\delta) := I + II \, .$$

Since by Lemma 3.7 $\|\tilde{\sigma}_*^\alpha f_0\|_\infty \le C_K\|f_0\|_\infty \le C_K\delta$ then we have $I = 0$. So,

$$\lambda(\tilde{\sigma}_*^\alpha(\sum_{i=1}^\infty f_i) > C_K\delta) \le |F| + \lambda(\bar{F} \cap \{\tilde{\sigma}_*^\alpha(\sum_{i=1}^\infty f_i) > C_K\delta\})$$
$$\le \frac{C_K\|f\|_1}{\delta} + \frac{C_K}{\delta}\sum_{i=1}^\infty \int_{I\setminus I^j} \tilde{\sigma}_*^\alpha f_j d\lambda =: \frac{C_K\|f\|_1}{\delta} + \frac{C_K}{\delta}\sum_{i=1}^\infty III_j,$$

where

$$III_j :=$$
$$\int_{I\setminus I^j} \tilde{\sigma}_*^\alpha f_j d\lambda \le \int_{I\setminus I_{k_j}(u^j)} \sup_{n\in N_{\alpha,K}} \left| \int_{I_{k_j}(u^j)} f_j(x)\tilde{K}_n^{\alpha_n}(y+x)d\lambda(x) \right| d\lambda(y).$$

The forthcoming estimation of III_j is given by the help Lemma 3.6

$$III_j \le C_K\|f_j\|_1.$$

That is, operator $\tilde{\sigma}_*^\alpha$ is of weak type (L^1, L^1). Next, we prove the estimation

$$|K_n^{\alpha_n}| \le \tilde{K}_n^{\alpha_n}. \tag{3.1}$$

To prove (3.1) recall again that $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \ge 0$ are integers. Since $n = 2^{h_s} + n^{(s-1)}$, then we have

$$\sum_{j=2^{h_s}}^{2^{h_s}+n^{(s-1)}} A^{\alpha_n-1}_{n^{(s-1)}+2^{h_s}-j} D_j = \sum_{k=0}^{n^{(s-1)}} A^{\alpha_n-1}_{n^{(s-1)}-k} D_{2^{h_s}+k}$$
$$= D_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A^{\alpha_n-1}_{n^{(s-1)}-k} + \omega_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A^{\alpha_n-1}_{n^{(s-1)}-k} D_k$$
$$= D_{2^{h_s}} A^{\alpha_n}_{n^{(s-1)}} + \omega_{2^{h_s}} A^{\alpha_n}_{n^{(s-1)}} K^{\alpha_n}_{n^{(s-1)}}.$$

So, by the help of the equalities above we get

$$K^{\alpha_n}_{n^{(s)}} = T^{\alpha_n}_{n^{(s)}} + \frac{A^{\alpha_n}_{n^{(s-1)}}}{A^{\alpha_n}_{n^{(s)}}} \left(D_{2^{h_s}} + \omega_{2^{h_s}} K^{\alpha_n}_{n^{(s-1)}} \right).$$

Apply this last formula recursively and Lemma 3.3. (Note that $n^{(-1)} = 0, T_0^{\alpha_n} = K_0^{\alpha_n} = 0, A_0^{\alpha_n} = 1$.)

$$
\begin{aligned}
|K_n^{\alpha_n}| &= |K_{n^{(s)}}^{\alpha_n}| \le |T_{n^{(s)}}^{\alpha_n}| + \sum_{l=0}^{s} \left(\prod_{j=l}^{s} \frac{A_{n^{(j-1)}}^{\alpha_n}}{A_{n^{(j)}}^{\alpha_n}} D_{2^{h_l}} + \prod_{j=l}^{s} \frac{A_{n^{(j-1)}}^{\alpha_n}}{A_{n^{(j)}}^{\alpha_n}} |T_{n^{l-1}}^{\alpha_n}| \right) \\
&= |T_{n^{(s)}}^{\alpha_n}| + \sum_{l=0}^{s} \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} |T_{n^{l-1}}^{\alpha_n}| \right) \\
&\le \tilde{K}_{n^{(s)}}^{\alpha_n} = \tilde{K}_n^{\alpha_n}.
\end{aligned}
$$

This completes the proof of inequality (3.1). This inequality gives that the operator σ_*^α is also of weak type (L^1, L^1) since

$$
\lambda(\sigma_*^\alpha f > 2C_K\delta) \le \lambda(\tilde{\sigma}_*^\alpha |f| > 2C_K\delta) \le C_K \frac{\||f|\|_1}{\delta} = C_K \frac{\|f\|_1}{\delta}.
$$

This completes the proof of Lemma 3.8. □

Proof of Theorem 3.1. (Abu Joudeh and Gát [6]) The proof is quite similar to the proof of Theorem 2.1 and that is why afew steps are omitted. Let $P \in \mathbf{P}$ be a polynomial where $P(x) = \sum_{i=0}^{2^k-1} c_i\omega_i$. Then for all natural number $n \ge 2^k$, $n \in \mathbb{N}_{\alpha,K}$ we have that $S_nP \equiv P$. Consequently, the statement $\sigma_n^{\alpha_n} P \longrightarrow P$ holds everywhere (of course not only for restricted $n \in \mathbb{N}_{\alpha,K}$). Now, let $\epsilon, \delta > 0,\ f \in L^1$. Let $P \in \mathbf{P}$ be a polynomial such that $\|f-P\|_1 < \delta$. Then

$$
\begin{aligned}
&\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} f - f| > \epsilon) \\
&\le \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} (f-P)| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} P - P| > \frac{\epsilon}{3}) \\
&+ \lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |P - f| > \frac{\epsilon}{3}) \\
&\le C_K \|P-f\|_1 \frac{3}{\epsilon} \\
&\le \frac{C_K}{\epsilon}\delta
\end{aligned}
$$

because σ_*^α is of weak type (L^1, L^1) (with any fixed $K > 0$). This holds for all $\delta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$
\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}} |\sigma_n^{\alpha_n} f - f| > \epsilon) = 0
$$

and consequently we also have

$$\lambda(\overline{\lim_{n\in\mathbb{N}_{\alpha,K}}}|\sigma_n^{\alpha_n}f-f|>0)=0.$$

This finally gives

$$\sigma_n^{\alpha_n}f\longrightarrow f\quad a.e.\quad (n\in\mathbb{N}_{\alpha,K}).$$

This completes the proof of Theorem 3.1. $\square$

Proof of Theorem 3.2. (Abu Joudeh and Gát [6]) The proof of this theorem are similar to those in the proof of Theorem 2.2 and we skip some steps. Inequality (3.1), Lemma 3.7 and Lemma 3.8 by the interpolation theorem of Marcinkiewicz [32] give that the operator σ_*^α is of type (L^p, L^p) for all $1<p\le\infty$. In the sequel we prove that operator $\tilde{\sigma}_*^\alpha f=\sup_{n\in\mathbb{N}_{\alpha,K}}|f*\tilde{K}_n^\alpha|$ is of type (H, L).

Let a be an atom ($a\neq 1$ can be supposed), $a\subset I_k(x)$, $\|a\|_\infty\le 2^k$ for some $k\in N$ and $x\in I$. Then, $n<2^k$, $n\in\mathbb{N}_{\alpha,K}$ implies $a*\tilde{K}_n^\alpha=0$ because $\tilde{K}_n^\alpha$ is $\mathcal{A}_k$ measurable for $n<2^k$ and $\int_{I_k(x)}a(t)d\lambda(t)=0$. That is,

$$\tilde{\sigma}_*^\alpha a=\sup_{\mathbb{N}_{\alpha,K}\ni n\ge 2^k}|\tilde{\sigma}_n^{\alpha_n}f|.$$

By the help Lemma 3.6 we have

$$\begin{aligned}\int_{I\setminus I_k(x)}\tilde{\sigma}_*^\alpha a\ d\lambda&=\int_{I\setminus I_k(x)}\sup_{\mathbb{N}_{\alpha,K}\ni n\ge 2^k}\left|\int_{I_k(x)}a(y)\tilde{K}_n^{\alpha_n}(z+y)d\lambda(y)\right|d\lambda(z)\\&\le C_K\int_{I_k(x)}|a(y)|d\lambda(y)\le C_K\|a\|_1\le C_K.\end{aligned}$$

Since the operator $\tilde{\sigma}_*^\alpha$ is of type (L^2, L^2) (i.e $\|\tilde{\sigma}_*^\alpha f\|_2\le C_K\|f\|_2$ for all $f\in L^2(I)$), then we have

$$\|\tilde{\sigma}_*^\alpha a\|_1=\int_{I\setminus I_k(x)}\tilde{\sigma}_*^\alpha a+\int_{I_k(x)}\tilde{\sigma}_*^\alpha a\le C_K.$$

That is $\|\tilde{\sigma}_*^\alpha a\|_1\le C_K$ and consequently the σ-sublinearity of $\tilde{\sigma}_*^\alpha$ gives

$$\|\tilde{\sigma}_*^\alpha f\|_1\le\sum_{i=0}^\infty|\lambda_i|\|\tilde{\sigma}_*^\alpha a_i\|_1\le C_K\sum_{i=0}^\infty|\lambda_i|\le C_K\|f\|_H$$

for all $\sum_{i=0}^\infty\lambda_i a_i\in H$. That is, the operator $\tilde{\sigma}_*^\alpha$ is of type (H, L). This by inequality (3.1) and then by $\|\sigma_*^\alpha f\|_1\le\|\tilde{\sigma}_*^\alpha|f|\|_1\le C_K\||f|\|_H$ completes the proof of Theorem 3.2. $\square$

Chapter 4

ALMOST EVERYWHERE CONVERGENCE OF CESÀRO MEANS OF TWO VARIABLE (C, β_n)

4.1 Cesàro means of two variable Walsh-Fourier series (C, β_n)

In this chapter, we formulate and prove that the maximal operator of some (C, β_n) means of cubical partial sums of two variable Walsh-Fourier series of integrable functions is of weak type (L^1, L^1). Moreover, the (C, β_n)-means $\sigma_{2^n}^{\beta_n} f$ of the function $f \in L^1$ converge a.e. to f for $f \in L^1(I^2)$, where I is the unit interval for some sequences $1 > \beta_n \searrow 0$.

In 1939, for the two-dimensional trigonometric Fourier partial sums $S_{j,j} f$ Marcinkiewicz [26] proved that for all $f \in \mathrm{L} \log \mathrm{L}\ ([0, 2\pi]^2)$ the relation

$$\sigma_n^1 f = \frac{1}{n+1} \sum_{j=0}^{n} S_{j,j} f \to f$$

holds a.e. as $n \to \infty$. Zhizhiashvili [42] improved this result and showed that for $f \in L([0, 2\pi]^2)$ the (C, α) means

$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{j=0}^{n} A_{n-j}^{\alpha-1} S_{j,j} f$$

converge to f a.e. for any $\alpha > 0$. Dyachenko [8] proved this result for dimensions greater than 2. In papers [22],[39] by Goginava and Weisz one can find that the $(C, 1)$ means $\sigma_n^1 f$ of the double Walsh-Fourier series of a function $f \in L^1([0, 1]^2)$ converges to f a.e. Recently, Gát [13] proved this result with respect to two-dimensional bounded Vilenkin systems. The d-dimensional Walsh-Fourier case is discussed in [21].

For the one dimensional trigonometric system it can be found in Zygmund [44] (Vol. I, p.94) that the Cesàro means or $(C, \alpha)(\alpha > 0)$ means $\sigma_n^\alpha f$ of the Fourier series of a function $f \in L^1([-\pi, \pi])$ converge a.e. to f as $n \to \infty$. Moreover, it is known that the maximal operator of the (C, α) means $\sigma_*^\alpha := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha|$ is of weak type (L^1, L^1), i.e.

$$\sup_{\gamma>0} \gamma\lambda(\sigma_*^\alpha f > \gamma) \le C\|f\|_1 \qquad (f \in L^1([-\pi, \pi])).$$

This result can be found implicitly in Zygmund [44] (Vol. I, pp. 154-156).

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [27] and the introduction of these (C, α_n) means of Fourier series is due to Akhobadze ([3], [4]) who investigated the behavior of the L^1-norm convergence of $\sigma_n^{\alpha_n}(f) \to f$ for the trigonometric system.

The a. e. divergence of Cesàro means with varying parameters of Walsh-Fourier series was investigated by Tetunashvili [43].

In 2007 Akhobadze [1] (see also [2]) introduced the notion of Cesàro means of Fourier series with variable parameters for one-dimensional functions. In the recent paper [6] we proved the almost everywhere convergence of the the Cesàro (C, α_n) means of integrable functions $\sigma_n^{\alpha_n} f \to f$, where $\mathbb{N} \supset \mathbb{N}_{\alpha,K} \ni n \to \infty$ for $f \in L^1(I)$, where I is the unit interval for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. The main aim of this chapter is to investigate to two-dimensional version of this issue.

Now, for the two variable case we have for $x = \left(x^1, x^2\right),\ y = \left(y^1, y^2\right) \in I^2,\ n = (n_1, n_2) \in \mathbb{N}^2$ the two-dimensional Fourier coefficients

$$\hat{f}(n_1, n_2) := \int\limits_{I\times I} f(x^1, x^2)\omega_{n_1}(x^1)\omega_{n_2}(x^2)d\lambda(x^1, x^2),$$

the rectangular partial sums of the two-dimensional Fourier series

$$S_{n_1,n_2}f(y^1,y^2) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\hat{f}(k_1,k_2)\omega_{k_1}(y^1)\omega_{k_2}(y^2)$$

and the rectangular Dirichlet kernels

$$D_{n_1,n_2}(z) := D_{n_1}(z^1)D_{n_2}(z^2) = \sum_{k_1=0}^{n_k-1}\sum_{k_2=0}^{n_k-1}\omega_{k_1}(z^1)\omega_{k_2}(z^2)$$

$$(z = (z^1, z^2) \in I^2).$$

We have the n^{th} Marcinkiewicz mean and kernel

$$\sigma_n^1 f(y) := \frac{1}{n+1}\sum_{k=0}^{n} S_{j,j}f(y), \quad K_n^1(z) = \frac{1}{n+1}\sum_{j=0}^{n} D_{j,j}(z)$$

and so we get

$$\sigma_n^1 f(y^1,y^2) = \int\limits_{I\times I} f(x^1,x^2)K_n^1(y^1+x^1,y^2+x^2)d\lambda(x^1,x^2).$$

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C,α_n)-Marcinkiewicz and call it the (C,α_n) kernel or the Cesàro-Marcinkiewicz kernel for $\alpha_n \in \mathbb{R}\setminus\{-1,-2,\dots\}$

$$K_n^{\alpha_n}(x_1,x_2) = \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n} A_{n-k}^{\alpha_n-1}D_{j,j}(x_1,x_2)$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n+1)(\alpha_n+2)...(\alpha_n+k)}{k!}.$$

The (C,α_n) Cesàro-Marcinkiewicz means of integrable function f for two variables are
$\sigma_n^{\alpha_n} f(y^1,y^2)$

$$= \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n} A_{n-k}^{\alpha_n-1}S_{k,k}f(y^1,y^2)$$

$$= \int\limits_{I\times I} f(x^1, x^2) K_n^{\alpha_n}(y^1 + x^1, y^2 + x^2) d\lambda(x^1, x^2).$$

$\sigma_n^{\alpha_n} f(y^1, y^2)$

$$= \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} \int\limits_{I\times I} A_{n-k}^{\alpha_n - 1} f(x^1, x^2) D_k(y^1 + x^1) D_k(y^2 + x^2) d\lambda(x^1, x^2).$$

Over all of this chapter we suppose that monotone decreasing sequences (α_n) and (β_n) satisfy

$$\beta_n = \alpha_{2^n}, \quad \frac{\alpha_N}{A_N^{\alpha_N}} \log^\delta \left(1 + \frac{N}{n}\right) \le C \frac{\alpha_n}{A_n^{\alpha_n}} \ (N \ge n, n, N \in \mathbb{P}) \tag{4.1}$$

for some $\delta > 1$ and for some positive constant C. We remark that from condition (4.1) it follows that sequence $(\frac{\alpha_n}{A_n^{\alpha_n}})$ is quasi monotone decreasing. That is, for some $C > 0$ we have $\frac{\alpha_N}{A_N^{\alpha_N}} \le C \frac{\alpha_n}{A_n^{\alpha_n}}$ $(N \ge n, n, N \in \mathbb{P})$.

The main aim of this chapter is to prove

Theorem 4.1. *(Abu Joudeh and Gát [7]) Suppose that monotone decreasing sequence $1 > \beta_n > 0$ satisfies the condition $\frac{A_{2^n}^{\beta_n}}{\beta_n} \frac{\beta_N}{A_{2^N}^{\beta_N}} (N + 1 - n)^\delta \le C$ for every $\mathbb{N} \ni N \ge n \ge 1$ and for some $\delta > 1$. Let $f \in L^1(I^2)$. Then we have the almost everywhere convergence*

$$\sigma_{2^n}^{\beta_n} f \to f.$$

Remark 4.2. (Abu Joudeh and Gát [7]) In the proof of Theorem 4.1 we define the sequence (α_n) in a way that $\alpha_{2^k} = \beta_k$ and for any $2^k \le n < 2^{k+1}$ let $\alpha_n = \alpha_{2^k} = \beta_k$. Then the sequence (α_n) satisfies that it is decreasing and $\frac{A_n^{\alpha_n}}{\alpha_n} \frac{\alpha_N}{A_N^{\alpha_N}} \log^\delta \left(1 + \frac{N}{n}\right) \le C$ for every $\mathbb{N} \ni N \ge n \ge 1$. That is, condition (4.1) is fulfilled.

- We give two examples for sequences (β_n) like above. Example one: $\beta_k = \alpha_{2^k} = \alpha_n = \alpha \in (0, 1)$ for every natural number k, n.

- Example two: Let $\alpha_n = 1/n$. Then it is not difficult to have that $A_n^{\alpha_n} \to 1$ and it should be fulfilled for sequence (α_n) that $CN/n \ge \log^\delta(1 + N/n)$ for some $\delta > 1$ and it trivially holds.

Introduce the following notations: for $a, n, j \in \mathbb{N}$ let $n_{(j)} := \sum_{i=0}^{j-1} n_i 2^i$, that is, $n_{(0)} = 0, n_{(1)} = n_0$ and for $2^B \leq n < 2^{B+1}$, let $|n| := B$, $n = n_{(B+1)}$. Moreover, introduce the following functions and operators for $n \in \mathbb{N}$ and $1 > \alpha_a > 0\,(a \in \mathbb{N})$ where $(x^1, x^2)\,, (y^1, y^2) \in I^2$ (Here we remark, that just for the proof of Theorem 4.1 $a = n$ could have been supposed, but in the future it will probably much more useful in the case when n is not a power of two.)

$$
\begin{aligned}
T_n^{\alpha_a}(x^1, x^2) &:= \frac{1}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n-j}^{\alpha_a-1} D_{j,j}(x^1, x^2),\\
\bar{T}_n^{\alpha_a}(x^1, x^2) &:= D_{2^B}(x^1) \frac{1}{A_n^{\alpha_a}} \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_n-1} D_j(x^2) \right|,\\
\bar{\bar{T}}_n^{\alpha_a}(x^1, x^2) &:= \bar{T}_n^{\alpha_a}(x^2, x^1),\\
\tilde{T}_n^{\alpha_a}(x^1, x^2) &:= \frac{1}{A_n^{\alpha_a}} D_{2^B,2^B}(x^1, x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1}\\
&+ \frac{(1-\alpha_a)}{A_n^{\alpha_a}} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left|K_j^1(x^1, x^2)\right|\\
&+ A_n^{\alpha_a-1} 2^B \left|K_{2^B-1}^1(x^1, x^2)\right|,\\
t_n^{\alpha_a} f(y^1, y^2) &:= \int_{I\times I} f(x^1, x^2) T_n^{\alpha_a}(y^1 + x^1, y^2 + x^2)\, d\lambda(x^1, x^2),\\
\tilde{t}_n^{\alpha_a} f(y^1, y^2) &:= \int_{I\times I} f(x^1, x^2) \tilde{T}_n^{\alpha_a}(y^1 + x^1, y^2 + x^2)\, d\lambda(x^1, x^2).
\end{aligned}
$$

We remark that is, in these definitions natural numbers a and n can vary independently. Now we need several Lemmas in the next section.

4.2 Proofs

Lemma 4.3. *(Abu Joudeh and Gát [7]) Let* $1 > \alpha_a > 0$, $(a \in \mathbb{N})$ $f \in L^1(I \times I)$ *such that supp* $f \subset I_k(u^1) \times I_k(u^2)$, $\int_{I_k(u^1)\times I_k(u^2)} f d\lambda = 0$ *for* *some dyadic rectangle, where $(u^1, u^2) \in I^2$. Then we have*

$$\int\limits_{I_k(u^1)\times I_k(u^2)} \sup_{n,a\in\mathbb{N}} |\tilde{t}_n^{\alpha_a} f| d\lambda \leq C\|f\|_1. \tag{4.2}$$

We also prove

$$|T_n^{\alpha_a}(x^1, x^2)| \leq \tilde{T}_n^{\alpha_a}(x^1, x^2) + \bar{T}_n^{\alpha_a}(x^1, x^2) + \bar{\bar{T}}_n^{\alpha_a}(x^1, x^2). \tag{4.3}$$

Proof. First, we start with the proof of the inequality

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a} + \bar{T}_n^{\alpha_a} + \bar{\bar{T}}_n^{\alpha_a}.$$

Recall that $B = |n|$. Then by equality $D_{2^B - j} = D_{2^B} - \omega_{2^B-1} D_j$ and $n_{(B)} = \sum_{j=0}^{B-1} n_j 2^j, n_{(B)} + 2^B = n$ we have:

$$\begin{aligned}
&A_n^{\alpha_a} T_n^{\alpha_a}(x) = \sum_{j=0}^{2^B-1} A_{2^B+n_{(B)}-j}^{\alpha_a-1} D_{j,j}(x) = \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{2^B-j,2^B-j}(x) \\
&= D_{2^B}(x^1) D_{2^B}(x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \\
&- \omega_{2^B-1}(x^1) D_{2^B}(x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^1) \\
&- \omega_{2^B-1}(x^2) D_{2^B}(x^1) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^2) \\
&+ \omega_{2^B-1}(x^1) \omega_{2^B-1}(x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1, x^2) \\
&=: (1) - (2) - (3) + (4).
\end{aligned}$$

So by the help of the Abel transform we get:

$$
\begin{aligned}
|(4)| &= \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1,x^2) \right| \\
&= \left| \sum_{j=0}^{2^B-1} (A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1}) \sum_{i=0}^{j} D_{i,i}(x^1,x^2) + A_{n_{(B)}+2^B}^{\alpha_a-1} \sum_{i=0}^{2^B-1} D_{i,i}(x^1,x^2) \right| \\
&= \left| (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1(x^1,x^2) + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1(x^1,x^2) \right| \\
&\leq (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left| K_j^1(x^1,x^2) \right| \\
&+ (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left| K_j^1(x^1,x^2) \right| \\
&+ A_n^{\alpha_a-1} 2^B \left| K_{2^B-1}^1(x^1,x^2) \right| =: I + II + III.
\end{aligned}
$$

By the above written we have

$$
\begin{aligned}
& A_n^{\alpha_a} \left| T_n^{\alpha_a}(x^1,x^2) \right| \\
& \qquad \leq D_{2^B,2^B}(x^1,x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} + D_{2^B}(x^1) \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^2) \right| \\
& \qquad + D_{2^B}(x^2) \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^1) \right| + \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1,x^2) \right|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left| T_n^{\alpha_a}(x^1,x^2) \right| \leq \tilde{T}_n^{\alpha_a}(x^1,x^2) &+ D_{2^B}(x^1) \frac{1}{A_n^{\alpha_a}} \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^2) \right| \\
&+ D_{2^B}(x^2) \frac{1}{A_n^{\alpha_a}} \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^1) \right| \\
&= \tilde{T}_n^{\alpha_a}(x^1,x^2) + \bar{T}_n^{\alpha_a}(x^1,x^2) + \bar{\bar{T}}_n^{\alpha_a}(x^1,x^2).
\end{aligned}
$$

For $n < 2^k$ and $(x^1, x^2) \in I_k(u^1) \times I_k(u^2)$ we have that $\tilde{T}_n^{\alpha_a}(y+x)$ depends (with respect to x) only on coordinates $x_0^1, \dots, x_{k-1}^1, x_0^2, \dots, x_{k-1}^2$, thus $\tilde{T}_n^{\alpha_a}(y+x) = \tilde{T}_n^{\alpha_a}(y+u)$ and consequently

$$\int\limits_{I_k(u^1)\times I_k(u^2)} f(x^1, x^2)\tilde{T}_n^{\alpha_a}(y^1 + x^1, y^2 + x^2)d\lambda(x^1, x^2)$$

$$= \tilde{T}_n^{\alpha_a}(y^1 + u^1, y^2 + u^2) \int\limits_{I_k(u^1)\times I_k(u^2)} f(x^1, x^2)d\lambda(x^1, x^2) = 0.$$

Observe that

$$\overline{I_k(u^1) \times I_k(u^2)} = \overline{I_k(u^1)} \times \overline{I_k(u^2)} \cup I_k(u^1) \times \overline{I_k(u^2)} \cup \overline{I_k(u^1)} \times I_k(u^2).$$

Since for any $j < 2^k$ we have that the kernel $K_j^1(y+x)$ depends (with respect to x) only on coordinates $x_0^1, \dots, x_{k-1}^1, x_0^2, \dots, x_{k-1}^2$, then

$$\int_{I_k(u^1)\times I_k(u^2)} f(x)|K_j^1(y+x)|d\lambda(x)$$

$$= |K_j^1(y+u)| \int_{I_k(u^1)\times I_k(u^2)} f(x)d\lambda(x) = 0.$$

gives $\int_{I_k(u^1)\times I_k(u^2)} f(x)I(y+x)d\lambda(x) = 0$. On the other hand,

$$II = (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} |K_j^1(y^1 + x^1, y^2 + x^2)|$$

$$\leq \sup_{j\geq 2^k} |K_j^1(x^1, x^2)|(1-\alpha_a) \sum_{j=0}^{n} A_j^{\alpha_a-1} = A_n^{\alpha_a}(1-\alpha_a) \sup_{j\geq 2^k} |K_j^1(x^1, x^2)|.$$

This by Lemma 3 in [13] gives

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} II d\lambda \leq \int\limits_{\overline{I_k\times I_k}} \sup_{j\geq 2^k} |K_j^1(x^1, x^2)|d\lambda \leq C.$$

The situation with III is similar. So, just as in the case of II we apply Lemma 3 in [13]:

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} III d\lambda \leq \int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k} |K_{2^{|n|}-1}^1| d\lambda \leq C.$$

Therefore, substituting $z^1 = (x^1 + y^1), z^2 = (x^2 + y^2)$, where $z \in \overline{I_k \times I_k}$ and consequently $D_{2^B,2^B}(z^1, z^2) = 0$ then

$$\begin{aligned}
&\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda \\
&= \int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int\limits_{I_k\times I_k} f(x^1, x^2)\tilde{T}_n^{\alpha_a}(y^1 + x^1, y^2 + x^2) d\lambda(x^1, x^2) \right| d\lambda(y^1, y^2) \\
&\leq \int\limits_{\overline{I_k\times I_k}} \int\limits_{I_k\times I_k} |f(x^1, x^2)| \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} [II(y^1 + x^1, y^2 + x^2) \\
&+ III(y^1 + x^1, y^2 + x^2)] d\lambda(x^1, x^2) d\lambda(y^1, y^2) \\
&= \int\limits_{I_k\times I_k} |f(x^1, x^2)| \int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} II(z^1, z^2) \\
&+ III(z^1, z^2) d\lambda(z^1, z^2) d\lambda(x^1, x^2) \\
&\leq C \int\limits_{I_k\times I_k} |f(x^1, x^2)| d\lambda(x^1, x^2).
\end{aligned}$$

This gives

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n, a\in\mathbb{N}} \left|\tilde{t}_n^{\alpha_a} f\right| d\lambda \leq C\|f\|_1.$$

This completes the proof of Lemma 4.3. □

Now, we just proved the Lemma which means that maximal operator $\sup_{n,a} |\tilde{t}_n^{\alpha_a}|$ is quasi-local. The following lemma shows that the one-dimensional operator which maps $f \in L^1(I)$ to $\sup_n \left| f * \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^n A_j^{\alpha_n - 1} |K_j| \right|$ is quasi-local. This lemma is interesting itself if one investigates Cesàro means with variable parameters and in the proof we introduce methods which will also be used later.

Lemma 4.4. *(Abu Joudeh and Gát [7]) Let (α_n) be a monotone decreasing sequence and $\left(\frac{\alpha_n}{n^{\alpha_n}}\right)$ be a quasi decreasing sequences with $1 > \alpha_n > 0\,(n \in$*

$\mathbb{N}$). *Then*

$$\int\limits_{\overline{I_k}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n} A_j^{\alpha_n-1} |K_j| \leq C.$$

Proof. Recall that K_n denotes the one-dimensional Fejér kernel. That is, $K_n = K_n^1$. By [14]

$$\begin{aligned}
&\int\limits_{\overline{I_k}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=2^k}^{n} A_j^{\alpha_n-1} |K_j(x)|\, dx \\
&\leq \int\limits_{\overline{I_k}} \sup_{j\geq 2^k} |K_j(x)| \sup_{n} \frac{1}{A_n^{\alpha_n}} \sum_{l=2^k}^{n} A_l^{\alpha_n-1} dx \\
&\leq \int\limits_{\overline{I_k}} \sup_{j\geq 2^k} |K_j(x)|\, dx \\
&\leq C.
\end{aligned}$$

On the other hand, if $j < 2^k$ by $\bar{I}_k = \bigcup_{a=0}^{k-1} (I_a \backslash I_{a+1})$ we have

$$\begin{aligned}
&\int\limits_{\overline{I_k}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{2^k-1} A_j^{\alpha_n-1} |K_j| \\
&\leq \sum_{a=0}^{k-1} \int\limits_{I_a\backslash I_{a+1}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=2^a}^{2^k-1} A_j^{\alpha_n-1} |K_j| \\
&+ \sum_{a=0}^{k-1} \int\limits_{I_a\backslash I_{a+1}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{2^a-1} A_j^{\alpha_n-1} |K_j| \\
&=: I + II.
\end{aligned}$$

For I we have

$$I \leq \sum_{a=0}^{k-1} \int\limits_{I_a \setminus I_{a+1}} \sup_{n \geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=a}^{k-1} \sum_{j=2^b}^{2^{b+1}-1} A_j^{\alpha_n - 1} |K_j|$$

$$\leq \sum_{a=0}^{k-1} \sum_{b=a}^{k-1} \int\limits_{I_a \setminus I_{a+1}} \sup_{j \geq 2^b} |K_j| \sup_{n \geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{l=2^b}^{2^{b+1}-1} A_l^{\alpha_n - 1},$$

where

$$\sup_{n \geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{l=2^b}^{2^{b+1}-1} A_l^{\alpha_n - 1} \leq \sup_{n \geq 2^k} \frac{A_{2^{b+1}-1}^{\alpha_n} - A_{2^b-1}^{\alpha_n}}{A_n^{\alpha_n}}$$

$$= \sup_{n \geq 2^k} \frac{A_{2^b-1}^{\alpha_n}}{A_n^{\alpha_n}} \left[\frac{(2^b + \alpha_n)...(2^{b+1} - 1 + \alpha_n)}{2^b(2^b+1)...(2^{b+1}-1)} - 1 \right]$$

$$= \sup_{n \geq 2^k} \frac{A_{2^b-1}^{\alpha_n}}{A_n^{\alpha_n}} \left[\left(1 + \frac{\alpha_n}{2^b}\right) \left(1 + \frac{\alpha_n}{2^b+1}\right) ... \left(1 + \frac{\alpha_n}{2^b + 2^b - 1}\right) - 1 \right]$$

$$\leq \sup_{n \geq 2^k} \frac{A_{2^b}^{\alpha_n}}{A_n^{\alpha_n}} \left[\left(1 + \frac{\alpha_n}{2^b}\right)^{2^b} - 1 \right]$$

$$\leq C \sup_{n \geq 2^k} \frac{A_{2^b}^{\alpha_n}}{A_n^{\alpha_n}} \alpha_n \leq C \sup_{n \geq 2^k} \left(\frac{2^b}{n}\right)^{\alpha_n} \alpha_n$$

$$\leq C \sup_{n \geq 2^k} \left(2^b\right)^{\alpha_{2^k}} \left(\frac{\alpha_n}{n^{\alpha_n}}\right)$$

$$\leq C \left(2^b\right)^{\alpha_{2^k}} \left(\frac{\alpha_{2^k}}{(2^k)^{\alpha_{2^k}}}\right),$$

where the inequality $\frac{A_{2^b}^{\alpha_n}}{A_n^{\alpha_n}} \leq C \left(\frac{2^b}{n}\right)^{\alpha_n}$ is given from [6, Lemma 2.4]. Besides, since (α_n) is a monotone decreasing sequences then $(2^b)^{\alpha_n} \leq (2^b)^{\alpha_{2^k}}$. Besides, sequence $\left(\frac{\alpha_n}{n^{\alpha_n}}\right)$ is quasi decreasing. Moreover, $\left(1 + \frac{\alpha_n}{2^b}\right)^{2^b} - 1 \leq C\alpha_n$, for any $0 < \alpha_n < 1, b \in \mathbb{N}$.

Thus, by (3) ([22])

$$I \leq C \sum_{a=0}^{k-1} \sum_{b=a}^{k-1} \frac{2^a}{2^b} (b-a) \alpha_{2^k} \left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}} = C \sum_{b=0}^{k-1} \sum_{a=0}^{b} \frac{2^a}{2^b} (b-a) \alpha_{2^k} \left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}}$$

$$\leq C \sum_{b=0}^{k-1} \alpha_{2^k} \left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}} \leq C \alpha_{2^k} \sum_{l=0}^{\infty} \frac{1}{2^{l\alpha_k}} \leq C \alpha_{2^k} \frac{1}{1 - 2^{\alpha_{2^k}}} \leq C.$$

We have to discuss II in the case when $j<2^a$ and thus $|K_j(x)|\le j$. Besides, $A_j^{\alpha_n-1}j=\alpha_n A_{j-1}^{\alpha_n}$ and this follows

$$\sum_{j=0}^{2^a-1} A_j^{\alpha_n-1}|K_j(x)|\le \alpha_n\sum_{j=0}^{2^a-1}A_j^{\alpha_n}\le \alpha_n A_{2^a}^{\alpha_n+1}=\alpha_n A_{2^a+1}^{\alpha_n}\left(\frac{2^a+1}{\alpha_n+1}\right).$$

Besides, by [6, Lemma 2.4] and by the fact that the sequence (α_n/n^{α_n}) is quasi decreasing we have

$$\sup_{n\ge 2^k}\frac{\alpha_n A_{2^a+1}^{\alpha_n}}{A_n^{\alpha_n}}.\frac{2^a+1}{\alpha_n+1}\le C2^a\sup_{n\ge 2^k}\alpha_n\left(\frac{2^a+1}{n}\right)^{\alpha_n}\le C2^a\alpha_{2^k}\left(\frac{2^a}{2^k}\right)^{\alpha_{2^k}}.$$

Then

$$II\le C\sum_{a=0}^{k-1}\frac{1}{2^a}2^a\alpha_{2^k}\left(\frac{2^a}{2^k}\right)^{\alpha_{2^k}}\le C\sup_k \alpha_{2^k}\sum_{l=0}^{\infty}\frac{1}{2^{l\alpha_{2^k}}}\le C.$$

This completes the proof of Lemma 4.4. $\square$

Next we prove the following lemma,

Lemma 4.5. *(Abu Joudeh and Gát [7]) Suppose that for the monotone decreasing sequence (α_n) the condition (4.1) is fulfilled. Let $a: I\setminus\{0\}\longmapsto\mathbb{N}$ be defined as $a(x)=a$ for $x\in(I_a\backslash I_{a+1})$. Then the inequality*

$$\int_{I_k\times\overline{I_k}}\sup_{n\ge 2^k}\frac{1}{A_n^{\alpha_n}}\sum_{s=k}^{|n|}\sum_{j=0}^{2^{a(x^2)}}A_j^{\alpha_n-1}\left|K_j(x^2)\right|D_{2^s}(x^1)d(x^1,x^2)\le C$$

holds.

Proof. Since $\int_{I_k\times\overline{I_k}}=\sum_{a=0}^{k-1}\int_{I_k\times(I_a\backslash I_{a+1})}$ then we have to check the values of the integrand on $I_k\times(I_a\backslash I_{a+1})$. That is, $x^2\in I_a\backslash I_{a+1}$. Thus, $\left|K_j(x^2)\right|\le Cj$ gives

$$A_j^{\alpha_n-1}.j=\frac{\alpha_n...(\alpha_n+j-1)}{j!}j=\alpha_n\frac{(1+\alpha_n)...(j-1+\alpha_n)}{(j-1)!}=\alpha_n A_{j-1}^{\alpha_n}.$$

This gives

$$\sum_{j=0}^{2^a} A_j^{\alpha_n-1}\left|K_j(x^2)\right| \leq C\sum_{j=1}^{2^a}\alpha_n A_{j-1}^{\alpha_n} = C\alpha_n A_{2^a-1}^{\alpha_n+1}$$
$$= C\alpha_n\frac{(2+\alpha_n)...(2^a+\alpha_n)}{(2^a-1)!} = C\alpha_n\left(\frac{2^a}{1+\alpha_n}\right)A_{2^a}^{\alpha_n} \leq C\alpha_n 2^a A_{2^a}^{\alpha_n}.$$

That is, we have to investigate

$$\sum_{a=0}^{k-1}\int_{I_k}\sup_{n\geq 2^k}\frac{\alpha_n}{A_n^{\alpha_n}}A_{2^a}^{\alpha_n}\sum_{s=k}^{|n|}D_{2^s}(x^1)d(x^1).$$

Recall that $\int\limits_{I_a\setminus I_{a+1}} 2^a \leq 1$, $A_{2^a}^{\alpha_n} \leq A_{2^a}^{\alpha_{2^k}}$ since $\alpha_n \searrow$ and $n \geq 2^k$. Also recall that

$$\frac{\alpha_n}{A_n^{\alpha_n}} \leq \frac{C}{\log^\delta\left(1+\frac{n}{2^k}\right)}\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}.$$

Which gives

$$\frac{\alpha_n}{A_n^{\alpha_n}}A_{2^a}^{\alpha_n} \leq C\alpha_{2^k}\frac{A_{2^a}^{\alpha_{2^k}}}{A_{2^k}^{\alpha_{2^k}}}\frac{1}{\log^\delta\left(1+\frac{n}{2^k}\right)}.$$

That is, we have to investigate :

$$\sum_{a=0}^{k-1}\alpha_{2^k}\frac{A_{2^a}^{\alpha_{2^k}}}{A_{2^k}^{\alpha_{2^k}}}\int_{I_k}\sup_{n\geq 2^k}\frac{1}{\log^\delta\left(1+\frac{n}{2^k}\right)}\sum_{s=k}^{|n|}D_{2^s}(x^1)d(x^1).$$

Check the integral above : $\int\limits_{I_k} = \sum_{t=k}^{\infty}\int\limits_{I_t\setminus I_{t+1}}$ and the integral on $I_t \setminus I_{t+1}$ can be estimated by

$$\int_{I_t\setminus I_{t+1}}\sup_{n\geq 2^k}\frac{C}{\left(1+|n|-k\right)^\delta}\sum_{s=k}^{\min(t,|n|)}2^s d(x^1) \leq \frac{C}{\left(t+1-k\right)^\delta}$$

and henceforth by $\delta > 1, \sum_{t=k}^{\infty}\frac{1}{(1+t-k)^\delta} \leq C$. We have by Lemma 2.4 in [6]

$$\sum_{a=0}^{k-1}\alpha_{2^k}\frac{A_{2^a}^{\alpha_{2^k}}}{A_{2^k}^{\alpha_{2^k}}} \leq 2\sum_{a=0}^{k-1}\alpha_{2^k}\left(\frac{2^a+1}{2^k}\right)^{\alpha_{2^k}} \leq C\sum_{a=0}^{k-1}\alpha_{2^k}\left(\frac{2^a}{2^k}\right)^{\alpha_{2^k}}$$
$$\leq C\alpha_{2^k}\sum_{j=0}^{\infty}\left(\frac{1}{2^{\alpha_{2^k}}}\right)^j = \frac{C\alpha_{2^k}}{1-\left(\frac{1}{2}\right)^{\alpha_{2^k}}} \leq C.$$

This completes the proof of Lemma 4.5. □

Let (α_n) be a monotone decreasing sequences such that $0 < \alpha_n < 1$ with property (4.1). That is, for some $\delta > 1, C > 0$ and

$$\frac{A_n^{\alpha_n}}{\alpha_n} \frac{\alpha_N}{A_N^{\alpha_N}} \log^\delta \left(1 + \frac{N}{n}\right) \leq C$$

for every $\mathbb{N} \ni N \geq n \geq 1$. We prove

Lemma 4.6. *(Abu Joudeh and Gát [7])*

$$\sum_{a=0}^{k-1} \int\limits_{I_k \times (I_a \setminus I_{a+1})} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{b=a}^{k-1} \sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n - 1} \left|K_j(x^2)\right| D_{2^s}(x^1) d(x^1, x^2) \leq C.$$

Proof. By the result of Goginava [22], that is by

$$\int\limits_{I_a \setminus I_{a+1}} \sup_{n \geq 2^b} \left|K_j(x^2)\right| d(x^2) \leq C \left(\frac{b-a}{2^{b-a}}\right) \tag{4.4}$$

we have to investigate

$$\mathbf{B_1} := \sum_{a<k} \int\limits_{I_k} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{b=a}^{k-1} \frac{b-a}{2^{b-a}} \sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n - 1} D_{2^s}(x^1) d(x^1).$$

So we have

$$\begin{aligned}
&\sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n - 1} = A_{2^{b+1}}^{\alpha_n} - A_{2^b}^{\alpha_n} = A_{2^b}^{\alpha_n} \left[\frac{(2^b + 1 + \alpha_n)...(2^{b+1} + \alpha_n)}{(2^b + 1)...(2^{b+1})} - 1\right] \\
&= A_{2^b}^{\alpha_n} \left[\left(1 + \frac{\alpha_n}{2^b + 1}\right) ... \left(1 + \frac{\alpha_n}{2^{b+1}}\right) - 1\right] \\
&\leq A_{2^b}^{\alpha_n} \left[0 \left(1 + \frac{\alpha_n}{2^b}\right)^{2^b} - 1\right] \\
&\leq C \alpha_n A_{2^b}^{\alpha_n}.
\end{aligned}$$

On the other hand, by $\int\limits_{I_k} = \sum_{t=k}^{\infty} \int\limits_{I_t \setminus I_{t+1}}$ it follows

$$\int\limits_{I_k} \sup_{n>2^k} \frac{1}{(|n|+1-k)^{\delta}} \sum_{s=k}^{|n|} D_{2^s}(x^1) d(x^1)$$
$$= \sum_{t=k}^{\infty} \int\limits_{I_t \setminus I_{t+1}} \sup_{n>2^k} \frac{1}{(|n|+1-k)^{\delta}} \sum_{s=k+1}^{min(t,|n|)} 2^s$$
$$\leq \sum_{t=k}^{\infty} \left(\int\limits_{I_t \setminus I_{t+1}} \sup_{t\geq|n|>k} \frac{1}{(|n|+1-k)^{\delta}} 2^{|n|} + \int\limits_{I_t \setminus I_{t+1}} \sup_{|n|>t} \frac{1}{(|n|+1-k)^{\delta}} 2^t \right)$$
$$=: \sum_{t=k}^{\infty} (\mathbf{B_{2,1}} + \mathbf{B_{2,2}}) .$$

Now we have :

$$\sum_{t=k}^{\infty} (\mathbf{B_{2,2}}) \leq \sum_{t=k}^{\infty} \frac{1}{(t+1-k)^{\delta}} \leq C,$$
$$\sum_{t=k}^{\infty} (\mathbf{B_{2,1}}) \leq \sum_{t=k}^{\infty} \sup_{t\geq|n|>k} \frac{2^{|n|+1-t}}{(|n|-k)^{\delta}} \leq \sum_{t=k+1}^{\infty} \frac{1}{(t-k)^{\delta}} \leq C.$$

That is, for $\mathbf{B_1}$ we get

$$\mathbf{B_1}$$
$$\leq C \sum_{a<k} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=a}^{k-1} \alpha_n A_{2^b}^{\alpha_n} \frac{b-a}{2^{b-a}} \log^{\delta}\left(1+\frac{n}{2^k}\right)$$
$$\times \sum_{t=k}^{\infty} \int\limits_{I_t \setminus I_{t+1}} \sup_{n>2^k} \frac{1}{(|n|+1-k)^{\delta}} \sum_{s=k}^{min(t,|n|)} D_{2^s}(x^1) dx^1$$
$$\leq C \sum_{a<k} \sup_{n>2^k} \frac{\alpha_n}{A_n^{\alpha_n}} \log^{\delta}\left(1+\frac{n}{2^k}\right) \sum_{b=a}^{k-1} A_{2^b}^{\alpha_n} \frac{b-a}{2^{b-a}}$$
$$\leq C \sum_{a<k} \sum_{b=a}^{k-1} A_{2^b}^{\alpha_{2^k}} \frac{b-a}{2^{b-a}} \sup_{n>2^k} \frac{\alpha_n}{A_n^{\alpha_n}} \log^{\delta}\left(1+\frac{n}{2^k}\right)$$
$$=: \mathbf{B_3}.$$

Recall that $A_{2^b}^{\alpha_n} \leq A_{2^b}^{\alpha_{2^k}}$ Since $n > 2^k$ and (α_n) is a monotone decreasing sequence. By the properties of (α_n) we have $\frac{\alpha_n}{A_n^{\alpha_n}} \log^\delta\left(1+\frac{n}{2^k}\right) \leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}$ and then by Lemma 2.4 for the Cesàro numbers in [6]

$$\mathbf{B_3} \leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \sum_{a<k} \sum_{b=a}^{k-1} A_{2^b}^{\alpha_{2^k}} \frac{b-a}{2^{b-a}} = C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \sum_{b=0}^{k-1} A_{2^b}^{\alpha_{2^k}} \sum_{a=0}^{b} \frac{b-a}{2^{b-a}}$$

$$\leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \sum_{b=0}^{k-1} A_{2^b}^{\alpha_{2^k}} \leq C\sum_{b=0}^{k-1} \alpha_{2^k}\left(\frac{2^b+1}{2^k}\right)^{\alpha_{2^k}} \leq C\sum_{b=0}^{k-1} \alpha_{2^k}\left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}} \leq C$$

again just as at the end of the proof of Lemma 4.5. This completes the proof of Lemma 4.6. □

Corollary 4.7. *(Abu Joudeh and Gát [7]) Let* $1 > \alpha_n > 0$ *fulfill property (4.1). Then by Lemmas 4.5 and 4.6 - as a direct consequence- we have*

$$\int_{I_k \times \overline{I_k}} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{j=0}^{2^k} A_j^{\alpha_n - 1} \left|K_j(x^2)\right| D_{2^s}(x^1) d(x^1, x^2) \leq C.$$

Moreover, we prove

Lemma 4.8. *(Abu Joudeh and Gát [7])*

$$\int_{I_k \times \overline{I_k}} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{j=2^k+1}^{2^{|n|}} A_j^{\alpha_n - 1} \left|K_j(x^2)\right| D_{2^s}(x^1) d(x^1, x^2) \leq C,$$

where $1 > \alpha_n > 0$ *is a decreasing sequence with property (4.1).*

Proof. By the result of Goginava [22] (see at (4.4)) we have $\int_{I \setminus I_k} \sup_{j \geq 2^b} \left|K_j(x^1)\right| d(x^1) \leq C\frac{b-k+1}{2^{b-k}}$ for any $b \geq k$. That is the integral in Lemma 4.8 is bounded by

$$C \int_{I_k} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{b=k}^{|n|-1} \frac{b-k+1}{2^{b-k}} \sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n - 1} D_{2^s}(x^1) d(x^1) =: \mathbf{B_4}.$$

As in the proof of lemma 4.6 we have $\sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n-1} \leq C\alpha_n A_{2^b}^{\alpha_n}$. In the proof of lemma 4.6 we can find inequality:

$$\int_{I_k} \sup_{n>2^k} \frac{1}{(|n|+1-k)^\delta} \sum_{s=k}^{|n|} D_{2^s}(x^1)d(x^1) \leq C$$

and henceforth

$$\begin{aligned}
\mathbf{B_4} &\leq \int_{I_k} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=k}^{|n|-1} \frac{b-k+1}{2^{b-k}} \alpha_n A_{2^b}^{\alpha_n} (|n|+1-k)^\delta \\
&\times \frac{1}{(|n|+1-k)^\delta} \sum_{s=k}^{|n|} D_{2^s}(x^1)d(x^1) \\
&\leq C \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=k}^{|n|} \frac{b-k+1}{2^{b-k}} \alpha_n A_{2^b}^{\alpha_n} \log^\delta\left(1+\frac{n}{2^k}\right) \\
&\times \int_{I_k} \sup_{n>2^k} \frac{1}{(|n|+1-k)^\delta} \sum_{s=k}^{|n|} D_{2^s}(x^1)d(x^1) \\
&\leq C \sup_{n>2^k} \frac{\alpha_n}{A_n^{\alpha_n}} \sum_{b=k}^{|n|} \frac{b-k+1}{2^{b-k}} A_{2^b}^{\alpha_n} \log^\delta\left(1+\frac{n}{2^k}\right) =: \mathbf{B_5}.
\end{aligned}$$

So by $\frac{\alpha_n}{A_n^{\alpha_n}} \log^\delta\left(1+\frac{n}{2^k}\right) \leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}$ we have

$$\mathbf{B_5} \leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \sup_{n>2^k} \sum_{b=k}^{|n|} \frac{b-k+1}{2^{b-k}} A_{2^b}^{\alpha_n}.$$

Since (α_n) is a monotone decreasing, then $A_{2^b}^{\alpha_n} \leq A_{2^b}^{\alpha_{2^k}}$.

Thus, by [6, Lemma 2.4] (second inequality below)

$$\begin{aligned}
\mathbf{B_5} &\leq C \frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \left[A_{2^k}^{\alpha_{2^k}} + \frac{2}{2} A_{2^{k+1}}^{\alpha_{2^k}} + \frac{3}{2^2} A_{2^{k+2}}^{\alpha_{2^k}} + + \frac{4}{2^3} A_{2^{k+3}}^{\alpha_{2^k}} + \ldots \right] \\
&\leq C\alpha_{2^k} \sum_{j=0}^{\infty} \left(\frac{2^{k+j}+1}{2^k} \right)^{\alpha_{2^k}} \frac{j}{2^j} \\
&\leq C\alpha_{2^k} \sum_{j=0}^{\infty} \frac{j}{2^{j(1-\alpha_{2^k})}} \\
&\leq C.
\end{aligned}$$

as it holds $0 < \alpha_{2^k} \leq 1 - \alpha_2 < 1$. That is, the of proof Lemma 4.8 is complete. □

Corollary 4.7 and Lemma 4.8 give the following consequence :

Corollary 4.9. *(Abu Joudeh and Gát [7]) Let $0 < \alpha_n < 1$ be a monotone decreasing sequence and*

$$\frac{\alpha_N}{A_N^{\alpha_N}} \frac{A_n^{\alpha_n}}{\alpha_n} \log^{\delta} \left(1 + \frac{N}{n} \right) \leq C$$

for every $N \geq n \geq 1$. Then

$$\int\limits_{\overline{I_k \times I_k}} \sup_{n > 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k+1}^{|n|} \sum_{j=0}^{2^{|n|}} A_j^{\alpha_n - 1} \left| K_j(x^2) \right| D_{2^s}(x^1) d(x^1, x^2) \leq C.$$

By the help of Corollary 4.9 and Lemma 4.3 we prove that operator

$$t^* f(y) := \sup_n |t_n^{*,\alpha_n} f(y)| := \sup_n \left| \int_{I \times I} f(x) |T_n^{\alpha_n}(x+y)| d\lambda(x) \right|$$

is quasilocal. That is,

Lemma 4.10. *(Abu Joudeh and Gát [7]) Suppose that sequence (α_n) fulfills the conditions of Corollary 4.9. Let $f \in L^1(I \times I)$ such that supp $f \subset I_k(u^1) \times I_k(u^2)$, $\int\limits_{I_k(u^1) \times I_k(u^2)} f d\lambda = 0$ for some dyadic rectangle. Then we have*

$$\int\limits_{\overline{I_k(u^1) \times I_k(u^2)}} t^* f d\lambda \leq C \|f\|_1.$$

Besides, operator t^* is of strong type (L^∞, L^∞).

Proof. Recall that for any $m, n \leq 2^k$ we have $\hat{f}(m,n) = 0$ and then $t^*f(y) := \sup_{n>2^k} |t_n^{*,\alpha_n} f(y)|$. The proof this lemma is based on Lemma 4.3. More precisely, on inequalities (4.2) and (4.3). That is,

$$\begin{aligned}
&\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}} t^* f d\lambda \\
&\leq \int\limits_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n>2^k} |\tilde{t}_n^{\alpha_n} f| d\lambda \\
&+ \int\limits_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n>2^k} |\bar{t}_n^{\alpha_n} f| d\lambda + \int\limits_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n>2^k} |\bar{\bar{t}}_n^{\alpha_n} f| d\lambda \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Lemma 4.3 means that $A_1 \leq C\|f\|_1$. Since the difference between terms A_2 and A_3 is only the interchange of variables therefore it is enough to discuss A_2 only. By the theorem of Fubini and the shift invariance of the Lebesgue measure we have

$$A_2 \leq \int\limits_{I_k(u^1)\times I_k(u^2)} |f(x^1,x^2)| \int_{\overline{I_k \times I_k}} \sup_{n>2^k} \bar{T}_n^{\alpha_n}(z^1,z^2) d\lambda(z) d\lambda(x).$$

Therefore, if we could prove the inequality $\int_{\overline{I_k\times I_k}} \sup_{n>2^k} \bar{T}_n^{\alpha_n}(z^1,z^2)d\lambda(z) \leq C$, then the proof of Lemma 4.10 would be complete.

By the help of the Abel transform we get:

$$\begin{aligned}
A_n^{\alpha_n}\bar{T}_n^{\alpha_n}(z^1,z^2) &= D_{2^B}(z^1)\left|\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(z^2)\right| \\
&= D_{2^B}(z^1)\left|\sum_{j=0}^{2^B-1} (A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1}) \sum_{i=0}^{j} D_i + A_{n_{(B)}+2^B}^{\alpha_n-1} \sum_{i=0}^{2^B-1} D_i(z^2)\right| \\
&= D_{2^B}(z^1)\left|(1-\alpha_n)\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1(z^2) + A_n^{\alpha_n-1} 2^B K_{2^B-1}^1(z^2)\right| \\
&\leq D_{2^B}(z^1)\sum_{j=0}^{2^B-1} A_j^{\alpha_n-1}\left|K_j^1(z^2)\right| + D_{2^B}(z^1) A_n^{\alpha_n-1} 2^B \left|K_{2^B-1}^1(z^2)\right|.
\end{aligned} \tag{4.5}$$

Use the facts that $\overline{I_k \times I_k} = \bar{I}_k \times I_k \cup \bar{I}_k \times \bar{I}_k \cup I_k \times \bar{I}_k$ and $D_{2^B}(z^1) = 0$ for $n > 2^k$, that is, $B = |n| \geq k$ in the case of $z^1 \in \bar{I}_k$. Moreover, $2^B A_n^{\alpha_n - 1}/A_n^{\alpha_n} \leq 1$ then by Corollary 4.9 the proof of the sublinearity of operator $t^* f$ is complete. On the other hand,

$$\|t^* f\|_\infty \leq \sup_n \left| \int_{I \times I} \|f\|_\infty |T_n^{\alpha_n}(x+y)| d\lambda(x) \right| \leq C\|f\|_\infty$$

as it comes from (4.5) and the fact that the L^1-norms of the Fejér kernels and also the Dirichlet kernels with indices of the form 2^m are uniformly bounded. This completes the proof of Lemma 4.10. □

Now, we can prove the main tool in order to have Theorem 4.1. for operators

$$\sigma_*^\beta f := \sup_{n \in \mathbb{N}} \left| \sigma_{2^n}^{\beta_n} f \right| = \sup_{n \in \mathbb{N}} \left| f * K_{2^n}^{\beta_n} \right|$$

and

$$\tilde{\sigma}_*^\beta f := \sup_{n \in \mathbb{N}} \left| \tilde{\sigma}_{2^n}^{\beta_n} f \right| = \sup_{n \in \mathbb{N}} \left| f * |K_{2^n}^{\beta_n}| \right|$$

Lemma 4.11. *(Abu Joudeh and Gát [7]) The operators $\tilde{\sigma}_*^\beta$ and σ_*^β are of weak type (L^1, L^1).*

Proof. First, we prove Lemma 4.11 for operator $\tilde{\sigma}_*^\beta$. We apply the Calderon-Zygmund decomposition lemma [32]. That is, let $f \in L^1(I^2)$ and $\|f\|_1 < \eta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that $\|f_0\|_\infty \leq C\eta$, $\|f_0\|_1 \leq C\|f\|_1$ and $I^j \times I^j = I_{k_j}(u^{j,1}) \times I_{k_j}(u^{j,2})$ are disjoint dyadic rectangles for which

$$f_j \subset I^j \times I^j, \ \int_{I^j \times I^j} f_j d\lambda = 0 \ , \ \lambda(F) \leq \frac{C\|f_1\|}{\eta}$$

$((u^{j,1}, u^{j,2}) \in I \times I \ , \ k_j \in \mathbb{N} \ , \ j \in \mathbb{P})$, where $F = \cup_{j=1}^\infty I^j \times I^j$. By the σ-sublinearity of the maximal operator with an appropriate constant C we have

$$\lambda(\tilde{\sigma}_*^\beta f > 2C\eta) \leq \lambda(\tilde{\sigma}_*^\beta f_0 > C\eta) + \lambda(\tilde{\sigma}_*^\beta (\sum_{i=1}^\infty f_i) > C_\eta) := I + II.$$

Notice that

$$K_{2^n}^{\beta_n}(x) = T_{2^n}^{\alpha_{2^n}}(x) + \frac{D_{2^n}(x^1)D_{2^n}(x^2)}{A_{2^n}^{\alpha_{2^n}}}$$

and keep in mind that operator $\sup_n |f*(D_{2^n} \times D_{2^n})|$ is quasi-local and it is of weak type (L^1, L^1) and it is also of type (L^p, L^p) for each $1 < p \leq \infty$ ([32]). Since by Lemma 4.10 $\|\tilde{\sigma}_*^\alpha f_0\|_\infty \leq C\|f_0\|_\infty \leq C\eta$ then we have $I = 0$. So,

$$\begin{aligned}
&\lambda(\tilde{\sigma}_*^\beta(\sum_{i=1}^\infty f_i) > C\eta) \leq \lambda(F) + \lambda(\bar{F} \cap \{\tilde{\sigma}_*^\beta(\sum_{i=1}^\infty f_i) > C\eta\}) \\
&\leq \frac{C\|f\|_1}{\eta} + \frac{C}{\eta}\sum_{i=1}^\infty \int\limits_{I^j \times I^j} \tilde{\sigma}_*^\beta f_j d\lambda =: \frac{C\|f\|_1}{\eta} + \frac{C}{\eta}\sum_{i=1}^\infty III_j,
\end{aligned}$$

where

$$III_j := \int\limits_{I^j \times I^j} \tilde{\sigma}_*^\beta f_j d\lambda$$

$$= \int\limits_{I_{k_j}(u^j) \times I_{k_j}(u^j)} \sup_{n \in \mathbb{N}} \left| \int\limits_{I_{k_j}(u^j) \times I_{k_j}(u^j)} f_j(x) \left| K_{2^n}^{\beta_n}(y+x) \right| d\lambda(x^1, x^2) \right| d\lambda(y^1, y^2).$$

The forthcoming estimation of III_j is given by the help Lemma 4.10

$$III_j \leq C\|f_j\|_1.$$

That is, operator $\tilde{\sigma}_*^\beta$ is of weak type (L^1, L^1) and same holds for operator σ_*^β. This completes the proof of Lemma 4.11. $\square$

Proof of Theorem 4.1. (Abu Joudeh and Gát [7]) Let $P \in \mathbf{P}$ be a two-dimensional Walsh polynomial, that is, $P(x) = \sum_{i,j=0}^{2^k-1} c_{i,j}\omega_i(x^1)\omega_j(x^2)$. Then for all natural number $m \geq 2^k$ we have that $S_{m,m}P \equiv P$. Consequently, the statement $\sigma_{2^n}^{\beta_n} P \to P$ holds everywhere. This follows from the fact that for any fixed j it holds $\frac{A_{2^n-j}^{\beta_n-1}}{A_{2^n}^{\beta_n}} \to 0$ since for instance for $j = 1$ we have $\frac{A_{2^n-1}^{\beta_n-1}}{A_{2^n}^{\beta_n}} = \frac{\beta_n 2^n}{(2^n-1+\beta_n)(2^n+\beta_n)} \to 0$.

Now, let $\eta, \epsilon > 0$, $f \in L^1(I^2)$. Let $P \in \mathbf{P}$ be a two-dimensional Walsh polynomial such that $\|f - P\|_1 < \eta$. Then by the already seen method we get

$$\begin{aligned}
&\lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} f - f| > \epsilon) \\
&\leq \lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n}(f - P)| > \frac{\epsilon}{3}) + \lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} P - P| > \frac{\epsilon}{3}) \\
&+ \lambda(\overline{\lim_{n\in\mathbb{N}}}|P - f| > \frac{\epsilon}{3}) \\
&\leq C\|P - f\|_1 \frac{3}{\epsilon} \\
&\leq \frac{C}{\epsilon}\eta
\end{aligned}$$

because σ_*^β is of weak type (L^1, L^1). This holds for all $\eta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$\lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} f - f| > \epsilon) = 0$$

and consequently we also have

$$\lambda(\overline{\lim_{n\in\mathbb{N}}}|\sigma_{2^n}^{\beta_n} f - f| > 0) = 0.$$

This finally gives $\sigma_{2^n}^{\beta_n} f \longrightarrow f$ a.e. This completes the proof of Theorem 4.1. $\square$

Chapter 5

Summary

The present thesis talks about convergence of Cesàro means with variable parameters for Walsh-Fourier series. It consists of an introduction, four chapters, an abstract and a bibliography. In the introduction, we present some important and well-known notions and definitions related to the new results appearing in the thesis. Moreover, we present some historical background.

From Chapter 2 to 4 we discuss some specific results with respect to the convergence of Cesàro means with variable parameters for the Walsh-Fourier series. Since in Chapter 1 we have already summarized our basic tools and concepts, we do not repeat them here. All results are quoted from the Thesis with the same numbering. If a theorem, lemma, corollary or proposition is not new, we mention the name of the original author right at the beginning of the statement. If a result of ours have been already published, we cite the publication also at the beginning of the theorem.

In 1800's Jean Baptiste Joseph Fourier began to work on the theory of heat. In 1822, he published book with tittle of Théorie Analytic de la Chaleur (The Analytic Theory of Heat).

A great deal of effort has been expended after this work in this research area. It became and called Fourier theory and field of harmonic analysis. Fourier theory gained exceptional importance in theoretical content and also enormous scope and great relevance everywhere in applications such as electrical engineering.

One of the greatest achievements of mathematics in the twentieth century is the result of Carleson. In 1966 he prved the almost everywhere convergence of the partial sums of the (trigonometric) Fourier series of a square integrable function. On the other, hand in 1926 Kolmogoroff [5] gave the construction of an integrable function with everywhere divergent trigonometric Fourier series. That is, if we want to have some pointwise convergence result for each function belonging to the Lebesgue space L^1 then it is needed to use some summation method. The invention of Fejér [11] was to use the arithmetical means of the partial sums. Among others, he proved for continuous functions that these means converge to the function in the supreumum norm. One year later, Lebesgue proved the almost everywhere convergence of these so-called Fejér means to the function for each integrable function. That is, the behavior of the Fejér (or also called $(C, 1)$) means is better than the behavior of the partial sums in this point of view. This fact also justifies the investigation of various summation methods of Fourier series. Later on, we write about the (C, α) summation - which is a generalization of the Fejér summation- of Fourier series. The result of Lebesgue above for the (C, α) case ($\alpha > 0$) is due to M. Riesz [33].

Moreover, Fourier analyis has been developed on other structures too. For example, the dyadic group is the simplest but nontrivial model of the complete product of finite groups. Representing the characters of the dyadic group ordered in the Paley's sense, we obtain the Walsh system.

A relatively new thing of the generalizations on the Walsh-Paley system is the Vilenkin system introduced by Vilenkin [37] in 1947. He used the set of all characters of the complete product of arbitrary cyclic groups to obtain the commutative case.

In Hungary a dyadic analysis team works leaded by F. Schipp having many results in this theory. For instance, he proved that the partial sums of the Vilenkin-Fourier series (even in the unbounded case) of a function in $L^p(G)$ $(1 < p < 1)$ converge in the appropriate norm to the function (Schipp [29], Simon [34]). And also Young [41] from Canada .

With respect to noncommutative Vilenkin groups (complete direct product of not necessarily Abelian groups) some studies were appeared in [14] by Gát and Toledo. They obtained not only negative results for this situation. They proved the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case.

In ***Chapter two***, we introduced the notion of Cesàro means of Fourier series with variable parameters. We proved the almost everywhere convergence of a subsequnce of the Cesàro (C, α_n) means of integrable functions. That is, $\sigma_{2^n}^{\alpha_{2^n}} f \to f$ for $f \in L^1(I)$, where I is the unit interval (representing the dyadic, or Walsh group) for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$.

The main theorems of this chapter was proving:
Theorem 2.1. Suppose that $1 > \alpha_n > 0$. Let $f \in L^1(I)$. Then we have the a.e convergence $\sigma_{2^n}^{\alpha_{2^n}} f \longrightarrow f$.

The method we used to prove Theorem 2.1 is to investigated the maximal operator $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_{2^n}^{\alpha_{2^n}} f|$. We also proved that this operator is of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,
Theorem 2.2. Suppose that $1 > \alpha_n > 0$. Let $f \in H(I)$. Then we have

$$\|\sigma_*^\alpha f\|_1 \leq C\|f\|_H.$$

Moreover, the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

$$\|\sigma_*^\alpha f\|_p \leq C_p\|f\|_p \ \text{ for all } \ 1 < p \leq \infty.$$

Basically, in order to proved Theorem 2.1 we verified that the maximal operator $\sigma_*^\alpha f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we got this, the investigation of kernel functions, and its maximal function on the unit interval I by making a hole around zero. To have the proof of Theorem 2.2 is the standard way after having the fact that $\sigma_*^\alpha f$ is of weak type (L^1, L^1).

In ***Chapter three*** we introduced the notion of Cesàro means of Fourier series with variable parameters. We proved the almost everywhere convergence of the Cesàro (C, α_n) means of integrable functions $\sigma_n^{\alpha_n} f \to f$ for each $f \in L^1(I)$, where $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. Provided that for some restriction set (discussed below) $\mathbb{N}_{\alpha,K} \ni n \to \infty$, where K is any but fixed natural number.

Set two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$. For instance $P(n, 1) = n$. Also set for sequences $\alpha = (\alpha_n)$ and positive reals K the subset of natural numbers

$$\mathbb{N}_{\alpha,K} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.$$

We can easily remark that for a sequence α such that $1 > \alpha_n \geq \alpha_0 > 0$ we have $\mathbb{N}_{\alpha,K} = \mathbb{N}$ for some K depending only on α_0. We also remark that $2^n \in \mathbb{N}_{\alpha,K}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $K \geq 1$.

In this chapter C denotes an absolute constant and C_K another one which may depend only on K. The introduction of (C, α_n) means of Fourier series due to Akhobadze (although for numerical series Kaplan published a paper in [27] 1960) investigated [1] the behavior of the L^1-norm convergence of

$\sigma_n^{\alpha_n} f \to f$ for the trigonometric system. In this chapter it is also supposed that $1 > \alpha_n > 0$ for all n.

The main theorems of this chapter was proving:

Theorem 3.1. Suppose that $1 > \alpha_n > 0$. Let $f \in L^1(I)$. Then we have the almost everywhere convergence $\sigma_n^{\alpha_n} f \to f$ provided that $\mathbb{N}_{\alpha,K} \ni n \to \infty$.

The method we used to proved Theorem 3.1 is to investigate the maximal operator $\sigma_*^\alpha f := \sup_{n\in\mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f|$. We also proved that this operator is a kind of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

Theorem 3.2. Suppose that $1 > \alpha_n > 0$. Let $|f| \in H(I)$. Then we have

$$\|\sigma_*^\alpha f\|_1 \leq C_K \||f|\|_H.$$

Moreover, the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

$$\|\sigma_*^\alpha f\|_p \leq C_{K,p}\|f\|_p, \;\; for\ all\ 1 < p \leq \infty.$$

For the sequence $\alpha_n = 1$ Theorem 3.2 is due to Fujii [12]. For the more general but constant sequence $\alpha_n = \alpha_1$ see Weisz [38].

Basically, in order to prove Theorem 3.1 we verified that the maximal operator $\sigma_*^\alpha f$ $(\alpha = (\alpha_n))$ is of weak type (L^1, L^1). The way we get this is the investigation of kernel functions and their maximal function on the unit interval I by making a hole around zero. Besides, some quasi locality issue (for the notion of quasi-locality see [32]). To have the proof of Theorem 3.2 is the standard way.

In ***Chapter four***, we formulated and proved that the maximal operator of some (C, β_n) means of cubical partial sums of two variable Walsh-Fourier series of integrable functions is of weak type (L^1, L^1). Moreover, the (C, β_n)-means $\sigma_{2^n}^{\beta_n} f$ of the function $f \in L^1$ converge a.e. to f for $f \in L^1(I^2)$, for some sequences $1 > \beta_n \searrow 0$.

We supposed that (α_n) and (β_n) sequences are monotone decreasing sequences and they satisfy:

$$\beta_n = \alpha_{2^n}, \quad \frac{\alpha_N}{A_N^{\alpha_N}} \log^\delta\left(1 + \frac{N}{n}\right) \leq C\frac{\alpha_n}{A_n^{\alpha_n}} \; (N \geq n, n, N \in \mathbb{P}) = \mathbb{N} \setminus \{0\}$$

for some $\delta > 1$ and for some positive constant C. We remark that from the condition above it follows that sequence $(\frac{\alpha_n}{A_n^{\alpha_n}})$ is quasi monotone decreasing. That is, for some $C > 0$ we have $\frac{\alpha_N}{A_N^{\alpha_N}} \leq C\frac{\alpha_n}{A_n^{\alpha_n}}$ $(N \geq n, n, N \in \mathbb{P})$.

The main theorem of this chapter is:

Theorem 4.1. Suppose that monotone decreasing sequence $1 > \beta_n > 0$ satisfies the condition $\frac{A_{2^n}^{\beta_n}}{\beta_n}\frac{\beta_N}{A_{2^N}^{\beta_N}}(N+1-n)^\delta \leq C$ for every $\mathbb{N} \ni N \geq n \geq 1$ and for some $\delta > 1$. Let $f \in L^1(I^2)$. Then we have the almost everywhere convergence

$$\sigma_{2^n}^{\beta_n} f \to f.$$

Remark 4.2. In the proof of Theorem 4.1 we defined the sequence (α_n) in a way that $\alpha_{2^k} = \beta_k$ and for any $2^k \leq n < 2^{k+1}$ let $\alpha_n = \alpha_{2^k} = \beta_k$. Then the sequence (α_n) satisfies that it is decreasing and $\frac{A_n^{\alpha_n}}{\alpha_n}\frac{\alpha_N}{A_N^{\alpha_N}} \log^\delta\left(1 + \frac{N}{n}\right) \leq C$ for every $\mathbb{N} \ni N \geq n \geq 1$. That is, condition above is fulfilled.

- We give two examples for sequences (β_n) like above. Example one: $\beta_k = \alpha_{2^k} = \alpha_n = \alpha \in (0,1)$ for every natural number k, n.

- Example two: Let $\alpha_n = 1/n$. Then it is not difficult to have that $A_n^{\alpha_n} \to 1$ and for the sequence (α_n) $CN/n \geq \log^\delta(1+N/n)$ trivially holds with some $\delta > 1$.

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   IF: 0.468

Idegen nyelvű tudományos közlemények külföldi folyóiratban (1)

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További közlemények

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A közlő folyóiratok összesített impakt faktora: 0,986
A közlő folyóiratok összesített impakt faktora (az értekezés alapjául szolgáló közleményekre): 0,986

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