SHORT THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (PHD)

Comparison and characterization problems in general classes of means

by

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The results described in the dissertation and in this thesis have been published in the following three papers: [**47**], [**48**], and [**49**].

Introduction

A function $M : I^d \to I$ is called a *d-variable mean* on an interval I if for all $(x_1, \dots, x_d) \in I^d$, the following so-called mean value property

$$\min(x_1, \dots, x_d) \le M(x_1, \dots, x_d) \le \max(x_1, \dots, x_d)$$

holds.

The goal of *Chapter 1* is to construct a general class of means based on a Chebyshev system, a measurable family of d-variable means, and a probability measure. In order to define this class of means, we recall the notions of Chebyshev system and measurable family of d-variable means, respectively.

We say that the pair (f, g) forms a *(two-dimensional) Chebyshev system on I* if, for any distinct elements x, y of I, the determinant

$$\mathcal{D}_{f,g}(x, y) := \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix}$$

is different from zero. If, for $x < y$, this determinant is positive, then (f, g) is called a *positive system*, otherwise we call (f, g) a *negative system*.

We say that $m : I^d \times T \to I$ is a *measurable family of d-variable means on I* if

(H1) I is a nonvoid open real interval.
(H2) $(T, \mathcal{A})$ is a measurable space, where $\mathcal{A}$ is the σ-algebra of measurable sets of T.
(H3) For all $t \in T$, $m(\cdot, t)$ is a d-variable mean on I.
(H4) For all $\boldsymbol{x} \in I^d$, the function $m(\boldsymbol{x}, \cdot)$ is measurable over T.

If (H1) and (H3) hold, and additionally we have that

(H2+) T is a topological space and $\mathcal{A}$ equals the σ-algebra $\mathcal{B}(T)$ of the Borel sets of T.

(H4+) For all $\boldsymbol{x} \in I^d$, the function $m(\boldsymbol{x}, \cdot)$ is continuous over T,

then $m : I^d \times T \to I$ will be called a *continuous family of d-variable means on I*.

The following lemma is the key to construct a mean in terms of a Chebyshev system, a measurable family of means, and a probability measure.

LEMMA. *Let $m : I^d \times T \to I$ be a measurable family of d-variable means, let μ be a probability measure on $(T, \mathcal{A})$ and let (f, g) be a Chebyshev system on I. Then, for all $\boldsymbol{x} \in I^d$, there exists a unique element $y \in I$ such that*

$$\int_T \mathcal{D}_{f,g}(m(\boldsymbol{x}, t), y) \, \mathrm{d}\mu(t) = 0. \tag{1}$$

In addition, if g is positive and f/g is strictly monotone, then

$$y = \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_T f\big(m(\boldsymbol{x}, t)\big) \, \mathrm{d}\mu(t)}{\int_T g\big(m(\boldsymbol{x}, t)\big) \, \mathrm{d}\mu(t)} \right).$$

The above lemma allows us to define a d-variable mean $M_{f,g,m;\mu} : I^d \to I$. Given $\boldsymbol{x} \in I^d$, let $M_{f,g,m;\mu}(\boldsymbol{x})$ denote the unique solution y of equation (1). In the particular case when g is positive and f/g is strictly monotone, for all $\boldsymbol{x} \in I^d$, we have that

$$M_{f,g,m;\mu}(\boldsymbol{x}) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_T f\big(m(\boldsymbol{x}, t)\big) \, \mathrm{d}\mu(t)}{\int_T g\big(m(\boldsymbol{x}, t)\big) \, \mathrm{d}\mu(t)} \right). \tag{2}$$

This mean will be called a *d-variable generalized Bajraktarević mean* in the sequel.

In the case when m is a two-variable family of weighted arithmetic means, this class of means was introduced and their comparison problem was also solved in the paper [**34**]. When $g = 1$, then

$$M_{f,1,m;\mu}(\boldsymbol{x}) = f^{-1} \left(\int_T f\big(m(\boldsymbol{x}, t)\big) \, \mathrm{d}\mu(t) \right) \qquad (\boldsymbol{x} \in I^d),$$

which will be termed a *d-variable generalized quasi-arithmetic mean*.

To define the *d-variable generalized Gini means*, let $p, q \in \mathbb{C}$ such that either $p, q \in \mathbb{R}$ or $p = \bar{q} = a + bi,\ a, b \in \mathbb{R},\ b \neq 0$ (this holds if and only if $p + q$ and pq are real numbers or, equivalently, if p and q are the roots of a second degree polynomial with real coefficients). For $x \in \mathbb{R}_+$ define

(3)
$$(f(x), g(x)) := \begin{cases} (x^p, x^q) & \text{if } p, q \in \mathbb{R},\ p \neq q, \\ (x^p \log(x), x^p) & \text{if } p = q \in \mathbb{R}, \\ \big(x^a \sin(b \log(x)), x^a \cos(b \log(x))\big) & \\ & \text{if } p = \bar{q} = a + bi \notin \mathbb{R}. \end{cases}$$

The d-variable generalized Gini mean $G_{p,q,m;\mu}$ is defined now to be the d-variable generalized Bajraktarević mean $M_{f,g,m;\mu}$, where f and g are given by (3). Let I be an open subinterval of $\mathbb{R}_+$ such that I is contained in the open interval $\big]\exp\big(-\frac{\pi}{2|b|}\big), \exp\big(\frac{\pi}{2|b|}\big)\big[$ if $p = \bar{q} = a + bi \notin \mathbb{R}$. Then the d-variable generalized Gini mean $G_{p,q,m;\mu}$ is of the form:

$$\begin{cases} \left(\dfrac{\int_T \big(m(\boldsymbol{x}, t)\big)^p \,\mathrm{d}\mu(t)}{\int_T \big(m(\boldsymbol{x}, t)\big)^q \,\mathrm{d}\mu(t)} \right)^{\frac{1}{p-q}} \\ \qquad\qquad\qquad \text{if } p, q \in \mathbb{R},\ p \neq q, \\ \exp\left(\dfrac{\int_T \big(m(\boldsymbol{x}, t)\big)^p \log\big(m(\boldsymbol{x}, t)\big) \,\mathrm{d}\mu(t)}{\int_T \big(m(\boldsymbol{x}, t)\big)^p \,\mathrm{d}\mu(t)} \right) \\ \qquad\qquad\qquad \text{if } p = q \in \mathbb{R}, \\ \exp\left(\dfrac{1}{b} \arctan\left(\dfrac{\int_T \big(m(\boldsymbol{x}, t)\big)^a \sin\big(b \log\big(m(\boldsymbol{x}, t)\big)\big) \,\mathrm{d}\mu(t)}{\int_T \big(m(\boldsymbol{x}, t)\big)^a \cos\big(b \log\big(m(\boldsymbol{x}, t)\big)\big) \,\mathrm{d}\mu(t)} \right) \right) \\ \qquad\qquad\qquad \text{if } p = \bar{q} = a + bi \notin \mathbb{R}. \end{cases}$$

If $q = 0$ in the above formula, then we get the definition of *d-variable generalized Hölder means* as follows: for all $\boldsymbol{x} \in \mathbb{R}_+^d$, we get

$$H_{p,m;\mu}(\boldsymbol{x}) := \begin{cases} \left(\dfrac{\int_T \big(m(\boldsymbol{x},t)\big)^p \,\mathrm{d}\mu(t)}{\int_T \mathrm{d}\mu(t)} \right)^{\frac{1}{p}} & \text{if } p \in \mathbb{R} \setminus \{0\}, \\ \exp\left(\dfrac{\int_T \log\big(m(\boldsymbol{x},t)\big) \,\mathrm{d}\mu(t)}{\int_T \mathrm{d}\mu(t)} \right) & \text{if } p = 0. \end{cases}$$

The particular cases of these general means have been extensively investigated by Aczél–Daróczy [**1**], Daróczy–Losonczi [**13**], Losonczi [**20**], [**21**], [**22**], [**23**], [**24**], Losonczi–Páles [**34**], [**35**], and Páles [**43**]. For further reading, we refer to [**5**], [**6**], [**15**], [**16**], [**19**], [**25**], [**36**], [**41**], [**45**], [**50**], [**51**].

Notations

For the sake of convenience and brevity, we introduce the following notations. The class $\mathcal{C}_0(I)$ consists of all those pairs of continuous functions $f, g : I \to \mathbb{R}$ that form a Chebyshev system over I.

If $n \geq 1$, then we say that the pair (f, g) is in the class $\mathcal{C}_n(I)$ if f, g are n-times continuously differentiable functions such that $(f, g) \in \mathcal{C}_0(I)$ and the *Wronski determinant*

$$\begin{vmatrix} f'(x) & f(x) \\ g'(x) & g(x) \end{vmatrix} = \partial_1 \mathcal{D}_{f,g}(x, x) \qquad (x \in I)$$

does not vanish on I.

For $(f, g) \in \mathcal{C}_2(I)$, the functions $\Phi_{f,g}, \Psi_{f,g} : I \to \mathbb{R}$ are defined by

$$\Phi_{f,g} := \frac{\begin{vmatrix} f'' & f \\ g'' & g \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}} \qquad \text{and} \qquad \Psi_{f,g} := -\frac{\begin{vmatrix} f'' & f' \\ g'' & g' \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}}.$$

If $\varphi : I^d \times T \to \mathbb{R}$, and for some $\boldsymbol{x} \in I^d$, the map $t \mapsto \varphi(\boldsymbol{x}, t)$ is μ-integrable, then we write

$$\langle \varphi \rangle_\mu(\boldsymbol{x}) := \int_T \varphi(\boldsymbol{x}, t)\, \mathrm{d}\mu(t).$$

For a μ-integrable function $\varphi : T \to \mathbb{R}$ define $\varphi^* : T \to \mathbb{R}$ by

$$\varphi^*(t) := \varphi(t) - \langle \varphi \rangle_\mu.$$

For a measurable family of d-variable means $m : I^d \times T \to I$, for all $\boldsymbol{x} \in I^d$, we introduce the notations:

$$\underline{m}(\boldsymbol{x}) := \inf_{t \in T} m(\boldsymbol{x}, t) \qquad \text{and} \qquad \overline{m}(\boldsymbol{x}) := \sup_{t \in T} m(\boldsymbol{x}, t).$$

Let $\mathcal{C}_1(I^d \times T)$ denote the class of measurable families of d-variable means $m : I^d \times T \to I$ with the following additional property:

(H5) For every $t \in T$, the function $m(\cdot, t)$ is continuously partially differentiable over I^d such that, for all $\boldsymbol{p} \in I^d$, $i \in \{1, \dots, d\}$, the function $\partial_i m$ is of L^1-type at $\boldsymbol{p}$.

Analogously, we define $\mathcal{C}_2(I^d \times T)$ to be the following subclass of $\mathcal{C}_1(I^d \times T)$:

(H6) For every $t \in T$, the function $m(\cdot, t)$ is twice continuously partially differentiable over I^d such that, for all $\boldsymbol{p} \in I^d$ and $i, j \in \{1, \dots, d\}$, the function $\partial_i m$ is of L^2-type and $\partial_i \partial_j m$ is of L^1-type at $\boldsymbol{p}$.

Similarly, we define $\mathcal{C}_3(I^d \times T)$ to be the following subclass of $\mathcal{C}_2(I^d \times T)$:

(H7) For every $t \in T$, the function $m(\cdot, t)$ is three times continuously partially differentiable over I^d such that, for all $\boldsymbol{p} \in I^d$ and $i, j, l \in \{1, \dots, d\}$, the function $\partial_i m$ is of L^3-type, $\partial_i \partial_j m$ is of $L^{\frac{3}{2}}$-type, and $\partial_i \partial_j \partial_l m$ is of L^1-type at $\boldsymbol{p}$.

For $m \in \mathcal{C}_1(I^d \times T)$ and $r \in \{1, \dots, d\}$, denote

$$\partial_r^* m(\boldsymbol{x}, t) := \partial_r m(\boldsymbol{x}, t) - \langle \partial_r m \rangle_\mu(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d,\ t \in T),$$

and, for $x \in I$, set

$$x^{(d)} := (x, \dots, x) \in I^d.$$

For the homogeneity problem, we will use the following notations. Given a nonempty open subinterval I of $\mathbb{R}_+$ and $c > 0$, introduce the following notations:

$$cI := \{cx \mid x \in I\} \qquad \text{and} \qquad I/I := \{x/y \mid x, y \in I\}.$$

These sets are also open subintervals of $\mathbb{R}_+$ and the interval I/I is logarithmically symmetric with respect to 1, i.e., $u \in I/I$ holds if and only if $1/u \in I/I$. It is also easy to see that the intersection $I_\lambda := I \cap \left(\frac{1}{\lambda} I\right)$ is nonempty if and only if $\lambda \in I/I$. For a function $f : I \to \mathbb{R}$ and number $\lambda > 0$, the function $f_\lambda : \left(\frac{1}{\lambda} I\right) \to \mathbb{R}$ is defined by

$$f_\lambda(x) = f(\lambda x).$$

Finally, for a real parameter $p \in \mathbb{R}$, introduce the sine and cosine type functions $S_p, C_p : \mathbb{R} \to \mathbb{R}$ by

$$\big(S_p(x), C_p(x)\big) := \begin{cases} (\sin(\sqrt{-p}x), \cos(\sqrt{-p}x)) & \text{if } p < 0, \\ (x, 1) & \text{if } p = 0, \\ (\sinh(\sqrt{p}x), \cosh(\sqrt{p}x)) & \text{if } p > 0. \end{cases}$$

Comparison problem of generalized Bajraktarević means

The aim of *Chapter 2* is to study the *global comparison problem*

$$M_{f,g,m;\mu}(\boldsymbol{x}) \le M_{h,k,n;\nu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d) \tag{4}$$

and also its *local* analogue. In terms of the Chebyshev systems (f, g) and (h, k), the measurable families of d-variable means $m : I^d \times T \to I$ and $n : I^d \times S \to I$, and the measures μ, ν, we give necessary conditions (which, in general, are not sufficient) and also sufficient conditions (that are also necessary in a certain sense) for (4) to hold. Our main results generalize that of the paper by Losonczi and Páles [**34**] and also many former results obtained in various particular cases of this problem, cf. [**11**], [**12**], [**13**], [**26**], [**28**], [**42**], [**44**].

Our first result offers a necessary as well as a sufficient condition for the local comparison of means. Given two d-variable means $M, N : I^d \to I$, we say that M *is locally smaller than* N *at* $x_0 \in I$ if there exists a neighborhood $U \subseteq I$ of x_0 such that

$$M(\boldsymbol{x}) \leq N(\boldsymbol{x})$$

holds for all $\boldsymbol{x} \in U^d$. The case $d = 1$ being trivial, we always assume that $d \geq 2$ holds in the subsequent considerations.

THEOREM. *Let $M, N : I^d \to I$ be d-variable means such that M is locally smaller than N at a point $x_0 \in I$. Assume that M and N are partially differentiable at the diagonal point $x_0^{(d)} = (x_0, \dots, x_0) \in I^d$. Then, for $x = x_0$ and for all $i \in \{1, \dots, d\}$,*

$$\partial_i M\big(x^{(d)}\big) = \partial_i N\big(x^{(d)}\big). \tag{5}$$

If, in addition, M and N are twice differentiable at $x_0^{(d)} \in I^d$, then the symmetric $(d-1) \times (d-1)$-matrix

$$\big(\partial_i \partial_j N\big(x_0^{(d)}\big) - \partial_i \partial_j M\big(x_0^{(d)}\big)\big)_{i,j=1}^{d-1} \tag{6}$$

is positive semidefinite.

On the other hand, if, for some $x_0 \in I$, the equality (5) *holds for all $i \in \{1, \dots, d\}$ and for all x in a neighborhood of x_0, furthermore, M and N are twice continuously differentiable at $x_0^{(d)}$ and the symmetric $(d-1) \times (d-1)$-matrix given by* (6) *is positive definite, then M is locally smaller than N at x_0.*

COROLLARY. *Let $(f, g), (h, k) \in \mathcal{C}_1(I)$, let $m \in \mathcal{C}_1(I^d \times T)$ and $n \in \mathcal{C}_1(I^d \times S)$ be measurable families of means, and let μ and ν be probability measures on the measurable spaces $(T, \mathcal{A})$ and $(S, \mathcal{B})$, respectively. Suppose that $M_{f,g,m;\mu}$ is locally smaller* *than $M_{h,k,n;\nu}$ at $x_0 \in I$. Then, there exists a neighborhood $U \subseteq I$ of x_0 such that for $x \in U$ and for all $i \in \{1, \dots, d\}$,*

$$\langle \partial_i m \rangle_\mu \big(x^{(d)}\big) = \langle \partial_i n \rangle_\nu \big(x^{(d)}\big). \tag{7}$$

If, in addition, $(f,g),(h,k) \in \mathcal{C}_2(I)$, $m \in \mathcal{C}_2(I^d \times T)$, and $n \in \mathcal{C}_2(I^d \times S)$, then the $(d-1) \times (d-1)$-matrix whose (i,j)th entry is given by

$$\begin{aligned} &\big\langle \partial_i^* n\, \partial_j^* n \big\rangle_\nu \big(x_0^{(d)}\big) \Phi_{h,k}(x_0) + \langle \partial_i \partial_j n \rangle_\nu \big(x_0^{(d)}\big) \\ &\qquad - \big\langle \partial_i^* m\, \partial_j^* m \big\rangle_\mu \big(x_0^{(d)}\big) \Phi_{f,g}(x_0) - \langle \partial_i \partial_j m \rangle_\mu \big(x_0^{(d)}\big) \end{aligned} \tag{8}$$

for $i, j \in \{1, \dots, d-1\}$ is positive semidefinite.

On the other hand, if $(f,g),(h,k) \in \mathcal{C}_2(I)$, $m \in \mathcal{C}_2(I^d \times T)$, $n \in \mathcal{C}_2(I^d \times S)$, and (7) *holds for all $i \in \{1, \dots, d\}$ and for all x in a neighborhood of x_0 and the $(d-1) \times (d-1)$-matrix whose (i,j)th entry is given by* (8) *is positive definite, then $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,n;\nu}$ at $x_0 \in I$.*

In the special setting when $T = [0,1]$, $d = 2$, m is given by $m((x,y),t) := tx + (1-t)y$, the above corollary simplifies to the result of [**34**, Theorem 5]. Now we consider the particular case when the families of means m and n as well as the measures μ and ν coincide.

COROLLARY. *Let $(f,g),(h,k) \in \mathcal{C}_2(I)$, let $m \in \mathcal{C}_2(I^d \times T)$ be a measurable family of means, and let μ be a probability measure on the measurable space $(T, \mathcal{A})$. Let $x_0 \in I$ and assume that there exists $i \in \{1, \dots, d-1\}$ such that, the map $t \mapsto \partial_i m(x_0^{(d)}, t)$ is not μ-almost everywhere constant on T. If $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,m;\mu}$ at $x_0 \in I$, then*

$$\Phi_{f,g}(x_0) \le \Phi_{h,k}(x_0). \tag{9}$$

On the other hand, if the functions

$$t \mapsto \partial_i m(x_0^{(d)}, t) - \langle \partial_i m \rangle_\mu \big(x_0^{(d)}\big) \qquad (i \in \{1, \dots, d-1\})$$

are μ-linearly independent and (9) *holds with strict inequality, then $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,m;\mu}$ at $x_0 \in I$.*

Now we consider the case when $\mu = \nu$ and $m = n$. In what follows, we give a condition containing two independent variables for the comparison problem which does not involve the measure μ and assumes first-order continuous differentiability of the Chebyshev system. In the special setting when $T = [0,1]$, $d = 2$, m is given by $m((x,y),t) := tx + (1-t)y$, the following theorem simplifies to the result of [**34**, Theorem 6].

THEOREM. *Let $(f,g),(h,k) \in \mathcal{C}_1(I)$ be Chebyshev systems, let T be a compact and connected topological space and let $m : I^d \times T \to \mathbb{R}$ be a continuous family of d-variable means. Define the set U_m by*

$$\begin{aligned} U_m :&= \{(u,v) \mid \exists \boldsymbol{x} \in I^d : u,v \in [\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})]\} \\ &= \bigcup_{\boldsymbol{x} \in I^d} [\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})]^2. \end{aligned}$$

The following three assertions are equivalent:

(i) For all Borel probability measures μ on T,

$$M_{f,g,m;\mu}(\boldsymbol{x}) \leq M_{h,k,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d).$$

(ii) There exists a nullsequence (γ_j) of positive numbers in $[0,1]$ such that, for all $t_0, t \in T$ and for all $j \in \mathbb{N}$,

$$M_{f,g,m;(1-\gamma_j)\delta_{t_0}+\gamma_j\delta_t}(\boldsymbol{x}) \leq M_{h,k,m;(1-\gamma_j)\delta_{t_0}+\gamma_j\delta_t}(\boldsymbol{x}) \quad (\boldsymbol{x} \in I^d).$$

(iii) For all $(u,v) \in U_m$,

$$\frac{\mathcal{D}_{f,g}(u,v)}{\partial_1 \mathcal{D}_{f,g}(v,v)} \leq \frac{\mathcal{D}_{h,k}(u,v)}{\partial_1 \mathcal{D}_{h,k}(v,v)}.$$

In the next result we offer 6 equivalent conditions for the comparison of d-variable generalized quasi-arithmetic means. The interesting feature of this result is the equivalence of the global and local comparability.

THEOREM. *Let $f, h : I \to \mathbb{R}$ be twice continuously differentiable functions with non-vanishing first derivatives, and let $m \in \mathcal{C}_2(I^d \times T)$ be a measurable family of d-variable means. Let μ_0 be a probability measure such that, for all $x_0 \in I$, there exists $i \in \{1, \dots, d-1\}$ such that $t \mapsto \partial_i m(\boldsymbol{x}_0, t)$ is not μ_0-almost everywhere constant on T. The following assertions are equivalent:*

(i) For all Borel probability measures μ on T,

$$M_{f,1,m;\mu}(\boldsymbol{x}) \le M_{h,1,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d).$$

(ii)

$$M_{f,1,m;\mu_0}(\boldsymbol{x}) \le M_{h,1,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d).$$

(iii) For all $x_0 \in I$, there exists a neighborhood $U \subseteq I$ of x_0 such that

$$M_{f,1,m;\mu_0}(\boldsymbol{x}) \le M_{h,1,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in U^d).$$

(iv) For all $x \in I$,

$$\frac{f''(x)}{f'(x)} \le \frac{h''(x)}{h'(x)}.$$

(v) The function $h \circ f^{-1}$ is convex (concave) on $f(I)$ provided that f is increasing (decreasing).

(vi) For all $(u, v) \in I^2$,

$$\frac{f(u) - f(v)}{f'(v)} \le \frac{h(u) - h(v)}{h'(v)}.$$

As an immediate consequence, we obtain the characterization of the comparison among generalized Hölder means.

COROLLARY. *Let $I \subseteq \mathbb{R}_+$, $p, q \in \mathbb{R}$, and let $m \in \mathcal{C}_2(I^d \times T)$ be a measurable family of d-variable means. Let μ_0 be a probability measure such that, for all $x_0 \in I$, there exists $i \in \{1, \dots, d-1\}$ such that $t \mapsto \partial_i m(\boldsymbol{x}_0, t)$ is not μ_0-almost everywhere constant on T. The following assertions are equivalent:*

(i) For all Borel probability measures μ on T,

$$H_{p,m;\mu}(\boldsymbol{x}) \le H_{q,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d).$$

(ii)

$$H_{p,m;\mu_0}(\boldsymbol{x}) \le H_{q,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d).$$

(iii) For all $x_0 \in I$, there exists a neighborhood $U \subseteq I$ of x_0 such that

$$H_{p,m;\mu_0}(\boldsymbol{x}) \le H_{q,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in U^d).$$

(iv) $p \le q$.

Equality and homogeneity of generalized Bajraktarević means

In *Chapter 3*, we study the equality and the homogeneity problems of these means, i.e., to find conditions for the generating functions (f, g) and (h, k), for the family of means m, and for the measure μ such that the functional equation

$$M_{f,g,m;\mu}(\boldsymbol{x}) = M_{h,k,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d)$$

and the homogeneity property

$$M_{f,g,m;\mu}(\lambda\boldsymbol{x}) = \lambda M_{f,g,m;\mu}(\boldsymbol{x}) \qquad (\lambda > 0,\, \boldsymbol{x}, \lambda\boldsymbol{x} \in I^d),$$

respectively, be satisfied. Our main results generalize that of the paper by Losonczi and Páles [**35**], Losonczi [**33**] and also many former results obtained in various particular cases of this problem, cf. [**1**], [**7**], [**8**], [**10**], [**24**], [**26**], [**27**], [**28**], [**29**], [**30**], [**31**], [**32**], [**37**], [**46**].

The following theorem characterize the equality of generalized Bajraktarević means under 3 times differentiability assumptions.

THEOREM. *Let $(f, g), (h, k) \in \mathcal{C}_3(I)$, let $m \in \mathcal{C}_3(I^d \times T)$ be a measurable family of means, and let μ be a probability measure* *on the measurable space* $(T,\mathcal{A})$*. Assume that, there exists a dense subset* $D\subseteq I$ *such that, for all* $x\in D$*,*

$$\mu\Big(\big\{t\in T \mid \partial_1^* m\big(x^{(d)},t\big)=\cdots=\partial_d^* m\big(x^{(d)},t\big)=0\big\}\Big)<1, \tag{10}$$

and there exist $i,j,l\in\{1,\dots,d\}$ *such that*

$$\big\langle \partial_i^* m\, \partial_j^* m\, \partial_l^* m\big\rangle_\mu\big(x^{(d)}\big)\neq 0.$$

Then the following assertions are equivalent:

(i) For all $\boldsymbol{x}\in I^d$*,*

$$M_{f,g,m;\mu}(\boldsymbol{x})=M_{h,k,m;\mu}(\boldsymbol{x}). \tag{11}$$

(ii) There exists an open set $U\subseteq I^d$ *containing the subdiagonal* $\{x^{(d)}\mid x\in D\}$ *such that, for all* $\boldsymbol{x}\in U$*, the equality* (11) *holds.*

(iii) The two identities

$$\Phi_{f,g}=\Phi_{h,k}\qquad \textit{and}\qquad \Psi_{f,g}=\Psi_{h,k}$$

hold.

(iv) The pairs (f,g) *and* (h,k) *are equivalent.*

In the next corollary, we consider the particular case of the previous theorem when the measurable family m is given in the form

$$m(\boldsymbol{x},t)=\varphi_1(t)x_1+\cdots+\varphi_d(t)x_d\qquad (\boldsymbol{x}\in I^d,\ t\in T). \tag{12}$$

COROLLARY. *Let* $(f,g),(h,k)\in\mathcal{C}_3(I)$*, let* μ *be a probability measure on the measurable space* $(T,\mathcal{A})$*, let* $\varphi_1,\dots,\varphi_d: T\to[0,1]$ *be* μ*-measurable functions with* $\varphi_1+\cdots+\varphi_d=1$ *and define the measurable family* $m: I^d\times T\to\mathbb{R}$ *by* (12)*. Assume that*

$$\mu\Big(\big\{t\in T\mid \varphi_1^*(t)=\cdots=\varphi_d^*(t)=0\big\}\Big)<1, \tag{13}$$

and there exist $i,j,l\in\{1,\dots,d\}$ *such that*

$$\big\langle \varphi_i^*\,\varphi_j^*\,\varphi_l^*\big\rangle_\mu\neq 0.$$

Then the following assertions are equivalent:

(i) *For all* $\boldsymbol{x} \in I^d$*, the equality* (11) *holds.*

(ii) *There exists a dense subset* $D \subseteq I$ *and an open set* $U \subseteq I^d$ *containing the subdiagonal* $\{x^{(d)} \mid x \in D\}$ *such that, for all* $\boldsymbol{x} \in U$*, the equality* (11) *holds.*

(iii) *The pairs* (f,g) *and* (h,k) *are equivalent.*

The next corollary concerns the case when $T = [0,1]$ and μ is a probability measure on the sigma algebra of Borel subsets of $[0,1]$. In this setting, define $\hat{\mu}_1$ to be the *first moment* and μ_n to be the n*th centralized moment* of the measure μ by

$$\hat{\mu}_1 := \int_{[0,1]} t \,\mathrm{d}\mu(t), \qquad \mu_n := \int_{[0,1]} (t - \hat{\mu}_1)^n \,\mathrm{d}\mu(t) \quad (n \in \mathbb{N}).$$

COROLLARY. *Let* $(f,g),(h,k) \in \mathcal{C}_3(I)$ *such that* g *and* k *do not vanish on* I*. Let* μ *be a probability measure on the sigma algebra of Borel subsets of* $[0,1]$ *with* $\mu_2 \neq 0$ *and* $\mu_3 \neq 0$*. Then the following assertions are equivalent:*

(i) *For all* $(x,y) \in I^2$*, the equality*

$$\begin{aligned}(14)\qquad &\left(\frac{f}{g}\right)^{-1}\left(\frac{\int_{[0,1]} f\big(tx + (1-t)y\big)\,\mathrm{d}\mu(t)}{\int_{[0,1]} g\big(tx + (1-t)y\big)\,\mathrm{d}\mu(t)}\right)\\ &\qquad = \left(\frac{h}{k}\right)^{-1}\left(\frac{\int_{[0,1]} h\big(tx + (1-t)y\big)\,\mathrm{d}\mu(t)}{\int_{[0,1]} k\big(tx + (1-t)y\big)\,\mathrm{d}\mu(t)}\right)\end{aligned}$$

holds.

(ii) *There exists a dense subset* $D \subseteq I$ *and an open set* $U \subseteq I^2$ *containing the subdiagonal* $\{(x,x) \mid x \in D\}$ *such that, for all* $(x,y) \in U$*, the equality* (14) *holds.*

(iii) *The pairs* (f,g) *and* (h,k) *are equivalent.*

The next corollary concerns the equality of nonsymmetric weighted two-variable Bajraktarević means.

COROLLARY. *Let $(f,g),(h,k) \in \mathcal{C}_3(I)$ such that g and k do not vanish on I. Let $s \in]0,\frac{1}{2}[\cup]\frac{1}{2},1[$. Then the following assertions are equivalent:*

(i) For all $(x,y) \in I^2$, the equality

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{sf(x)+(1-s)f(y)}{sg(x)+(1-s)g(y)}\right) = \left(\frac{h}{k}\right)^{-1}\left(\frac{sh(x)+(1-s)h(y)}{sk(x)+(1-s)k(y)}\right)$$

holds.

(ii) There exists a dense subset $D \subseteq I$ and an open set $U \subseteq I^2$ containing the subdiagonal $\{(x,x) \mid x \in D\}$ such that, for all $(x,y) \in U$, the above equality holds.

(iii) The pairs (f,g) and (h,k) are equivalent.

In the following results, we are going to characterize the equality of generalized quasi-arithmetic means in various settings.

THEOREM. *Let $f,g : I \to \mathbb{R}$ be twice continuously differentiable functions such that f' and g' do not vanish on I. Let $m \in \mathcal{C}_2(I^d \times T)$ be a measurable family of means, and let μ be a probability measure on the measurable space $(T,\mathcal{A})$. Assume that, there exists a dense subset $D \subseteq I$ such that, for all $x \in D$, condition* (10) *holds. Then the following assertions are equivalent:*

(i) For all $\boldsymbol{x} \in I^d$,

$$f^{-1}\left(\int_T f\big(m(\boldsymbol{x},t)\big)\,\mathrm{d}\mu(t)\right) = g^{-1}\left(\int_T g\big(m(\boldsymbol{x},t)\big)\,\mathrm{d}\mu(t)\right).$$

(ii) There exists an open set $U \subseteq I^d$ containing the subdiagonal $\{x^{(d)} \mid x \in D\}$ such that, for all $\boldsymbol{x} \in U$, the above equality holds.

(iii) The functions f''/f' and g''/g' are identical on I.

(iv) There exist real constants a,b such that $g = af + b$.

COROLLARY. *Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions such that f' and g' do not vanish on I, let μ be a probability measure on the measurable space $(T, \mathcal{A})$, let $\varphi_1, \dots, \varphi_d : T \to [0,1]$ be μ-measurable functions with $\varphi_1 + \cdots + \varphi_d = 1$ such that condition* (13) *holds. Then the following assertions are equivalent:*

(i) For all $(x_1, \dots, x_d) \in I^d$,

$$f^{-1}\left(\int_T f\big(\varphi_1(t)x_1 + \cdots + \varphi_d(t)x_d\big)\,\mathrm{d}\mu(t)\right) = g^{-1}\left(\int_T g\big(\varphi_1(t)x_1 + \cdots + \varphi_d(t)x_d\big)\,\mathrm{d}\mu(t)\right).$$

(ii) There exists an open set $U \subseteq I^d$ containing the subdiagonal $\{x^{(d)} \mid x \in D\}$ such that, for all $(x_1, \dots, x_d) \in U$, the above equality holds.

(iii) There exist real constants a, b such that $g = af + b$.

The following consequence of the previous corollary has been dealt with in the paper [**37**, Theorem 7]. There f and g are assumed only to be continuous, however, the equivalence to condition (ii) is missing.

COROLLARY. *Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions such that f' and g' do not vanish on I. Let μ be a probability measure on the sigma algebra of Borel subsets of $[0,1]$ with $\mu_2 \neq 0$. Then the following assertions are equivalent:*

(i) For all $(x, y) \in I^2$, the equality

$$f^{-1}\left(\int_T f\big(tx + (1-t)y\big)\,\mathrm{d}\mu(t)\right) = g^{-1}\left(\int_T g\big(tx + (1-t)y\big)\,\mathrm{d}\mu(t)\right)$$

holds.

(ii) There exists a dense subset $D \subseteq I$ and an open set $U \subseteq I^2$ containing the subdiagonal $\{(x,x) \mid x \in D\}$ such that, for all $(x,y) \in U$, the above equality holds.

(iii) There exist real constants a, b such that $g = af + b$.

The next statement is related to the equality problem of weighted two-variable quasi-arithmetic means. We note that the equivalence of conditions (i) and (iii) can be obtained under the assumption of continuity of the generating functions f and g. For further and important particular cases of the previous corollary, we refer to the examples elaborated in the paper [**37**].

COROLLARY. *Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions such that f' and g' do not vanish on I. Let $s \in]0,1[$. Then the following assertions are equivalent:*

(i) For all $(x,y) \in I^2$, the equality

$$f^{-1}(sf(x) + (1-s)f(y)) = g^{-1}(sg(x) + (1-s)g(y))$$

holds.

(ii) There exists a dense subset $D \subseteq I$ and an open set $U \subseteq I^2$ containing the subdiagonal $\{(x,x) \mid x \in D\}$ such that, for all $(x,y) \in U$, the above equality holds.

(iii) There exist real constants a, b such that $g = af + b$.

In the second part of Chapter 3, we characterize the homogeneity of generalized Bajraktarević and generalized quasi-arithmetic means under 3 times and 2 times differentiability assumptions, respectively.

A d-variable mean $M : I^d \to \mathbb{R}$ is called *homogeneous* if, for all $\lambda \in I/I$ and for all $\boldsymbol{x} \in I^d_\lambda$,

$$M(\lambda \boldsymbol{x}) = \lambda M(\boldsymbol{x}).$$

THEOREM. *Let $(f,g) \in \mathcal{C}_3(I)$, let $m \in \mathcal{C}_3(I^d \times T)$ be a homogeneous measurable family of means, and let μ be a probability measure on the measurable space $(T, \mathcal{A})$. Assume that there* *exists a point $x_0 \in I$ such that*

(15)
$$\mu\Big(\big\{t \in T \mid \partial_1^* m\big(x_0^{(d)}, t\big) = \cdots = \partial_d^* m\big(x_0^{(d)}, t\big) = 0\big\}\Big) < 1,$$

and there exist $i, j, l \in \{1, \ldots, d\}$ such that

$$\big\langle \partial_i^* m \, \partial_j^* m \, \partial_l^* m \big\rangle_\mu \big(x_0^{(d)}\big) \neq 0.$$

Then the following assertions are equivalent:

(i) $M_{f,g,m;\mu}$ *is homogeneous.*

(ii) For all $\lambda \in I/I$ and for all $\boldsymbol{x} \in I_\lambda^d$,

$$M_{f,g,m;\mu}(\boldsymbol{x}) = M_{f_\lambda, g_\lambda, m;\mu}(\boldsymbol{x}).$$

(iii) For all $\lambda \in I/I$, the pairs (f, g) and (f_λ, g_λ) are equivalent on the interval I_λ.

(iv) For all $\lambda \in I/I$ and for all $x \in I_\lambda$,

$$\Phi_{f,g}(x) = \Phi_{f_\lambda, g_\lambda}(x) \qquad \text{and} \qquad \Psi_{f,g}(x) = \Psi_{f_\lambda, g_\lambda}(x).$$

(v) There exist two real numbers α, β such that $y = f$ and $y = g$ are solutions of the second-order linear differential equation

$$y''(x) = \frac{\alpha}{x} y'(x) + \frac{\beta}{x^2} y(x) \qquad (x \in I).$$

(vi) There exists a pair $(p, q) \in \{(z, w) \in \mathbb{C}^2 \mid z + w, zw \in \mathbb{R}\}$ such that $M_{f,g,m;\mu}$ is equal to the d-variable generalized Gini mean $G_{p,q,m;\mu}$.

THEOREM. *Let $f : I \to \mathbb{R}$ be a twice continuously differentiable function such that f' does not vanish on I. Let $m \in \mathcal{C}_2(I^d \times T)$ be a homogeneous measurable family of means, and let μ be a probability measure on the measurable space $(T, \mathcal{A})$. Assume that condition* (15) *holds. Then the following assertions are equivalent:*

(i) $M_{f,1,m;\mu}$ *is homogeneous.*

(ii) *For all* $\lambda \in I/I$ *and for all* $\boldsymbol{x} \in I_\lambda^d$,

$$M_{f,1,m;\mu}(\boldsymbol{x}) = M_{f_\lambda,1,m;\mu}(\boldsymbol{x}).$$

(iii) *For all* $\lambda \in I/I$, *there exist real constants* a_λ, b_λ *such that* $f_\lambda(x) = a_\lambda f(x) + b_\lambda$ *holds for all* $x \in I_\lambda$.

(iv) *For all* $\lambda \in I/I$, *the functions* f''/f' *and* f_λ''/f_λ' *are identical on* I_λ.

(v) *There exists a real number* α *such that* $y = f$ *is a solution of the second-order linear differential equation*

$$y''(x) = \frac{\alpha}{x} y'(x) \qquad (x \in I).$$

(vi) *There exists a real number* p *such that* $M_{f,1,m;\mu}$ *is equal to the* d*-variable generalized Hölder mean* $H_{p,m;\mu}$.

Invariance equation

Given three strict means $M, N, K : \mathbb{R}_+^2 \to \mathbb{R}_+$, we say that the triple (M, N, K) satisfies the *invariance equation* if

$$K(M(x,y), N(x,y)) = K(x,y) \qquad (x, y \in \mathbb{R}_+)$$

holds. If this equation is valid, then we say that K is invariant with respect to the mean-type mapping (M, N). The easiest example when the invariance equation is satisfied is the well-known identity

$$\sqrt{xy} = \sqrt{\frac{x+y}{2} \cdot \frac{2xy}{x+y}} \qquad (x, y \in \mathbb{R}_+).$$

The last identity means that

$$G(x,y) = G(A(x,y), H(x,y)) \qquad (x, y \in \mathbb{R}_+),$$

where A, G, and H are the two-variable arithmetic, geometric, and harmonic means, respectively.

In *Chapter 4*, we investigate the invariance of the arithmetic mean with respect to two weighted Bajraktarević means, i.e., to

solve the functional equation
(16)
$$\left(\frac{f}{g}\right)^{-1}\left(\frac{tf(x)+sf(y)}{tg(x)+sg(y)}\right)+\left(\frac{h}{k}\right)^{-1}\left(\frac{sh(x)+th(y)}{sk(x)+tk(y)}\right)=x+y,$$

where $f,g,h,k:I\to\mathbb{R}$ are unknown continuous functions such that g,k are nowhere zero on I, the ratio functions f/g, h/k are strictly monotone on I, and $t,s\in\mathbb{R}_+$ are constants different from each other. By the main result of this chapter, the solutions of the above invariance equation can be expressed either in terms of hyperbolic functions or in terms of trigonometric functions and an additional weight function. For the necessity part of this result, we will assume that $f,g,h,k:I\to\mathbb{R}$ are four times continuously differentiable.

The invariance equation in more general classes of means has been studied by several authors in a large number of papers. The invariance equation of Hölder mean solved completely by Daróczy and Páles [**14**]. The more general invariance equation for quasi-arithmetic means was first solved under infinitely many times differentiability by Sutô [**52**], [**53**] and later by Matkowski [**38**] under twice continuous differentiability. Without imposing unnecessary regularity conditions, this problem was finally solved by Daróczy and Páles [**14**].

Burai [**9**] and Jarczyk-Matkowski [**18**] studied the invariance equation involving three weighted arithmetic means. Jarczyk [**17**] solved this problem without additional regularity assumptions. The invariance of the arithmetic mean with respect to Lagrangian mean has been investigated by Matkowski (cf. [**40**]). The invariance of the arithmetic, geometric, and harmonic means has been studied by Matkowski [**39**].

Baják and Páles [**3**] describes the invariance of the arithmetic mean with respect to generalized quasi-arithmetic means. Recently, Baják and Páles [**2**], [**4**] have solved the invariance equations of two-variable Gini and Stolarsky means, respectively.

Now we are in the position to formulate the main result of this chapter.

THEOREM. *Let $p \in \mathbb{R}$, let $\varphi : I \to \mathbb{R}_+$ be a positive continuous function and let $(f,g),(h,k) \in \mathcal{C}_0(I)$ such that*

(17)
$$(f,g) \sim (S_p/\varphi, C_p/\varphi) \qquad \text{and} \qquad (h,k) \sim (S_p \cdot \varphi, C_p \cdot \varphi).$$

Then, for all $s,t \in \mathbb{R}_+$, the invariance equation (16) *holds.*

Conversely, let $(f,g),(h,k) \in \mathcal{C}_4(I)$ and $t,s \in \mathbb{R}_+$ with $t \neq s$ such that the functional equation (16) *be valid. Then there exist a positive 4 times continuously differentiable function $\varphi : I \to \mathbb{R}_+$ and a real parameter $p \in \mathbb{R}$ such that the equivalences* (17) *are satisfied.*

As a consequence of the previous theorem, Głazowska Dorota in the 56th International Symposium on Functional Equations presented the invariance of the two-variable quasi-arithmetic mean with respect to two weighted Bajraktarević means, i.e., to solve the functional equation

(18)
$$A_\psi \circ \big(B_{f,g}((x,y),(t,s)), B_{h,k}((x,y),(s,t))\big) = A_\psi(x,y),$$

where, $x,y \in I$, or equivalently,

$$\psi\big(B_{f,g}((x,y),(t,s))\big) + \psi\big(B_{h,k}((x,y),(s,t))\big) = \psi(x) + \psi(y).$$

COROLLARY. *Let $p \in \mathbb{R}$, let $\varphi : I \to \mathbb{R}_+$ be a positive continuous function, let $\psi : I \to \mathbb{R}$ be a continuous strictly monotone function, and let $(f,g),(h,k) \in \mathcal{C}_0(I)$ such that*

$$(19) \qquad \begin{aligned} &(f,g) \sim \big((S_p/\varphi) \circ \psi, (C_p/\varphi) \circ \psi\big), \\ &(h,k) \sim \big((S_p \cdot \varphi) \circ \psi, (C_p \cdot \varphi) \circ \psi\big). \end{aligned}$$

Then, for all $s,t \in \mathbb{R}_+$, the invariance equation (18) *holds.*

Conversely, let $\psi : I \to \mathbb{R}$ be a 4 times continuously differentiable function, let $(f,g),(h,k) \in \mathcal{C}_4(I)$ and $t,s \in \mathbb{R}_+$ with $t \neq s$ such that the functional equation (18) *be valid. Then there exist a* *positive 4 times continuously differentiable function* $\varphi : I \to \mathbb{R}_+$ *and a real parameter* $p \in \mathbb{R}$ *such that the equivalences* (19) *are satisfied.*

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List of talks

(1) *Characterization of the equality of Bajraktarević means to quasiarithmetic means*, 16th International Student Conference on Analysis Síkfőkút, Hungary, February 1–4, 2020.
(2) *On the equality of Cauchy means to quasiarithmetic means*, 20th Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities Hajdúszoboszló, Hungary, January 29–February 1, 2020.
(3) *Equality problems related to Cauchy means*, 18th International Conference on Functional Equations and Inequalities, Bedlewo, Poland, June 9-15, 2019.
(4) *On the equality of Bajraktarević means to quasi-arithmetic means*, 57th International Symposium on Functional Equations, Jastarnia, Poland June 2–9, 2019.
(5) *Equality problems related to Cauchy means*, Annual Seminar on Analysis, Síkfőkút, Hungary, May 24–26, 2019.
(6) *On the equality of Bajraktarević means to quasi-arithmetic means*, Analysis Research Seminar, Institute of Mathematics, University of Debrecen, May 8, 2019.
(7) *On the equality of Bajraktarević means to quasi-arithmetic means*, 2nd Debrecen Spring Workshop on Analysis, Institute of Mathematics, Debrecen, Hungary, April 4, 2019.

(8) *On the equality problem of two variable Cauchy means*, 15th International Student Conference on Analysis Ustroń, Poland, February 2–5, 2019.
(9) *Equality of Bajraktarević means with Cauchy means*, 19th Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Zakopane, Poland, January 30–February 2, 2019.
(10) *Invariance of a symmetric Bajraktarević mean with respect to two nonsymmetric Bajraktarević means*, Numbers, Functions, Equations 2018, Hajdúszoboszló, Hungary, August 26–September 01, 2018.
(11) *On the local and global comparison of generalized Bajraktarević means*, Annual Seminar on Analysis, Síkfőkút, Hungary, June 8–10, 2018.
(12) *On the invariance equation for two-variable weighted nonsymmetric Bajraktarević means*,1st Debrecen Spring Workshop on Analysis, Institute of Mathematics, Debrecen, Hungary, April 4, 2018.
(13) On the invariance equation for two-variable weighted nonsymmetric Bajraktarević means, 14th International Student Conference on Analysis Síkfőkút, Hungary, February 3–6, 2018.
(14) *Equality and homogeneity of generalized Bajraktarević means*, 18th Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities Hajdúszoboszló, Hungary, January 31–February 3, 2018.
(15) *On the local and global comparison of generalized Bajraktarević means*, 12th International Symposium on Generalized Convexity and Monotonicity, Hajdúszoboszló, Hungary, August 27–September 02, 2017.
(16) *On the local and global comparison of generalized Bajraktarević means*, Analysis Research Seminar, Institute of Mathematics, University of Debrecen, May 7, 2017.

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List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (1)

1. Páles, Z., **Zakaria, A.**: Equality and homogeneity of generalized integral means.
   Acta math. Hung. Epub ahead of print, [1-32], 2019. ISSN: 0236-5294.
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Foreign language scientific articles in international journals (2)

2. Páles, Z., **Zakaria, A.**: On the invariance equation for two-variable weighted nonsymmetric Bajraktarević means.
   Aequ. Math. 93 (1), 37-57, 2019. ISSN: 0001-9054.
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3. Páles, Z., **Zakaria, A.**: On the local and global comparison of generalized Bajraktarević means.
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   DOI: http://dx.doi.org/10.1016/j.jmaa.2017.05.073
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List of other publications

Foreign language international book chapters (3)

4. **Zakaria, A.**: Measuring Soft Roughness of Soft Rough Sets Induced by Covering.
   In: Thriving Rough Sets. Ed.: Guoyin Wang, Andrzej Skowron, Yiyu Yao, Dominik Dominik Ślęzak, Lech Polkowski, Springer International Publishing Ag, Cham, 269-281, 2017.
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9. **Zakaria, A.**, John, S. J., Girish, K. P.: Multiset filters.
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