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Developing Mathematical Problem Solving Abilities and Skills

With Cooperative Teaching Techniques

Matematikai problémamegoldó képességek és készségek fejlesztése

kooperatív tanulásszervezési technikákkal

Doktori (PhD) Értekezés

Barczi - Veres Krisztina

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Tanúsítom, hogy Barczi-Veres Krisztina doktorjelölt 2009 - 2016 között a fent megnevezett Doktori Iskola Matematika-Didaktika programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Nyilatkozom továbbá arról, hogy a tézisekben leírt eredmények nem képezik más PhD disszertáció részét.

Az értekezés elfogadását javasolom.

Debrecen, 20. . . . .

Dr. Ambrus András

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Köszönöm témavezetőmnek, Dr. Ambrus Andrásnak a dolgozat megírásához nyújtott inspiráló segítségét.
Köszönöm édesanyámnak a matematika és a matematika tanítása iránti szeretetet.
Köszönöm férjemnek a dolgozat megírásához nyújtott támogatását.

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1 Introduction

In Hungarian classrooms we face the following problem more and more often: students are less independent when it comes to solving mathematical problems individually, furthermore based on discussions with teachers of other science subjects we can say that this tendency is unfortunately present not only in mathematics lessons. Many students are passive listeners when the teacher presents a new material, they try to follow the explanations or the solution of a sample problem but when they have to solve similar or totally different problems on their own the majority looks helpless. A lot of students find it challenging to see which idea leads to the solution and many have difficulties with communicating their ideas correctly and checking their solutions or their way of thinking which according to Pólya [65] is a crucial step in problem solving. In spite of this these steps are usually completely missing from the students' work.

In Hungarian secondary schools mathematics is mainly taught in heterogeneous classes which means that talented students work together with the average ability ones and the low-achievers. In this environment the teacher needs to ensure some kind of success for all students regardless of their mathematical ability. For helping all students experience success effective differentiation is needed in the classroom, which is not always easy to accomplish. For this recognizing talented students in mathematics is definitely necessary. As Miller [58] says students who are talented in mathematics have an ability to understand even complex mathematical ideas and can use mathematical reasoning very well, but they are not necessary the students who are good at arithmetic calculations or have top grades in Maths. Furthermore, maths talents often have emotions related to numbers and they have an interest in the subject in their early years already. They like solving puzzles and they are successful in completing tests that measure memory or visual skills. Moreover, these students often use specific procedures for solving problems.[35]

Furthermore, differentiation can be difficult as different students have different problem solving preferences, which does not only depend on their mathematical ability but also on their personality, their motivation ... etc, so there is no rule which tells us that for talented students or low-achievers we should use a specific approach. For high ability students fast track or acceleration programs might be beneficial especially if talent is matched with motivation. On the other hand, slower or less motivated students might do better if the learning pace is slower and the learning focuses deliberately on the mathematical concepts being taught.[58]

The main aims of teaching mathematics include teaching effective problem solving skills; to make students understand mathematical concepts and their relations and to help them acquire thinking skills that can be used in problem solving regardless of the content, both in the field of mathematics and in other areas as well.[9] Understanding and learning a mathematical concept is only the beginning of a process. In order to become successful problem solvers students need to be able to apply what they learnt. Moreover, the ability to generalize certain statements and the need for proof needs to be developed in secondary school mathematics, as well. One of the many reasons for this is that the previously mentioned thinking skills are required in higher education.

Besides helping students understand mathematics and helping them be able to use what they learnt in different problem solving situations it is important that they experience the joy of thinking.[81] For average ability students it is slightly more difficult to show that mathematical thinking can be similar to playing with different thoughts but if the teacher manages to make these students feel that doing maths can be creative and enjoyable then they can become more successful.

Talented or average ability ones, all students gain from learning in a mathematically enriched environment. For this social interactions between students should be encouraged, the appearance of emotional connections should be facilitated, the students' individual style should be taken into consideration and the development of students' self-confidence should be emphasized.[22]

Learning will be maximized when: 1) students are actively engaged in mathematical activity and exploration; 2) students rely on their existing knowledge and experience; 3) students are encouraged to use and develop their own strategies; 4) students are not afraid of making mistakes; 5) students can communicate with the others effectively about mathematics.[78]

Taking everything into consideration choosing a right teaching method is not always easy. Many teachers try using various teaching techniques to help their students' development. Although more and more alternative teaching methods are appearing in Hungarian schools the most widely used teaching method is still frontal teaching where differentiation is not impossible but sometimes is really challenging. Even students with similar ability can have different working pace or some are slower in grasping concepts and ideas even if they are good at actually solving problems.

It was mentioned before that having great problem solving skills is important not only in school but also later, in everyday life. The students' future life and work will include endless problems that need to be solved. Furthermore, having experience with working in groups and cooperating with others will also be beneficial in their future life as most workplaces require team work. These are the reasons why the idea of experimenting with cooperative teaching techniques, problem solving skills and open problems occurred.

In Hungary cooperative teaching techniques are already widely used and are becoming more and more wide spread, however, there are not enough experiments examining the effectiveness of this method in case of secondary school students. That is why the question occurred how effectively cooperative techniques could be used in secondary mathematics education within this in teaching problem solving skills.

This dissertation describes an experiment that was carried out in a secondary school class. The duration of the experiment was one academic year and there were 16 secondary school students participating. The teacher of the class was the writer of this dissertation. In the first part of the experiment there were 12 consecutive lessons planned only with cooperative methods. In the second part of the experiment the students took part in lessons planned with cooperative methods once every two - three weeks depending on the scheme of work. Furthermore, the students filled out a mathematical test and an attitude test before the first part, a mathematical test between the two parts and a mathematical test, an attitude test and a questionnaire related to cooperative work at the end of the school year.

The main aims of the experiment were:

- to develop students' problem solving skills
- to encourage mathematical communication between students
- to try teaching techniques - cooperative teaching, guiding questions - that allow students with different mathematical ability to do their best in the subject
- to try teaching techniques that allow students to work and understand mathematics in their own pace.

2 Research questions

1. How does the regular use of cooperative teaching techniques affect
   - the mathematical problem solving skills of secondary school students?
   - the attitude of the students to mathematics?
   - the relationship of the students and their attitude to each other?
2. How does the use of open problems affect the students' problem solving ability?
3. How do cooperative techniques affect the working memory?
4. How do open problems affect the working memory?
5. How does combining cooperative techniques with using open problems affect the students' problem solving ability?

3 Theoretical background

3.1 Problem solving

3.1.1 Problems

There are many definitions of a *problem*. For example Fisher [33] defines a problem as a task in which we know some data and some conditions based on which we try to find a solution. Finding the solution is often impeded by different obstacles that are not always obvious to recognize. In sum, Fisher says that a problem is a composition of:

- What is given - conditions, data, context
- Obstacles - the solution process is often unknown at the beginning which is an obstacle
- Aims - what we want to reach; the solution
- Effort - trial or conscious work that helps us solve a problem

It can be seen that for Fisher *problem* is a rather broad concept. Based on the above mentioned definition an argument, any small everyday affair but also a dangerous situation can be considered as a problem.

Another definition for a *problem* is from Lénárd [53] who says that a problem is a situation in which we want to reach a certain aim but the way of reaching this aim is hidden. Furthermore, solving one task might be a problem for someone but not a problem for another person, therefore the concept of a problem is subjective.

Mayer and Hegarty [56] define a *problem* as a task where on the basis of some known data and conditions we are trying to find a solution. First of all, we need to distinguish between routine and non-routine problems. Routine problems are the ones where the problem solver knows and recognizes the correct process for the solution at once, whereas in case of non-routine problems the solver does not see the way of solution immediately. Of course, the final goal is to find the solution although it is often impeded by different obstacles which depend on many factors such as the age, the ability etc. of the problem solver. Recognizing and overcoming these obstacles is not always obvious.

Pólya [67] describes *problems* similarly. He says that if we have a problem it means that we try to find some means with the help of which we can reach a clearly stated but not necessary easy to reach aim. All the above mentioned definitions regard problem as quite a broad concept and they agree on that in problem solving we know where to start and in most cases we know where we want to end up but the "HOW" is yet unknown for the problem solver.

Differentiating between tasks and problems is also essential. The former means a situation in which the way to the solution is well-known, however the latter refers to a situation in which the solver does not know the way leading to the solution. This indicates that the concept of a problem is person and time specific as the conditions, the aptitude vary from time to time and from person to person. When teaching problem solving problems often transform to tasks with practice.[51]

Types of problems

Problems can be classified based on different points of view. Fisher [33] groups problems in the following way:

- **Real problems** - eg.: A delivery boy has to deliver pizza to 20 different homes in our town. Which route is the shortest?
- **Realistic problems** - eg.: Find and mark the possible routes on the map.
- **Concrete problems** - eg.: Make a model of the streets that can be found in our town and "walk" along them with a marker.
- **Indirect problems** - eg.: Summarize in a story where exactly the pizza has to be delivered and what the best way of doing it is.
- **Abstract problems** - eg.: Mark and number the possible routes in our street model.

Kirkley [50] differentiates between types of problems based on how well the pieces of information needed for the solution are described in the text of the problem and places problems on a *problem continuum* spreading from well-structured through moderately structured to ill-structured problems.

- **Well - structured** problems always use the same step - by - step solution. Their solution is similar to following a recipe thus their solution strategy is predictable. There is one right solution and all necessary information can be found in the starting statement.
- **Moderately structured** problems require varying problem solving strategies. There is only one right answer, however the necessary information need to be collected and there are more solution strategies.
- **Ill - structured** problems are vague with unclear goals and not constrained solution strategies. There are multiple solutions and necessary information need to be collected.

In Kontra's [51] work we can find several other ways of differentiating between problems. He talks about **semantically rich** and **semantically poor** problems depending on the amount of relevant knowledge the problem solver possesses. Furthermore, problems can be grouped based on how well the following three aspects are defined: a) initial conditions; b) aim; c) possible means of reaching the aim. When all three of these are clearly given then we speak about **well-defined**, otherwise about **ill-defined** problems. He also quotes the work of Borasi, R. – *On the nature of problems. Educational Studies in Mathematics, 17. 125–141.(1986)* – who categorized educational related problems listing their main features as well in the following way:

- **Practice:** The context of these problems is non-existing; the wording is straightforward and explicit; the solution is straightforward and exact; solution strategies contain the combination of known algorithms.
- **Word problem:** The context is explicit; the wording is straightforward and explicit; the solution is straightforward and exact; solution strategies contain the combination of known algorithms.
- **Puzzle problem:** The context is explicit; the wording is straightforward and explicit; the solution is straightforward and exact; solution strategies contain working out new algorithms, realizations and reinterpretations.
- **Proving a conjecture:** The context can be found in the text only partially and requires theoretical background knowledge; the wording is straightforward and explicit; the solution is usually straightforward but not necessarily; solution strategies contain exploring the context, reinterpretations and creating new algorithms.
- **Real problem:** The context can be found in the text only partially; the wording is partially given with many possible alternatives; many solutions are possible which are mainly approximate solutions; solution strategies contain exploring the context, reinterpretations and modelling.
- **Problem situation:** The context can be found in the text only partially and is problematic; the wording is mainly implicit; many solutions are possible; solution strategies contain exploring the context, reinterpretations and problem posing.
- **Situation:** The context can be found in the text and is not problematic; the wording is non-existing; the solution is creating a problem; solution strategies contain problem posing.

Yet another way of differentiating between problems is to look at the knowledge required to solve them. Based on this problems can be **knowledge-free**, like puzzles or **knowledge -filled** like geometry problems, various physics problems, programming problems etc.[23]

Pólya[65], [66], [67], who focused on mathematical problems said that the type of a problem should already refer to its solution method, so he mentioned that there are **problems to find** and **problems to prove**. Problems to find can be theoretical or practical problems but they can be concrete or abstract ones, too. The aim when solving these types of problems is to find the unknown, like a number in an algebra problem or a figure in a geometry problem. The main component of such problems are *the unknown, the data* and *the conditions*. On the other hand, the aim of problems to prove is to show the truth or the falseness of a clearly stated assertion. The main components here are *the hypothesis* and *the conclusion*.

Problems also can be grouped based on how well-explained their components are. Problems can be **open** or **closed**. As the majority of the problems used in the first part of our experiment are either open - problems or there is a possibility to make them open these types of problems play a vital role in our research. That is why we discuss them in more detail.

Open-problems

To define what open problems are consider closed problems. A problem is closed if both its starting situation and its goal situation are exactly explained. If at least one of the afore mentioned is not clearly explained, i.e. open than we talk about an open problem. (Figure 1)[61] With other words, open problems are problems (1) that are unsolved or (2) whose solution depends on the interpretation of the problem solver or (3) where more ways of solving are possible or (4) that suggest further questions and possible generalizations.[13] Investigations, real – life situations, projects or problem fields are considered as open problems according to Pehkonen [62] and he says that open-ended tasks can be created by problem posing, problem variations or by working with problem fields, as well. Open problems are often referred to as exploratory problems. In our experiment we used many investigations and often let the problem solvers interpret the given problems in their own ways.

starting situation \ goal situation	CLOSED (i.e. exactly explained)	OPEN
CLOSED (i.e. exactly explained)	closed problems	open-ended problems real-life situations investigations problem fields problem variations
OPEN	real-life situations problem variations	real-life situations problem variations projects problem posing

Figure 1: Open and closed problems [61]

The following problems are good examples of open-problems:

1. My vegetable garden is shaped like a rectangle. The perimeter of the garden is 30 meters. What might be the area of my garden?

2. Find two objects with the same mass but different volume.[78]

The main characteristics of open problems are: 1) they involve significant mathematics; 2) they elicit a range of responses; 3) their solution requires communication among students; 4)they should be clearly stated and 5) their assessment is not restricted to being *right* or *wrong*.[25]

The main benefits of using open mathematical problems are that they give an opportunity for children with different mathematical abilities to experience success; they allow students to progress in their own pace and with multiple solutions they provide a great base for mathematical discussions.[85] Solving open problems may increase students' needs for proving and justifying ideas that occur during problem solving so that they can return to the original problem and investigate it from a different aspect thus creating kind of a problem solving cycle.[38] Through solving open-problems students become active learners of mathematics and combining the use of open-problems with group work gives the opportunity to satisfy students curiosity to different extents.[78] The occurrence or development of the previously mentioned factors was important in our experiment.

The different ways of opening up problems are:[85]

- Remove the restriction

- Remove what is known
- Swap the known to the unknown and remove the restriction
- Remove the known and the restriction and change the unknown
- Change all the known, the unknown, and the restriction

3.1.2 Solving problems

Problem solving models

For developing problem solving skills the knowledge of different phases of problem solving is indispensable. There are many models that summarize and describe the steps required in a problem solving process. In one of the earliest models Wallas [84] outlined four phases of the process:

- **Preparation** - this phase means collecting resources, and pieces of information needed to solve the problem; it involves research, planning and preparing the problem solver's mind for the thinking process; it is conscious;
- **Incubation** - this is an unconscious process during which no direct effort is put into solving the problem at hand
- **Illumination** - it is the moment of "flash" or "click" when to idea of the solution is found ("Aha!")
- **Verification** - this phase is the conscious justification of the idea that occurred in the previous phase and the expression of this idea in an exact form (e.g. calculations)

Another well-known and effective model for problem solving is that of Dewey, who defined the following steps in reflective thinking:[2]

- Defining the problem
- Analysing the problem
- Determining criteria for optimal solution
- Proposing solution
- Evaluating proposed solution
- Selecting a solution

- Suggesting strategies to implement solution

The two models mentioned above describe problem solving in general but in the next section we focus on solving mathematical problems. When solving mathematical problems the following models need to be taken into consideration.

- Pólya's model

  Pólya [65] gave the following phases for problem solving:

  1. **Understanding the problem** - The first very important step in problem solving is understanding the given data, the conditions and understanding what needs to be found. Drawing figures to represent data and introducing suitable notations also belong to this step.
  2. **Devising a plan** - In this phase the problem solver tries to find the solution of the problem. For devising a plan the problem solver can choose from different problem solving and heuristic strategies like thinking backwards, indirect proof etc.
  3. **Carrying out the plan** - Going through the planned steps very carefully, always checking what we do.
  4. **Looking back** - This step does not simply mean checking whether the obtained answer is correct or not but also checking the strategy that was used in the solution.

- Schoenfeld's model

  Schoenfeld [9] added some extra suggestions to Pólya's model as it is often considered as too general.

  1. **Understanding the problem** - Here, this is also an important step but Schoenfeld gives more specific suggestions about how to start (see below: Guiding questions)
  2. **Draft for the plan of the solution** - First, try to solve simpler problems planning each step very thoroughly.
  3. **Finding a plan for more difficult problems** - Choosing the right heuristic strategy, again with more specific suggestions.
  4. **Review the solution** - Check whether you used every data and check for different solutions.

- Mason's model

  Mason's [54] model is similar to the above mentioned two models. The phases have different names, however their functions are familiar from

Processes	**Phases**	**Activities**
		I know
	Entry	I'm looking for
Specialization		Introduction
	Attack	STUCK! AHA!
Generalization		Check
	Review	Reflecting
		Extension

Table 1: Mason's problem solving phases

Pólya's and Schoenfeld's lists. It can be summarized in the following table:

The **Entry** phase involves reading the problem and preparing for solving the problem which means understanding what I know and what I want and choosing a notation. The **Attack** phase includes more complex mathematical activities and basically it lasts until the problem is solved or the solution is given up. Finally, the **Review** phase means checking the result, reflecting on the important ideas that occurred during the solution and thinking of possible extensions. In our experiment we paid special attention to the "Reflection" phase so we designed the lessons so that the students have plenty of opportunity to reflect on their own or on their classmates' work.

Problem solving strategies [9]

The knowledge of problem solving strategies is vital in the problem solving process. The following list contains problem solving strategies.

1. *Thinking forward:* starting from the conditions and the given data the problem solver proceeds towards the solution through a chain of steps. For example: solving equations, working with algebraic expressions, calculating the value of trigonometric expressions.

2. *Thinking backwards:* Starting from the aim/solution of the problem the problem solver works backwards trying to find the intermediate steps until reaching the starting conditions. For example: word problems like, *I think of a number then double it and add 4 to the result. The answer is 24. Which number did I think of?*

3. *Systematic trial:* the problem solver examines concrete values trying to find the solution with their help. For example: difficult equations can be solved with this method. [1]

The problems used in the first part of the experiment were selected so that all the aforementioned problem solving strategies could be used at some point of the problem solving process.

Heuristic principles

Heuristics is a discipline whose aim is to study the methods and rules of discovery and invention.[65] Heuristic is often regarded as "a simple procedure that helps find adequate, though often imperfect, answers to difficult questions." [48] Pólya, in his book **How To Solve It** mentions many heuristic principles that can be used effectively in problem solving.

1. *Analogy:* The best examples for using analogy are geometry problems involving solids. When solving these problems the solver often tries to work with analogous problems from plane geometry.

2. *Tracing back to a known problem:* A good example for using this principles is when solving higher degree equations the solver often introduces new variables thus reducing the equation to a quadratic equation whose solutions is known.

3. *The principle of invariance:* Applying this principle means looking for functional relationship between quantities and qualities.

4. *Case distinction:* One example for using this principle is when trying to eliminate cases that are irrelevant or impossible for the solution. Another example is examining different cases when solving divisibility problems.

5. *The principle of optimality:* In mathematical problems we often need to find the biggest possible/the smallest possible value for certain quantities.

6. *Special cases:* Especially when solving geometrical problems examining special cases or arrangements often helps think of the idea that will lead to the final solution.

7. *The principle of symmetry:* A very useful principle when working with algebraic expressions for example. The problem solver notices the symmetrical relationship of the variables in the task and uses this feature to arrive to the final solution. Another example is from geometry when we use line or point symmetry to solve problems.

[1] NB: this method is neglected in Hungarian mathematics education.

8. *The principle of transformation:* This principle means trying to solve algebraic problems by geometric means or trying to solve geometric problems by algebraic means.

Among many educators Pólya assumed that problem solving strategies and heuristic principles are learnable and teachable. However, there is no research that supports the theory that teaching these strategies leads to better problem solving.[80] Schoenfeld described Pólya's problem solving strategies as being too broad. He mentioned that the strategies seem right but nobody managed to teach students how to use these strategies effectively.[70]

3.1.3 Teaching and developing problem solving skills

When teaching problem solving the following aspects need to be taken into consideration. First, most classes are heterogeneous containing students with mixed abilities. It is the teacher's task to develop the problem solving skills of not only the talented but also the average ability students which is usually a more challenging task. Furthermore, problem solving plays an important role in every area of life that is why necessary skills should be taught not only for the sake of passing an exam. To become successful problem solvers in their future lives students need to acquire ways of thinking and need to develop a "thinking tool kit" which contains problem solving strategies. Solving countless practice problems without reflecting on the problem solving process is not enough.

Mathematical thinking

The essence of mathematical thinking does not lie in the knowledge of definitions, theorems, proof etc. but in solving problems and understanding the main points of problem situations. Mathematical thinking includes specializing, generalizing and convincing. Therefore learning to think mathematically is more than using mathematical tools. Moreover, having knowledge of certain thinking skills is still not enough, students have to be able to decide which thinking skill to use in a given situation. Recognizing, searching for and applying the relevant knowledge are also essential thinking skills.[82]

According to Schoenfeld [69]: *"Learning to think mathematically means (a) developing a mathematical point of view - valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure - mathematical sense-making."*

Mathematical thinking is one of the eight competencies required for "doing" mathematics efficiently and it includes *posing questions, understanding* *and handling concepts, abstracting and generalizing concepts, distinguishing between mathematical statements.* Mathematical competence means the ability to understand and use mathematics in different contexts or situations in which mathematics plays or could play a role. Mathematical competencies are the constituents of mathematical competence: 1) thinking mathematically; 2) posing and solving mathematical problems; 3) modelling mathematically; 4) reasoning mathematically; 5) representing mathematical objects and situations; 6) handling mathematical symbols and formulae; 7) communicating in, with and about mathematics; 8) use of aids and tools.[60]

As presented above there are a number of things that are required to do mathematics. The most significant attributes are: 1) number sense; 2) numerical and algorithmic ability; 3) ability to handle abstraction; 4) a sense of cause and effect; 5) the ability to construct and follow a casual chain of facts or events; 6) logical reasoning ability; 7) relational reasoning ability; 8) spatial reasoning ability.[32]

When teaching problem solving developing mathematical thinking should definitely be an aim as well.

Learning theories

To understand a bit more about teaching problem solving effectively consider some learning theories. There are many theories about how learning happens and what learning is.[28] One of the most significant theories is that of Pavlov's classical conditioning which says that learning is the process of exposing students to stimuli and their response to stimuli. Another theory says that external stimuli is not enough for successful learning but the active participation of the learner is needed. Dewey, however, says that learning happens through continuous problem solving. Related to this constructivist learning theory says that learners construct new knowledge for themselves connecting those to their already exiting knowledge thus forming a network of knowledge. If we accept the idea that learning happens through constructing meaning then the following two consequences need to be examined: 1) we have to focus on how the learner thinks about learning; 2) there is no knowledge independent of the meaning constructed by the learner.[37]

A further learning theory can be connected to Galperin [8] who says that cognitive processes and thinking skills develop through four phases:

1. **Material or materialized phase:** the learner uses concrete objects for performing operations or representing thinking processes

2. **Phase of loud speech:** the learner performs operations step by step while explaining aloud to someone else what he/she is doing

3. **Phase of loud speech for himself:** it is the same as the previous phase except for the learner talking to himself rather than explaining to someone else

4. **Phase of internal speech:** Performing the operations happens only in the mind

Teaching problem solving should happen through solving various problems with our students. The knowledge acquired this way is not only reproductive but also creative and applicable:[28] As students construct knowledge for themselves - find out new pieces of information, discover new connections - it will be more stable and lasting, moreover they gain proficiency in problem solving.

When talking about teaching problem solving Kirkley [50] mentions two types of knowledge, *declarative knowledge* and *procedural knowledge*. Declarative knowledge is the "know what", it includes factual knowledge, the knowledge of principles and concepts in a discipline, while procedural knowledge is the "know how". When teaching problem solving both types of knowledge should be developed. Furthermore, problem solving skills should be taught in a context and not as independent skills. Kirkley also suggests that the teacher should help students understand the problem; use students' errors to detect and correct misconceptions; ask questions to promote problem solving and to encourage reflection on problem solving strategies.

Transfer

The concept of transfer is necessary to be mentioned when talking about teaching problem solving skill. There are two modes of thought. One of them is an implicit mode which is effortless but inaccessible while the other mode is more explicit and there is a greater access to knowledge but it is more demanding. As the effortless processes are more effective the aim of learning should be their development. However, transferring from implicit to explicit knowledge requires high cognitive effort. There are two types of transfer: "low road" and "high road" transfer. The former refers to relatively automatic generalizations which are the results of continuous practice while the latter refers to transfer which is obtained through conscious abstraction.[82]

Metacognition

The concept of metacognition refers to the knowledge of our own cognitive processes and products like learning, or relevant information or data. Furthermore, it refers to active monitoring and consequent organization and variation of these processes.[86] Metacognition also means thinking about our own thinking which includes knowledge of our own thinking processes, control and beliefs, intuitions.[69] In problem solving metacognition plays a vital role as problem solvers need to decide how they divide their time between understanding the problem, creating a plan, deciding what to do and carrying out plans. To be able to do this they have to monitor and reflect on their cognitive processes. If an idea seems to work continue using it and checking the steps. If an idea does not seem to work make changes or try alternatives. For developing problem solving skills metacognitive activities should be taught as well but not only ideas in general but concrete methods should be presented to students showing them how they can control continuously their problem solving activities.[64]

Affective factors

When teaching problem solving the affective side is not negligible either. Secondary school students already have an attitude towards mathematics but this can be formed and changed. The way they react in problem solving situations depends on many factors including their previous experience in similar situations.[57] To help students develop a positive relationship with mathematics understanding plays an important role as it is very difficult and might be frustrating to accept a rule that they don't understand. Another important factor is the teacher-student relationship.[73]

According to Szendrei [81] to like mathematics 1) students need to feel that they are able to do it which is a source of feeling happy; 2) students need to feel that they have a chance to solve the set task and it is not too difficult for them; 3) students internal motivation should be developed; 4) the need for self-check has to be formed; 5) students should worry about themselves but focus on the task at hand.

As we can read in [9] Wittman suggests that when teaching problem solving 1) students should be set open-problems thus be encouraged to discover ideas on their own; 2) divergent way of thinking should be encouraged; 3) teachers should have a positive attitude, a constructive behaviour to mistakes; 4) students should be encouraged to use their intuition in problem solving.

The teaching technique used in our experiment has a greater potential to form positive feeling in the students' towards mathematical problem solving than the more traditional frontal teaching.

3.1.4 Creating more problems

When teaching problem solving it is good to let students try and have their own experience with the task at hand and encourage them not only to solve the problem but also to formulate and to find new problems, since formulating problems plays a vital role in problem solving. For example, the creation of a new problem often might lead to finding the solution for an already existing problem.

Furthermore, additional questions or problems can deepen the understanding of a solved problem. Allowing students to create their own problems might be a motivational factor and it also facilitates creative thinking.[87] The "new" problems should engage students in mathematical inquiry and activate them during problem solving. Problems with high level of cognitive demand contribute more to learning in class.[27] The problems used in the first part of our experiment could be extended which means that there was the opportunity for the students to change the original problem thus creating their own problems.

Clearly, the easiest way to start with is using already existing problems. One option for making new problems is to look back at an already solved problem. Here, the solution can suggest new problems or we can check whether changing the conditions of the original problem results in something new. Another way of posing problems from old ones is when we have not found the solution. In this case the problem solver usually breaks up the original task into smaller parts and attempts to solve these new, probably easier problems.[49], [65]

Problem solving does not need to end when a solution is found for a set problem. On the other hand, finding a solution can be the first step in the problem solving process. As Kilpatrick [49] agrees formulating new problems is also an important part of problem solving and it should be part of students' education as well. Finding sources for new problems is easier than we might think. First of all, if we look around in our life we can come up with many interesting problems. However, we need to be careful when trying to create a real-life like problem not to make it a "false" problem, for example talking about painting a cube-shaped room in a geometry problem has nothing to do with reality [81]. Moreover, solved problems can be used as sources for new problems, too and we can meet new problems being in the middle of a solution of another problem. According to Kilpatrick [49] the processes of problem formulating are 1) *Association*; 2) *Analogy*; 3) *Generalization* and 4) *Contradiction*. Association uses the idea that knowledge is represented as a network of ideas which can be used to generate further problems. The idea of using analogy can be found among Pólya's problem solving steps, furthermore it can be a good source of creating new problems. For example, checking whether Pythagoras' theorem applies in solid geometry.[65] Generalization is also a problem solving strategy but the process of inducing a general idea from several concrete instances can lead to numerous new problems. A very simple example: $2 \cdot 3 \cdot 4$ is divisible by 6; $3 \cdot 4 \cdot 5$ is divisible by 6; is it true that the product of any three consecutive number is divisible by 6? Contradictions are for example the *what-if-not* [77] types of problem which means that from a given statement we can obtain new problems by contradicting one or more parts of the assertion. The *what- if- not* strategy can contribute to revealing important aspects of mathematical thinking to our students which is often delayed in problem solving situations in which we take the given for granted.[72]

Solving problems, reflecting on their solutions and generating more problems can create a problem solving cycle - as mentioned above - which is demonstrated on Figure 2:

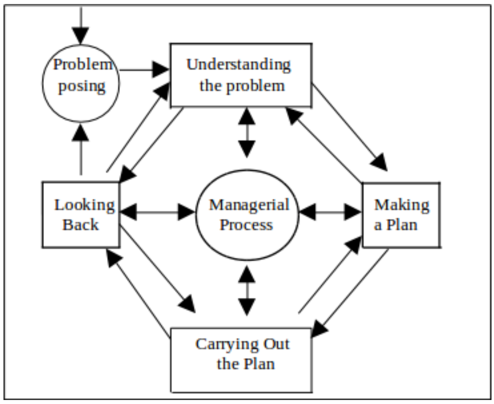


Figure 2: The problem solving cycle [38]

3.1.5 Guiding questions

The best way to summarize why asking questions is important in teaching mathematical problem solving is to cite Reinhart [68] who says:

"Never say anything a kid can say! This one goal keeps me focused. Although I do not think that I have ever met this goal completely in any one day or even in a given class period, it has forced me to develop and improve my questioning skills. It also sends a message to students that their participation is essential. Every time I am tempted to tell students something, I try to ask a question instead."

When solving problems most students have difficulty with finding the first steps, organising the given pieces of information and choosing a right solution method. Using questions - let us call them *"guiding questions"* - instead of telling students instructions helps students more organize their thoughts and grasp the idea of the solution. Furthermore, finding *"the big idea"* might become more simple when students are guided through the steps with questions.

Pólya [65], [66], [67] also suggests using questions in teaching problem solving. As mentioned above, in his famous book **How to Solve It** he defines four phases of problem solving and for each phase he provides a list of questions and recommendations that problem solvers should pose to themselves or the teachers should pose to their students.

1. *What is the unknown? What are the data? What is the condition?*

2. *Do you know a related problem? Try to think of a familiar problem having the same or a similar unknown. Could you restate the problem? Did you use all the data? Did you use the whole condition?*

3. *Can you see clearly that the step is correct?*

4. *Can you check the result? Can you check the argument? Can you derive the result differently? Can you see it at a glance?*

Schoenfeld also suggested using questions or helping guidelines in the different phases of problem solving. His questions and suggestions are more specific than those of Pólya. Some examples are[9]:

- *Examine special cases. Examine special values.*

- *Substitute natural numbers so that you can apply an inductive conclusion.*

- *Try to simplify the problem. Try to use symmetry. Try without the restriction of generalization.*

- *Choose a partial goal in which some of the conditions are fulfilled.*

All the above mentioned questions and recommendations are rather general. The following eight tips should be taken into consideration when trying to ask questions effectively: *1) Anticipate students' thinking; 2) Link to learning goals; 3) Pose open questions; 4) Pose questions that need to be answered; 5) Use verbs like* explain, describe, predict, consider, evaluate ... etc.; *6) Pose questions that encourage other students to take part in the conversation; 7) Keep questions neutral; 8) Provide wait time.*[7]

3.1.6 Expert and novice problem solvers

Several findings related to problem solving derive from researchers examining the problem solving of novice and expert problem solvers. There are quantitative and qualitative differences between the ways they solve problems. Quantitative differences are: the speed and the time of the solution; the way of eliciting formulae; making errors; problem solving strategies. Qualitative differences are: experts carry out a qualitative analysis before collecting necessary formulae, experts hardly use metastatements (comments about the problem solving process); experts tend to work forward while novices tend to work backward.[23]

Another description of the differences between novice and expert problem solvers focuses on three areas. First, *the memory of problem-state configurations* differ in the sense that experts store their memory in larger chunks, however, the number of chunks in case of novices was not different. Second, as mentioned before, *problem solving strategies* used by experts and novices are various. Third, experienced problem solvers are able to *categorize problems* with a high degree of intersubject agreement.[79]

As in our research one of the aims was to reach not only the expert problem solvers but also the less talented, the less experienced ones, the afore mentioned differences are important.

3.2 Memory structures

Different learning theories assign different roles to memory in the learning process. For example cognitive theories say that memory is critical for learning and the way pieces of information are learned has a great effect on how they are stored in and then later how they are retrieved from the memory.[71]

Most neuroscientists accept Baddeley's [15] model of memory structures in which working memory and long term memory play a vital role. In this section we analyze these components of the memory systems in detail.

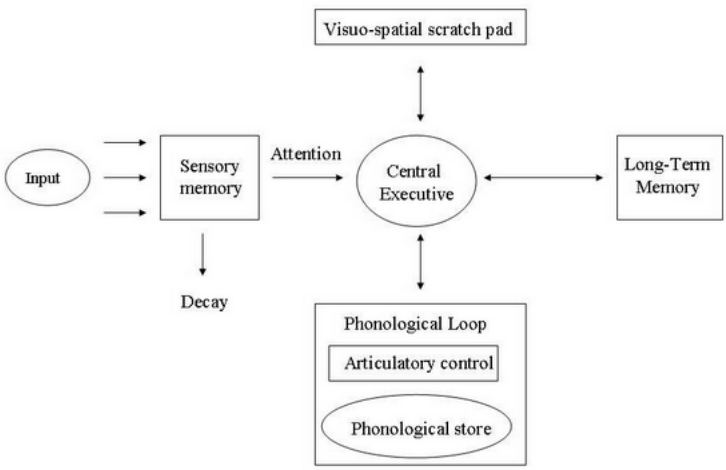


Figure 3: Baddeley's model of memory structures [15]

3.2.1 Working Memory

In problem solving, the role of Working Memory is vital. It is called the workbench of our brain; it is the active problem space. It has four components:

phonological loop to hold and rehears verbal information; visual-spatial sketch-pad to hold and rehears visual and spatial information; episodic buffer which connects the verbal and visual-spatial information directed by the central executive with the help of the information taken from the long term memory. The central executive is the so called supervisory attention system, because it monitors and controls the information processing in our brain.

Our WM constructs plans, uses transformation strategies, analogies, metaphors, brings together things in thought, abstracts and externalizes mental representations. In problem solving students need a clear mental representation of the task (Understanding the problem). While seeking a strategy (solution method), students need to hold the conditions and the goal and the possible transformation (solution) steps in their memory, and taken this into consideration they should monitor their progress in the solution, inhibit wrong, unsuccessful ideas and check their results. These steps need too much resource from the WM, for novices it exceeds the whole capacity and they learn nothing. It is one reason why problem-oriented teaching does not work for average students. The WM has a very limited capacity holding 7 ± 2 info units, its time limit is 18 - 30 sec without rehearsal, goal maintenance and inhibition of irrelevant information.

3.2.2 Long Term Memory (LTM)

The LTM contains information in form of schemas. Schemas are abstract, structured, dynamic representations of information. Schema automaticity means a skill, procedure is learned so that it does not place demand on the WM to deal with it. It has a very important consequence: we may extend the capacity of the WM with recalling relevant schema from the LTM it functions as only one information unit in the WM. (See capacity limit!) In the WM novel information is incorporated into existing schema(s), or similar schema(s) are produced and altered or new schema(s) are recoded back into the LTM. A huge difference between experts and novices is that experts have a lot of solution schemas, which they can apply in traditional problem solving but novices do not have such schemas.

3.2.3 Overcoming the limits of WM: Cognitive Load Theory (CLT)

[14]

Cognitive load can be defined as the load imposed on the WM by information being presented. It is based on the following assumptions: 1) The capacity of the working memory is limited. 2) Pieces of information are stored as schemas in the long term memory. 3) Schemas represent units of information. 4) Automaticity of schemas in the long term memory can be achieved. 5) Learning requires active conscious process in the working memory.

Types of cognitive loads

- *Intrinsic cognitive load*

  Intrinsic cognitive load depends on the elements that must be processed simultaneously. For example when solving word problems: reading the problem, concluding what the problem asks, solving the problem are elements which interact. The intrinsic cognitive load is embedded in the problem, we as teachers cannot influence them. Examples: low intrinsic cognitive load: $5 + 6$; high intrinsic load: $2\frac{3}{4} + 5\frac{6}{7}$.

- *Extraneous cognitive load*

  It depends on the way the information is presented. May include superfluous information that is not necessary for learning the presented material such as background music or holding mental representations of facts or figures. For example, for some students it might be hard to understand that a geometrical figure and the corresponding statements are separated.

- *Germane cognitive load*

  It means the cognitive load placed on the WM at schema formation, integration and automation. It explains the differences between students in terms of experience, ability level and content knowledge.

  The actual cognitive load imposed on the working memory is composed from the intrinsic load, the extraneous load and the germane load. The possible cognitive loads need to be taken into consideration when planning our teaching. For successful problem solving reduction of the cognitive load is an essential aspect which can be achieved by 1) Applying goal free (open) problems. In case of some problems the distance between the starting phase and the goal is very big and students are asked to find as many pieces of data as they can. Example: *"In a triangle two sides are 7 cm and 11 cm long, the angle between them is 73°. Find all the missing information about this triangle that you can."* In our experiment opening problems goes in this direction; or 2) Applying cooperative teaching methods as research experiments show that in group work the WM capacities of the members are added together, so the cognitive load is not very high for the individuals.[24] Our classroom experiment is based on group work.

3.3 Cooperative teaching and learning

In our everyday life and in every community there are competitive and cooperative situations. Competitive situations occur when the resources are limited and an order need to be decided while in cooperative situations the resources are unlimited and can be shared out equally. An example for the former situation can be any competitive game in which there is only one winner, the latter can be a learning situation in which there is enough knowledge for everyone.[36]

Competition and cooperation are both forms of behaviour of at least two individuals who are aiming for the same goal. In a competitive situation the goal can be achieved by some but not by all of the individuals. Whereas in a cooperative situation the goal can be achieved by all or almost all individuals involved.[55]

The educational significance of cooperation was first pointed out by Dewey but became wide-spread following the work of the American social psychologist, M. Deutsch. Since then there was many cooperative learning related research carried out and they all supported the effectiveness of cooperative learning especially in case of low-achievers or students with difficult social background.[83]

3.3.1 What is cooperative teaching and learning?

Many people think that cooperative work is merely arranging students in small groups and asking them to work this way. However, cooperative teaching and learning involves much more.[30] Cooperative teaching and learning is an arrangement where people work together in order to achieve a common goal which often means solving a problem. During this work the group members depend on each other, the success of the team depends on their ability to cooperate. They must support each other, trust each other and respect each other if they want to overcome the difficulties that might hinder them.[46] There are many different forms of cooperative learning but a common feature of them is that all of them involve students working in small groups or teams to discover or learn something together.

Cooperative learning methods can be grouped into two main categories. First, there is **structured team learning** which rewards teams based on the members' progress in learning but individual accountability also plays an important role. Second, there are **informal group learning methods** which focus more on the discussion among team members, projects and social dynamics. In our research we focused on the second type of learning methods.[76]

As mentioned above cooperative teaching and learning is not a simple group work. According to Kagan [46] in cooperative learning the following principles should always be present: positive interdependence, individual accountability, equal participation and simultaneous interactions. Furthermore, for effective work the team members should possess social skills, which they can apply appropriately. These need to be taught to the students in advance if necessary.[39]

- **Positive Interdependence** [40]

In cooperative learning it is one of the most important requirements that if students have to learn something the group members ensure that each member of the group learns the assigned material or if they have to solve a problem each member understands every step of the solution. As Johnson and Johnson say the group members "sink or swim together". This element exists in a learning situation when students realize that they cannot succeed without the help of their group mates and also the success of their group mates depends on their contribution to the set task. Positive interdependence establishes that each member's efforts are necessary for the group's success (no place for "free-riders") and each member is unique as his/her role or resources are indispensable.

- **Individual Accountability** [40]

  This principle is closely related to the one mentioned before. Each student needs to put an effort into achieving the common goal of his/her group because the purpose of cooperative learning is also to develop the individuals and help them to get stronger through group work. The main aim is that after students participated in a cooperative task in the future they will be able to solve a similar task on their own. To ensure the presence of individual accountability teachers need to assess each student's effort, provide feedback to groups and individuals, too. Students must work and learn together and be able to perform alone later.

- **Equal Participation** [43]

  Equal participation is not an attribute of traditional methods. In a frontal lesson, now matter how hard the teacher tries to involve each student in a discussion, it is always the "good" students or the eager ones who participate while the average students or the shy ones just sit and listen. To avoid some students doing all the talking and to involve every member of a class in cooperative learning the teacher needs to ensure the followings: 1) giving different roles to different students; 2) dividing the work load. Some Kagan - structures, like the *RoundRobin* ensure equal talking time and therefore equal participation for each group member.

- **Simultaneous Interactions** [46]

  In a traditional classroom it is usually one person, either the teacher or one student, who talks. This scenario results in little or no active participation, which might cause lack of attention or daydreaming on the student side. Cooperative learning allows more interactions to happen at the same time, for example there are structures that require pair discussions, which means that half of a class is talking thus participating actively at the same time.

Moreover, the other half is listening actively since they are also responsible for the success of their team.

- **Interpersonal and Group skills** [40]

  For successful cooperation the members of a group must know and trust each other; their communication must be accurate and unambiguous; they must be tolerant and supportive with each other and they must be able to resolve their conflicts effectively. With students who have low interpersonal skills cooperative work will be less effective. That is why these skills need to be taught first if necessary.

3.3.2 Why is cooperation necessary in class?

In our fast moving world it is becoming more and more difficult to decide how we should teach our students. There are many valuable pieces of information, however at the moment us, teachers have no idea what the world is like in which our students will have to live. That is why one of the main aims of education should be to prepare our students to find necessary information, to process new pieces of information, to solve problems creatively and to cooperate with others. There are many changes that affect the future of our students and therefore need to be taken into consideration from educational point of view.

First of all, the social environment of our students has changed a lot. Students have different social values when they start attending school. Many of them are disrespectful, find it difficult to communicate with each other or are unable to listen to each other. That is why it is the task of the school to teach them social behaviour as well. Cooperative work is a good means of this as students must communicate effectively and must pay attention to each other if they want to succeed in their work.

Moreover, the economical environment is changing as well. Our society relies more and more on consumerism and on handling information. The Internet is a inexhaustible source of information. But not each piece of information is relevant or necessary for solving for example a given problem. So, students need to learn to select necessary information, to be able to work independently but at the same time to be able to work as a member of a team as well, because working together might help in processing an enormous amount of information.

In our world travelling abroad and being able to work with people from different cultures is becoming more and more natural. Cooperative learning also plays a vital role in teaching how to accept differences between people.[46]

The above mentioned changes inspire us to change our current teaching - learning culture. Learning should become a dynamic process in which students are active participants. Communication skills of the students need to be improved and the demand for student - student and student - teacher communication need to be reinforced. The atmosphere of the learning situation should be changed so that students are not afraid of making mistakes and asking questions. Working in small groups gives the opportunity to realize the aforementioned innovations.[63]

3.3.3 Difficulties with cooperative learning

As with any other method some things might go wrong with cooperative learning. In this section we show some problems that might occur in a cooperative classroom and suggest some solutions.[74]

- Failing to get along

  Sometimes students can end up as group mates of other students who they do not like. First of all, they should be given time to try to get used to working with each other and accept each other. The first team success will show them that they need to put their personal feelings aside because they are able to work as a team effectively. However, it might happen that the differences are not resolved after a longer period then new groups need to be assigned trying to avoid problematic arrangements.

- Misbehaving students

  This problem can be handled by assessing each team in front of the whole class emphasizing positive behaviour. However, misbehaviour can be reduced if the teacher chooses to apply cooperative learning in a class that she/he can handle very well.

- Noise

  Noise is a natural consequence of cooperative learning but it should be kept as low as possible. Students need to learn to control their voices and talk so that only their team members can hear them. To help them do this if the noise level rises the activity needs to be stopped and the students need to be reminded to keep quiet. Another solution is to set a noise level criteria.

- Ineffective use of team practice time

  As most students are used to working alone on practice tasks the team might not use their team practice time effectively. Choosing a right structure in which each team member has a specific role for problem solving is a good solution for this problem.

- Too much difference between the “best” and the “worst” students

Again, the solution to this situation is to chose a right cooperative structure which makes the differences in ability a strength rather than a weakness.

3.3.4 Cooperative structures

One of the most important features of cooperative teaching is that we change HOW we teach as we cannot change WHAT we teach.

In this section we present some examples of cooperative structures that were found useful and efficiently applicable in Maths classes. The structures were given catchy, easy-to-remember names so that students and teachers can remember the structures better and it gives an opportunity to use the same structure in many different contexts.[44]

Pairs Check

In this structure two students work together. One of them is the "coach" who only checks the work of the other student or if necessary gives advice on how to carry on. The second student has to write everything down while explaining aloud what he is doing.[47]

Jigsaw Expert Groups or Jigsaw

The main idea of this structure is that every group is an expert in a topic or a task. The teams are given some time to prepare - either collect ideas or solve a task - then new groups are formed in a way that each new group contains one person from the original groups. As a result of this the new groups contain students who are experts in each task. In the new groups students share their topics with each other. Notes are made and comments are discussed.[47], [74]

Jigsaw Problem Solving

Each student has a part of the answer; team mates must put their information together to solve the problem.[74]

Gallery Walk

Students have to collect information or solve a task in groups of four. What they gather is written on a poster that is displayed in the classroom. After this everybody walks around and they can check the work of the different groups. The mingling students are allowed to write comments or ideas on each other's posters.[46]

Think-pair-share

Students work in groups of four and are paired up within their groups. Each group is presented a problem then the students are given some time to think about possible solutions or ideas that can be used in finding the solution. The next step is that students discuss their answers either with their face or with their shoulder partners. This is followed by a class discussion.[46]

Timed Pair Share

The steps of this structure seem identical to the steps of *Think-pair-share*, however, this structure is stronger in ensuring equal participation. While when using *Think-pair-share* after a set time students are asked to discuss their solutions - which might result in one student talking the other listening only - when using *Timed Pair Share* there is a set time for sharing ideas as well, so both student has the opportunity to participate in the discussion.[42]

Think-pair-square

The steps of this structure are the same as that of *Think-pair-share* but instead of a class discussion students share their answers with the other pair at their table.[46]

RoundRobin

The students are given a question or a topic or a problem. Each student is given a set time to provide an answer or to present part of the topic or the solution to the problem. The team mates help each other if necessary and provide appropriate praise or constructive criticism.[46]

Switched Task Cards

This structure can be used with ready made task cards but here we mention the version we used in our experiment. Students or groups prepare task cards then they swap cards with another student or group. Each student/group has to solve the newly received task.

As mentioned above, one of the objectives of our education is to prepare our students to be able to succeed in their future life. The difficulty with fulfilling this aim is that we do not have any idea of what the future world will look like but one thing is sure, our students will have to cope with rapid change. To prepare them for this teaching must include the development of a full range of thinking skills. The above mentioned structures help improve different types of thinking. Kagan [45] summarizes the 15 fundamental types of thinking in the following way:

Understanding Information	Manipulating Information	Generating Information
A. Recalling	A. Analysing	A. Brainstorming
B. Summarizing	B. Applying	B. Synthesizing
C. Symbolizing	C. Inducing	C. Predicting
D. Categorizing	D. Deducing	D. Evaluating
E. Role - Taking	E. Problem Solving	E. Questioning

Table 2: 15 fundamental types of thinking

The above listed Kagan structures can be applied for developing the following thinking skills: 1) *RoundRobin*: symbolizing, categorizing, problem solving, brainstorming, synthesizing; 2) *Timed Pair Share*: role-taking, evaluating; 3) *Jigsaw*: analysing, problem solving; 4) *Think-pair-share/square*: inducing.

3.3.5 Teacher's role in cooperative learning

Obviously, when using cooperative teaching techniques it is not only the classroom setting, the students' role that change but also the teacher's role. From an instructor the teacher becomes a tutor, someone who guides students in the teaching – learning situation. In class while the students work cooperatively the teacher's task is to monitor and observe their work to make sure that they do some progress, help them if they are stuck on a problem and cannot carry on alone and have extension exercises ready for those groups who finish faster.[21] The teacher still should be the leader and it is her/his responsibility to maintain an environment where work can be done, furthermore she/he needs to explain the guidelines of cooperative work. In cooperative classwork students are supposed to talk to each other, so the classroom becomes noisy. The teacher should prevent the classroom turning into a chaotic environment.[31]

The teacher does not lecture about the teaching material, but provides the groups with tasks that help them end up in situations from which they can learn.The traditional role in which the teacher tells information to students is changed. There is more opportunity to focus on personal relationships between the teacher and individual students. The role of the teacher becomes dual: in the small groups the teacher is someone who helps and supports the groups' work while in whole class discussions the teacher plays the role of a chairman. This changed role has numerous advantages, for example, the teacher can examine the students' behaviour and learn more about individual learning preferences and the teacher has more opportunities to help individual students in their development.[20]

3.3.6 Groups

Groups might be formed in many different ways. Students can be allowed to choose who they want to work with or grouping can be made random or based on some principle of the teacher. The most important aspect however is that the number of students in one group should be four. The reason for this is that in a groups of four nobody feels left out. If two students start a discussion the other two can engage in a conversation as well (Fig.4.). Moreover, some structures (like Pairs Check) are based on students working in pairs rather than in fours. Another advantage is that an ideal seating arrangement can be used where two-two students sit at the opposite sides of a table (Fig. 5.) thus allowing communication between students sitting next to each other and also between students sitting opposite to each other. Of course, in a classroom grouping students into groups of four is not always possible. In this case a group of five students is preferable to a group of three.[26]

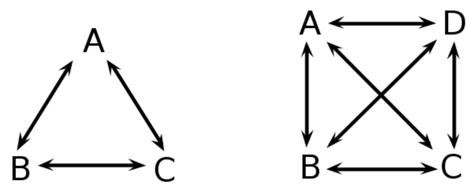


Figure 4: Number of interactions in groups with three and four members

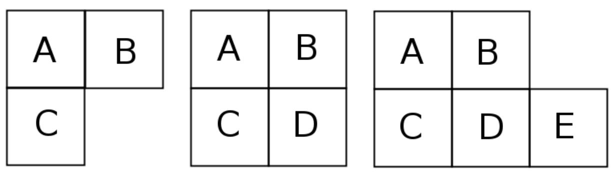


Figure 5: Possible seating arrangement for teams with three, four or five members

In addition to the number of students in a group the combination of the students in terms of ability is another important issue. One option is arranging them to work in heterogeneous groups which might be beneficial in a problem solving lesson as the thinking range varies within a group.[21] On the other hand, homogeneous groups might be advantageous in situations when the focus is on giving the opportunity for the slow students to catch up and to the fast students to extend their knowledge even more.

There are some rules that should be observed by each student when working in groups which are in line with the principles of cooperative teaching. 1) Everybody is responsible for their own work and behaviour. 2) Everybody must be willing to help their group mates who need help. 3) A student is allowed to ask for the teacher's help only when none of his/her group mates can help and they all have the same question.[21]

In small groups each student has an opportunity to communicate his/her ideas, they have the chance to speak more often participating in the class activity more actively. Many students are more willing to take a risk of saying something that is incorrect or making a mistake when working in a group of four or five. However, teachers have to keep in mind that students need to be taught the rules of working in groups and they need to be given time, practice and encouragement until they learn how to work successfully together.[21]

3.3.7 Cooperative teaching in Hungary

In Hungarian education cooperative learning appeared as a result of the work of József Benda. He thought that cooperative learning could cause a positive change in the way Hungarian schools work which would result in better achievement, integration and development of students.[41]

Benda started working with cooperative teaching and learning as a result of his personal experience which showed that there was a tendency in Hungarian education which tried to force students not to move around or talk with each other and help each other thus withdrawing their basic needs. Since there was no tradition of cooperative teaching and learning in Hungarian education Benda and his colleagues started experiments based on this method. As a result of his work the method became more and more wide spread, first only in private education - due to financial possibilities. Nowadays the method is taught at universities for future teachers and it is becoming more and more well-known and frequently used in primary and secondary education, too. When Benda developed his teaching program he based his ideas on humanistic psychology the gist of which is that students are involved in the teaching - learning process thus their willingness will help reach the set goals. This new program places more emphasis on students' socialization; instead of teaching learning through experience becomes more important; knowledge can be obtained not only form the teacher or from the books but also from the complex educational situation.[19]

3.3.8 Cooperative techniques in mathematics education

The following list provides more arguments in favour of using cooperative teaching techniques in mathematics education.[30]

- Small groups contribute to the development of the social aspects of learning mathematics and provide a sense of security as students are not left alone but can rely on the help of their team mates.
- Since there is no competition within the groups every student has the opportunity to experience success in learning.
- Working in small groups gives the opportunity to the members to discuss mathematical problems logically.
- Many mathematical problems have more solutions. Group work gives the opportunity to find and discuss different solutions and ideas.
- Team mates can help each other to learn the basic concepts and numerical methods.
- Students reinforce their understanding by explaining processes or concepts to each other.
- Together students are able to handle situations and problems that would exceed the capabilities of individual students.

4 Research methodology

4.1 Background information

The experiment was an action research, which means that the researcher was the teacher of the class as well. This type of experiment has become popular amongst those who are practitioners and would like to carry out professional development related research. Koshy [52] defines action research as a kind of an enquiry whose aim is to constantly refine practice and finally contribute to the teacher's professional development. The writer says that action research means researching your own practice, therefore it is participatory and situation – based, it is emergent and it is mainly about improvement. Action research is also a good tool of bringing mathematics education and methodology related research closer together.[88]

Since the researcher was interested in the effect of cooperative techniques on the development of students' problem solving skills the question of using control groups can arise. However, according to Slavin [75] when comparing the outcomes of cooperative teaching and learning to other programs the problem is that there are many factors that differ in the two alternative programs such as the subjects, the duration, etc. which can account for the differences in the outcomes.

4.2 The school

The school where the experiment took place is a mixed comprehensive secondary school whose strength lies in scientific subjects and computer science. The students here are between 12 and 19 years old and they have to sit a so called entrance exam before starting their studies here. The school is a selective school taking the more talented students from the area.

4.3 The students

The students taking part in the action research were 16 - 17 years old. It was 16 of them and they attended a class that prepares students for tertiary education in technology and specializes in foreign languages. Following a preparatory year these students had four years to complete their secondary school studies. As mentioned above, the writer of this dissertation was the teacher of this group. The academic year 2012/2013 was the students' third year in our school. In their first year the number of Maths lessons a week was three which increased to four in the following two years. In the year of the experiment they followed the year 10 scheme of work for secondary school students. Since they had more Maths lessons than a "normal" class we often had the opportunity to discuss a topic in more detail or to solve problems from Maths competitions. These students were not necessary gifted in Mathematics but the majority certainly had a great interest in Maths and other scientific subjects. Their Maths grades were good (4) or excellent (5), only one of them had satisfactory (3). These students were mainly motivated although it was not always easy to make them active in class. Some of them regularly took part in Maths competitions and they attended group study sessions weekly.

As for the ethical considerations of the research since the students were underage their parents were asked to sign a statement of consent at the beginning of the school year.

4.4 Methods of data collection

The students participating in the experiment started with filling in a tests about attitude to Maths and a mathematical pre-test. The first part of the experiment was followed by a mathematical post-test and at the end of the school year the attitude test was repeated and the students filled in a questionnaire about cooperative work and they had to complete a mathematical delayed test, too.[9] The mathematical tests were constructed so that the chosen problems tested the mathematical knowledge and thinking methods that were needed during the experiment.

Furthermore, half of the lessons were recorded on video and the discussions of the different groups were recorded with a voice recorder as well. During the lessons the teacher observed the groups' work and the reaction of the individual students. In addition, each student had a so called "reflection book". As we saw one of the most important steps of problem solving is reflecting on both the solution and the strategy that was used. To record all the ideas, thoughts, comments that might occur during the experiment this so called "reflection book" was introduced which was a small exercise book. Every student had one and they were asked to note everything in this book. Not only mathematical ideas and ways of problem solving but also personal feelings, reactions ... etc. First the students described their working method with their own words, but in the discussions we named some heuristic strategies as well.

5 Learning trajectory

5.1 The lessons

5.1.1 12 lessons - 5 problems

There were 5 problems selected, each of which was curriculum based and it was made sure that the problems are from different fields of Maths (algebra, geometry, number theory, combinatorics). As Hungarian textbooks hardly contain open problems or investigations, these problems were either open problems already or they were chosen so that they could be "opened". The possible ways of extension were suggested either by the teacher or by the students. To discuss these tasks we had twelve 45 minute long lessons. The main aim was to explore and discuss each problem in great detail and if possible extend it as well. That is why two or three lessons were planned to be used for one problem.

Since our plan was to examine the effect of regular use of cooperative teaching techniques on the development of problem solving skills, each of the above mentioned 12 lessons were planned with this method. The main aim of this experiment was not to provide fun Maths for the students – although the tasks might suggest this – that is why all problems are curriculum based and develop mathematical competencies.

It is said that the best groups size is four.[26] There were 16 students who took part in this research, so groups of four were created. Before deciding on how to form the groups the following issues were taken into consideration: Shall we put friends in a groups or is it better to group students with those who they do not know that well? Some students are more patient and more tolerant than the others. How should they be distributed among the groups? Shall the groups be homogeneous or heterogeneous according to the students' mathematical ability? How often shall we change the grouping arrangement during the experiment?

After careful thinking the following arrangements were made: each group was put together so that it had a weaker student, a more able student, a quiet student and a more talkative student, moreover the "difficult" students had to be grouped with patient ones. The group – settings were changed once during the twelve lessons. The reason for this was to give the students opportunity to work with as many fellow students as possible. However, for efficient cooperative work the group members needed time to get used to each other, so more frequent change in the group – settings was avoided.

The problems

1. **Matchstick game**

   Two players and 27 matchsticks are needed. The two players take turns and remove 1 or 2 or 3 matchsticks. The winner is who removes the last matchstick. Task for the groups: to find a winning strategy for both players.[9]

   Curriculum Reference: logical thinking, thinking backwards, combinatorial thinking

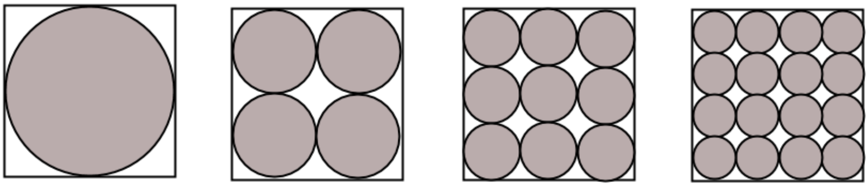
2. **Number magic**

   In this problem field the individual problems are related to simple number tricks that can be explained using number theory. For example: type the number 15873 in your calculator. Select a number between 1 and 9 – including 1 and 9 – and multiply 15873 by that number. Now multiply the product by 7. What do you notice? Try with different digits. Can you explain what is going on?[34]

   Curriculum Reference: number theory, divisibility, algebra, generalization

3. **Area investigation**

   From a square measuring 60 cm x 60 cm we cut out circles as you can see on the figure. What percentage of the square is wasted in each case? Do you notice a pattern? Can you generalize your idea? Can you prove your conjecture for n circles?



Curriculum Reference: area, percentages, sequences, pattern recognition, generalization, problem posing

4. **More beads**

   Three beads are threaded on a circular wire and they are coloured either red or blue. You repeat the following actions over and over again. Between any two of the same colour put a red and between any two of different colours put a blue, then remove the original beads. Discuss all the possible outcomes. What happens when you do the same thing with 4 beads, 5 beads or 6 beads?"[4]

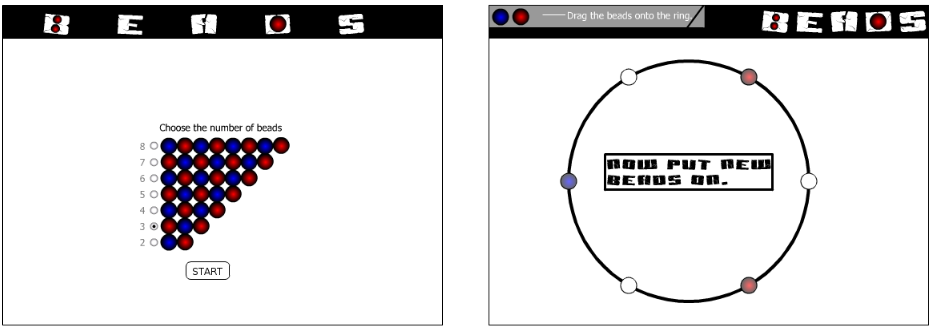


Table 3: Interactive representation of the problem

Curriculum Reference: combinatorics, permutations, listing all options logically, pattern recognition, generalization

5. **Primes and factors**

   This problem field contains algebraic problems that can be solved using factorisation, special products and other algebraic modifications.

   - Think of a two digit number. Reverse its digits to obtain a new number and subtract the smaller number from the bigger one. eg: $42 - 24 = 18$ Try with more numbers. Can you get a prime number as the result? Why/Why not? Can you explain your idea? Prove that the answer is never a prime. What if you work with three digit numbers? With four digit numbers? How about n digit numbers?[5]

- Find the biggest whole number which is a factor of each term of the following sequence:
  $1^5 - 1, 2^5 - 2, 3^5 - 3, 4^5 - 4, ..., n^5 - n$ [1]
- Take any pair of two digit numbers ab and cd where, without loss of generality, ab $>$ cd. Form two 4 digit numbers abcd and cdab and calculate:

  $$\frac{abcd^2 - cdab^2}{ab^2 - cd^2}$$

  Repeat this with other choices of ab and cd. There is a common feature of all the answers. What is it? Why does this occur? Generalise this to n digits for other values of n.[3]
- Think of a prime number that is bigger than 3. Square it then subtract 1 from the result. Do these operations with more prime numbers. What do you notice? Can you prove your assumption?[6]

Curriculum Reference: number theory, primes, factors, multiples, algebra, generalization

5.1.2 Further lessons

In this section we provide a list of further lessons that were planned using cooperative teaching techniques. As mentioned above, these lessons were held during the school year and they were in line with the topic being actually taught. The type of these lessons varied from introducing a new material to revision and summary.

The following table summarizes the topic of the lessons, their type and gives the cooperative structures that were used.

Topic of the lesson	**Type**	**Cooperative structures used**
Quadratic equations	Practice	Pairs Check
Word problems (2 lessons)	Problem solving	Think-pair-share, Think-pair-square, Group discussion, Jigsaw
Parallel intersecting lines	Practice (consolidation)	Think-pair-share, , Think-pair-square, Group discussion
Similarity	Introducing new material	Think-pair-share, Think-pair-square, Group discussion, Jigsaw
Introducing trigonometric ratios	New topic	Jigsaw
Trigonometry	Practice	Pairs Check, Group discussion
Trigonometry, similarity	Problem solving	Jigsaw Problem Solving
Generalising trigonometric ratios	Practice	Jigsaw
Combinatorics	Revision	Jigsaw
Probability	Problem solving	Think-pair-share, Pairs Check
Probability	Problem solving	Think-pair-share, Pairs Check

Table 4: Lessons in the second part of the experiment

The detailed list of the problems and techniques used in the second part of the experiment can be found in *Appendix 1*.

6 The experiment

6.1 Pre-tests

Before starting the first stage of the experiment the students who participated were asked to fill out different pre- tests. One of them was related to their attitude to learning mathematics and there was a mathematical pre-test that measured the mathematical knowledge and the knowledge of heuristic strategies needed for solving the problems that were to be presented in the first part of the experiment.

6.1.1 The Mathematical Pre - Test

To complete the mathematical test the students had 45 minutes. The problems were the followings:

1. The bigger cog-wheel of a bike has 35 teeth and the smaller one has 15. How many times do we need to turn the pedal so that both cog-wheels get back to their original position? (The pedal is on the bigger wheel.)

   Testing: Systematic thinking, Lowest Common Multiple

2. The sides of a cuboid are whole numbers in centimetre. The area of two of the faces are 24 cm^2 and 36 cm^2. Find the volume of the cuboid.

   Testing: Thinking backwards, Calculating area, Volume, Metric units

3. In a jewellery shop on Monday half of the stock and four pieces of jewellery were sold. On Tuesday half of what was left and further two pieces were sold. On Wednesday the shop assistant sold five pieces of jewellery. On Thursday 2 less than half of what was left was sold. At the end there were 8 pieces of jewellery in the shop. How many pieces of jewellery did the shop have on Monday?

   Testing: Thinking backwards, Ratio, Equations

4. How many different four digit numbers can we form so that each digit is the element of the set $\{1; 2; 3; 4; 5; 6; 7\}$?

   Testing: Listing all options logically, Permutations

5. Snow White and the seven dwarfs have dinner around a round table. In how many different ways can they sit down next to each other?

   Testing: Listing all options logically, Permutations in circular arrangements

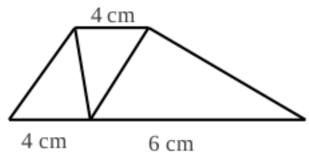
6. The sum of two numbers is 2250. 12% of the first number equals to 18% of the second. Find the two numbers.

   Testing: Systematic thinking, Ratio, Percentages

7. Laci got a pay rise of 15% so his current salary is 241500 HUF. How much was his original salary?

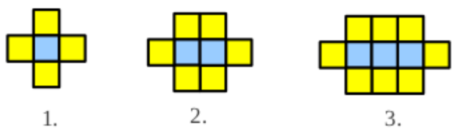
   Testing: Systematic thinking, Ratio, Percentages

8. Work out the area of the triangles on the figure if you know that the area of the trapezium is 21 cm^2. (The shape is not drawn to scale.)



Testing: Thinking backwards, Area

9. Look at the following shapes. Find the number of yellow squares in the 4., 5., 6. shape. How many yellow squares are there in the 100. shape? How many in the n^{th} shape?



Testing: Pattern recognition, Generalization, Sequences

6.1.2 The results

In the mathematical pre - test each task was worth maximum 5 points so the students received from 0 to 5 points for each task depending on how well they managed to solve it. For correct ideas but incomplete solutions they could receive 1, 2, 3 or 4 points depending on how far they proceeded with the solution. The table below summarises the sum of the points individual students achieved on the test; the mean average of the points they obtained for the separate tasks and the standard deviation of the points they obtained for the separate tasks. To make sure that the students took these tests seriously they were awarded with a grade [2] for each test which were taken into consideration in their half year and end of year assessment.

[2]in Hungarian education we use a 1 (fail) to 5 (outstanding) scale

Student	Sum	Grade		Mean average	Standard deviation
BZS	32	5		3.56	1.94
BT	20	3		2.22	1.99
BP	38	5		4.22	1.72
HM	40	5		4.44	1.13
HP	23	4		2.56	2.40
KR	34	5		3.78	1.99
KA	16	3		1.78	2.44
KMM	19	3		2.11	1.45
KD	29	4		3.22	2.05
ML	18	3		2.00	2.40
MSZ	10	2		1.11	1.69
MT	24	4		2.67	2.24
NB	30	5		3.33	2.18
NSZ	21	4		2.33	1.87
OR	19	3		2.11	2.03
PR	27	4		3.00	2.18

Table 5: Results of the mathematical pre-test

The graph below displays the sum of the points each student achieved on the test. The red line indicates the mean average of the sums achieved on the mathematical pre-test.

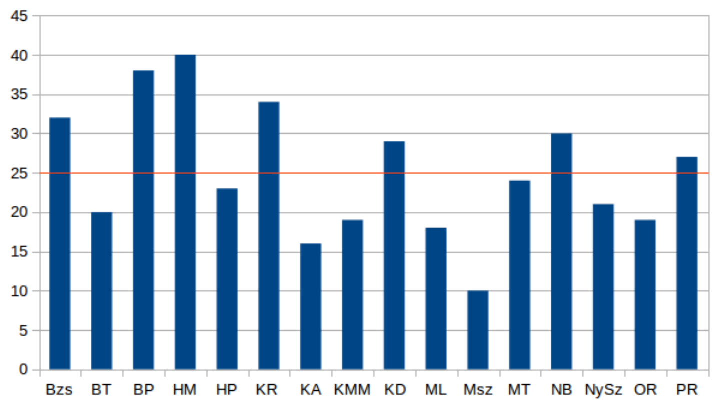


Figure 6: Total points of each student on the mathematical pre-test

The second graph shows the number of students who achieved sums in a given group.

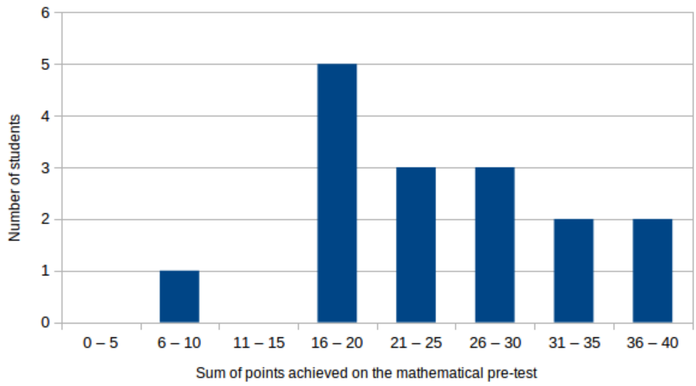


Figure 7: Total points on the mathematical pre-test

6.1.3 Interpreting the results

First of all, let us consider the total points achieved on the mathematical pre-test. The bar charts clearly show that seven students managed to receive points that are above the mean average of the sums (25 points) and the sum of nine students was below the mean average. The median of the sums is 23,5. The highest sum achieved on the pre-test was 40 while the lowest was 10. These results indicate that more students found the test difficult.

Comparing the test results to the students general ability and previous performance in mathematics we can say that these results are in line with the students previous achievements and ability. BP and KR are indeed the best achievers in class with great potential to do well in mathematical competitions, too. HM and NB who also had high sums are not outstanding students but are very keen and have a great interest in the subject. The table indicates that MSZ is a low ability student in mathematics but from experience we can say that he would be capable of achieving more if he had an appropriate attitude. Based on their previous mathematics test results and activity in class we can say that KA and OR are the lowest ability students in the class (N.B. as this class is a selective one these students would be considered as average ability students in an other class).

The mean average and the standard deviation of the points indicate the number of tasks the students attempted to solve and how successful they were in solving these tasks. The mean average of BP and HM were both above 4 (4.22 and 4.44) and their standard deviations were rather low (1.72 and 1.13). These values imply that both students attempted most or all tasks and they were successful in solving them correctly. On the other hand, the two lowest achievers, KA and MSZ had mean averages 1.78 and 1.11 with standard deviations 2.44 and 1.69. In case of KA the mean average was low but the standard deviation was rather high which suggests that he attempted only a few tasks but there is a great difference between the points he received for them. However, the mean average of MSZ was also low and the standard deviation was low as well. This hints that he attempted more tasks but received low points for all of them. In general we can say that:

- low mean average and low standard deviation means that most tasks were attempted mainly with a little success
- low mean average and high standard deviation means that less tasks were attempted but with more success
- high mean average and low standard deviation means that most tasks were attempted mainly with success
- high mean average and high standard deviation means that less tasks were attempted mainly with more success

Finally, let us discuss the achievements in the separate tasks. As mentioned before the mathematical tests were formed so that they tested both mathematical knowledge and thinking skills necessary for the subsequent experiment. Here, we discuss the students' achievement on the tasks grouped based on the skills needed to solve them. The horizontal red line on each bar graph indicates the mean average of the points received for solving the given problem.

Systematic thinking

The bigger cog-wheel of a bike has 35 teeth and the smaller one has 15. How many times do we need to turn the pedal so that both cog-wheels get back to their original position? (The pedal is on the bigger wheel.)

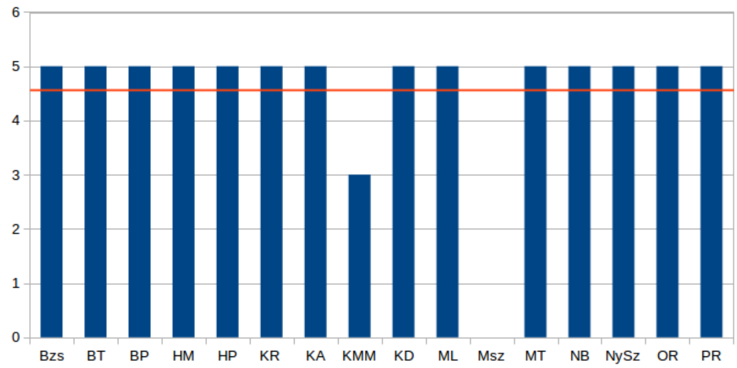


Figure 8: Systematic thinking (pre-test)

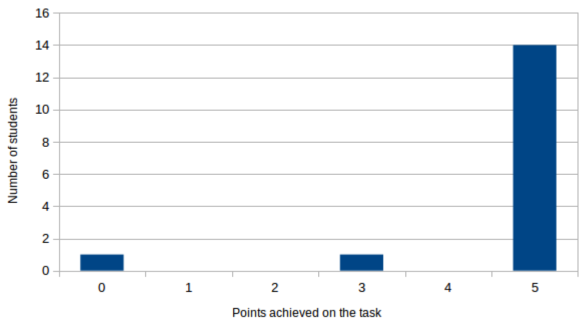


Figure 9: Systematic thinking (pre-test) - grouped results

It can be clearly seen from the bar charts that solving this problem was not difficult for the vast majority of the students. Only one student who can be considered an average ability student in the group completed the problem partially and there was one student, MSZ (see above), who did not solve the problem.

Thinking backwards

In jewellery shop on Monday half of the stock and four pieces of jewellery were sold. On Tuesday half of what was left and further two pieces were sold. On Wednesday the shop assistant sold five pieces of jewellery. On Thursday 2 less than half of what was left was sold. At the end there were 8 pieces of jewellery *in the shop. How many pieces of jewellery did the shop have on Monday?*

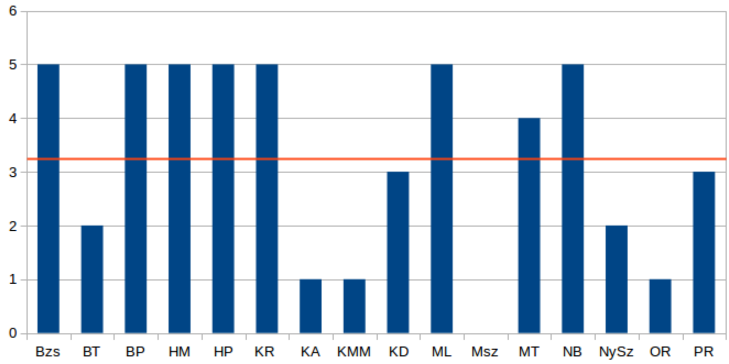


Figure 10: Thinking backwards (pre-test)

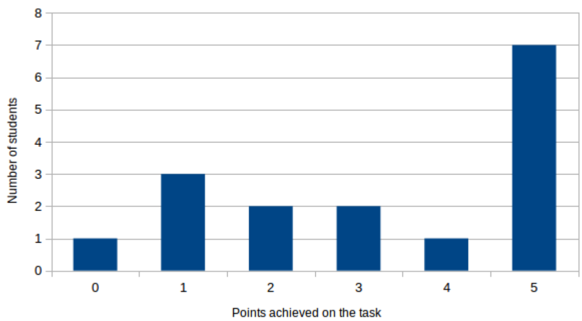


Figure 11: Thinking backwards (pre-test) - grouped results

According to the graphs half of the group managed to solve this problem successfully. Three students received only 1 point for this task which means that they had some idea about solving the problem but they were not able to progress with the solution. It has already been pointed out that two of these students (KA and OR) are low achievers compared to the rest of the group. It can be seen that MSZ did not gain any points for this problem either.

Listing all options logically

I. *How many different four digit numbers can we form so that each digits are elements of the set* $\{1; 2; 3; 4; 5; 6; 7\}$*?*

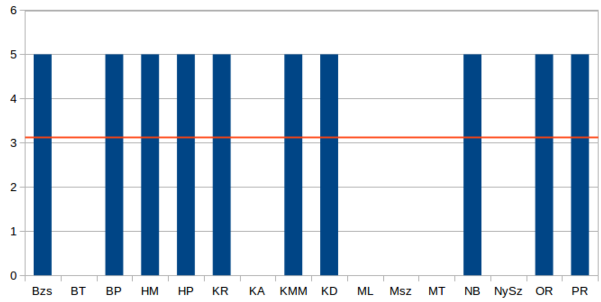


Figure 12: Listing all options logically I. (pre-test)

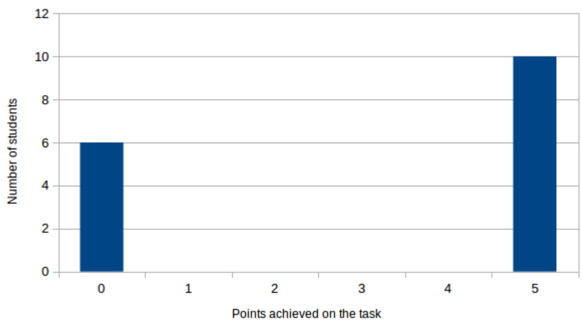


Figure 13: Listing all options logically I. (pre-test) - grouped results

II. *Snow White and the seven dwarfs have dinner around a round table. In how many different ways can they sit down next to each other?*

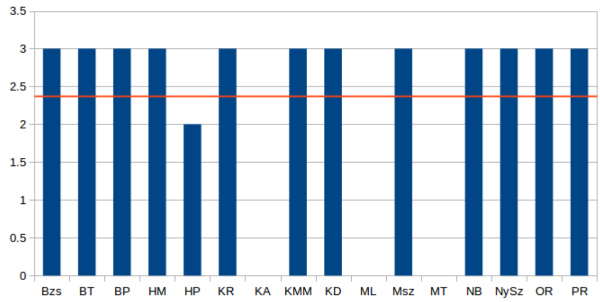


Figure 14: Listing all options logically II. (pre-test)

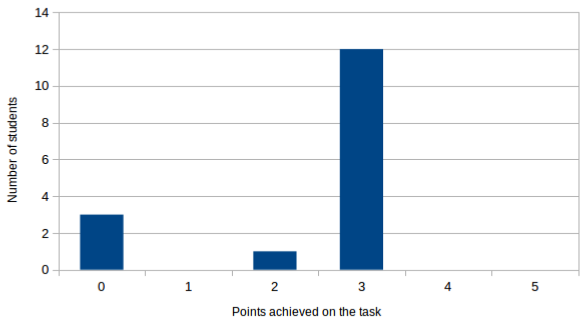
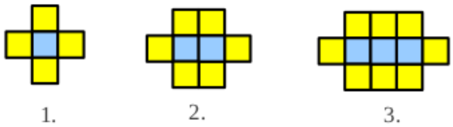


Figure 15: Listing all options logically II. (pre-test) - grouped results

The first problem type in this section is more well-known for students in this age group than the second one. As the first two graphs show ten students completed the problem successfully, however the rest of the group did not even attempt to solve it. It can be seen from the results that the second problem was less familiar for the students, the highest point gained on this problem was three. Three of the students did not write anything that could be evaluated - KA was again among the low achieving students.

Pattern recognition

Look at the following shapes. Without drawing find the number of yellow squares in the 4., 5., 6. *shape. How many yellow squares are there in the* 100. *shape? How many in the* n^{th} *shape?*



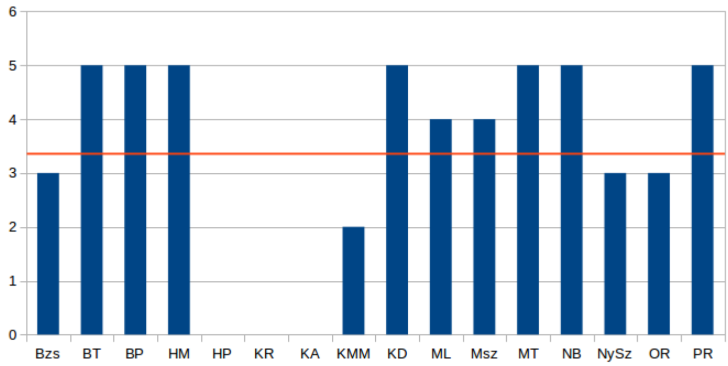


Figure 16: Pattern recognition (pre-test)

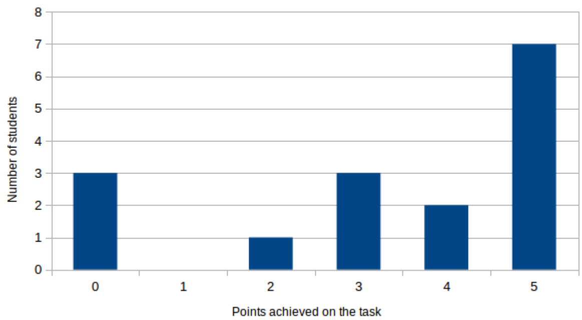


Figure 17: Pattern recognition (pre-test) - grouped results

Despite the fact that pattern recognition should not have been a problem for students in this group only seven students managed to achieve full points on this problem. The reason for this can be that generalizing the pattern was part of the problem as well which is a rather unusual task for the students. Unfortunately in Hungarian text books there are hardly any mathematical problems that require generalization. Although HP and KR are talented students they did not gain any points on this problem. The reason for this might be misunderstanding.

6.1.4 Attitude to mathematics test - pre-experiment

1. Mathematics is among my three favourite subjects. (+)	Yes	No
2. Mathematics is among my three least favourite subjects. (-)	Yes	No
3. Mark on the line how much you are interested in maths. not at all ______________________________very much		
4. If I had to choose I would choose maths. (+)	Yes	No
5. If I were a teacher I would like to teach maths. (+)	Yes	No
6. I think, after finishing school I won't use maths. (-)	Yes	No
7. If I had a lot of time I would do maths for fun. (+)	Yes	No
8. In maths I always worry about saying something stupid. (-)	Yes	No
9. In written tests I always worry about making an arithmetic mistake. (-)	Yes	No
10. I'm afraid of maths tests.(-)	Yes	No
11. I'm not afraid of maths tests. (+)	Yes	No
12. When I have to go to the board in a maths class I'm very nervous. (-)	Yes	No
13. I like solving:		
a) equations (+)	Yes	No
b) inequalities (+)	Yes	No
c) word problems (+)	Yes	No
d) geometry problems (+)	Yes	No
e) more difficult problems (+)	Yes	No
14. I'd rather solve 5 equations than one very difficult problem.	Yes	No
15. Maths is easy, I don't have difficulties with it. (+)	Yes	No
16. Maths is so difficult, there are a lot of things I hardly understand. (-)	Yes	No
17. I like solving a difficult problem alone. (+)	Yes	No
18. Sometimes/often I understand the material only if my parents or classmates explain it.		
19. Sometimes/often I cannot solve maths problems without help.		
20. What do you like in the way you are taught maths?		
21. Anything you do not like?		
22. What could your maths teacher do better?		
23. What could you do better?		

Table 6: Attitude to mathematics test

6.1.5 The results

Answers for the first 17 questions

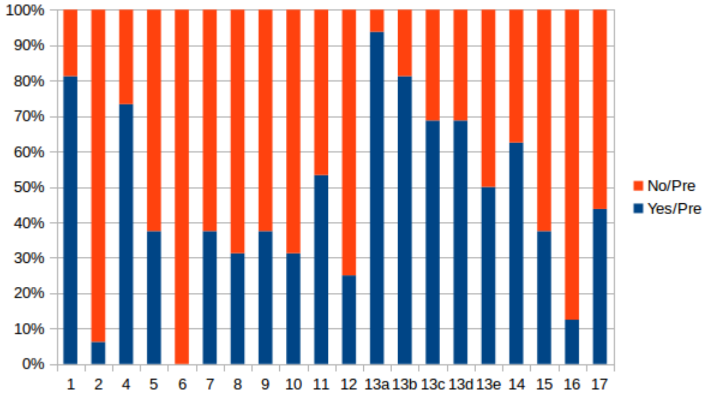


Figure 18: Answers for the pre-experiment mathematical attitude test

6.1.6 Interpreting the results

For a better insight into the students' attitude to mathematics before the experiment we divided the statements into two groups, one of them containing statements that reflect positive attitude to mathematics while the other one containing statements showing negative attitude to the subject. Some statements will be discussed separately due to their nature.

Positive statements

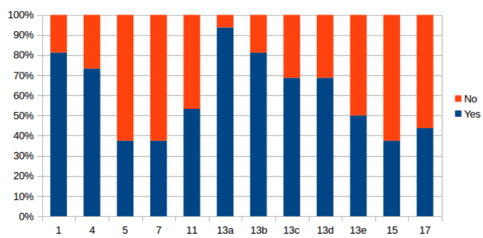


Figure 19: Students' answers for positive statements

As the graph clearly shows there were eight questions for which half of the answers or more was positive (yes). Positive answers reflect positive attitude to the subject in this case. The questions for which less than 40 % of the students answered yes can be related to either future plans with mathematics (question 5) or to doing extra mathematical problems and not only the compulsory "school" mathematics (questions 7 and 17). Taking everything into consideration we can say that the overall attitude of the students in this group to mathematics is positive.

Negative statements

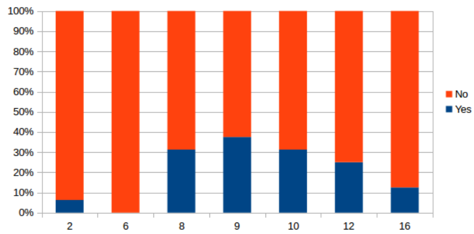


Figure 20: Students' answers for negative statements

The data presented on this graph appears to confirm the general positive attitude to mathematics. In case of the negative statements the answer "No" reflects a positive attitude to the subject. It is noticeable that for each question the number of "No" answers outweighs the number of "Yes" answers. The fact that for question 6 every student answered with a "No" suggests that each of them have plans for their future in which mathematics plays an important role.

14. I'd rather solve 5 equations than one very difficult problem.

This statement should be examined separately as it does not reflect positive or negative attitude to mathematics but it examines whether the students prefer solving routine problems to solving more challenging mathematical problems. Ten students out of sixteen answered that they would rather solve 5 equations than one very difficult problem. This suggests that although this class contains students with mathematical interest the majority of the students feel more confident when solving problems that do not require too much thinking. This phenomena can be explained by the nature of problems average Hungarian students are familiar with. In class they hardly meet problems that require creative thinking, creative problem solving. Unfortunately, often school mathematics is mainly about practising solving certain types of problems. To develop problem solving skills more open problems or real-life like problems would be needed.

Suggestions listed for questions 20. - 23.

In this section we present some suggestions made by the students. The comments were chosen so that they are meaningful and they reflect general attitude.

- Likes and dislikes

Likes	**Dislikes**
equations	geometry
combinatorics	word problems
statistics	special products
the teacher's explanations	solving difficult problems alone
the lessons	proof
working together	homework

Table 7: What students like and dislike in mathematics

- Suggestions for the teacher: 1) solving more problems within a topic for better understanding (OR - weak student); 2) more examples when introducing new material (NB, PR); 3) more interesting problems (KR)

- Suggestions for themselves: 1) "I should do my calculations on paper instead of doing it in my head." (KD); 2) practice more (NB, BZS, KMM, HP, MSZ); 3) "I should pay more attention" (NYSZ, KR, PR)

As the table suggests students tend to dislike those problems and areas of mathematics that require independent and creative thinking. For example, in case of geometrical problems it is more difficult to find one typical problem, discuss it with the students and then solve many similar problems to make sure that students understood and learnt the idea. Solving problems alone and doing homework also require individual work and good problem solving skills.

6.2 The lessons

This section contains the detailed presentation and analysis of the lessons from the first part of the experiment. In case of each problem we included the lesson plan and also the way students handled the problem and the problem solving situation. The teacher's notes, the video and the voice recordings were also used in the analysis of the lessons and the students' work.

6.2.1 Matchstick game

The problem: Two players and 27 matchsticks are needed. The two players take turns and remove 1 or 2 or 3 matchsticks. The winner is who removes the last matchstick. Task for the groups: to find a winning strategy for both players.[9]

Since this problem was the first one that we wanted to tackle with using cooperative teaching techniques the first task was to explain this working format to the students and to organize them in groups. (Grouping principles: see above) This was followed by distributing the rules of the "game" and giving each group two boxes of matches.

Finding the winning strategy

Each group was divided into two pairs so that the students could try the game and see what happens. The cooperative structure *Think-pair-share* was used. The students experimented with different methods trying to win the game. They were asked to record their assumed strategies in their reflection books and try whether they really work. A winning strategy had to be found for both players.

The students soon realized that for finding the winning strategy they have to use the heuristic strategy "Thinking backwards" [9], [65]. Many solutions contained the idea that if 4 matchsticks were left on the table so that it is your opponent's turn then you are definitely the winner - of course every pair put this idea in words differently, but the main point was the same. The pairs found out the importance of number 4 quite fast. The next step was finding the other "winning" numbers starting from 4 matchsticks working backwards. The pairs of students used different approaches to determine the "important" numbers. Some examined the number of matchsticks left on the table after the player who wants to win drew; others examined the number of matchsticks on the table before they drew; and some students worked with the total number of drawn matchstick in a round. Here are some examples of how the students worded their ideas:[17]

"If you want to win you have to try leaving the multiples of 4 for your opponent: 4, 8, 12, 16, 20, 24."

"You have to leave 4 matchsticks for your opponent. From 24 counting back you have to complete the number of drawn matchsticks to 4. If your opponent draws 3 sticks at the beginning he will win (if he knows how to play.)"

"We start with 3 then slow down to 1 and we try to leave 4 for our opponent."

"The aim at the beginning is to take away 3 as a sum, then this should always be 4."

"At the end you have to be careful to leave 7 or more matchsticks because otherwise your opponent can leave you 4. The person who starts and draws 3 always leaves the multiples of 4 for his opponent. The number drawn by the

opponent must be completed to 4."

Class discussion

After a set time the pairs in the groups had to come together and share their ideas with each other. This was followed by a whole class discussion where a spokesman from each group presented the group's winning strategies. These strategies were compared and the differences were discussed.

Problem variation - problem posing

The first variation of the problem was suggested by the teacher. The task was to find the "losing" strategy without changing the rules of the original game. The cooperative structure used was *Think-pair-share* again.

The second variation of the problem was suggested by the groups. First, each group had to create a game that is similar to the original one by changing one or more factors in the game - the number of matchsticks, how many can be taken away, who wins ... etc. After writing new rules for the new games the groups swapped their games (Switched Task Cards). Using *Think-pair-share* the pairs in the groups had to figure out a winning strategy for the new games.

Finding ideas for different variations of the problems was rather straightforward for the students. Many of them suggested changes like "We could play with 5 sticks." or "Next time play with 10." right after the distribution of the rules of the original game.

Reflection - feedback

The final stage in the discussion of this problem was a whole class discussion in which we not only checked the solutions for the problem but also discussed the difficulties the students had to deal with while trying to find the solution. We collected those stages where the groups got stuck during the solution process and discussed what helped them to get through and what hindered them in finding a strategy.

Most of the students found finding the starting idea the most difficult. They wrote in their reflection books that once they had found a right way of thinking it was easy to follow it.

6.2.2 Number magic [10]

Starter problem

Type the following mystic number in your calculator: 15873. Chose a number between 1 and 9 including 1 and 9 then multiply 15873 by the chosen number. Multiply the result by 7. What do you notice? Try with more numbers. Can you explain what is going on?

The students were organized into groups of four[26], [21] and they had to solve the starter activity using the cooperative structure *Think–pair-share*. When applying this structure students were grouped at tables and the problem was presented to them. The team members were given a specific amount of time to think on their own about possible answers. Following this students discussed their answers with each other.

The problems [34]:

1. Type your age in your calculator and multiply it by 1443 then multiply the result by 7. What do you notice? What can the explanation be?

2. Type 12 345 679 in your calculator and multiply it by 9 then by a positive one digit number. What do you notice? What can the explanation be?

3. Type an arbitrary three digit number in your calculator then type the same digits again (so you can see a six digit number of the form ABCABC). Divide this number by 13, divide the result by 11. Finally, divide this result by 7. What do you notice? What can the explanation be?

4. For the next trick type 999 999 in your calculator then multiply it by a number between 1 and 6 (1 and 6 included). Divide the result by 7. What happens? Why?

The students worked in groups of four and we used cooperative techniques again. The structure applied this time is called *Pairs Check*. As mentioned before, the steps for this method are: *1. Divide student into groups of four and have them work with their shoulder partner. 2. Give each team one worksheet with problems or questions. (see problems above) 3. Partner A works the first problem or question while Partner B coaches and praises when necessary. 4. Partners reverse the roles. 5. After two problems are completed, pairs check with their partners across the table (face partners). Team discussion. 6. Steps 3 – 5 are repeated for every two problems/questions.* [15]

More number magic!!! (Time filler)

Write the last 7 digits of your phone number on a piece of paper. Use these digits to form a seven digit number that is different from your phone number. Subtract the smaller seven digit number from the bigger one then divide the result by 9. Check the remainder. What do you notice? What can the explanation be?

Students' solutions [3]

Most of the students followed the next pattern when trying to solve the problems: 1) systematic trial; 2) some kind of generalizations; 3) attempt to

[3]For more solutions from the students see Appendix 1

prove the statement. Let us examine some examples of students' work showing the aforementioned steps.

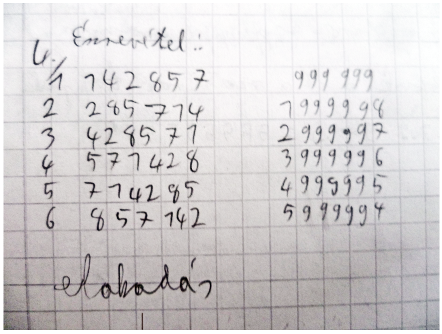


Figure 21: Systematic thinking

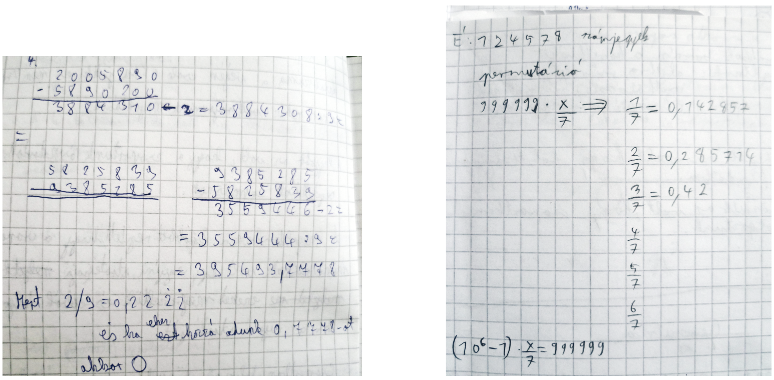


Figure 22: Attempts to prove

6.2.3 Area investigation [16]

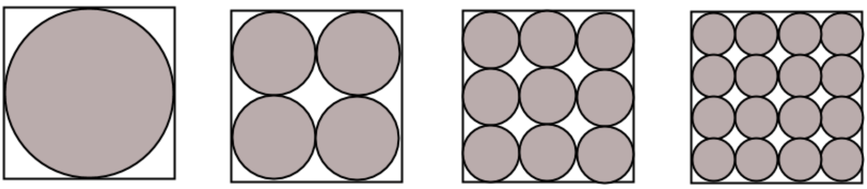
The detailed discussion of the whole problem took three consecutive 45-minute lessons. Solving and reflecting on the original problem was followed by the students' modification of the problem and the solution and discussion of a part of this latter problem.

Starter activity

For the starter activity the groups were given a worksheet on which they listed all mathematical knowledge – this could be a mathematical word or a formula – that they thought will be useful in solving the problem. An interesting point needs to be mentioned, namely: all groups listed that they will use percentages during the solution as it was mentioned in the question of the problem but at the end there was only one group that really worked out the percentage of the wasted material, the rest worked with fractions. Instead of using posters the mathematical information that the groups collected was written on the white board which was divided into four parts. Each group had a section to write their formulae in.1 After the groups had finished collecting useful mathematical ideas they sent one of their members to the white board who shared his group's notes with the others. Following this each group had time to modify they notes on the basis of what was written on the board and then they started solving the problem.

Solving the original problem

The problem: From a square measuring 60 cm x 60 cm we cut out circles as you can see on the figure. What percentage of the square is wasted in each case? Do you notice a pattern? Can you generalize your idea? Can you prove your conjecture for n circles?



To start with, the easiest part for the groups was to figure out that the area of the wasted material is the same for all four arrangements. Admittedly, calculating the area of a square and some circles and doing some basic arithmetic should not be a challenge for a 16 or 17 year old student, so no wonder that they succeeded in this part of the task quite fast and none of the groups needed extra help.

Moreover, when I asked the groups:*"How about n circles?"*, all of them said that the waste must be the same in that case too. Then I asked them to prove their statement, to generalize what they found. The reason for this was that students usually do not feel the need for proof and they have little experience even with generalizing not to mention proving statements. So, this was the point where the students became puzzled. Nobody had an idea how to start therefore I had to provide some guidelines, some guiding questions and comments.[7] These were the following ones:

- What can you say about the number of circles?

- What is the relationship between the number of circles and the length of the side of the square? Can you write an expression for the side length in terms of the number of circles?
- Can you write an expression for the area of the circles?
- Can you express the wasted area?

Since the different groups were working in different paces the above mentioned discussion happened four times.

Why did the students struggle with generalizing their ideas? The reason for this could be that in everyday Maths lessons problem solving usually ends here: You did some calculations on the basis of the given data, you do have some results that seems correct, job done, you can go to the next exercise. So no wonder when I encouraged them to read the whole question again and they found out that they have to find an answer in case of the n^{th} figure, they looked a bit puzzled.

They were asked to continue working in pairs using the *Pairs Check* method after the groups received the above mentioned "guiding questions".

The pairs were given some time to brainstorm ideas but they were asked to continue the solution in the original groups. After some productive thinking and discussion the groups came up with some seemingly different formulae for the amount of wasted material in the n^{th} square. Although it was a lesson based on cooperative teaching methods, to ensure that everybody was on the right track an occasional whole class discussion was unavoidable. To do this in an effective way the white board was divided into four sections again, and after every group had worked out a formula for the n^{th} square. These formulae were written in the different sections. Following this each formula was interpreted with the help of the groups. In this discussion the teacher led the students through their explanations with the help of questions.

The groups came up with four different formulae that were meant to express the same thing, so first of all we had to clarify how the groups created these formulae and what the different parts meant. Thus, a speaker form each groups was selected who had to explain their formula. It was not surprising that all formulae turned out to be correct and they were equivalent. After generalizing the problem the question how the task could be modified arose.

Students' solutions

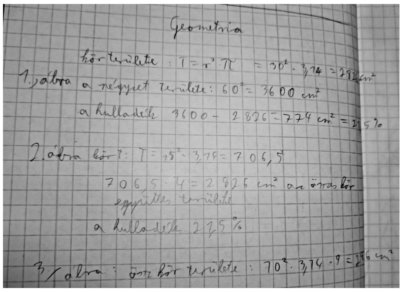


Figure 23: Trying with exact values

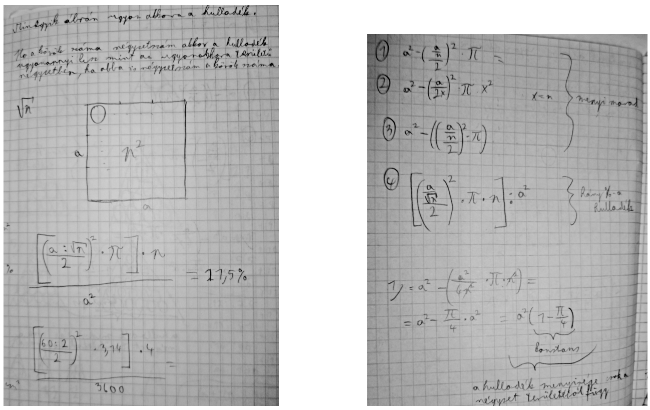


Figure 24: Trying to generalize and simplify the formula

Modifying the problem

The students came up with different ideas, however, due to lack of time we settled for the one that suggested replacing the square with an equilateral triangle and trying to arrange congruent circles inside it. Since on the basis of my experience it often does help the students if they receive a task in chunks which are easy to understand at first they only had to work out the possible arrangements.

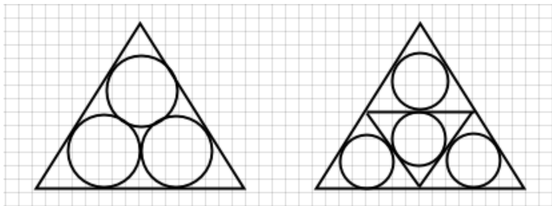


Figure 25: Students' ideas for the solution of the modified problem

The new problem

Now that we had some pictures to work with the groups received the following questions to answer about both arrangements (see picture above):

- How many congruent circles fit into each triangle? Do you notice a pattern?
- How many circles would be in the n^{th} triangle?
- If we cut out the circles how much of the triangle would be wasted?

The students soon noticed that in the first arrangement the total number of circles can be calculated from the following pattern:

1
1, 2
1, 2, 3
1, 2, 3, 4

After each group had discovered this sequence we had to stop for another whole class discussion as it was a great opportunity to introduce the concept of triangle numbers. Furthermore, students noticed that having found this pattern leads to yet another problem, i.e. how to find the sum of the first n natural numbers.

Once the pattern of the circles was sorted out and the formula for the n^{th} triangle number was explained the groups found a new obstacle. Clearly, this time a new method was needed for deriving the radius of the circle that was inscribed an equilateral triangle. Therefore the groups had to work together again using *Pairs Check*. The procedure was the same as the one used before. Following the pair work the groups came together and compared their results then the different ways of calculating the length of the radius were presented on the board that had been divided into four sections like before.

Unfortunately, due to lack of time we could not go any further with the detailed analysis of the problem in class. Therefore the rest of the problem was left to be discussed in group study sessions.

6.2.4 More beads

The problem: Three beads are threaded on a circular wire and they are coloured either red or blue. You repeat the following actions over and over again. Between any two of the same colour put a red and between any two of different colours put a blue, then remove the original beads. Discuss all the possible outcomes. What happens when you do the same thing with 4 beads, 5 beads or 6 beads?

After organising the students into new groups each group received the description of the task. As the original task was found on the *nrcih* (http://nrich.maths.org) website which also contained an interactive program that showed how the problem can be approached, with the help of the interactive whiteboard the whole class watched a demonstration. This visualization helped understand the problem.

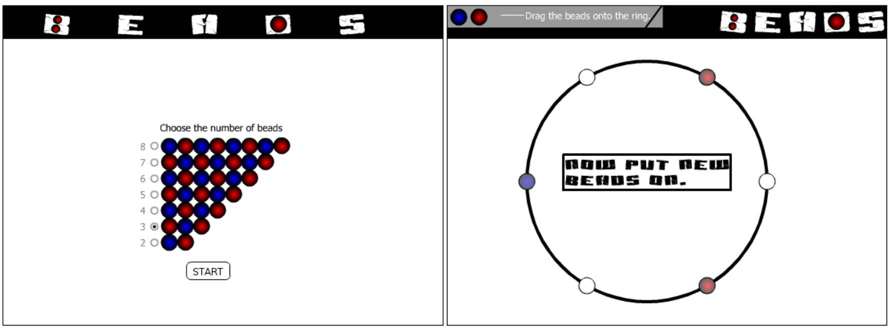


Figure 26: Interactive representation of the problem

Solving for three beads

To help students see and try the different arrangements they worked with concrete materials. Each group received small discs. One side of each disc was painted red the other side was painted blue. In this phase the cooperative structures *Think-pair-square* and *RoundRobin* were used.

After reading the instructions and watching the demonstration on the whiteboard the pairs in the groups tried to understand what their task was. Despite the fact that they saw an example on the board there were some pairs who had no idea how to start solving the problem. - The reason for this might be that investigation type open problems are hardly ever used in Hungarian mathematics classrooms. - Using the discs the students started creating different arrangements and experimented with them to try to recognize some pattern. However, listing all arrangements in a logical order and making notes of the occurring patterns caused a problem for most of them. Not to mention explaining a connection between the starting arrangement and the resulting pattern. It seemed that the students just try the arrangements at random so the teacher decided to have a class discussion.

The first point that had to be discussed was that since the beads are on a circle the red - blue -red, red - red - blue and blue - red - red arrangements are identical. Following this based on the suggestion of the majority of the class we collected the different patterns for three beads like this:

1. $rrr \rightarrow rrr$
2. $bbb \rightarrow rrr$
3. $rrb \rightarrow rbb \rightarrow rrb$
4. $bbr \rightarrow rrb \rightarrow brr$

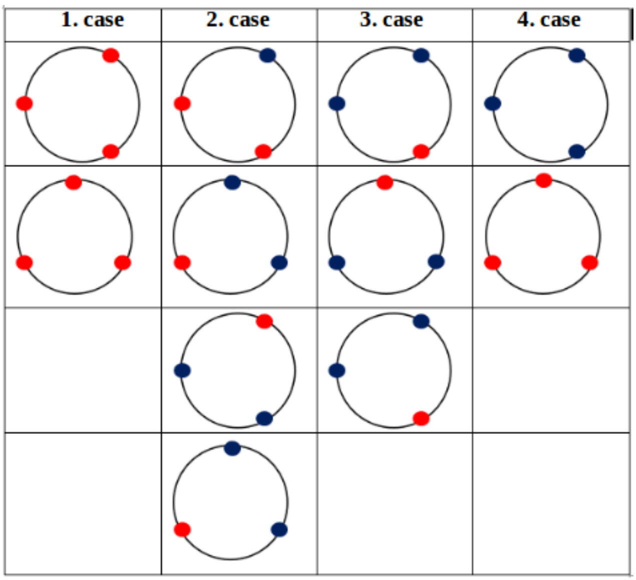


Figure 27: Summary of the possible arrangements for three beads

The observations about the change of patterns formulated by the whole class together were the following:

- if the beads are the same colour at the beginning then the final pattern is all red beads
- if the beads are different colour at the beginning then the same patterns keep being repeated

Solving for more beads

After the class discussion every student had an idea about how to make sensible notes. 2 - 2 groups had to investigate for 4 beads and another 2 - 2 groups for 5 beads. The applied cooperative structure was *Think-pair-square*. The class was given a set time to try to discover a pattern and to formulate a rule. The next phase was to share information among the groups. To do this efficiently we used the *Jigsaw* method.

While working on the modified problems being able to use concrete material was a huge help for the students. It enabled them to see which arrangements are identical and made it easier to do the changes. As a result of working in pairs and using cooperative techniques more students took part actively than in an average lesson. Thanks to the nature of the cooperative structures everybody had to participate equally if they wanted to achieve success. Even the usually quiet students took an active part in making notes, creating arrangements and formulating, explaining ideas. Moreover, communication among students was more lively than usually. Sometimes with more, sometimes with less accuracy, but they always tried to explain their own views while the listeners often corrected those who spoke or tried to convince them that they are not right. As a result of this their formulations became more and more accurate and their arguments became more and more meaningful. However, there was one group whose members kept communication to the necessary minimum which slowed down the working pace but did not impede the successful problem solving.

The final phase was class discussion whose aim was to make sure that every student had understood the different ways of thinking and solutions. At this point each student had the description of patterns for 3, 4 and 5 beads so their homework task was to try to formulate a general rule for the change of patterns. Furthermore, due to lack of time investigations for 6, 7 or 8 beads were left for homework and only a short discussion of discovered patterns was part of the next lesson. Detailed analysis was left for group study sessions again.

Students' work:

The following figures show how students recorded their work:

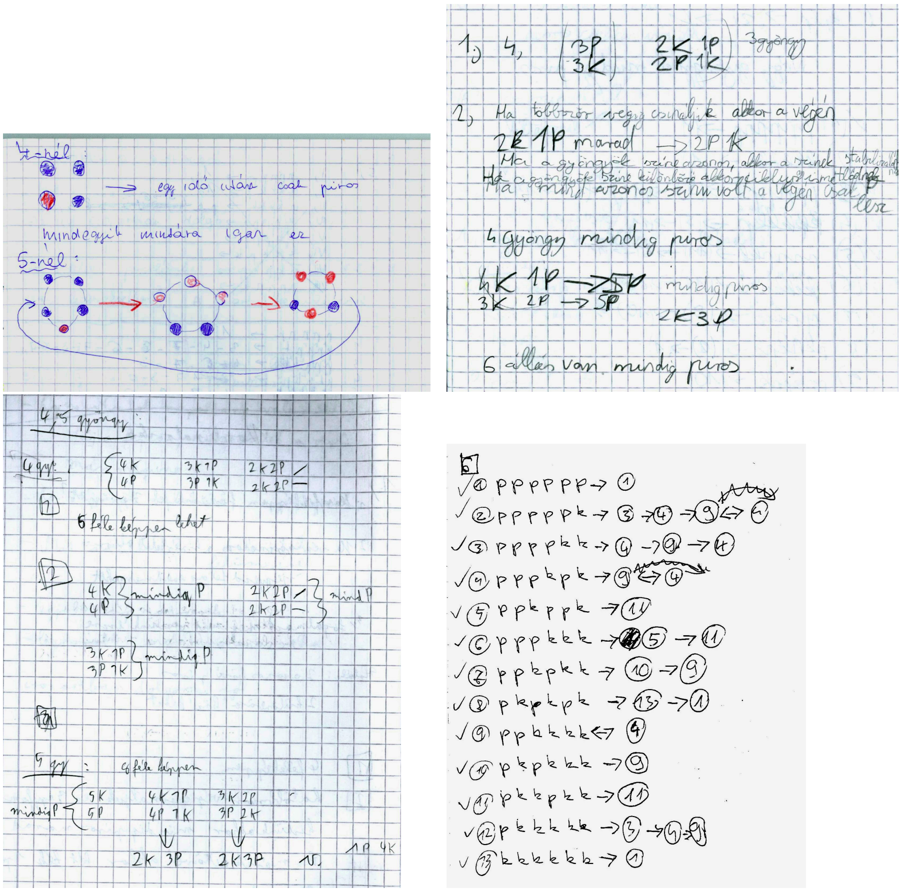


Figure 28: Students representing the arrangements of beads

The example above shows that investigation type problems are effective tools for demonstrating the creative nature of mathematics. Many people think that mathematics is a rigid system of rules and facts but with investigations we can prove them wrong. Mathematics can be an exciting subject for students if they are allowed to discover connections even if these connections are new only for them.[81]

6.2.5 Primes and factors

In the discussion of this problem field our aim was to combine open problems and guiding questions and to present these with cooperative teaching techniques.

Starter problem

Think of a two digit number. Reverse its digits to obtain a new number and subtract the smaller number from the bigger one. eg: $42 - 24 = 18$ Try with more numbers. Can you get a prime number as the result? Why/Why not? Can you explain your idea? Prove that the answer is never a prime. What if you work with three digit numbers? With four digit numbers? How about n digit numbers?

After organising the students into groups of four they were divided into two pairs and we used the *Pairs Check* structure to solve this problem. They were given 10 minutes to answer as many questions as they could. This ten-minute period was followed by a class discussion.

All pairs started trying out the operation with different two digit numbers. After a while the roles suggested by the *Pairs Check* structure were broken. Although only one of the students made notes, the ideas came from both of them and they mutually guided each other in the solution. The majority of the pairs soon realized that it is impossible to obtain a prime number as the result of the calculation. The difficulties occurred when they had to prove this statement. As none of the pairs realized that if they label the digits with different variables and express the place values as well they obtain an expression that clearly shows that the difference is divisible by 9 therefore it cannot be a prime number.

First number: $\overline{xy} = 10x + y$

Second number: $\overline{yx} = 10y + x$

Difference: $10x + y - (10y + x) = 9x - 9y = 9(y + x)$

A problem

Find the biggest whole number which is a factor of each term of the following sequence:

$1^5 - 1, 2^5 - 2, 3^5 - 3, 4^5 - 4, ..., n^5 - n$

This problem was discussed applying the *Pairs Check* structure in the following way. One of the students in each pair had to start solving the problem while saying every thought, every idea aloud and commenting on everything he wrote down. If this student got stuck it was the other student's responsibility to help him/her carry on. The "helper" student could come up with his own ideas or he could use the guiding questions. The guiding questions were printed on separate pieces of paper in advance and were placed on the tables in order, face down so that the top most question was the first one. When the "solver" student run out of ideas the “helper” turned the top most piece of paper over and read out the question which was meant to guide the students through the steps of the solution.

The guiding questions:

1. Examine the second term. Can you factorize it?
2. Look at the term in factorized form. Which numbers divide this term?
3. Is it divisible by 2, 3 or 5?
4. Do the previous steps with the third, fourth and fifth terms.
5. Can you carry out the steps for the general term $n^5 - n$?

After trying to solve the problem for a set time the pairs came together in the original groups of four and shared and compared their ideas. Most of the groups managed to carry out the steps suggested in the guiding questions, furthermore nearly all of them found that the factorized form of the general term is:

$n(n-1)(n+1)(n^2+1)$

Finding factors of terms that contained concrete numbers was not a problem for the students. For example, for the third term:

$3^5 - 3 = 3 \cdot (3-1) \cdot (3+1) \cdot (3^2+1) = 3 \cdot 2 \cdot 4 \cdot 10$

It was easy to notice that this number is divisible by 2, by 3, by 5 and by 4. However, hardly any students were able to connect divisibility to the factorized form of the general term that is why we had to change to whole class discussion.

Another problem

The third problem we discussed in this problem field was:

Take any pair of two digit numbers ab and cd where, without loss of generality, ab $>$ cd. Form two 4 digit numbers abcd and cdab and calculate:

$$\frac{abcd^2 - cdab^2}{ab^2 - cd^2}$$

Repeat this with other choices of ab and cd. There is a common feature of all the answers. What is it? Why does this occur? Generalise this to n digits for other values of n.

The discussion of this problem was carried out in the same way as that of the first problem. To summarize the experience with this problem field we can say that investigation-type problems help the average ability or less able students to get started, as we could see here, everybody could try to solve the problem with exact numbers but when it came to generalization and proving ideas the students were stuck.

Time filler problem

Think of a prime number that is bigger than 3. Square it then subtract 1 from the result. Do these operations with more prime numbers. What do you notice? Can you prove your assumption?

Students' work:

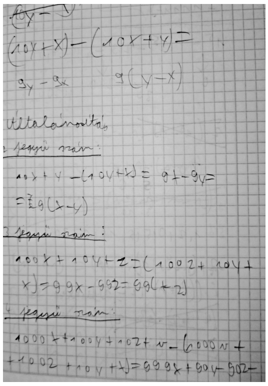


Figure 29: Attempt to generalize the starter problem

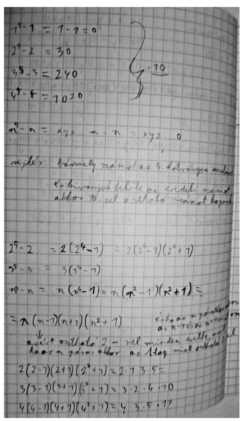


Figure 30: Following the guiding questions - the second problem

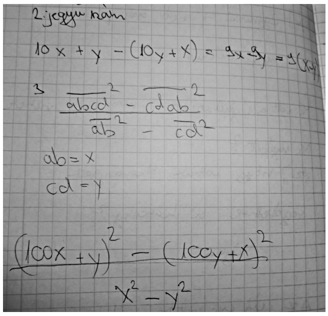


Figure 31: Attempting the starter and the second problem

6.2.6 Students' comments on the first part of the experiment

In the following section we present some comments that students wrote in their reflection books. The comments were listed in three groups based on their content. The first group contains comments on cooperative work, the second group contains comments on the problems and the third group contains comments on cognitive load. Some of these comments could have been listed under more than one heading.

On cooperative work

Positive comments

- "The easiest thing in solving this problem was that we could work in groups." NYSZ
- "Our new group was better." ML
- "Summarizing the previous ideas at the beginning of the lesson and working in groups were useful." OR
- "Our new group took the problem more seriously, I enjoyed working with them better." KA
- "The best thing was working in groups." HP
- "It helped that we could work in groups." MT
- "I learned how to plan ahead and that working in groups is often more effective but also difficult." NB

- “I learned that it is good to work in groups and that mathematical puzzles can be interesting.” NYSZ
- “It is easier to work in groups than alone.” OR and KMM
- “It was good to work in groups because everybody had an idea and it was exiting to test each of them.” KD
- “Working in groups has good and bad aspects as well. It is good because the others might point out mistakes I didn’t notice, but also bad because everybody has a different way of thinking.” KA
- “Working in groups helped in finding the solution.” BZS and MT
- “Working in groups was good because more people managed to find the solution faster.” HP
- “I really liked working in groups however towards the end the others were too much.” MT
- “Working in groups was good, we often completed each other’s ideas, but the behaviour of BP was often annoying.” HM
- “Personally, I don’t like working in groups but it often came handy that I was able to discuss my results with someone.” KR
- “There was always someone to help. ... I don’t really want to work in this format more.” KA
- “I learned a lot from my group mates.”MSZ

Negative comments

- “The most difficult thing was to work in a noisy classroom.” MSZ
- “Working in groups has good and bad aspects as well. It is good because the others might point out mistakes I didn’t notice, but also bad because everybody has a different way of thinking.” KA
- “The most difficult was to make others understand our way of thinking.” HM
- “We could work in groups again but not too often.” PR
- “Working in groups would have been better if we could have chosen our groups mates.” KD

- "Group work confirmed that I prefer working alone. I would rather work following my teacher's explanations." NYSZ

On the problems

Positive comments

- "It (*the Matchstick problem*) wasn't a difficult problem. Working together and discussing ideas helped in the solution." ML
- "It (*Number magic*) was easy because we could use a calculator and try some numbers." BZS
- "At first the problems (*Number magic*) looked like magic, but after thinking about them we saw that we had to use maths to solve them." KA
- "The guiding questions made the situation more complicated, but to some extent they helped get started." MSZ
- "First the guiding questions seemed useless but soon we realized their importance." NB and OR
- "The guiding questions were really helpful." BT, KA, KMM and HP
- "The guiding questions helped start solving the problem." MT
- "When I manage to solve a problem it makes me happy. I feel I can cross another thing of from the list of challenges." NB
- "It was good to see new types of problems." KD
- "The problems were interesting and it was good that we didn't follow the scheme of work." HP
- "The problems were diverse." KA
- "The problems were interesting and they helped me develop new thinking strategies." KD

Negative comments

- "It was difficult that sometimes after many trials I still couldn't see the solution." BT
- "It was difficult to see the logic of the game when the number of matchsticks was still big." NB

- “The guiding questions were useless.” BP
- “It was difficult to find out how to start solving the problem.” NYSZ
- “We didn’t use the guiding questions.” KD, HM and PR
- “I hope we won’t have more problems like this (*More beads*).” MSZ
- “Sometimes it was boring doing the same task for a long time.” NYSZ and BT
- “The first problem was very engaging, but at the end the problems became boring.” KMM, PR
- “The modifications of the problems often became boring.” NYSZ

On the cognitive load

- “After trying more options everybody had an idea and we only had to choose the best one.” KMM
- “The easiest thing in solving this problems was that everybody had an idea about the task.” OR
- “What helped most in the solution was that everybody shared their ideas.” MSZ
- “It helped that we could use real “beads”.” MSZ, BP, KD and BT
- “The best feeling was when the others used our solution.” OR
- “It felt good that often I was the first one in our group who saw the solution but sometimes I had difficulties with explaining my ideas to the others.” HM
- “The best thing was when we were thinking together and compared each other’s ideas.” PR
- “It helped in the solution that everybody had a strategy which we discussed and using these we formulated the winning strategy.“OR
- “Working in groups was good because more people managed to find the solution faster.” HP
- “It was easier to work in groups than alone because we could use the thinking of more people.” KMM, OR
- “Working in groups was good, we often completed each other’s ideas, but the behaviour of BP was often annoying.” HM

6.3 Post-test

6.3.1 Mathematical post-test

As mentioned before the students had to complete a mathematical pre – test (see above) before the experiment that measured the mathematical knowledge and the knowledge of heuristic strategies needed for solving the problems that were presented in the first part of the experiment. After finishing the first part of the experiment the students completed a mathematical post – test. Some tasks of the two tests were identical, however the post – test included questions in which the problem solving skills learnt during the first part had to be applied.

1. The bigger cog-wheel of a bike has 35 teeth and the smaller one has 15. How many times do we need to turn the pedal so that both cog-wheels get back to their original position? (The pedal is on the bigger wheel.)

2. Which numbers are always factors of the product of three consecutive numbers? Why?

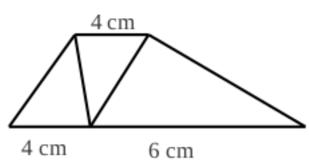
3. In a jewellery shop on Monday half of the stock and four pieces of jewellery were sold. On Tuesday half of what was left and further two pieces were sold. On Wednesday the shop assistant sold five pieces of jewellery. On Thursday 2 less than half of what was left was sold. At the end there were 8 pieces of jewellery in the shop. How many pieces of jewellery did the shop have on Monday?

4. In how many ways can we arrange 3 red and 3 blue beads in a circle?

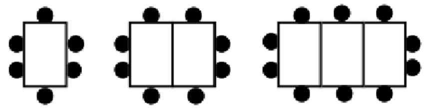
5. How many different four digit numbers can we form so that each digit is an element of the set $\{1; 2; 3; 4; 5; 6; 7\}$?

6. Think of a three digit number, write it down then reverse the digits (eg.: 756 and 657). Subtract the smaller number from the bigger one. What do you notice? Prove your assumption.

7. Work out the area of the triangles on the figure if you know that the area of the trapezium is 21 cm^2. (The shape is not drawn to scale. $A = \frac{a+c}{2} \cdot m$)



8. Work out the area of the inscribed circle if the triangle is equilateral and its sides are 1 m long.
9. In a restaurant the guests sit around tables as you can see on the figure. Continuing the pattern how many guests can sit around four tables? Around 5 tables? Around 100 tables? Around n tables?



6.3.2 The results

Student	Sum	Grade		Mean average	Standard deviation
BZS	27	4		3.00	2.35
BT	24	4		2.67	2.00
BP	41	5		4.46	0.73
HM	44	5		4.89	0.33
HP	32	5		3.56	1.59
KR	36	5		4.00	1.80
KA	18	3		2.00	2.12
KMM	33	5		3.67	1.94
KD	42	5		4.67	0.50
ML	27	4		3.00	2.29
MSZ	23	4		2.56	2.30
MT	38	5		4.22	1.64
NB	19	3		2.11	2.52
NSZ	35	5		3.89	1.83
OR	17	3		1.89	2.09
PR	21	4		2.33	2.06

Table 8: Results of the mathematical post-test

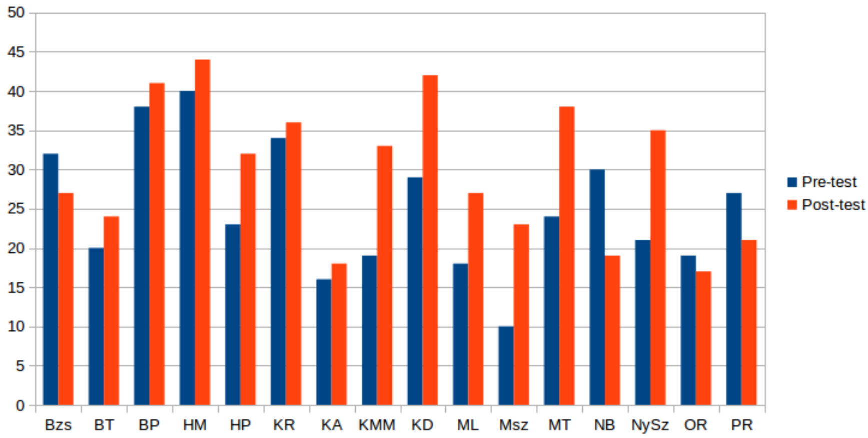


Figure 32: Total marks achieved on the pre- and post-mathematical tests

6.3.3 Interpreting the results

The pre- and the post test

In the following section we compare the students' achievement on the mathematical pre-test and post-test. The analysis was published in [12].

It can be clearly seen on the graph that 12 students out of the 16 improved their results and only four of them achieved a lower sum on the post – test than on the pre – test. Students BZS , BP, HM and KR already had a sum above 30 points on the pre – test. Two of these students (BP and KR) based on their previous achievement in mathematics, their previous contribution to classwork and their achievement on previous mathematics competitions can be considered as talented students in mathematics. On the other hand, KA, OR and MSZ are weaker students and their pre – test results are lower compared to the whole group. However, after solving mathematical problems with cooperative techniques two of these students (KA and MSZ) managed to improve their test results, especially MSZ, who doubled his total points on the post – test. Out of the four students who received lower points on the post – test one (OR) can be considered as a weak student, while the others (BZS, NB and PR) are average ability students.

For each task the students were awarded with maximum of 5 points. The table below shows the mean average point of each student on the two tests and the standard deviation (SD in the table) of the received points on each test.

	Pre-test mean	Pre-test SD*	Post-test mean	Post-test SD
Bzs	3.56	1.94	3.00	2.35
BT	2.22	1.99	2.67	2.00
BP	4.22	1.72	4.56	0.73
HM	4.44	1.13	4.89	0.33
HP	2.56	2.40	3.56	1.59
KR	3.78	1.99	4.00	1.80
KA	1.78	2.44	2.00	2.12
KMM	2.11	1.45	3.67	1.94
KD	3.22	2.05	4.67	0.50
ML	2.00	2.40	3.00	2.29
Msz	1.11	1.69	2.56	2.30
MT	2.67	2.24	4.22	1.64
NB	3.33	2.18	2.11	2.52
NySz	2.33	1.87	3.89	1.83
OR	2.11	2.03	1.89	2.09
PR	3.00	2.18	2.33	2.06

Figure 33: Statistical measures of the pre- and post-tests

On the pre – test the mean average score was the highest for HM and BP and their standard deviation was among the low ones. This means that these students received high points for each task on the whole test and they attempted to answer each question. As mentioned before BP is a talented student, so it is not surprising that he made an improvement on his mean average as well. KMM and MSZ also had a low standard deviation but their mean averages were also rather low. These data suggest that these two students attempted most tasks but received low points for them. MSZ, who is one of the least talented students in the group managed to raise his mean average point to 2.56 on the post – test. The stronger students (BP, HM, KR) achieved better average points on the post – test; one of the weaker students (OR) achieved a lower, the others (KA and MSZ) achieved a higher mean average point on the post – test.

As in case of the pre-test let us examine the separate problems based on which thinking skills their solution requires. The horizontal lines on the graphs indicate the mean average point gained for the given problem while the numbers after the words "Pre" and "Post" show the number of the problem in the given test.

Systematic thinking

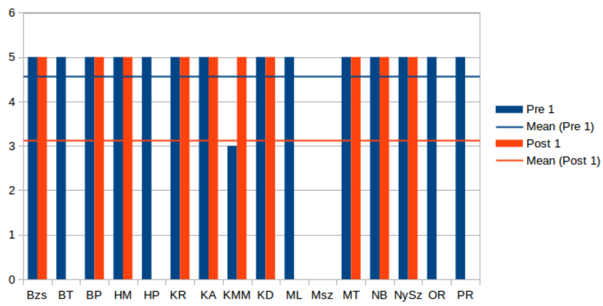


Figure 34: Systematic thinking (pre- and post-tests)

It is interesting to see that the mean average point of the post-test problem was lower than that of the pre-test problem. It is noticeable that fewer students managed to solve the problem successfully on the post-test but those who attempted it achieved full points for this problem. There are five students who solved the problem on the pre-test but could not obtain any points for it on the post-test. KMM is the only student who improved his performance. It is difficult to explain the reason behind this lapse of achievement as the majority of the problems discussed during the first part of the experiment require systematic thinking. However, lack of motivation to complete the post-test to ones best might be one reason.

Thinking backwards

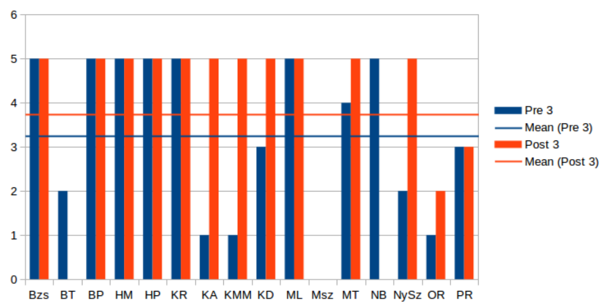


Figure 35: Thinking backwards (pre- and post tests)

The graph shows that there was an improvement in the achievement of the students in this problem. Those students who were able to solve this problem on the pre-test also managed to complete it on the post-test except for NB who did not gain any points for the problem on the post-test despite the fact that he is one of the more talented students. Six students improved their achievement, among them KA and OR who are the least talented in this group. The data presented on this graph suggests that the first part of the experiment had a positive effect on improving the thinking skill thinking backwards.

Listing all options logically

I.

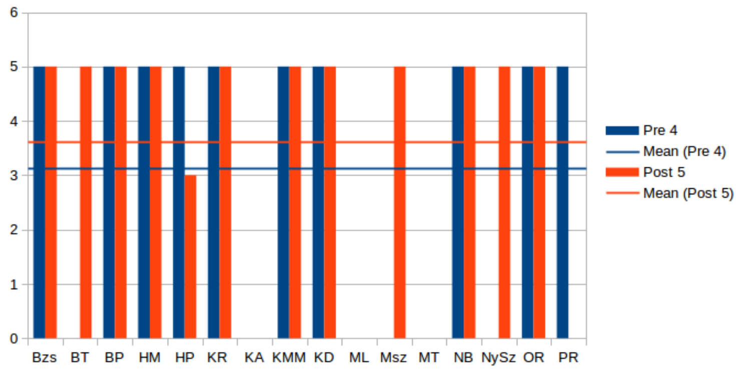


Figure 36: Listing all options logically I. (pre- and post-tests)

II.

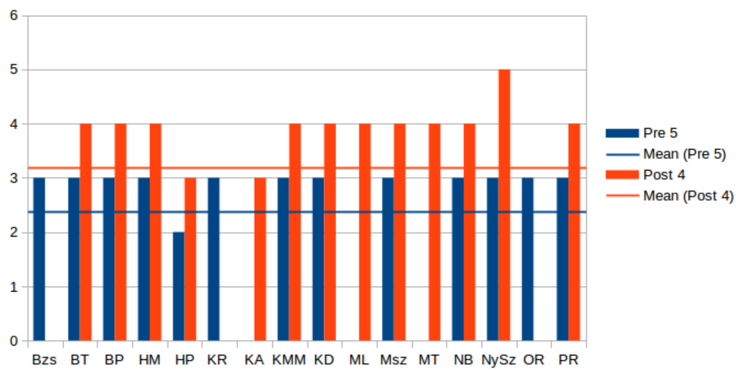


Figure 37: Listing all options logically II. (pre- and post-tests)

It is noticeable that there was an improvement in the achievement in case of both problems. In case of the first one there were three students (MSZ is one of them) who did not gain any points for the problem on the pre-test but obtained full points on the post-test and there were only two students who gained less points for the post-test problem. It can be seen that three students did not obtain any points for this problem on any of the tests. The reason for this in case of KA can be that he belongs to the less talented students. However, ML and MT are quite able in mathematics so it is surprising that they were not successful in solving this problem. The improvement is more obvious in case of the second task. Thirteen students managed to obtain more points for this problem on the post-test than on the pre-test (KA and MSZ are among them). This improvement can be explained by the discussion of the *More Beads* problem that includes listing all options logically in case of a circular arrangement.

Pattern recognition

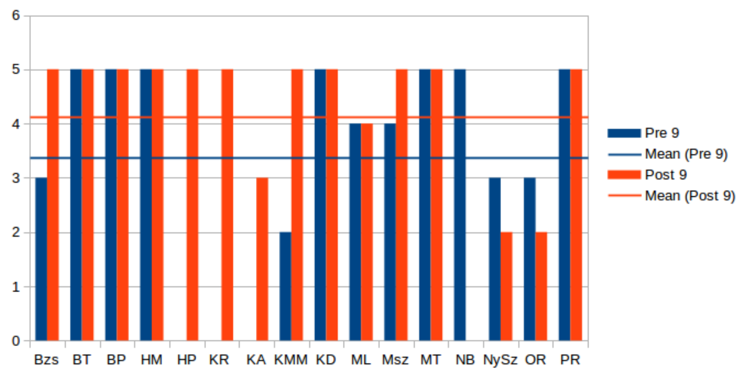


Figure 38: Pattern recognition (pre- and post-tests)

It can be clearly seen that the majority of the students gained the same or more points on the post-test than on the pre-test and all students managed to obtain some points for this problem on the post-test. As mentioned above this problem includes a part that asks for generalization which is unusual for Hungarian students. More problems in the first part of the experiment also required generalization and that is why this technique was thoroughly discussed with the students. The improvement in their achievement suggests that cooperative techniques were useful tools for developing pattern recognition and generalization skills.

6.4 The delayed test

The questions in the delayed test were exactly the same as the questions in the post-test.

6.4.1 The results

Student	Sum	Grade		Mean average	Standard deviation
BZS	26	4		2.89	2.09
BT	24	4		2.67	2.24
BP	41	5		4.56	0.73
HM	42	5		4.67	0.71
HP	21	4		2.33	2.40
KR	31	5		3.44	2.35
KA	26	4		2.89	2.03
KMM	27	4		3.00	2.12
KD	37	5		4.11	1.69
ML	27	4		3.00	2.24
MSZ	27	4		3.00	1.87
MT	32	5		3.56	2.13
NB	24	4		2.67	2.40
NSZ	39	5		4.33	1.32
OR	21	4		2.33	1.58
PR	29	4		3.22	1.72

Table 9: Results of the mathematical delayed test

Comparing the three mathematical tests

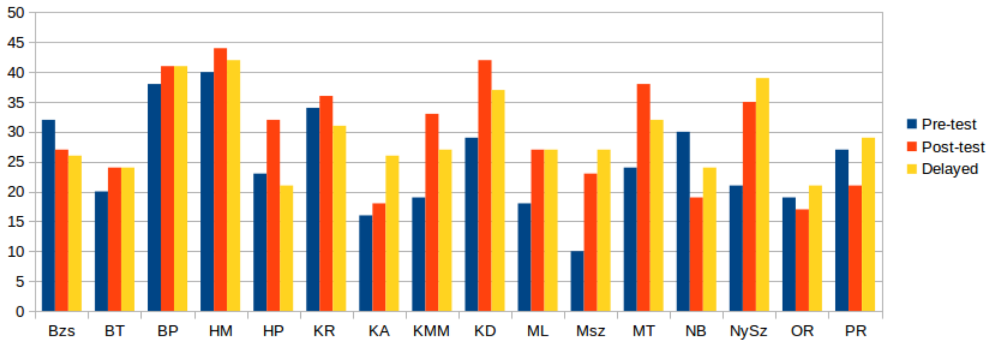


Figure 39: Total points achieved on the three tests

6.4.2 Interpreting the results

It can be clearly seen on the graph that 12 students out of the 16 improved their results on the post – test, however there are only three students who kept improving their results on the delayed test and there were further three students who obtained a lower sum on the post-test but managed to improve their achievement on the delayed test. The aforementioned talented students, BP and KR did not produce an outstanding result neither on the post- nor on the delayed test. The sum of BP stayed the same on these two test, however, the sum of KR decreased by four points. The bar chart indicates that the total points of the previously mentioned weaker students, KA, OR and MSZ were higher on the delayed test than on the previous two tests - there was a significant change in the achievement of KA. It is noticeable that there were four students (HP, KMM, KD and MT) whose total points increased highly from the pre-test to the post test. In comparison, their total points on the delayed test seemingly decreased. All four students can be considered as average ability students.

The table below shows the mean average point of each student on the three tests and the standard deviation (SD in the table) of the received points on each test.

	Pre-test mean	Pre-test SD*	Post-test mean	Post-test SD	Delayed mean	Delayed SD
BZS	3.56	1.94	3.00	2.35	2.89	2.09
BT	2.22	1.99	2.67	2.00	2.67	2.24
BP	4.22	1.72	4.56	0.73	4.56	0.73
HM	4.44	1.13	4.89	0.33	4.67	0.71
HP	2.56	2.40	3.56	1.59	2.33	2.40
KR	3.78	1.99	4.00	1.80	3.44	2.35
KA	1.78	2.44	2.00	2.12	2.89	2.03
KMM	2.11	1.45	3.67	1.94	3.00	2.12
KD	3.22	2.05	4.67	0.50	4.11	1.69
ML	2.00	2.40	3.00	2.29	3.00	2.24
MSZ	1.11	1.69	2.56	2.30	3.00	1.87
MT	2.67	2.24	4.22	1.64	3.56	2.13
NB	3.33	2.18	2.11	2.52	2.67	2.40
NySz	2.33	1.87	3.89	1.83	4.33	1.32
OR	2.11	2.03	1.89	2.09	2.33	1.58
PR	3.00	2.18	2.33	2.06	3.22	1.72

Figure 40: Statistical measures of the three tests

On the three tests the mean average score was the highest for HM and BP and their standard deviation was among the low ones. As mentioned before this means that these students received high points for each task on the whole test and they attempted to answer each question. The most significant change can be noticed in KD's (average ability student but with potential to do well in mathematics) standard deviation through the three tests. The standard deviation was 2.05 on the pre-test which dropped to 0.5 on the post-test and raised to 1.69 on the delayed test while the mean average of the points raised. These numbers suggest that KD attempted nearly all problems on the post-test and he received high points for each and he was successful in solving more problems on the delayed test as well, however, the points he received for the separate problems vary more (see Delayed SD). As for the weaker students, the mean average points of KA showed a steady improvement from the pre-test through the post-test to the delayed test and at the same time the standard deviation was decreasing which indicates that his performance became more and more stable and successful. In comparison, the mean average points of OR were fluctuating with a barely changing standard deviation. These data suggest that his performance improved then dropped and that he attempted more problems in all three tests but gained different points for them. Finally, let us examine the results of MSZ. There was a significant raise in his mean average points while the standard deviation increased then decreased reflecting an improvement in his overall performance.

Finally, let as discuss the separate tasks on the three test.

Systematic thinking

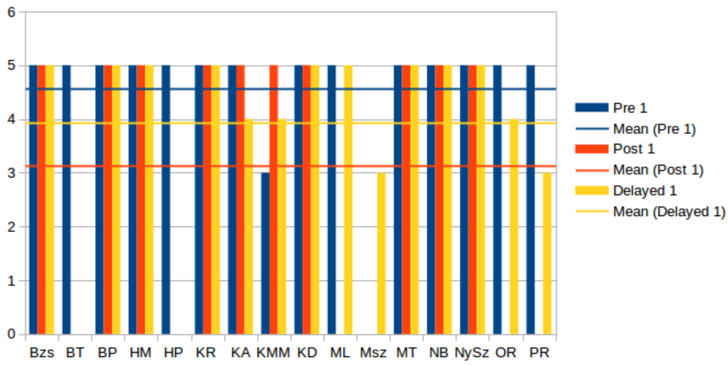


Figure 41: Systematic thinking (pre-, post- and delayed tests)

The bar chart shows that after a decrease of the total points on the post-test there was an increase of the sums on the delayed test. Eight students obtained full points for this task on the delayed test seven of whom had full points on all three tests. There was a great improvement in the achievement of MSZ, OR and PR, two of these students belong to the weaker ones.

Thinking backwards

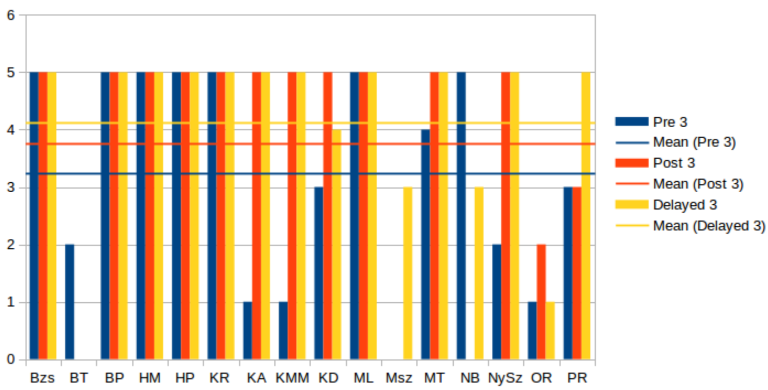


Figure 42: Thinking backwards (pre, post- and delayed tests)

It can be seen from the graph that there was a steady increase in the mean average of points throughout the three tests. There were seven students who gained full points for this problem on the pre-test - each of them can be considered either talented or quite able in mathematics - and this number rose to eleven on the post- and delayed tests. KA (weak student), KMM and NYSZ (average ability students) showed a great improvement between the pre- and the post-test and managed to keep their good achievement on this problem on the delayed test as well. The bars representing the achievement of BT and MSZ stand out. BT, who can be regarded as a weaker average ability student, gained 2 points for this problem on the pre-test but did not gain any points on the following two tests. In contrast, MSZ (weaker student) did not obtain any points for this problem on the first two tests but gained 3 points on the delayed-test.

Listing all options logically

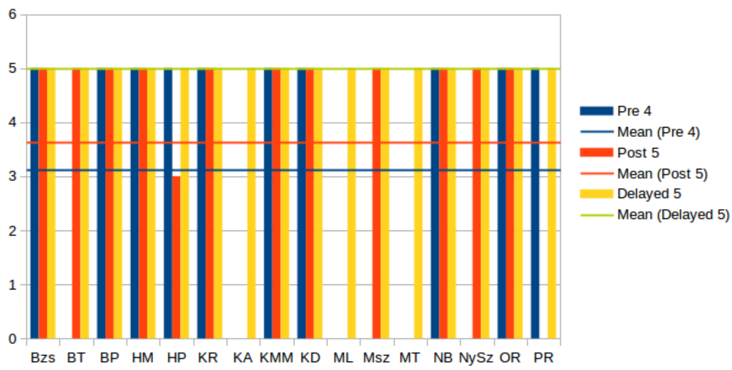


Figure 43: Listing all options logically I. (pre, post- and delayed tests)

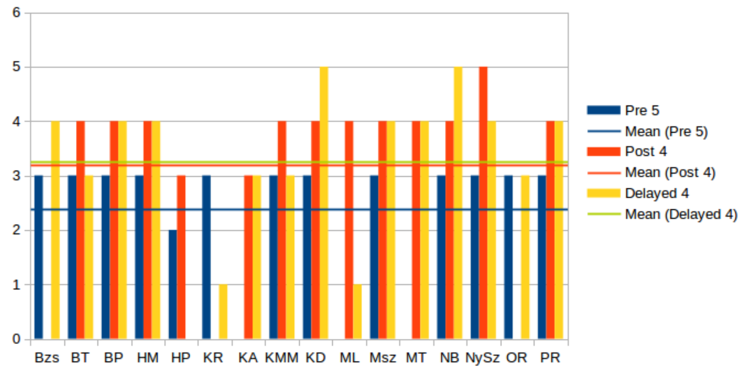


Figure 44: Listing all options logically II. (pre, post- and delayed tests)

The first graph indicates a significant improvement in solving this type of problems, especially between the post- and the delayed test. The reason for this is that combinatorics, including permutations with and without repetition is part of the year 10 scheme of work. Many problems like the first one were discussed towards the end of the school year both with cooperative techniques and individual work.

Problems related to combinatorics also involved examining circular arrangements, yet, not all students gained full points for this problem. Thirteen students improved their results between the pre- and the post-tests and eleven students improved or kept their results between the post- and delayed tests. KD and NB (both of whom are stronger average ability students) managed to solve this problem fully. It is noticeable that the results of KR (talented student) decreased throughout the three tests.

Pattern recognition

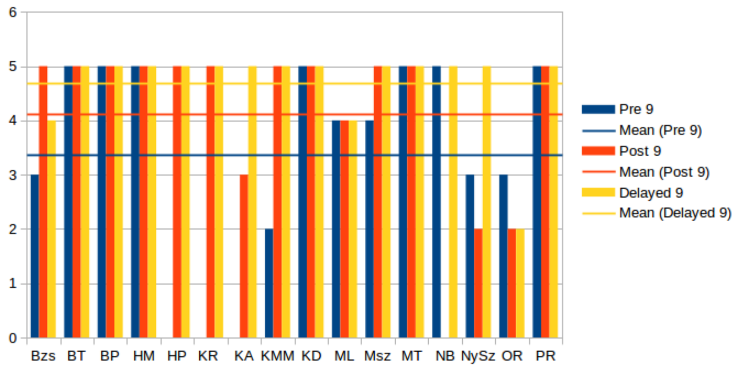


Figure 45: Pattern recognition (pre, post- and delayed tests)

The data shows that there was a steady increase in the mean average points gained for this problem. Seven (pre-test), eleven (post-test) and thirteen (delayed test) students obtained full points for this problem. Both talented students (BP and KR) gained full points on the post- and delayed tests. One of the weaker students, KA, showed a great improvement between the pre-test (no points) and the delayed test (full points).

6.5 Tests at the end of the school year

6.5.1 Attitude to mathematics test - post-experiment

As the attitude to mathematics pre- and post-tests contained exactly the same questions in this section we present only the answers given to the post-test.

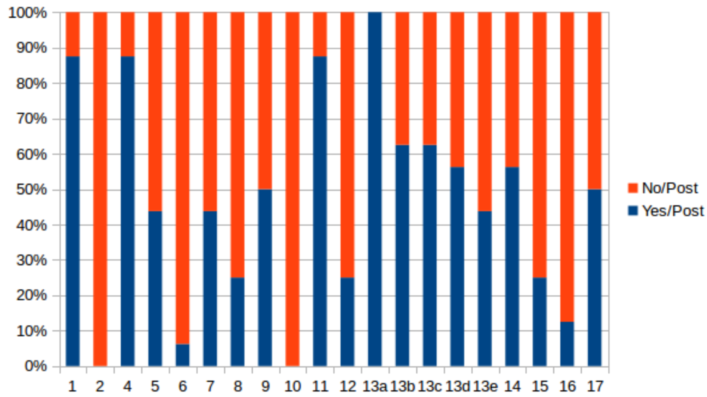


Figure 46: Answers for the post-experiment mathematical attitude test

6.5.2 Interpreting the results

Similarly to the analysis of the pre-experiment attitude test we divided the statements of the post-test into two groups, one of them containing statements that reflect positive attitude to mathematics while the other one containing statements showing negative attitude to the subject. Some statements will be discussed separately due to their nature.

Positive statements

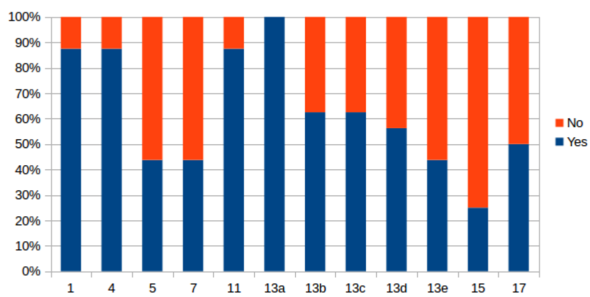


Figure 47: Positive statements

As in the pre-experiment attitude test there were eight questions for which the students gave a positive answer. It is also noticeable that the percentage of students giving a positive answer increased between the two tests. For example, for question 13 a) every student answered with a "Yes". It was assumed at the beginning of the experiment that the group had a positive attitude to mathematics. Based on the graph above it can be said that this attitude improved during the school year.

Negative statements

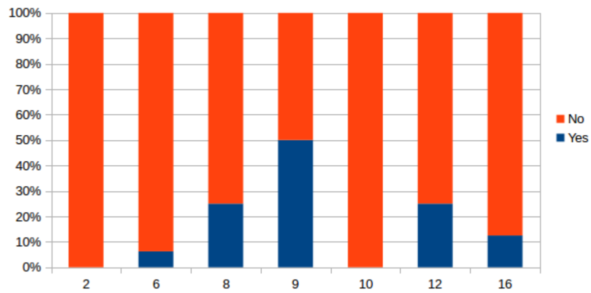


Figure 48: Negative statements

The graph also clearly indicates a positive attitude to mathematics as the number of "No" answers outweighs the number of "Yes" answers again. Furthermore, there are two questions for which each student answered with a "No". There were no students in this group at the end of the school year who would consider mathematics as one of their three least favourite subjects and none of the students in this group were afraid of maths tests. The latter result is very important since knowing mathematics is often not enough in the Hungarian education system but students have to prove their knowledge on different tests (eg: maturity exam, tests at the beginning of their university studies etc.)

14. I'd rather solve 5 equations than one very difficult problem.

As mentioned before this statement refers to the type of mathematical problems students feel confident solving. Comparing the result of the post-experiment test to the pre-test it can be said that there was no significant change in the attitude of the students to solving more challenging mathematical problems. Nine out of sixteen said that they would rather solve equations than a difficult problem. This number is only one less than the number of students preferring equations in the pre-test.

Suggestions listed for questions 20. - 23.

In this section we present some suggestions made by the students. The comments were chosen so that they are meaningful and they reflect general attitude.

- Likes and dislikes

Likes	**Dislikes**
the more interesting problems (more students)	difference between easy and hard topics
exact explanations	group work, sometimes
group work	being in the same group with the "wrong" people
new types of problems	word problems
equations	geometry
the atmosphere of the lessons	
guiding questions	

Table 10: What students like and dislike in mathematics after the experiment

- Suggestions for the teacher: 1) check homework more often
- Suggestions for themselves: 1) practice more (KA, MT, NB, ML, HP, OR, MSZ, NySz, PR, KMM, BZS, HM); 2) more homework (KR)

A conspicuous difference between the Likes-Dislikes table of the pre-experiment test and the post-experiment test is that in the latter the students did not mentions topics but working formats, types of problems, problem solving strategies etc. It seems from the list that the students developed a positive attitude to working in groups and they appreciated solving "unusual" problems. However, for future work it needs to be taken into consideration that unfortunate grouping arrangements can have negative effects on the attitude of the students.

6.5.3 Questionnaire about cooperative learning

The questionnaire

The questionnaire asking about students' attitude to learning in cooperative groups was composed based on a questionnaire in [59]. Students were asked to evaluate the statements from 1 to 6 in the following way: 1 means not true for me and 6 means totally true for me.

Statements	**1**	**2**	**3**	**4**	**5**	**6**
1. The other students' explanations helped.						
2. I liked working together.						
3. The noise was disruptive.						
4. I would have been faster alone.						
5. I enjoy lessons with group work more.						
6. I understood maths better than in the previous year.						
7. I liked talking to people to whom I had never spoken before.						
8. I'm not afraid of maths.						
9. I could explain the solutions to others.						
10. I dare to ask questions from my teacher.						
11. I paid more attention and solved more tasks in maths.						
12. Doing homework is easier now.						
13. I prefer my teacher's explanations.						
14. I understand my classmates' explanations better.						
15. I don't mind sharing my ideas with the whole class.						
16. I prefer sharing my ideas in small groups.						
17. I would like to work in groups in Maths next year.						
18. I prefer doing maths tasks alone.						
19. I am more active when working in groups.						
20. I became more patient with others.						

Table 11: Cooperative questionnaire

The results

- Average scores and standard deviation of individual students

Student	Mean average	Standard deviation
BZS	2.90	1.15
BT	3.86	2.04
BP	4.10	1.98
HM	3.62	1.15
HP	3.52	1.34
KR	3.00	1.63
KA	2.86	1.08
KMM	3.62	1.06
KD	3.90	1.41
ML	2.76	1.55
MSZ	4.43	1.57
MT	3.81	1.21
NB	3.48	1.09
NSZ	2.86	1.41
OR	3.90	1.25
PR	3.24	1.96

Table 12: Results of the cooperative questionnaire

- "Positive" and "Negative" statements

  The questionnaire contained 20 statements. For analysing the students' answers the statements were organized into three groups on the basis of the following aspects: (1) statements related to cooperative work; (2) statements related to attitude towards Mathematics; (3) statements about relationship to the others (students or the teacher).

  Statements related to cooperative work

  "Positive" statements

Statement (+)	1	2	5	7	14	15	16	17	19
Mean average	3.81	4.25	4.25	2.69	2.81	3	4.06	4.5	3.5
Standard deviation	1.28	1.48	1.39	1.30	0.83	1.41	1.39	1.67	1.46

  "Negative" statements

Statement (-)	3	4	13	18
Mean average	3.56	3.31	5.13	3.44
Standard deviation	1.79	1.62	0.81	1.55

Statements related to attitude to Maths

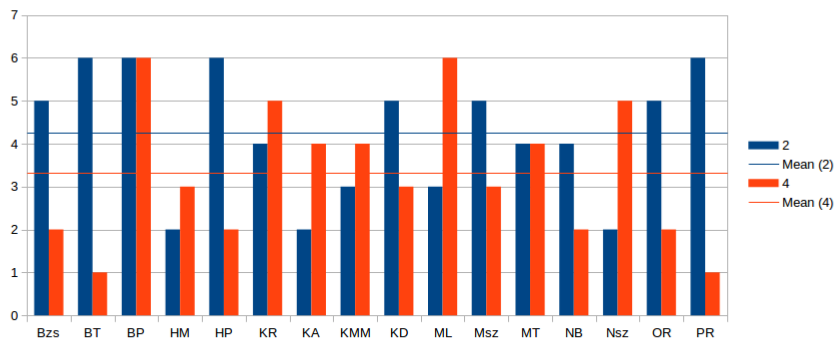
Statement	6	8	9	11	12	16
Mean average	3.56	3	4.0	3.44	3.88	4.06
Standard deviation	1.09	1.37	1.18	1.59	1.82	1.39

Statements related to relationship to others

Statement	10	20
Mean average	3.88	3.19
Standard deviation	1.59	1.72

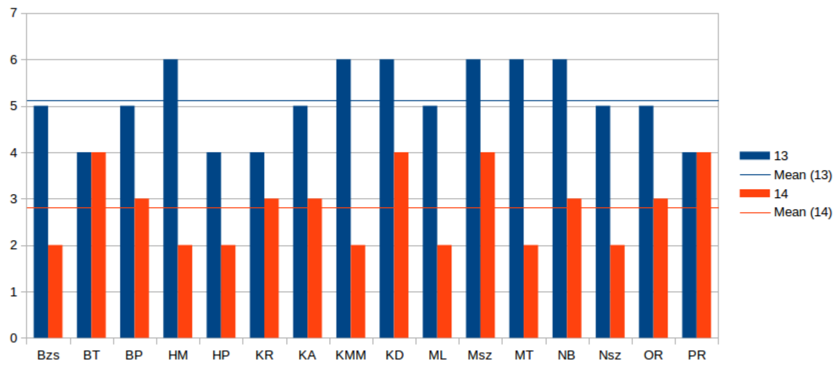
- Comparing some statements

  "I liked working together."(2) and *"I would have been faster alone. (4)"*



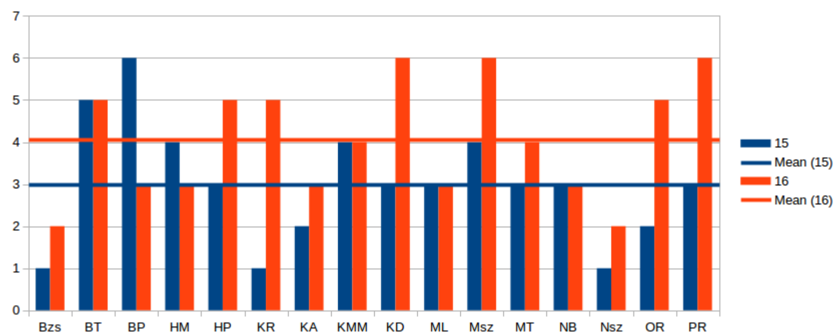
Although eight students out of sixteen agreed more or equally with the first statement, the graph suggests that students tended prefer working in groups to individual work. It is interesting to see that one of the talented students (KR) and also one of the the weakest students in the group (KA) agreed more with the second statement.

"I prefer my teacher's explanations."(13) and *"I understand my classmates' explanations better."(14)*



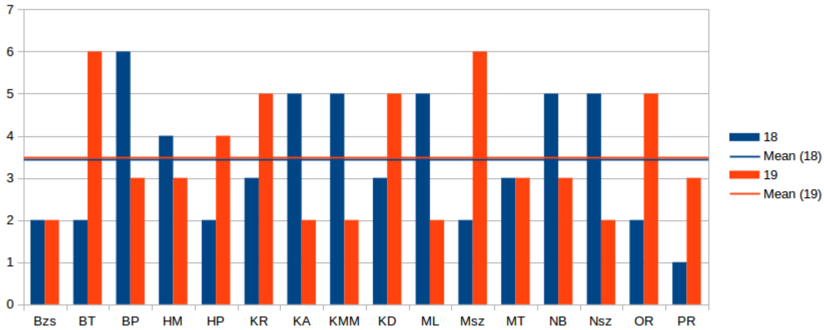
The data appears to confirm that the majority of the students prefers the teacher's explanations to their classmates' explanations.

"I don't mind sharing my ideas with the whole class."(15) and *"I prefer sharing my ideas in small groups."(16)*



Again, it is clearly demonstrated that the majority of the students is more comfortable with speaking in front of a smaller group than with sharing ideas with the whole class.

"I prefer doing maths tasks alone."(18) and *"I am more active when working in groups."(19)*



The graph shows that there are two students who do not mind the working format (BZS and MT, both of them are better average ability students), seven students prefer solving mathematical problems alone (both BP, the talented one and KA, the weaker student belong to this group). The data suggests that working format preference does not depend on the mathematical ability.

6.6 Other aspects

6.6.1 Open problems

Since all the problems used in the first part of the experiment and some used in the second part were open problems we can say that there are several benefits of using open problems in teaching mathematical problem solving.

First of all, open problems give the opportunity for the less able students to get started since if nothing comes to their mind related to the solution they can start experimenting with specific values or cases. Generalization and proving statements have always been a critical point for Hungarian students. When using open problems students can examine the different options which helps them formulate a general statement and helps find an idea for proof. The students taking part in the experiment were rather creative in terms of formulating general statements and giving a vague outline of the proof but they needed the teacher's assistance in presenting their ideas mathematically. Because of the nature of open problems there might be more solutions or the same solution can be expressed in many different ways.

Handling this in class is not always easy and is definitely time consuming. Using cooperative techniques seemed useful as working in groups provided opportunity for the students to be creative, to test their own ideas and to discuss different ways of solution. Taking everything into consideration using open problems contributes well to developing most students' problem solving skills and even if it is time consuming the problems can be discussed in greater detail.

6.6.2 Guiding questions

Using questions when teaching mathematics is an effective tool regardless of whether we introduce a new material or teach problem solving. As mentioned in the theoretical background many researches (e.g. Pólya, Schoenfeld) suggest using questions in problem solving and also in teaching problem solving. Experience shows that in a frontal lesson even if the teacher asks numerous questions students do not remember these questions, however the main idea is to teach students ask these questions from themselves when solving problems or even better if asking these questions becomes an unconscious process.

Combining guiding questions with cooperative techniques gave the students the opportunity to focus more on the nature and the order of these guiding questions. Moreover, some tasks included the students asking their own guiding questions from each other. Learning the technique of effective questioning is a long process but cooperative methods can be used to shorten this process and to draw students attention to the importance of these guiding questions in problem solving.

6.6.3 Talented and Average Ability Students [18]

During the experiment we also had the opportunity to examine the following questions : 1) How do talented/average ability students solve problems in frontal classwork? 2) How do talented/average ability students solve problems in cooperative work? 3) How do they feel about the different scenarios?

Taking the whole school year into consideration the following observations and thoughts can be phrased.

For talented students there was no significant difference between the results in the pre-, post- or delayed Mathematics test. Of course, since their achievement was already quite good on the first test there was not too much area for improvement. However, there were tasks where they showed some improvement, for example in problems that involved permutations one of them had higher total points in the post- and delayed – tests.

On the other hand there was definitely a positive change in their attitude towards the others. They became more patient with the other students when they had to explain something or when they presented their ideas. From the teacher's observations it is clear that they were more and more willing to be "team players". Moreover, there was a positive change in terms of their classroom activity, too.

Some average ability students whose mathematical achievement had never been steady during this experiment achieved better result in some post-test tasks and delayed test points were much better, too. Comparing the post- and the delayed tests there definitely was an improvement in problems that required pattern recognition (9.) or in which permutations had to be applied (4. and 5.). Besides these tests there was an improvement in their school grades as well which might be a result of their personality becoming more mature, too.

As for the attitude of average ability students to cooperative work from the observations it is clear that their self-confidence changed in the positive direction. Some of them became more confident presenters and they needed less reassurance by the end of the school year.

Since using cooperative techniques definitely had a positive impact on each type of student, in my future classes I will use this method to encourage the average ability students to share their mathematical ideas and to help talented students to be more willing to share their knowledge with their classmates.

7 Conclusion - Summary

This thesis examined the effects of using cooperative techniques on secondary school students' problem solving skills, on their attitude to mathematics and on their relationship with fellow students. Moreover, the reaction to solving open problems and the reduction of cognitive load during problem solving were also examined. In this section we will reflect on the different aspects of the experiment described in the thesis: cooperative techniques, open problems and working memory and also answer the research questions. The answers for the research questions will be printed in bold. The following section was published in [11].

During the experiment cooperation allowed students to work in their own pace, slower students had time to understand the task better while faster ones could be given a so called time filler activity – for example in a trigonometry task they had to justify why certain triangles are possible or not. Moreover, working in groups facilitated communication among students and encouraged them to use mathematical language when explaining ideas to each other, furthermore it inspired even the less able students to participate more actively in the problem solving process. This working format contributed to a kind of personal development of the students as well. Many of them became more patient and more tolerant with their classmates. As for the teacher, she had a better opportunity to help students individually and had a better insight into the students' way of thinking and methods of problem solving.

However, applying cooperative techniques was definitely time-consuming. In class fewer problems could be discussed. This working format is noisy which was disruptive for some students (see the questionnaire above) and individual personalities should be taken into consideration too as some students perform better when working alone. Planning a cooperative lesson requires more preparation and more creativity from the teacher and when teaching such a lesson she/he must be in full control of the class because it is the teacher's responsibility to maintain an atmosphere which in spite of being noisy is suitable for efficient work.

The investigation tasks and the other types of **open problems used in the experiment gave the opportunity for the less able students to get started with concrete values or with visual representations thus reducing the "pressure" on the working memory**. In case of problems from the first part of the experiment they could try the calculations with special values and then try to form a general rule or calculate the area of the given arrangements with given values before trying to come up with a general formula for the wasted area. However, students often needed the teacher's assistance in listing all options in a logical order and justifying their ideas for the general rule.

In case of an investigation more solutions can occur or the same solution can be found in different ways. Checking each student's solution and discussing everyone's idea in class takes a lot of time and is difficult to manage. For tackling open problems and investigations cooperative techniques were useful as the discussion of many ideas was much easier in small groups.

As mentioned previously working memory plays a vital role in problem solving. First, it has an important part in understanding the task. **When using cooperative work students** could discuss the original question with each other and they had the opportunity to clarify what needs to be done. Second, when looking for a solution method they **could rely not only on their knowledge but also on the knowledge of the other members of the group**. The following comments from the students' reflection booklets support this statement:

- *"In the solution it helped that each group member had a winning strategy and we could choose the right one from these."*
- *"It was effective to work in groups as everybody had an idea."*
- *"It helped in the solution that we were thinking about it together."*
- *"It made the solution easier that I didn't have to do it alone."*

Moreover, in the checking progress and when noticing occasional mistakes the students could also help each other. All the above mentioned factors **contributed to the extension of the working memory of the participants**. One of the students wrote that working together was more effective than working alone as there was always someone who had an idea or knew how to proceed. The students said that their knowledge was put together this way. As for the cognitive load, when working in groups there were less elements to process simultaneously since the work was shared between the members of the groups.

It was the task of more students to remember facts and figures related to the problem and the groups contained students with mixed mathematical abilities and experience in problem solving. So their cognitive load was reduced. From the voice recordings it was clear that the students often completed each others ideas or new ideas were mentioned based on what one of the group members said. The teacher's observations also support the idea that the students' cognitive load was reduced. In class the students were on task and each students contributed to the solution process.

To sum up, we can state that **cooperative techniques can be considered as effective tools for developing problem solving skills but they should be used alongside and mixed with other methods**. These methods **also contribute to forming a positive attitude to mathematics** and also **aid the development of positive relationship and a more accepting atmosphere among the students**. These techniques also **help reducing the cognitive load of individual students and extending the working memory.** Moreover, **open problems and investigations also contribute well to developing problem solving skills and although they were time-consuming, combined with cooperative teaching techniques they provided an opportunity for discussing fewer problems but in greater detail**. As mentioned before we were looking for methods with which we can reach not only the top 10 – 15 % of our students and we can involve the average students in active learning. Cooperative techniques and open problems were efficient tools for these.

8 Future work

As mentioned above cooperative techniques either on their own or combined with open problems and investigations can be used effectively in teaching mathematical problem solving and developing students problem solving skills. Moreover, these methods contribute well to reducing the cognitive load of individual students which is also important for becoming successful problem solvers.

Because our experiment confirmed many positive aspects of using cooperative teaching techniques and open problems I find it important to include them in my future professional work. There are three areas I would like to concentrate on in the future: 1) dissemination; 2) developing my everyday practice; 3) designing appropriate teaching resources.

Dissemination

First, although cooperative techniques are becoming more and more well-known and wide spread there are still many teachers who either have not heard about these techniques or are not sure how to start implementing them in their own practice. That is why sharing the outcomes and experience related to this experiment is important.

There are several options for sharing ideas with colleagues both in my school and in other school too. First of all, in my school there is a tradition of organizing a two-day event every school year during which interesting lectures and workshops are held related to different scientific or contemporary topics. For the last couple of years a workshop was organised specially for teachers where we had the opportunity to share our ideas and experience related to alternative teaching methods with other colleagues. The positive aspect of these events was that we managed to convince many colleagues to try cooperative techniques in their own classroom and later share their experience with us. However, unfortunately the number of colleagues turning up for the event was rather low which suggests that those who have innovative thinking in terms of teaching is not high. Taking everything into consideration these workshops should be organized again calling the colleagues' attention to the opportunities these discussions offer. The workshops can be expanded and colleagues from other school can be invited either to participate or if they have experience with cooperative techniques then to share ideas and views. Conference presentation and articles are also good tools for spreading how effective cooperative techniques in mathematics educations are.

However, there are some difficulties when it comes to trying to make cooperative methods widely used. Unfortunately, the workload of teachers nowadays is rather high and there are many quite demanding schools. It can be rather difficult to plan a cooperative lesson as it requires more creativity from the teacher's side than planning a frontal lesson. In mathematics individual practice is a vital part of learning that is why some colleagues might be difficult to convince that solving problems in groups can also have a positive effect on the development of problem solving skills.

Applying cooperative techniques in everyday practice

As mentioned above planning a cooperative lesson can be more demanding than planning a frontal lesson since it requires creativity from the teacher; it is time consuming and often the resources need to be created as well. In spite of the difficulties it is worth including cooperative methods now and again in our practice. The outcomes of the experiment described in this thesis support this idea.

Both in the year of the experiment and in the following academic years I had the opportunity to apply cooperative techniques in my other classes. First of all, there was a class where the main difficulty for me as the teacher was to make my students participate actively, so I decided to try to teach a cooperative lesson in that class. The used structure was the *Jigsaw* structure and the lesson had definitely a positive outcome. There was plenty of communication among the students and all of them participated well.

Although for acquiring mathematical thinking skill and problem solving abilities individual practice is unavoidable, there are many topics and thinking skill that can be practised through cooperative work. In the following I provide a list of ideas which I have already used in classes different from the one in which the experiment took place or ideas which I find worth considering for using cooperative techniques.

- Transformations of functions - practice
- Congruence transformations - summary or revision
- Special products - practice, summary
- Differentiation- practice, summary
- Integration - practice, summary
- Calculating the limits of progressions or functions - practice, summary
- Basic geometry (Angles, triangles, quadrilaterals ...) - summary
- Solving higher degree equations - practice

Based on experience making cooperative techniques part of our everyday practice can happen only step by step. First of all, students have to learn how to behave in a cooperative lesson and many of them need to learn how to work together with others. They also need to get used to active participation. Furthermore, in mathematics not every topic and not every lesson is suitable for cooperative work so frontal teaching and individual practice are often vital. So, the teacher has to choose when and how to apply cooperative techniques in mathematics taking one of the results of the experiment into consideration which says that we should not overuse cooperative techniques. Moreover, the workload can be shared out if the teacher plans one cooperative lesson at a time and reuses the resources in another lesson.

However, preparation time and lack of resources are not the only difficulties when it comes to applying cooperative techniques. In Hungarian secondary schools students have to be prepared for the school leaving matura exam in mathematics which means that there is a set material that has to be covered in a given time. As discussed before in a cooperative lesson, although students are usually more active, the number of problems that can be discussed is often less than in a frontal lesson.

Teaching resources

Finally, preparing teaching resources is part of our future plans. The already existing books and task collections can be used for gathering ideas but beside these the teacher's creativity is needed. In planning cooperative mathematics lessons online resources also provide a great help. An advice from Kagan [46] for those who want to start using cooperative techniques is to choose one or two structures and keep using them in different scenarios until you feel confident with the chosen structure. Following this you can experiment with a new structure thus enriching your methodological toolbox step by step.

For helping future work with cooperative structures worksheets that fit cooperative lessons can be designed and mathematical problems can be altered so that they can be discussed in group work. Already used mathematical problems can be changed so that they become open problems.

Finally, it is important to emphasize that the aim of using cooperative techniques and open problems is not to provide fun mathematics for the students. All resources and each lesson planned to support the methods will be strictly curriculum based.

9 Összefoglalás

A disszertáció azt vizsgálta, hogy milyen hatással van a kooperatív tanulás-szervezési technikák rendszeres használata középiskolás tanulók problémamegoldó készségeire, a matematikához való hozzáállásukra és tanulótársaikkal való kapcsolataikra. Ezenkívül azt is megfigyeltük, hogy a nyílt problémák megoldása milyen reakciót vált ki a tanulókból, illetve hogy csökken-e a tanulók kognitív terhelése. A következő részben a disszertációban leírt kísérlet eredményeire reflektálunk kiemelve a kooperatív technikák hatásait, a nyílt problémákat és a munkamemóriát és választ adunk a kutatási kérdésekre. A kutatási kérdésekhez tartozó válaszok a nyomtatásban vastagon szedve jelennek meg. Az elemezés angol nyelven a LUMAT [11] nevű online folyóiratban jelent meg.

A kísérlet során a kooperatív technikák használata minden tanuló számára lehetővé tette, hogy a saját ütemében dolgozzon. A lassabb tanulóknak volt ideje alaposan átgondolni és megérteni a feladatot, ugyanakkor a gyorsabbak úgy nevezett időkitöltő feladatokat oldhattak meg ha hamarabb elkészültek a megoldással. Például egy trigonometria feladatban a gyorsabbaknak meg is kellett indokolni, hogy bizonyos háromszögek megrajzolása miért lehetséges vagy lehetetlen. Továbbá a csoportmunka elősegítette a diákok közötti kommunikációt és bátorította őket a matematikai nyelv használatára, amikor egymásnak magyaráztak. Mindezeken felül még a kevésbé tehetséges tanulókat is a problémamegoldó folyamatban való aktív részvételre inspirálta. Ez a munkaforma hozzájárult a tanulók személyiségének fejlődéséhez is. Sokan türelmesebbé és elfogadóvá váltak osztálytársaikkal szemben. A tanárnak pedig sokkal több lehetősége nyílt arra, hogy a tanulóknak egyenként segítsen, ezáltal jobban beleláthatott abba, hogy hogyan gondolkodnak és oldanak meg problémákat az egyes tanulók.

Mindezek mellett a kooperatív technikák használatának voltak hátrányai is. Elsősorban a módszerek használata igen időigényes. Az órán kevesebb feladat megbeszélésére van lehetőség. Ez a munkaforma zajos, ami néhány tanuló számára zavarónak bizonyult (ld. kooperatív kérdőív eredményei) és a tanulók személyiségét is figyelembe kell venni, hiszen sokan jobban teljesítenek, ha egyedül dolgozhatnak. Egy kooperatív technikákkal tartott óra megtervezése is időigényes és a tanár részéről nagyobb kreativitást igényel. Egy ilyen óra tanítása során a tanárnak teljes mértékben kontrollálnia kell az osztályt, hiszen az ő feladata egy olyan légkör biztosítása, ami zajos ugyan, de még hatékony munkavégzésre alkalmas.

A vizsgálódással megoldható és az egyéb típusú **nyílt problémák, amiket a kísérletben használtunk lehetővé tették a kevésbé tehetséges tanulók számára, hogy konkrét értékek kipróbálásával vagy vizuális**

reprezentációkkal kezdjenek hozzá a problémák megoldásához így csökkentve a munkamemóriájukra "nehezedő nyomást" A kísérlet első felében használt problémák esetében konkrét értékekkel próbálhatták ki a számításokat és ezt követően megpróbálhattak egy általános szabályt megfogalmazni vagy konkrét alakzatok területét tudták kiszámolni mielőtt egy általános képletet írtak volna fel a kérdéses terület kiszámítására. Mindezek mellett a tanulóknak gyakran volt szükségük tanári segítségre az összes lehetőség logikus felsorolásában, illetve az általuk felírt általános szabály matematikai bizonyításában.

Egy vizsgálódással megoldható feladat esetében több megoldás és több megoldáshoz vezető út is felmerülhet. Minden egyes tanuló megoldását ellenőrizni és mindenki ötletét meghallgatni egy hagyományos tanítási óra keretein belül lehetetlen. Ezért a nyílt problémák megvitatását kooperatív technikák segítségével terveztük. Kis csoportokban sokkal egyszerűbb volt ellenőrizni és megvitatni a problémák megoldásának menetét.

Mint említettük, a munkamemória szerepe igen jelentős a problémamegoldásban. Elsősorban a feladat megértésében van fontos része. **Kooperatív technikák alkalmazásakor a diákok** meg tudták beszélni egymással az eredeti problémában felmerülő kérdéseket és lehetőségük volt tisztázni, hogy mit kér a feladat. Másrészt a megoldás keresésekor a tanulók **nemcsak a saját tudásukra hagyatkoztak, hanem a csoport többi tagjának tudása is rendelkezésükre állt**. Továbbá a tanulók segítették egymást a megoldás ellenőrzésében és az esetleges hibák felfedezésében. Az imént említett tényezők **a résztvevők munkamemóriájának kibővítéséhez járultak hozzá.** Az egyik tanuló le is írta, hogy együtt dolgozni hatékonyabb volt mint egyedül, mivel a csoportban mindig volt valakinek ötlete a továbbhaladást illetően. A tanuló szerint ilyen módon a csoporttagok tudása összeadódott. Ami a kognitív terhelést illeti, csoportmunkában kevesebb dologra kell egyszerre odafigyelni, mert a munkát a csoporttagok elosztják egymás között. Több tanuló feladata volt, hogy észben tartson bizonyos számú adatot és képletet, ami az adott probléma megoldásához szükséges. Ezenkívül a csoportok különböző matematikai képességekkel és problémamegoldási tapasztalattal rendelkező tanulókból álltak. Tehát az egyes tanulók kognitív terhelése csökkent.

Mindent egybevetve úgy találtuk, hogy **a kooperatív technikák hatékony eszközök a problémamegoldó készségek fejlesztésére, de mindenképpen más tanulásszervezési módszerekkel együtt kell őket használni**. Ezek a módszerek **szintén hozzájárulnak ahhoz, hogy a tanulók matematikához való hozzáállása pozitív legyen**, továbbá **segítik a tanulók közötti pozitív kapcsolatok kialakulását**. Összességében **a nyílt problémák és a vizsgálódással megoldható feladatok nagymértékben hozzájárultak a problémamegoldó készségek fejlesztéséhez és annak ellenére, hogy megoldásuk időigényes volt, kooperatív technikákkal**

együtt alkalmazva őket, az egyes problémákat sokkal részletesebben meg tudtuk beszélni. Mint említettük, egy olyan módszert kerestünk, amivel nem csak a tanulók legjobbjait (kb. 15-20 %) tudjuk bevonni a problémamegoldó gondolkodás tanulásába, hanem az átlagos képességű tanulókat is. A kooperatív technikák és a nyílt problémák hasznos eszközöknek bizonyultak ehhez.

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10 Appendix 1 - Lesson plans form the second part of the experiment

- Quadratic equations

  Pairs Check As mentioned before in this structure two students - student A and student B - work together. One of them is the "coach", he only checks the work of the other student or, if it is necessary, he gives advice on how to carry on. The second student has to write everything down while explaining aloud what he is doing.[47]

  Student A

  Solve the following equations on the set of real numbers. Check your answers.

  1. $3x^2 + 6x = 8x^2 - 9x$
  2. $(2x + 2)(x - 1) = 5x + 6$
  3. $\frac{6x+8}{x^2-4} - \frac{x-2}{x+2} = \frac{x+2}{x-2}$

  Student B

  Solve the following equations on the set of real numbers. Check your answers.

  1. $\frac{3x^2-11}{4} + \frac{74-2x^2}{6} = 20$
  2. $x(2x + 3 = -12x - 6)$
  3. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{64}{x^2-16}$

- Word problems (2 lessons)

  Structures used: *Think-pair-share, Group discussion, Jigsaw*

  Four groups were given two word problems each one of which was easier the other one a bit more challenging. The different groups received problems of four different types:

  Geometry problems

  The perimeter of a rectangle is 40 cm. The sum of the areas of the squares written on the sides of the rectangle is 208 cm^2. Find the sides of the rectangle.

  Distance-Time-Velocity problems

  Two towns along the river are 240 km apart. A return journey of a ship takes 25 hours. Find the speed of the ship in still water if the speed of the current is 4 $\frac{km}{h}$.

Mixing problems

A sulphuric acid solution contains 0.8 kg pure sulphuric acid, another contains 0.6 kg pure sulphuric acid. When mixing the two solutions we obtain 10 kg solution. Find the weight of the first and the second solution if you know that the concentration of the sulphuric acid in the first one was 10% more than in the second one.

"Work" problems

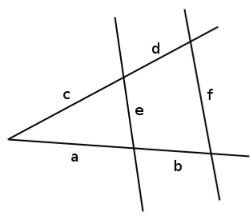
It takes 6 hours for two taps to fill a pool. The first tap needs 5 hours less to fill the pool than the second. How much time do each taps need to fill the pool on their own?

- Parallel intersecting lines

  Structures used: *Think-pair-share, Group discussion,*

  Each group had the following problems to work with:

  1. The arms of an angle were intersected by parallel lines as can be seen on the figure below. Fill in the table with the correct measures.[29]



a	7 cm	10 cm	
b	4 cm		5 cm
c	9 cm	11 cm	
d		6 cm	4 cm
e	6 cm	8 cm	7 cm
f			10 cm

  2. A 50 meters wide football pitch is surrounded by a 2 meters tall fence. There is a ten-storey building 500 meters from the fence each of whose floors are 3 meters high. From which floor can the football pitch be seen? (can be considered as an open problem)[29]

- Similarity

  Structures used: *Think-pair-share, Group discussion, Jigsaw*

  Task 1: Work out the area of the following shapes:

1. a square with 6-cm long sides
2. an equilateral triangle with 10-cm long sides
3. a rectangle with 7-dm and 9-dm long sides
4. a symmetrical trapezium with bases 10 cm and 5 cm and height 6 cm
5. a rhombus with 5-cm and 9-cm long diagonals

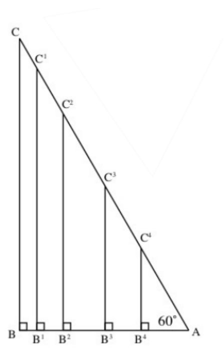
Task 2: Enlarge the shape using the scale factor X and calculate the area of the new shapes. (NB: The value of X was different in the four different groups.) Summarize your results in a table and calculate the ratio of the areas of the new shapes to the areas of the old ones. What do you notice?

- Introducing trigonometric ratios

  Structures used: *Jigsaw*. As mentioned above the main idea of this structure is that every group is an expert in a topic or a task. They are given some time to prepare - either collect ideas or solve a task - then new groups are formed in a way that each new group contains one person from the original groups. As a result of this the new groups contain students who are experts in each task. In the new groups students share their topics with each other. Notes are made and comments are discussed.[47]

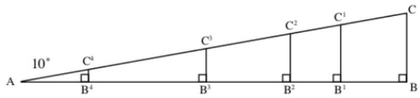
  Group 1 - The tangent ratio

Angle	$\frac{BC}{AB}$	$\frac{B^1C^1}{AB^1}$	$\frac{B^2C^2}{AB^2}$	$\frac{B^3C^3}{AB^3}$	$\frac{B^4C^4}{AB^4}$	Average ratio
Ratio						



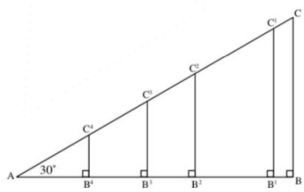
Group 2 - The sine ratio

Angle	$\frac{BC}{CA}$	$\frac{B^1C^1}{C^1A}$	$\frac{B^2C^2}{C^2A}$	$\frac{B^3C^3}{C^3A}$	$\frac{B^4C^4}{C^4A}$	Average ratio
Ratio						



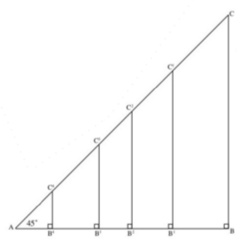
Group 3 - The cosine ratio

Angle	$\frac{AB}{CA}$	$\frac{AB^1}{C^1A}$	$\frac{AB^2}{C^2A}$	$\frac{AB^3}{C^3A}$	$\frac{AB^4}{C^4A}$	Average ratio
Ratio						



Group 4 - The cotangent ratio

Angle	$\frac{AB}{CB}$	$\frac{AB^1}{C^1B^1}$	$\frac{AB^2}{C^2B^2}$	$\frac{AB^3}{C^3B^3}$	$\frac{AB^4}{C^4B^4}$	Average ratio
Ratio						



- Trigonometry

  Structures used: *Pairs Check, Group discussion*

  Student A

1. The diagonal of a rectangle is 14,3 cm long. This diagonal makes a $23,2^\circ$ angle with one of the sides. Find the sides of the rectangle.
2. There is a poplar tree in the middle of a clearing. We stand 32 m from the tree and from a 1,7 m height the tree can be seen at 33°. Find the height of the tree.

Student B

1. The shadow of a tower is 42,5 m long. The rays of the sun make a $38,6^\circ$ angle with the ground. Find the height of the tower.
2. There is a chapel on the top of a small hill. The straight road leading to the chapel from the bottom of the hill is 120 m long and its angle of elevation is 15°. From the starting point of the road the chapel can be seen at 7°. Find the height of the chapel.

- Trigonometry, similarity

  Structures used: *Jigsaw Problem Solving*

  The travelling salesman: Each group receives 12 cards which need to be shared out equally between the team members. Each card contains different instructions. The task is to draw the journey of a travelling salesman and calculate the total distance he covers. The students are not allowed to show the instructions to each other only read them out.

  Sample cards:

  1. Dick is a travelling Dictionary Salesman who is about to retire and he is showing Harry, his replacement, his territory in Trigland.
  2. On the 1 : 250 000 map the distance from Dick's home to Axisminster measures 8 cm.
  3. All towns in Trigland are joined by perfectly straight roads.
  4. Bodmastown lies 40 km East and 30 km South of Axisminster
  5. From Dividingham they return home by the shortest route.

- Generalising trigonometric ratios

  Structures used: *Jigsaw*

  Given two segments and the trigonometric ratio of an angle. Are there any triangles whose sides are the segments and whose angle belongs to the given trigonometric ratio? Try to find and draw all possible triangles.

	a	b	the sine of the angle
First group	8 cm	6 cm	0,8
Second group	7 cm	5 cm	0,4
Third group	10 cm	7 cm	0,2
Fourth group	9 cm	6 cm	0,7

- Combinatorics

  Structures used: *Jigsaw*

  Group 1 - Permutations without repetition

  1. Anna, Bea, Csilla and Dóra go to the cinema together. How many different ways are there in which they can sit next to each other? How many different arrangements are there if Anna and Bea want to sit next to each other? How many arrangements are there if Csilla does not want to sit next to Dóra?
  2. How many different 6 digit numbers can be formed using the digits 1, 3, 2, 4, 5, 9? Each digit can be used only once. How many of these numbers are even? How many is divisible by 12?

  Group 2 - Permutations with repetition

  1. How many different 4 digit numbers can be formed using the digits 1, 1, 2, 4? Each digit can be used only one. How many of these numbers are even?
  2. How many different "words" can be formed using the letters of CURRICULUM? How many of these start and end in a letter C?

  Group 3 - Combinations without repetition

  1. From a deck of French cards (contains 52 cards) in how many different ways can we choose 5 cards? In how many different ways can we choose 5 cards so that 2 of them are red and 3 are black? In how many different ways can we choose 5 cards so that one of them is a king?
  2. In a box there are 100 CD players 10% of which is faulty. In how many different ways can we choose 8 players so that a) all of them are good; b) 3 of them are faulty; c) at most one of them is faulty?

  Group 4 - Variations without and with repetition

1. Tamás wants to buy ice cream. He has the following options: chocolate, lemon, vanilla, strawberry and melon. In how many different ways can he choose 3 scoops (the order of the scoops does matter) if a) each scoop is different; b) the scoops can be the same flavour?
2. In how many different ways can you fill in a football pool?

- Probability - problems with guiding questions - 2 blocks of lessons

  Structures used: *Think-pair-share, Pairs Check*

  First block

  1. We toss three coins. What is the probability of all three landing on the same side?
  2. A science test contains 5 multiple choice questions, each with 4 options. The number of your correct answers is your grade (Hungary: five grade system). If all your answers are wrong you still get a one. As you didn't prepare for the test, your answers are random. What is the probability that you get a five? What is the probability that you don't get a one?
  3. In a box there are 20 pieces of 40 watt light bulbs and 30 pieces of 60 watt light bulbs. If we choose two bulbs at random what is the probability that *a)* both of them are 60 watt bulbs; *b)* the bulbs are the same; *c)* the bulbs are different?
  4. How many 5 digit numbers can we make using the digits 0, 1, 2, 3, 4, 5, 6, 7 if we use each digit only once? What is the probability that we obtain an even number?

  Second block

  1. There are 9 guest sitting around a table. Everybody orders something to drink. 3 people ask for beer, 4 for red wine and 2 for white wine. If the waiter distributes the drinks at random what is the probability that everyone gets what he ordered?
  2. Two siblings are students in the same class of 27 students. If they stay in single file in a random order what is the probability that exactly 10 students stand between the siblings? How does the answer change if they stand in a circle?
  3. Knight Chevaler de Méré asked the following question from Pascal: If you throw a fair six-sided dice four times what is the probability that you obtain at least one 6?

4. Out of 100 apples 10 is maggoty. If we choose five apples at random what is the probability that at least one of them is maggoty?
5. In a box there are 3 red, 3 white and 3 green balls. If we choose six balls at random what is the probability that we have at least one from each colour?
6. There are 11 counters in a box numbered from 1 to 11. We choose six counters without replacement. What is the probability that the sum of the chosen numbers is odd?

11 Appendix 2 - Students' work

1. Matchstick game

    - "Four is the magic number and we try with different solutions to reach that our opponent has 4 at the end. For example: 24, 20, 16, 12, 8, 4."
    - "It is important that we leave 5 for our opponent. If we start we need to draw two, so 25 is left. Then we have to aim for the numbers 21, 17, 13, 9, 5."
    - "You have to complete the number of matchsticks drawn by your opponent to a number divisible by four."
    - "We start with taking three. After half of the matchsticks have been drawn we remove them one by one and try to leave the last four for our opponent."
    - "It does not matter who starts. For winning the game you need to leave 4 matchsticks for the other person. Before that you have to leave more than 7 but less than 11 for your opponent."

2. Number magic

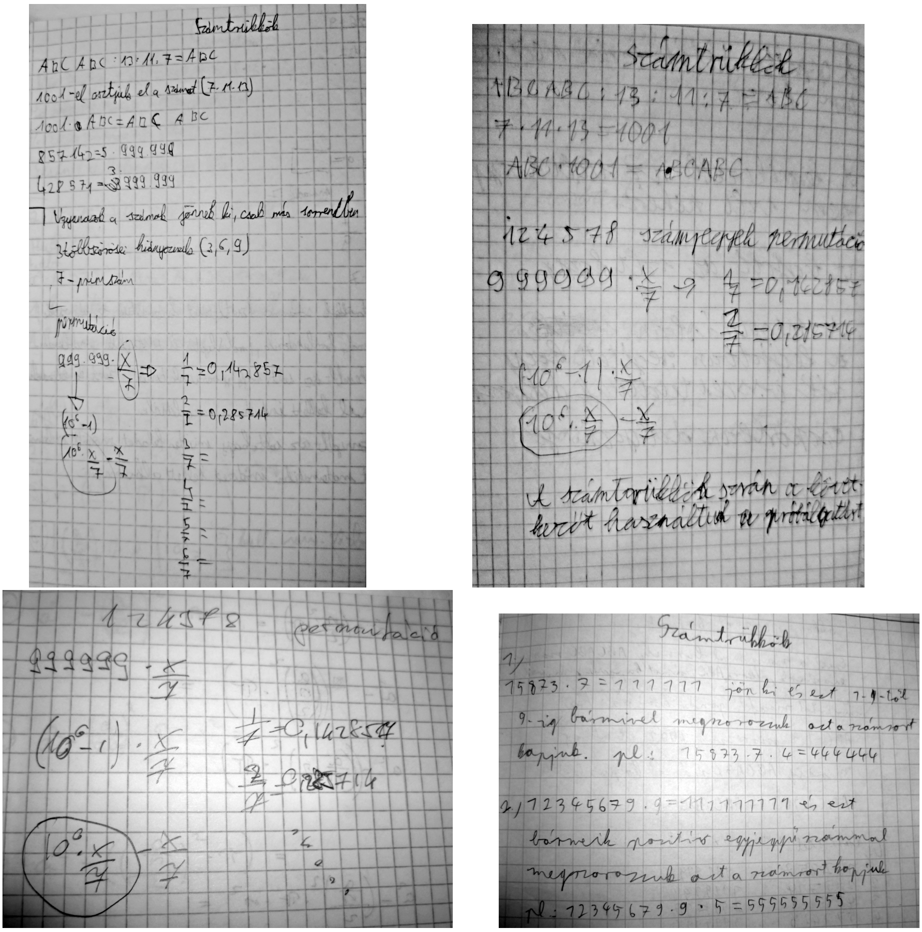
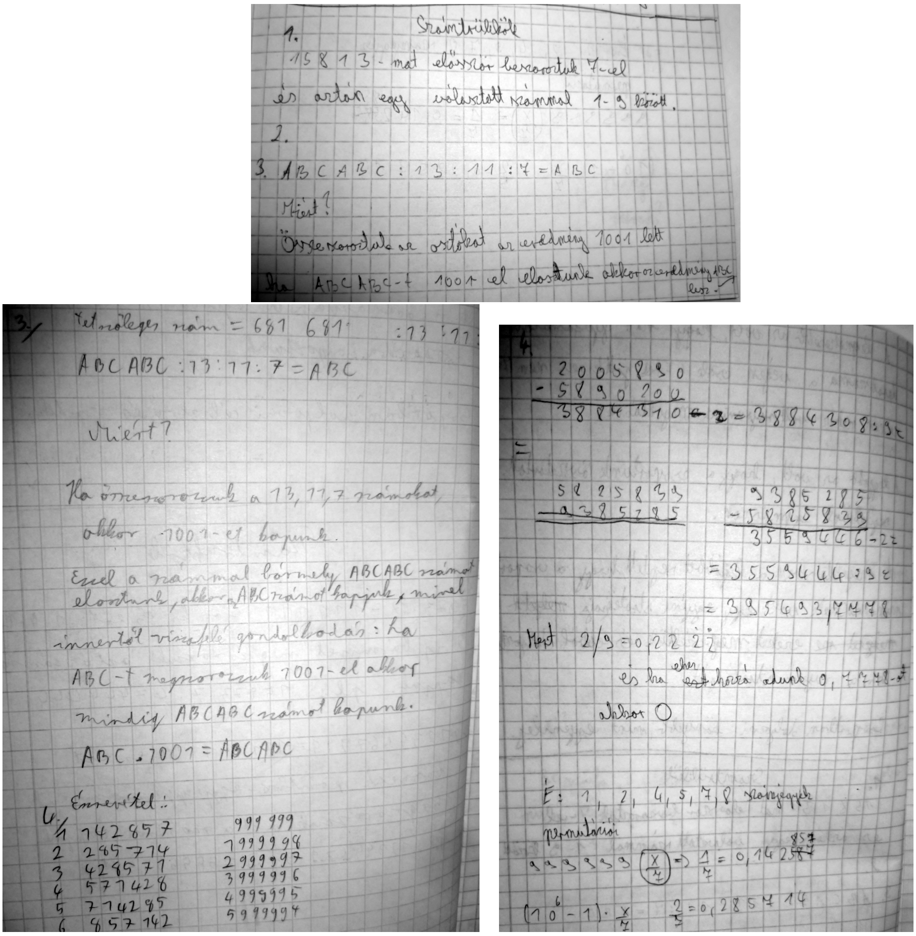


Figure 49: Solutions of the problems from the *Number magic* problem field I.



Számtrükkök
1.
15813-mat először beszoroztuk 7-el
és aztán egy választott számmal 1-9 között.
2.
3. ABCABC : 13 : 11 : 7 = ABC
Miért?
Összeszoroztuk az osztókat az eredmény 1001 lett
Ha ABCABC-t 1001-el elosztunk akkor az eredmény ABC lesz.

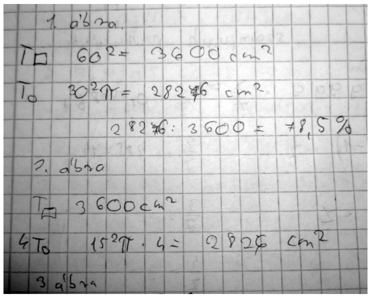
3./ Tetszőleges szám = 681 681 :13 :11
ABCABC : 13 : 11 : 7 = ABC
Miért?
Ha összeszorozzuk a 13, 11, 7 számokat akkor 1001-et kapunk.
Ezzel a számmal bármely ABCABC számot elosztunk, akkor ABC számot kapjuk, mivel innentől visszafelé gondolkodás: ha ABC-t megszorozzuk 1001-el akkor mindig ABCABC számot kapunk.
ABC · 1001 = ABCABC

4/
1 142857 999999
2 285714 1999998
3 428571 2999997
4 571428 3999996
5 714285 4999995
6 857142 5999994

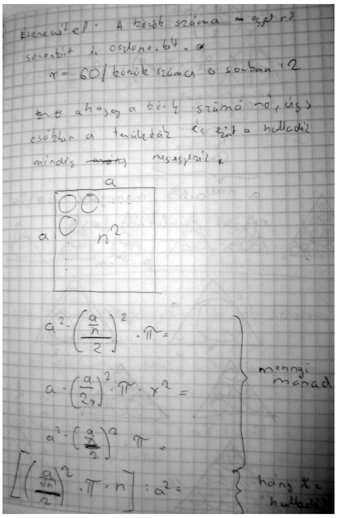
4.
2005890
− 5890200
3884310

Figure 50: Solutions of the problems from the *Number magic* problem field II.

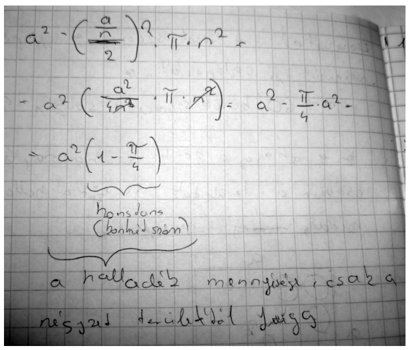
3. Area investigation



(a) Trying with exact values



(b) Creating a general formula



(c) Simplifying the formula

Figure 51: Solutions of the problems from the *Area investigation* problem field

4. More beads

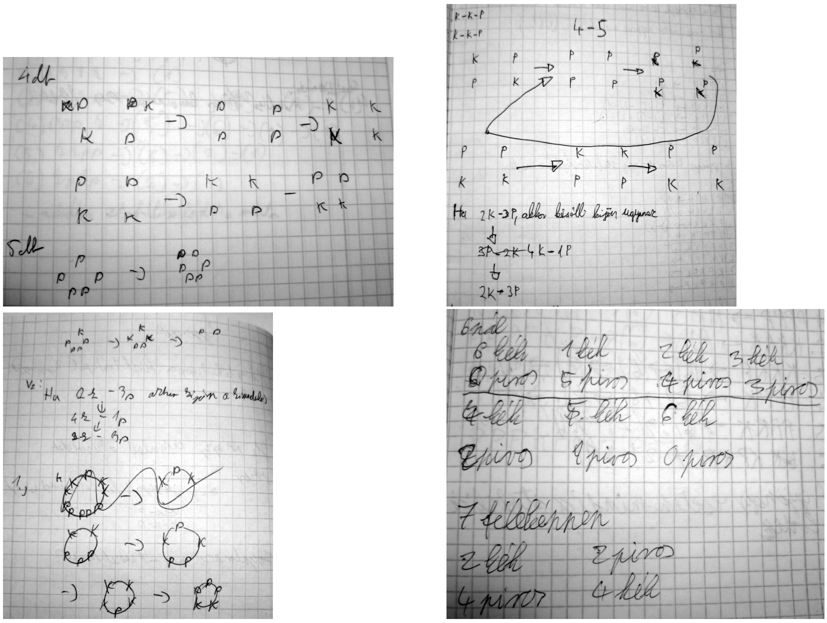


Figure 52: Solutions of the problems from the *More beads* problem field

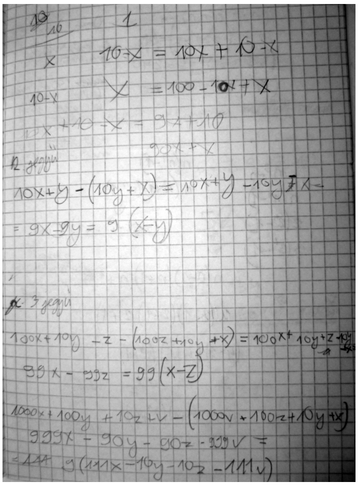


Figure 53: Solution of a problem from the *Primes and factors* problem field

12 Appendix 3 - Pictures

Figure 54: Students working in groups I.

Figure 55: Students working in groups II.

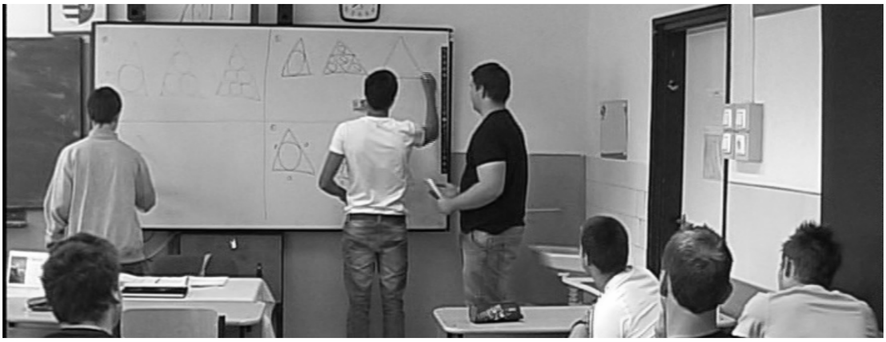


Figure 56: Students working on the board - More beads problem

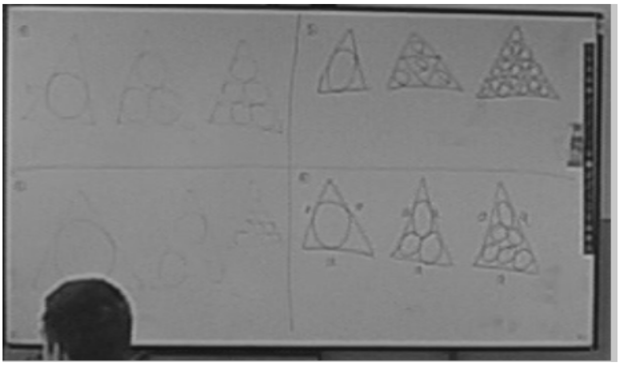


Figure 57: Ideas for modifying the Area investigation

13 Appendix 4 - Publikációk

Az értekezés alapjául szolgáló közlemények

Idegen nyelvű, hazai könyvrészlet(ek) (1)

Barczi, K.: How do they solve problems? Mathematical problem solving of the average and the talented.
In Problem Solving in Mathematics Education: Proceedings of the 15th ProMath conference 30 August - 1 September, 2013 in Eger. Ed.: András Ambrus, Éva Vásárhelyi. Eötvös Loránd University, Faculty of Science, Institute of Mathematics, Budapest, 18-34, 2014.

Idegen nyelvű, külföldi könyvrészlet(ek) (2)

Ambrus, A., **Barczi-Veres, K.**: Teaching mathematical problem solving in Hungary for students who have average ability in maths.
In: Posing and Solving Mathematical Problems: Advances and Perspectives. Patricio Felmer, Erkki Pehkonen, Jeremy Kilpatrick, Springer, New York, 137-156, 2016.

Barczi, K.: A study on how Hungarian students solve problems that are unusual for them.
In: Handbook of Mathematics Teaching Improvement: Professional Practices that Address PISA, Ed.: Stefan Turnau, University of Rzeszow, Rzeszow, 119-130, 2008. ISBN 9788373383944

Magyar nyelvű tudományos közlemény(ek) hazai folyóiratban (1)

Barczi K.: Nyílt végű és vizsgálódással megoldható feladatok a matematika órán.
Mat. tan. (Szeged). 2, 19-24, 2013. ISSN: 1216-6650

Idegen nyelvű tudományos közlemény(ek) külföldi folyóiratban (3)

Ambrus, A., **Barczi-Veres, K.**: Using open problems and cooperative methods in mathematics education.

LUMAT: Research and Practice in Math, Science and Technology Education 3, 3-18, 2015. ISSN 2323-71122

Ambrus, A., **Barczi-Veres, K.**: Opening Problems Is One Step Forward to Reach More Students.
Journal of Mathematical Sciences 1, 37-45, 2014. ISSN 2372-5214

Barczi, K.: Applying cooperative techniques in teaching problem solving.
Center for Educational Policy Studies Journal 3, 61-78, 2013. ISSN 1855-9719

További közlemények

Idegen nyelvű, külföldi könyvrészlet(ek) (1)

Nagy, B., **Barczi, K.**: Isoperimetrically optimal Polygons in the Triangular Grid.
In: Combinatorial Image Analysis: 14th International Workshop IWCIA 2011, Madrid, Spain, May 23 - 25, 2011. Proceedings. Eds.: K. Aggarwal, Reneta P. Barneva, Valentin E. Brimkov, Kostadin E. Koroutchev, Elka R. Korutcheva, Springer, Berlin, 194-207, 2011. ISBN 9783642210723.

Idegen nyelvű közlemény(ek) külföldi folyóiratban (1)

Nagy, B., **Barczi, K.**: Isoperimetrically Optimal Polygons in the Triangular Grid with Jordan-type Neighbourhood on the Boundary.
International Journal of Computer Mathematics 90 (8), 1629-1652, 2013. ISSN 0020-7160

14 Appendix 5 - Előadások

1. Third Annual PDTR Conference. Siedlce. 2008. August. Problems in real-life context based on PISA experinece.

2. Varga Tamás Módszertani Napok. ELTE Budapes. 2009. november. Teaching Mathematics in English in Hungarian Classrooms.

3. Matematika és Informatikai Didaktikai Konferencia (MIDK). Debrecen. 2010. A jelenkori angol és magyar matematika oktatás összehasonlítása.

4. Matematika és Informatikai Didaktikai Konferencia (MIDK). Lőcse. 2012. Matematikában tehetséges tanulók és a kooperatív technikák használata – egy tervezett kísérlet.

5. Problem Solving in Mathematics Education (ProMath) Conference. Ljubljana. 2012. Some Problems in Teaching Mathematical Problem Solving.

6. Matematika és Informatikai Didaktikai Konferencia (MIDK). Nagyvárad. 2013. Irányító kérdések alkalmazása a matematikaoktatásban.

7. Problem Solving in Mathematics Education (ProMath) Conference. Eger. 2013. How do they solve problems? Comparing the mathematical problem solving of the talented and the average.

8. Varga Tamás Módszertani Napok. ELTE Budapes. 2013. november. Solving Problems – Together.

9. Matematika és Informatikai Didaktikai Konferencia (MIDK). Eger. 2014. Tanulói magyarázatok.